ON SOME EQUATIONS RELATED TO DERIVATIONS IN RINGS

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Let m and n be positive integers with $m+n\neq 0$, and let R be an $(m+n+2)!$-torsion free semiprime ring with identity element. Suppose there exists an additive mapping $D: R\to R$, such that $D(x^{m+n+1})=(m+n+1)x^mD(x)x^n$ is fulfilled for all $x\in R$, then D is a derivation which maps R into its center.

Throughout this paper, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n>1$ is an integer, in case $nx=0$, $x\in R$ implies $x=0$. As usual the commutator $xy-yx$ will be denoted by $[x,y]$. We will use basic commutator identities $[xy,z]=[x,z]y+x[y,z]$ and $[x,yz]=[x,y]z+y[x,z]$. Recall that a ring R is prime if $aRb=(0)$ implies that either $a=0$ or $b=0$, and is semiprime if $aRa=(0)$ implies $a=0$. An additive mapping $D:R\to R$ is called a derivation if $D(xy)=D(x)y+xD(y)$ for all pairs $x,y\in R$, and is called a Jordan derivation in case $D(x^2)=D(x)x+xD(x)$ for all $x\in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [11, Theorem 3.1] asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation (see [7] for an alternative proof). Cusack [9, Corollary 5] has generalized Herstein's theorem to 2-torsion free semiprime rings (see [4] for an alternative proof). A mapping F of a ring R into itself is called commuting (centralizing) on R in case $[F(x),x]=0$ ($[F(x),x]\in Z(R)$) holds for all $x\in R$. The theory of commuting and centralizing mappings was initiated by a result of Posner [12, Theorem 2] (Posner's second theorem), which states that the existence of a nonzero centralizing derivation $D:R\to R$, where R is a prime ring, forces the ring to be commutative.

Vukman has proved the following result.

Theorem 1 [13, Theorem 3]. *Let R be a 2- and 3-torsion free noncommutative prime ring with identity element, and let $D:R\to R$ be an additive mapping such that $D(x^3)=3xD(x)x$ holds for all $x\in R$. In this case $D=0$.*

Let us point out that any commuting derivation on an arbitrary ring satisfies the relation $D(x^3)=3xD(x)x$. Theorem 1 was the motivation for the result.

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THEOREM 2. *For integers m, n with $m \geq 0$, $n \geq 0$, and $m+n \neq 0$, let R be an $(m+n+2)!$-torsion free semiprime ring with identity element. Suppose there exists an additive mapping $D : R \to R$, such that $D(x^{m+n+1}) = (m+n+1)x^m D(x)x^n$ is fulfilled for all $x \in R$. In this case, D is a derivation, which maps R into its center. In case R is a noncommutative prime ring, we have $D = 0$.*

In case $m = 1$, $n = 0$ (we adopt the convention $x^0 = e$, for all $x \in R$, where e denotes the identity element), we have an additive mapping satisfying the relation $D(x^2) = 2xD(x)$, $x \in R$. Such mappings are called left Jordan derivations (see [8, 10, 15]). Brešar and Vukman [8, Corollary1.3] have proved that the existence of a nonzero Jordan derivation on a 2- and 3-torsion free prime ring forces the ring to be commutative. For the proof of Theorem 2, we need Theorem 4, which is of independent interest. For the proof of Theorem 4 the lemma below will be needed. We refer the reader to [3] for the definitions and an account of the theory of the extended centroid and central closure as well as related topics and to [6] for an introductory survey on functional identities.

LEMMA 3. *Let R be a 2-torsion free prime ring and let A be its central closure. Suppose that an additive mapping $F : R \to A$ satisfies $[[F(x),x],x] = 0$ for all $x \in R$. Then, $[F(x),x] = 0$ holds for all $x \in R$.*

Proof. In the case when F maps into R, the lemma was first proved by Brešar in [5, Theorem 2]. Fortunately, the same proof works in the case when F maps into A (on the other hand, see, e.g., [2] for a more general result). □

THEOREM 4. *Let R be a 2-torsion free semiprime ring. Suppose that an additive mapping $F : R \to R$ satisfies $[[F(x),x],x] = 0$ for all $x \in R$. Then, $[F(x),x] = 0$ holds for all $x \in R$.*

Proof. Since R is semiprime, there exists a family of prime ideals $\{P_\alpha;\ \alpha \in A\}$ such that $\cap_\alpha P_\alpha = (0)$. Moreover, without loss of generality, we may assume that the prime rings $R_\alpha = R/P_\alpha$ are 2-torsion free (see, e.g., [1, page 459]). Now fix some $P = P_\alpha$, $\alpha \in A$. The theorem will be proved by showing that $[F(x),x] \in P$ for every $x \in R$. Given $x \in R$, we will write $\overline{x}$ for the coset $x + P \in R/P$. By C, we denote the extended centroid of the prime ring R/P, and by A the central closure of R/P . One can consider A as a vector space over the field C. Since C can be regarded as a subspace of A, there exists a subspace B of A such that $A = B + C$. We denote by π the canonical projection of A onto B. Substituting $x + p$ for x in $[[F(x),x],x] = 0$, it follows at once that $[[F(p),x],x] \in P$ for all $x \in R$, $p \in P$, that is, $[[\overline{F(p)},\overline{x}],\overline{x}] = 0$. Using Posner's theorem [12, Theorem 2] (or just [5, Lemma 2] for that matter), it follows that $[\overline{F(p)},\overline{x}] = 0$ for all $x \in R$, $p \in P$, that is, $\overline{F(p)}$ lies in the center of R/P. In particular, $\pi\overline{F(p)} = 0$. Using this, we see that the mapping $\overline{F} : R/P \to A$, $\overline{F}(\overline{x}) = \pi\overline{F(x)}$ is well defined. Note that $\overline{F}$ is additive and satisfies $[[\overline{F}(\overline{x}),\overline{x}],\overline{x}] = 0$ for all $x \in R$. But then the lemma shows that $[\overline{F}(\overline{x}),\overline{x}] = 0$ for all $x \in R$, which implies that $[F(x),x] \in P$. The proof of the theorem is complete. □

Theorem 4 generalizes Theorem 2 proved by Brešar [5] and Theorem 2 proved by Vukman in [14].

Proof of Theorem 2. From the relation

$$D(x^{m+n+1}) = (m+n+1)x^m D(x)x^n, \quad x \in R, \tag{1}$$

it follows immediately that

$$D(e)=0, \tag{2}$$

where e denotes the identity element. Putting $x+e$ for x in the relation (1) and using (2), we obtain

$$\begin{aligned}&\sum_{i=0}^{m+n+1}\binom{m+n+1}{i}D\left(x^{m+n+1-i}\right)\\&\quad=(m+n+1)\left(\sum_{i=0}^{m}\binom{m}{i}x^{m-i}\right)D(x)\left(\sum_{i=0}^{n}\binom{n}{i}x^{n-i}\right),\quad x\in R.\end{aligned} \tag{3}$$

Using (1) and collecting together terms of (3) involving the same number of factors of e, we obtain

$$\sum_{i=1}^{m+n}f_i(x,e)=0,\quad x\in R, \tag{4}$$

where $f_i(x,e)$ stands for the expression of terms involving i factors of e.

Replacing x by $x+2e, x+3e,\dots,x+(m+n)e$ in turn in (1) and expressing the resulting system of $m+n$ homogeneous equations, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix}1 & 1 & \cdots & 1\\ 2 & 2^2 & \cdots & 2^{m+n}\\ \vdots & \vdots & \vdots & \vdots\\ m+n & (m+n)^2 & \cdots & (m+n)^{m+n}\end{bmatrix}. \tag{5}$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution.

In particular,

$$\begin{aligned}f_{m+n-1}(x,e)=&\binom{m+n+1}{m+n-1}D\left(x^2\right)\\&-(m+n+1)\left(\binom{m}{m-1}\binom{n}{n}xD(x)+\binom{m}{m}\binom{n}{n-1}D(x)x\right)=0,\quad x\in R,\end{aligned} \tag{6}$$

$$\begin{aligned}f_{m+n-2}(x,e)=&\binom{m+n+1}{m+n-2}D\left(x^3\right)\\&-(m+n+1)\left(\binom{m}{m-2}\binom{n}{n}x^2D(x)+\binom{m}{m-1}\binom{n}{n-1}xD(x)x\right.\\&\left.+\binom{m}{m}\binom{n}{n-2}D(x)x^2\right)=0,\quad x\in R.\end{aligned} \tag{7}$$

Since R is a $(m+n+2)!$-torsion free ring, the above equations reduce to

$$(m+n)D(x^2) = 2mxD(x)+2nD(x)x, \quad x \in R, \tag{8}$$

$$\begin{aligned}(m+n)(m+n-1)D(x^3) = {} & 3m(m-1)x^2D(x)+6mnxD(x)x \\ & +3n(n-1)D(x)x^2, \quad x \in R,\end{aligned} \tag{9}$$

respectively. We intend to prove that the mapping $x \mapsto [D(x),x]$ is commuting on R. For this purpose, we write in $x+y$ for x in (8), which gives

$$(m+n)D(xy+yx) = 2mxD(y)+2myD(x)+2nD(x)y+2nD(y)x, \quad x,y \in R. \tag{10}$$

Putting $y=(m+n)x^2$ in the relation above, we obtain

$$\begin{aligned}(m+n)^2D(x^3) = {} & m(m+n)xD(x^2)+m(m+n)x^2D(x) \\ & +n(m+n)D(x)x^2+n(m+n)D(x^2)x, \quad x \in R.\end{aligned} \tag{11}$$

According to (8), the above relation reduces to

$$(m+n)^2D(x^3) = (3m^2+mn)x^2D(x)+4mnxD(x)x+(3n^2+mn)D(x)x^2, \quad x \in R. \tag{12}$$

Subtracting (9) from (12), we obtain

$$(m+n)D(x^3) = m(n+3)x^2D(x)-2mnxD(x)x+n(m+3)D(x)x^2, \quad x \in R. \tag{13}$$

From the above relation, we obtain

$$\begin{aligned}(m+n)^2D(x^3) = {} & (m+n)m(n+3)x^2D(x)-2(m+n)mnxD(x)x \\ & +(m+n)n(m+3)D(x)x^2, \quad x \in R.\end{aligned} \tag{14}$$

Subtracting (14) from (12), one obtains

$$mn(m+n+2)x^2D(x)-2mn(m+n+2)xD(x)x+mn(m+n+2)D(x)x^2=0, \quad x \in R. \tag{15}$$

Since R is $(m+n+2)!$-torsion free ring, the above relation reduces to

$$D(x)x^2+x^2D(x)-2xD(x)x=0, \quad x \in R, \tag{16}$$

which can be written in the form

$$[[D(x),x],x]=0, \quad x \in R. \tag{17}$$

Now Theorem 4 makes it possible to conclude that

$$[D(x),x]=0, \quad x \in R. \tag{18}$$

In other words, D is commuting on R. The fact that D is commuting on R makes it possible to replace $D(x)x$ in (8) by $xD(x)$. The relation (8) reduces to $D(x^2) = 2xD(x)$, $x \in R$. Using again the fact that D is commuting, we obtain $D(x^2) = D(x)x + xD(x)$, $x \in R$. In other words, D is a Jordan derivation. Let us recall that any Jordan derivation on a 2-torsion free semiprime ring is a derivation. It is well known and easy to prove that any commuting derivation on a semiprime ring R maps R into $Z(R)$ (see [15]). In case R is a noncommutative prime ring, Posner's second theorem completes the proof of the theorem. □

In the proof of Theorem 2, we met an additive mapping D satisfying the relation below

$$(m+n)D(x^2) = 2mD(x)x + 2nxD(x). \tag{19}$$

In case $n = 0$ and R is an m-torsion free ring, we have an additive mapping D satisfying the relation $D(x^2) = 2xD(x)$, $x \in R$. In other words, D is a left Jordan derivation. It was proved (see [15, Theorem 1]) that left Jordan derivations on a 2- and 3-torsion free semiprime ring are derivations which map the ring into its center. These observations lead to the conjecture.

Conjecture 5. *Let R be a semiprime ring with suitable torsion restrictions. Suppose there exists an additive mapping $D : R \to R$ satisfying the relation*

$$(m+n)D(x^2) = 2nD(x)x + 2mxD(x), \tag{20}$$

for all $x \in R$ and some integers $m \geq 0$, $n \geq 0$, $m+n \neq 0$. In case $m \neq n$, the mapping D is a derivation which maps R into $Z(R)$.

Our next result is related to the conjecture above.

Theorem 6. *Let R be a 2, m, n, $m+n$, and $|m-n|$-torsion free semiprime ring, and let $D : R \to R$ be an additive mapping satisfying the relation*

$$(m+n)D(xy) = 2mD(x)y + 2nxD(y), \tag{21}$$

for all pairs $x, y \in R$ and some integers $m \geq 0$, $n \geq 0$, $m+n \neq 0$. In case $m \neq n$, we have $D = 0$.

Proof. We have the relation

$$(m+n)D(xy) = 2mD(x)y + 2nxD(y), \quad x, y \in R. \tag{22}$$

We compute the expression $(m+n)^2D(xyx)$ in two ways. First we obtain (using (22))

$$\begin{aligned}(m+n)^2D(x(yx)) &= 2m(m+n)D(x)yx + 2n(m+n)xD(yx)\\ &= 2m(m+n)D(x)yx + 2nx(2mD(y)x + 2nyD(x)), \quad x, y \in R.\end{aligned} \tag{23}$$

Thus we have

$$(m+n)^2D(xyx) = 2m(m+n)D(x)yx + 4mnxD(y)x + 4n^2xyD(x), \quad x, y \in R. \tag{24}$$

On the other hand, we have (using (22))

$$(m+n)^2D((xy)x) = 2m(m+n)D(xy)x+2n(m+n)xyD(x)$$
$$= 2m(2mD(x)y+2nxD(y))x+2n(m+n)xyD(x), \quad x,y \in R. \tag{25}$$

Thus we have

$$(m+n)^2D(xyx) = 4m^2D(x)yx+4mnxD(y)x+2n(m+n)xyD(x), \quad x,y \in R. \tag{26}$$

Subtracting the relation (24) from the relation (26), we obtain

$$m(m-n)D(x)yx+n(m-n)xyD(x) = 0, \quad x,y \in R, \tag{27}$$

which reduces to

$$mD(x)yx+nxyD(x) = 0, \quad x,y \in R. \tag{28}$$

Putting yx for y in the relation (28), we obtain

$$mD(x)yx^2+nxyxD(x) = 0, \quad x,y \in R. \tag{29}$$

Right multiplication of the relation (28) by x gives

$$mD(x)yx^2+nxyD(x)x = 0, \quad x,y \in R. \tag{30}$$

Subtracting the relation (29) from the relation (30), we obtain

$$n(xy(D(x)x-xD(x))) = 0, \quad x,y \in R, \tag{31}$$

which gives

$$xy[D(x),x] = 0, \quad x,y \in R. \tag{32}$$

Writing in the relation (32) $D(x)y$ for y, then multiplying the relation (32) by $D(x)$ from the left-hand side and comparing the relations so obtained, we obtain

$$[D(x),x]y[D(x),x] = 0, \quad x,y \in R, \tag{33}$$

whence it follows

$$[D(x),x] = 0, \quad x \in R, \tag{34}$$

by semiprimeness of R. Putting $y = x$ in the relation (22) and using the relation (34), we obtain $D(x^2) = 2D(x)x$, $x \in R$, which can be written in the form

$$D(x^2) = D(x)x + xD(x), \quad x \in R, \tag{35}$$

because of (34). In other words, D is a Jordan derivation. As we have already mentioned, any Jordan derivation on a 2-torsion free semiprime ring is a derivation. Now one can replace $D(xy)$ with $D(x)y + xD(y)$ in the left-hand side of (22), which gives

$$D(x)y = xD(y), \quad x, y \in R. \tag{36}$$

Substituting zx for x in (36) gives

$$D(z)xy = 0, \quad x, y, z \in R, \tag{37}$$

whence it follows first $D(z)xD(z) = 0$ for all $x, z \in R$, and then by semiprimeness $D = 0$. The proof of the theorem is complete. □

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References

[1] W. E. Baxter and W. S. Martindale, III, *Jordan homomorphisms of semiprime rings*, J. Algebra **56** (1979), no. 2, 457–471.
[2] K. I. Beidar, Y. Fong, P.-H. Lee, and T.-L. Wong, *On additive maps of prime rings satisfying the Engel condition*, Comm. Algebra **25** (1997), no. 12, 3889–3902.
[3] K. I. Beidar, W. S. Martindale, III, and A. V. Mikhalev, *Rings with Generalized Identities*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 196, Marcel Dekker, New York, 1996.
[4] M. Brešar, *Jordan derivations on semiprime rings*, Proc. Amer. Math. Soc. **104** (1988), no. 4, 1003–1006.
[5] ———, *On a generalization of the notion of centralizing mappings*, Proc. Amer. Math. Soc. **114** (1992), no. 3, 641–649.
[6] ———, *Functional identities: a survey*, Algebra and Its applications (Athens, Ohio, 1999), Contemp. Math., vol. 259, American Mathematical Society, Rhode Island, 2000, pp. 93–109.
[7] M. Brešar and J. Vukman, *Jordan derivations on prime rings*, Bull. Austral. Math. Soc. **37** (1988), no. 3, 321–322.
[8] ———, *On left derivations and related mappings*, Proc. Amer. Math. Soc. **110** (1990), no. 1, 7–16.
[9] J. M. Cusack, *Jordan derivations on rings*, Proc. Amer. Math. Soc. **53** (1975), no. 2, 321–324.
[10] Q. Deng, *On Jordan left derivations*, Math. J. Okayama Univ. **34** (1992), 145–147 (1994).
[11] I. N. Herstein, *Jordan derivations of prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1104–1110.
[12] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.
[13] J. Vukman, *Commuting and centralizing mappings in prime rings*, Proc. Amer. Math. Soc. **109** (1990), no. 1, 47–52.
[14] ———, *Derivations on semiprime rings*, Bull. Austral. Math. Soc. **53** (1996), no. 3, 353–359.

[15] ______, *On some additive mappings in semiprime rings and Banach algebras*, Aequationes Math. **58** (1999), 1–10.

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