ESSAYS ON GAME THEORY

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Declaration

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

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Contents

Summary

In this thesis, we study equilibrium existence issues in games whose components (players, types, actions, etc.) have general structures that allow uncountably many elements. Specifically, we focus on the existence of perfect equilibria in large games, Bayesian Nash equilibria in general Bayesian games, and stationary Markov perfect equilibria in stochastic games. We provide a complete characterization for the existence of these equilibria and discuss their related properties in general frameworks.

It has been pointed out that a large game with infinitely many actions does not necessarily have a pure-strategy Nash equilibrium, much less a pure-strategy perfect equilibrium. In Chapter 2, we formulate the concept of pure-strategy perfect equilibrium under such setting, and restore its existence by providing a necessary and sufficient condition. This work complements, and extends, Rath (1994, 1998), who considered the case of finite actions. Our analysis borrows insights from Simon and Stinchcombe (1995), who showed that, unlike Selten's well-known result, perfect equilibrium is not necessarily admissible and characterized limit admissibility, and He, Sun and Sun (2016), who utilized a crucial condition, namely nowhere equivalence, to study relevant existence problems. We prove that nowhere equivalence is not only sufficient but also necessary for the existence of limit admissible perfect equilibrium. Further, we establish a condition requiring certain level of robustness on best strategies responding to small social aggregates fluctuations, and discuss its related properties.

Since Harsanyi (1967–68), the literature on theory of games with incomplete information has not yet yielded a general existence result for pure-strategy Bayesian Nash equilibria. To obtain such existence result, it is crucial for the information structures in Bayesian games to be absolutely continuous, since Simon (2003) showed a non-existence example without such continuity. In Chapter 3, we study Bayesian games with general action space, interdependent payoffs and correlated general type space. We introduce a condition of coarser density weighted payoff-relevant information, a general version of the absolute continuity condition mentioned above, and prove that it is a necessary and sufficient condition for the existence of pure-strategy Bayesian Nash equilibria in every general Bayesian game. This condition is obtained by distinguishing two different diffuseness of information and characterizing the relation between them. Our result brings new insights into the longstanding issue in the literature and provides a complete characterization in a more general framework compared to previous studies. We also show that under such condition, every behavioral strategy profile possesses a strong purification.

Stochastic games are first introduced by Shapley (1953). Since then, the existence of stationary Markov perfect equilibria in stochastic games has been studied extensively. The literature on stochastic games developed substantially in the last twenty years, but except for several special classes, no general result has been known. In Chapter 4, we consider large stochastic game with general state and action spaces. The main result establishes the existence of behavioral-strategy stationary Markov perfect equilibria in large stochastic games without any specific condition. Since pure-strategy stationary Markov perfect equilibrium may fail to exist, we further identify a condition, namely the nowhere equivalence condition, which completely characterizes the existence of pure-strategy stationary Markov perfect equilibria in general. Under the same condition, every behavioral-strategy stationary Markov perfect equilibrium has a strong purification, which means, for every behavioral-strategy stationary Markov perfect equilibrium, there is an equivalent pure-strategy stationary Markov perfect equilibrium with the same total discounted payoffs for almost all the players. In addition, the closed-graph properties for the correspondences of both behavioral-strategy and pure-strategy stationary Markov perfect equilibria are presented.

Chapter 1

Introduction

Games whose components have general structure have shown their value in modelling realistic social phenomena. Think of describing "perfect competition" in a competitive market where the amount of participants is huge and each of them has negligible influence, one may need to consider games with a continuum of players as suggested in Aumann (1964). Allowing other components such as feasible type or action space to be uncountablely many is also more relevant to the reality. Therefore, the study of games in a general framework receives significant attentions in the literature on game theory.

In contrast to finite games, equilibria often fail to exist in a general framework due to the limitation of Lebesgue measurability. Various methods have been proposed to tackle this non-existence problem, such as distributional equilibria, standard representations, hyperfinite agent spaces, saturated probability spaces, and agents spaces with the condition of "many more agents than strategies", but they only focus on special circumstances.[1] To obtain such existence in a general framework, we borrow the insights from He *et al.* (2016), who utilized the " nowhere equivalence" condition to study relevant equilibrium existence issues, and He and Sun (2013); He and Sun Y (2014), who provided the theory of regular conditional distributions of correspondences as a theoretical foundation for strategic analysis, and we study the characterizations of equilibrium existences in various environments.

[1] See for example, Mas-Colell (1984), Hart, Hildenbrand and Kohlberg (1974), Khan and Sun (1999), Keisler and Sun (2009) and Rustichini and Yannelis (1991).

In this thesis, we consider equilibrium existences as well as their related properties in three typical classes of games with general structure in their components regarding player, type, action or state spaces. Specifically, the focus is on large deterministic games, general Bayesian games and large stochastic games, as elaborated in the following.

1.1 Large deterministic games

Selten (1975) introduced the notion of perfect equilibrium into finite games, for the purpose of precluding “irrational” Nash equilibria in which some players use dominated strategies. But similar notion has not yet been established in large games with infinitely many actions. Technical issues arise as soon as one wants to fomulate this concept properly in large environment. Such issue, to be specific, is about the possible failure of the standard fixed point arguments due to a lack of regular properties[2] of conditional distributions/expectations of correspondences in general. This was elaborated by Khan, Rath and Sun (1997) in the early literature, who showed that a large game with infinitely many actions does not necessarily have a pure-strategy Nash equilibrium. Hence the existence of its refinement–a pure strategy perfect equilibrium–cannot be guaranteed either.

In the literature, there are two independent strands relevant to the study of perfect equilibra in games with a continuum of players and infinitely many actions: one is to work with finite-player games with compact, metrizable sets of normal form actions and continuous payoff (“infinite normal-form games” for short), while the other is to study large games with finite actions. In contrast to finite games, irrational strategies may survive in perfect equilibria of games with large structures regarding either player or action spaces. Still, we can retrieve a similar but weaker property, namely the limit admissible property, for perfect equilibria in the large environment.

Simon and Stinchcombe (1995) made a breakthrough in generalizing Selten’s idea to infinite normal-form games.[3] Their work formulated the notion of perfect equilibrium, and showed its existence in infinite normal-form games. As shown by Simon and Stinchcombe (1995), it is not

[2]Regular properties here means convexity, compactness and preservation of upper hemicontinuity.

[3]Specifically, “infinite normal-form game” is referred to the finite-player games with compact, metrizable sets of normal form actions and continuous payoff.

generally true that each perfect equilibrium in their setting precludes dominated strategies, or in other words, each perfect equilibrium is admissible. In fact, they constructed a specific finite-player game with infinitely many actions where any admissible perfect equilibrium fails to exist. To compensate and improve this result, they proposed the concept of "limit admissibility" as an appropriate alternative, which enlarged admissible strategy sets by including their boundaries, and showed each perfect equilibrium in infinite normal-form games is limit admissible. Note that these two concepts ("limit admissible" and "admissible") coincide in games with finitely many actions.

In the context of large games, Rath (1994, 1998) formulated and established the existence of perfect equilibria in games with a continuum of players and finitely many actions. To illustrate that perfect equilibria in large games may fail to preclude irrational strategies, he constructed a specific large game with finite actions which possesses a non-admissible perfect equilibrium. Nevertheless, he further pointed out that every large game with finite actions always has an admissible perfect equilibrium.

In Chapter 2, we formulate the notion of perfect equilibrium in large games with infinitely many actions. In order to establish its existence, we adopt the nowhere equivalence condition introduced by He *et al.* (2016), and show it is not only sufficient but also necessary for the general existence of perfect equilibria in large game with infinitely many actions. As with Simon and Stinchcombe (1995) and Rath (1994, 1998), perfect equilibria under our setting may not be admissible either, due to the large structure of both player and action spaces. In fact, we give non-existence examples to illustrate such failure, which will continue even when we relax the requirement from admissible to limit admissible. Nevertheless, we prove that under nowhere equivalence condition, every large game with infinitely many actions has a limit admissible perfect equilibrium, which extends Rath's result. This strengthened existence result also deduces the necessity of nowhere equivalence condition. Finally we study a so-called "best response robustness condition", analogue to Rath's boundary condition, under which every Nash equilibrium is a perfect equilibrium. This condition actually demands a strong level of robustness on best responses with respect to small fluctuations of social aggregates, and hence yields nice properties as well as constraints which we will discuss in the last section of this chapter.

1.2 General Bayesian games

Bayesian games, or games with incomplete information, capture a common phenomenon in many realistic cases, the information asymmetry among individuals. In this literature, many progresses have been made, for example, Harsanyi (1967–68). Nevertheless, a fundamental question about the existence of pure-strategy Bayesian Nash equilibria in Bayesian games in a general framework still remains unanswered, which calls for further studies.

In fact, the general existence of pure-strategy equilibria will not be possible without a condition of absolute continuity on information structure. As a counterexample, a special Bayesian game with no pure-strategy equilibria was brought up by Simon (2003). Hellman (2014) further showed that without such continuity, a Bayesian game may not even have measurable approximate pure-strategy equilibria.

In Chapter 3, we study the existence of pure-strategy Bayesian Nash equilibria in Bayesian games with general action spaces, interdependent payoffs and correlated types.[4] We introduce a condition of coarser density weighted payoff-relevant information, and prove that it is sufficient for pure-strategy Bayesian Nash equilibria to exist in every general Bayesian game. This condition is obtained by distinguishing two different diffuseness of information and characterizing the relation between them, which borrows technical insights from He and Sun X (2014), He *et al.* (2016) and He and Sun (2014a). Such condition is also shown to be a minimal requirement given the validity of the pure-strategy equilibrium existence in every Bayesian games of an identical information structure with the same set of players and actions.

Previous studies in the literature has proposed various assumptions to settle this existence issue in general, including supermodular in strategies (see Vives (1990)), Spence-Mirrlees single crossing property (see Athey (2001)), multidimensional and partially-ordered type and action spaces (see McAdams (2003)), and compact locally complete metric action semilattics and

[4]We called Bayesian games with general action spaces, interdependent payoffs and correlated types to be general Bayesian games for short.

partially-ordered type spaces (see Reny (2011)). But all these results are built upon a basic assumption that the information is independent and diffusive among players.

However, the case of Bayesian games with correlated types and interdependent payoffs is different. Though positive result was obtained in general finite-action Bayesian games by He and Sun (2014a), who proposed a condition of ”coarser inter-player information” that completely charaterlizes such existence, Khan *et al.* (1999) pointed out that no general existence result is valid in Bayesian games with general action space even with independent and diffusive information. Therefore, a suitable condition to restore such existence is desirable.

In searching of such condition, evidence in extant literature suggests that an enriched information structure may be able to support a general existence result. With the underlying assumption of independent information, relevant studies on such enriched information structure includes the atomless Loeb space (see Khan and Sun (1999)), the saturated probability type spaces (see Loeb and Sun (2006) or Wang and Zhang (2012)), and the “relative diffuseness” information assumption (see He and Sun X (2014)). Each one of these information structures can establish the existence of pure-strategy equilibria in Bayesian game in general.

Our result brings new insights to the longstanding fundamental issue by establishing the existence of pure-strategy Bayesian Nash equilibria in Bayesian games in a general framework, and provides a complete characterization for information structure that supports such existence. A purification result is further presented.

1.3 Large stochastic games

Begining from Shapley (1953), the literature on stochastic games has been substantially developed, providing theoretical foundations for modelling dynamic strategic interactions in a stochastic environment. To further study such interactions in a framework with many players that fits into a wide range of applications, it is natural to consider stochastic games with a continuum of players. Previous studies proposed various solution concepts under such setting. In this paper, we focus on a standard one: the stationary Markov perfect equilibrium, and explore its related properties.

There is extensive research on establishing the existence of stationary Markov perfect equilibria in stochastic games with general state spaces; see Nowak and Raghavan (1992), Duffie *et al.* (1994), Duggan (2012) and Levy (2013). Though recent development by He and Sun (2015) is able to cover previous results in Nowak and Raghavan (1992), Duffie *et al.* (1994), Nowak (2003) and Duggan (2012), the extant literature has not yielded a general result. In fact, examples constructed in Levy (2013) and Levy and McLennan (2015) show that such equilibria may not exist in the case of general state spaces.

This paper studies stationary Markov perfect equilibria in stochastic games with a continuum of players and general state spaces. The main result in Chapter 4 establishes the existence of behavioral-strategy stationary Markov perfect equilibria without any specific condition, and provides a sufficient and necessary condition for the existence of pure-strategy stationary Markov perfect equilibria in such games.[5]

Our work offers a new technique to circumvent a measurability issue arisen in the establishment of our main result This technical development therefore is crucial to support the general existence of behavioral-strategy stationary Markov perfect equilibria. For pure-strategy stationary Markov perfect equilibria, we identify a crucial condition, namely the nowhere equivalence condition,[6] and show that it is not only sufficient but also necessary for the existence of pure-strategy stationary Markov perfect equilibria. Under this particular condition, we also show that every behavioral-strategy Markov perfect equilibrium can be purified into an equivalent pure-strategy stationary Markov perfect equilibrium with identical payoffs for almost all players, which reveals that the virtue contribution of such condition is to validate the purification.

This work also studies the closed-graph property for stationary Markov perfect equilibria in large stochastic games with a prerequisite that transition probabilities are uniformly norm-continuous. We show that in such setting, this property holds generally for the correspondence of stationary Markov perfect equilibria in behavioral strategies. To obtain an analogous result for

[5]To be specific, we study stochastic games with a continuum of players, general state spaces (Polish spaces) and action spaces (compact metric spaces). We also adopt action distributions as societal summaries in this setting.

[6] As mentioned previously, this condition is introduced by He *et al.* (2016)

equilibria in pure strategies, we impose the nowhere equivalence condition, which guarantees the pure-strategy equilibrium existence, and show that such condition actually provides a complete characterization of the closed-graph property for the correspondence of pure-strategy stationary Markov perfect equilibria in our setting.

1.4 Organization

The main results in Chapters 2, 3 and 4 are based on the papers Sun and Zeng (2014), He, Sun, Sun, and Zeng (2014) and He, Sun, and Zeng (2016), respectively.

This thesis is organized as follows. In Chapter 2, we formulate the notion of perfect equilibria in large games with infinitely many actions, and provide a complete characterization of the existence of perfect equilibria. The property of limit admissibility as well as robustness are further discussed. In Chapter 3, we study pure-strategy Bayesian Nash equilibria in games with incomplete information, interdependent payoffs and correlated types. We identify a necessary and sufficient condition for such existence, followed by a related purification result. In Chapter 4, we establish the existence of behavioral-strategy stationary Markov perfect equilibria in every large stochastic game, and provide a necessary and sufficient condition for the existence of pure-strategy stationary Markov perfect equilibria in this setting. We further present a purification result as well as the closed graph property for such equilibria.

Chapter 2

Perfect Equilibria in Large Games

2.1 Introduction

Selten (1975) introduced (trembling hand) perfect equilibrium to restrict the set of Nash equilibria in finite games (*i.e.*, games with finite players and finite actions). This refinement precludes weakly dominated actions by requiring some notion of neighborhood robustness to small perturbations of the original game. Based on Selten (1975)'s original idea, Simon and Stinchcombe (1995) formulated perfect equilibrium in finite-player games with infinitely many actions, and showed its existence and several properties. Rath (1994, 1998) was the first to consider perfect equilibrium in large games.[1] In particular, Rath (1994, 1998) formulated the notion of perfect equilibrium in large games with finite actions and established its existence. One may ask whether perfect equilibria exist in the large games with infinitely many actions.[2] This motivates this chapter.

Structurally, we work with the framework of large games, where the action spaces are compact metric spaces and each player's payoff function continuously depends on her/his own action and on the societal aggregate of other players' actions. In this framework, a perfect

[1]The large games and their applications have been extensively studied; see the survey by Khan and Sun (2002) for example. For some recent developments and applications of large economies/games, see Einy *et al.* (2000), Einy and Shitovitz (2001, 2003), Fu and Yu (2015), Hammond (1979, 2015), Khan *et al.* (2013), Qiao and Yu (2014), Rauh (2007), Sun *et al.* (2012), Sun and Yannelis (2007), Yannelis (2009) among others.

[2]It is well known that a large game with infinitely many actions does not necessarily have a pure strategy Nash equilibrium, and a pure strategy perfect equilibrium either; see Section 2 in Khan *et al.* (1997) for example.

equilibrium is defined as a "limit" of a sequence of ϵ-perfect equilibria describing the possibility of making mistakes. In each ϵ-perfect equilibrium, each player adopts a completely mixed strategy, which is a full support probability distribution on the action set assigning at least $1 - \epsilon$ weight to the set of best response actions. Then we show that the set of perfect equilibria is generally a proper subset of the set of Nash equilibria; see Proposition 1, Examples 1 and 2. This result is compatible with the analogue in finite games.

The notion of mixed/behavioral strategy which involves randomization is widely used in game theory. However, one may not observe individuals to make decisions by using various randomization devices in many practical situations.[3] Then the question is in what kind of games, randomization essentially plays no useful role. Namely, under what conditions a game has a pure strategy equilibrium. To obtain the existence of pure strategy perfect equilibria, we turn to the nowhere equivalence condition introduced in He *et al.* (2016).[4] This condition distinguishes the player space from the characteristics type space which is generated by the mapping specifying the individual payoff functions and action sets.[5] In Theorem 1, we prove that a large game always has a pure strategy perfect equilibrium whenever the underlying player space satisfies the nowhere equivalence condition. By distinguishing the player space from the characteristics type space, we allow different players with the same characteristics (payoff and action set) to choose different optimal actions in equilibria. This leads to possible purifications of behavioral strategy equilibria[6] and the existence of pure strategy equilibria.[7] In contrast, the traditional set-up where the spaces of players and their characteristic types are identical may fail to have such a heterogeneity and hence could not ensure the existence of pure strategy equilibria. Moreover, it follows from Theorem 2 of He *et al.* (2016) that the condition of nowhere equivalence is also necessary for obtaining the existence of pure strategy perfect equilibria.

[3]See Aumann (1987), Milgrom and Weber (1985), Radner and Rosenthal (1982) and Rubinstein (1991) among others.

[4]The condition of nowhere equivalence is one of the four equivalent conditions identified in He *et al.* (2016) that provide a unified and minimal framework for handling the failure of the Lebesgue interval in in various problems associated with competitive equilibria in large economies and Nash equilibria in large games.

[5]The nowhere equivalence condition requires that give any non-trivial collection of players, when the player space and the characteristics type space are restricted to such a collection, the former one contains the latter one strictly in terms of measure spaces.

[6]A pure strategy profile is said to be a purification of a behavioral strategy profile if the expected payoffs/distributions of these two strategy profiles are the same for all the players.

[7]For more details, see Example 3 and Remark 4 in He *et al.* (2016).

The admissibility[8] of perfect equilibria was studied extensively. Though Selten (1975) showed that every perfect equilibrium in finite games is admissible, this is not generally true. Simon and Stinchcombe (1995) constructed a finite-player game with infinitely many actions, where an admissible perfect equilibrium fails to exist; see Example 2.1 therein. In the context of large games, Rath (1998) constructed a large game with finite actions, in which there is a perfect equilibrium that is not admissible; see Section 5 therein. Nevertheless, Rath (1998) further pointed out that every large game with finite actions always has an admissible perfect equilibrium in Theorem 2(a). However, we construct a large game with infinitely many actions, in which an admissible perfect equilibrium does not exist. This suggests that the existence and the admissibility of perfect equilibria may not be compatible in large games with infinitely many actions. In finite-player games (with infinitely many actions), as suggested by Simon and Stinchcombe (1995), the limit admissibility could be an appropriate alternative: an equilibrium is limit admissible if it puts no mass on the interior of the set of weakly dominated strategies. It is of interest to see if such a limit admissible perfect equilibrium exists or not in large games with infinitely many actions. We provide a full characterization on this existence in Theorem 2: the nowhere equivalence condition is sufficient and necessary for the existence of limit admissible pure strategy perfect equilibrium.

We also discuss the concept of proper equilibrium in large games with infinitely many actions. Such a notion was introduced by Myerson (1978) for finite games. It further refines perfect equilibrium by assuming that more costly trembles are made with significantly smaller probability than less costly ones. The notions of proper equilibrium for finite-player games with infinitely many actions, and for large games with finite actions have been formulated by Simon and Stinchcombe (1995) and Rath (1994, 1998) respectively. They also established the existence of proper equilibria in the corresponding settings. For large games with infinitely many actions, we follow the approach of finite partitions adopted by Simon and Stinchcombe (1995) to define proper equilibrium. In Theorem 3, we show that the nowhere equivalence condition is sufficient and necessary for the existence of pure strategy proper equilibria for large games with infinitely many actions.

[8]An equilibrium is admissible if it puts no mass on weakly dominated strategies.

Motivated by Rath (1998), we establish a condition requiring certain level of robustness on best responses while the social aggregate fluctuating within a small range. This condition, called "best response robustness condition" in the following, is examined in Section 2.6. We show that every Nash equilibrium is a perfect equilibrium when the game satisfies the best response robustness condition; see Proposition 2. For almost every player, the perfect equilibrium strategy is required to be in the lim sup of the ϵ-perfect equilibrium strategies. Based on the best response robustness condition, we have a stronger result: almost every perfect equilibrium strategy can be a limit of a sequence of ϵ-perfect equilibrium strategies under the condition of best response robustness; see Proposition 3.

The rest of this chapter is organized as follows. In Section 2.2, we present the set-up of large games with infinitely many actions and formulate the notions of ϵ-perfect equilibrium and perfect equilibrium. We also consider the relation between a Nash equilibrium and a perfect equilibrium in Section 2.2. In Section 2.3, we establish our main existence result via the nowhere equivalence condition; see Theorem 1. In Sections 2.4 and 2.5, we strengthen the existence result of pure strategy perfect equilibria to the existence of their refinement that satisfies the limit admissibility or the properness; see Theorems 2 and 3, respectively. Section 2.6 studies the best response robustness condition. All the proofs are collected in Section 2.7.

2.2 Perfect equilibria of large games

In this section, we shall specify the formulation of large games with general action spaces. The player space is modeled by an atomless probability space $(I, \mathcal{F}, \lambda)$.[9] Each player i has to restrict her/his actions to a certain (nonempty) subset of a compact metric space A, denoted by A_i.[10] This action correspondence $\mathcal{A}\colon i \mapsto A_i$ is supposed to be measurable and compact-valued.[11] The set of Borel probability measures on A, denoted by $\mathcal{M}(A)$, will serve as the space of

[9] A probability space $(I, \mathcal{F}, \lambda)$ (or its σ-algebra) is atomless if for any non-negligible subset $E \in \mathcal{F}$, there is a $\mathcal{F}$-measurable subset E' of E such that $0 < \lambda(E') < \lambda(E)$.

[10] One can take A to be the compact metric space $[0,1]^{\mathbb{N}}$ with the product topology, since every compact metric space is homeomorphic to a compact subset of $[0,1]^{\mathbb{N}}$; see Theorem 3.40 in Aliprantis and Border (2006).

[11] We are grateful to the anonymous referee for suggesting us to work with an action correspondence rather than a common action space.

societal summaries. Note that $\mathcal{M}(A)$ is also a compact metric space under weak topology. Each player's payoff continuously depends on her/his own action as well as on a societal summary that describes the action distribution of all the players. For simplicity, the payoff function for each player is assumed to be a bounded continuous function on $A \times \mathcal{M}(A)$.[12] The space of payoffs $\mathcal{U}_A$ is then defined as the space of all bounded continuous functions on the product space $A \times \mathcal{M}(A)$ with its sup-norm topology and the resulting Borel σ-algebra. Now, we are ready to present the definition of large games.

Definition 1 (Large game)**.** A *large game* is a measurable function G from I to $\mathcal{U}_A$.

In a large game G, player i's payoff function $G(i)$ is usually rewritten as u_i for simplicity.

A *behavioral strategy profile* (resp. *a pure strategy profile*) g is an $\mathcal{F}$-measurable function from I to $\mathcal{M}(A)$ (resp. A) such that $g(i; A_i) = 1$ (resp. $g(i) \in A_i$) for λ-almost all $i \in I$, where $g(i; A_i)$ is the value of the probability measure $g(i)$ on the subset $A_i \subseteq A$. Let Ψ be the set of all the behavioral strategy profiles.

Moreover, let $\mathcal{M}(A_i)^{fs}$ be the set of Borel probability measures on A_i assigning strictly positive mass to every nonempty open subset of A_i. Then an ($\mathcal{F}$-measurable) behavioral strategy profile g with $g(i) \in \mathcal{M}(A_i)^{fs}$ for λ-almost all $i \in I$ captures the notion that any player may "tremble" and play any one of her actions. Such a behavioral strategy profile is called a *behavioral strategy profile with full support*. Let Ψ^{fs} be the set of all the behavioral strategy profiles with full support.

Given a behavioral strategy profile g, player i's expected payoff is

$$U_i(g) = \int_A u_i \left(a, \int_I g(i) \, \mathrm{d}\lambda(i) \right) g(i; \mathrm{d}a),$$

where $\int_I g(i) \, \mathrm{d}\lambda(i)$ is the Gelfand integral of g,[13] denoting the average action distributions of all the players. Note that when g is a pure strategy profile, the social summary $\int_I g(i) \, \mathrm{d}\lambda(i)$ is

[12]The results in this paper still hold even if we assume that each player i's payoff function is on $A_i \times \mathcal{M}(A)$. In this formulation, a large game G is defined to be a function from $D \times \mathcal{M}(A)$ to $\mathbb{R}$ such that $G(\cdot, \mu)$ is $\mathcal{F} \times \mathcal{B}(A)$-measurable and $G(i, \cdot, \cdot) \colon A_i \times \mathcal{M}(A) \to \mathbb{R}$ is continuous, where $D = \{(i, a) \in I \times A \mid a \in A_i\}$.

[13]The Gelfand integral of g is a probability measure such that $\left(\int_I g(i) \, \mathrm{d}\lambda(i) \right)(B) = \int_I g(i; B) \, \mathrm{d}\lambda(i)$ for each $B \in \mathcal{B}(A)$.

λg^{-1}, which is the action distribution induced by g. Let $\mathrm{Br}_i(g)$ be the set of player i's (pure) best response actions to the behavioral strategy profiles g, that is,

$$\mathrm{Br}_i(g) = \arg\max_{a \in A_i} u_i \left(a, \int_I g(i) \, \mathrm{d}\lambda(i) \right).$$

Since A_i is a compact metric space and u_i is continuous on $A \times \mathcal{M}(A)$, we have that $\mathrm{Br}_i(g)$ is nonempty and compact for each g.

Next, we present the definition of Nash equilibria.

Definition 2 (Nash equilibrium)**.** A *pure strategy Nash equilibrium* g is an $\mathcal{F}$-measurable pure strategy profile such that for λ-almost all $i \in I$,

$$u_i\big(g(i), \lambda g^{-1}\big) \geq u_i\big(a, \lambda g^{-1}\big) \text{ for all } a \in A_i.$$

The distance between two probability distributions μ and ν on A will be measured by the strong metric ρ^s and the weak metric ρ^w:

$$\rho^s(\mu, \nu) = \sup \Big\{ |\mu(B) - \nu(B)| \;\Big|\; B \in \mathcal{B}(A) \Big\} \text{ and}$$
$$\rho^w(\mu, \nu) = \inf \left\{ \epsilon > 0 \;\middle|\; \text{for all } B \in \mathcal{B}(A), \begin{array}{l} \mu(B) \leq \nu(B^\epsilon) + \epsilon \\ \nu(B) \leq \mu(B^\epsilon) + \epsilon \end{array} \right\},$$

where B^ϵ is the ϵ-neighborhood of the Borel measurable set B. Clearly, the topology on $\mathcal{M}(A)$ induced by the strong metric is finer than the topology induced by the weak metric, and these two metrics induce the same topology when A is a finite set. We turn now to the definitions of (strong/weak) ϵ-perfect equilibria and (strong/weak) perfect equilibria, noting the central role that different distances play.

Definition 3 (ϵ-perfect equilibrium)**.** A *strong ϵ-perfect equilibrium* g^ϵ is an $\mathcal{F}$-measurable behavioral strategy profile with full support such that for λ-almost all $i \in I$,

$$\rho^s\big(g^\epsilon(i), \mathcal{M}(\mathrm{Br}_i(g^\epsilon))\big) := \inf_{\mu \in \mathcal{M}(\mathrm{Br}_i(g^\epsilon))} \rho^s(g^\epsilon(i), \mu) < \epsilon,$$

whereas a *weak ϵ-perfect equilibrium* satisfies

$$\rho^w\big(g^\epsilon(i), \mathcal{M}(\mathrm{Br}_i(g^\epsilon))\big) := \inf_{\mu \in \mathcal{M}(\mathrm{Br}_i(g^\epsilon))} \rho^w(g^\epsilon(i), \mu) < \epsilon.$$

Definition 4 (Perfect equilibrium)**.** A pure (resp. behavioral) strategy *strong perfect equilibrium* g is an $\mathcal{F}$-measurable pure (resp. behavioral) strategy profile such that there exists a sequence $\{g^n\}_{n\in\mathbb{Z}_+}$ where

(1) each g^n is a strong ϵ_n-perfect equilibrium with $\epsilon_n \to 0$ as n goes to infinity,

(2) for λ-almost all $i \in I$, $g(i) \in \operatorname{supp} \operatorname{Ls} g^n(i)$ (resp. $\operatorname{Ls} g^n(i)$),[14]

(3) $\lim_{n\to\infty} \int_I g^n(i)\,\mathrm{d}\lambda(i) = \lambda g^{-1}$ (resp. $\int_I g(i)\,\mathrm{d}\lambda(i)$).

A pure (resp. behavioral) strategy *weak perfect equilibrium* g is an $\mathcal{F}$-measurable pure (resp. behavioral) strategy profile such that there exists a sequence $\{g^n\}_{n\in\mathbb{Z}_+}$ where

(1') each g^n is a weak ϵ_n-perfect equilibrium with $\epsilon_n \to 0$ as n goes to infinity,

(2') for λ-almost all $i \in I$, $g(i) \in \operatorname{supp} \operatorname{Ls} g^n(i)$ (resp. $\operatorname{Ls} g^n(i)$),

(3') $\lim_{n\to\infty} \int_I g^n(i)\,\mathrm{d}\lambda(i) = \lambda g^{-1}$ (resp. $\int_I g(i)\,\mathrm{d}\lambda(i)$).

The above notions of ϵ-perfect equilibrium and perfect equilibrium are straightforward adaptations of the corresponding notions for games with finite players (Selten (1975), Simon and Stinchcombe (1995)) and for large games with finite actions (Rath (1994, 1998)). In games with a finite number of players, a perfect equilibrium is required to be a limit of a sequence of ϵ-perfect equilibria; see Selten (1975) and Simon and Stinchcombe (1995). In the presence of a continuum of players, this (almost everyone) convergence may break down; see Rath (1994). Hence, one has to settle for something less by weakening the requirement from almost

[14] $\operatorname{Ls} g^n(i)$ denotes the topological lim sup of the sequence of the subsets $\big\{\{g^n(i)\}\big\}_{n\in\mathbb{Z}_+}$; see Definition 3.80 in Aliprantis and Border (2006). This limit as well as the notion of limit $\lim_{n\to\infty}$ are with respect to the usual weak convergence of distributions. For a probability μ on a topological space X endowed with the Borel σ-algebra $\mathcal{B}(X)$, $\operatorname{supp}\mu$ denotes its support, which is required to satisfy (1) $\mu\big((\operatorname{supp}\mu)^c\big) = 0$ and (2) if G is open and $G \cap \operatorname{supp}\mu \neq \emptyset$, then $\mu(G \cap \operatorname{supp}\mu) > 0$; see Section 12.3 in Aliprantis and Border (2006). For a set of probability measures $\mathcal{N}$ on $(X, \mathcal{B}(X))$, we use $\operatorname{supp}\mathcal{N}$ to denote the set $\{\operatorname{supp}\mu \mid \mu \in \mathcal{N}\}$.

everyone convergence. Then, it is a natural one that a perfect equilibrium is in the lim sup of strategies of almost every player. Furthermore, as discussed in Rath (1994), the requirement that $\int_I g^n(i)\,\mathrm{d}\lambda(i)$ converges to λg^{-1} is crucial as this assumption guarantees that the limit of ϵ-best responses is a best response in the limit.

The following proposition shows that a perfect equilibrium is always a Nash equilibrium in large games. This result is compatible with the fact that the perfect equilibrium is a refinement of Nash equilibria in games with finite players.

Proposition 1. *In any large game, a pure strategy weak/strong perfect equilibrium is a pure strategy Nash equilibrium.*

In games with finite players, it is well known that the perfect equilibrium is a *strict* refinement of Nash equilibrium. We present an example to illustrate that this statement also holds in large games.

Example 1. Let the Lebesgue unite interval $(L, \mathcal{L}, \eta)$ be the player space, where $L = [0, 1]$, $\mathcal{L}$ is the Lebesgue σ-algebra, and η is the Lebesgue measure. Let the set $A = \{0, 1\}$ be the common action set. For each player $i \in L$, given a societal summary $\xi \in \mathcal{M}(A)$, the payoff function is defined as follows:

$$u_i(1, \xi) = \xi(\{1\}) \text{ and } u_i(0, \xi) = 0.$$

Claim 1. For the game in Example 1, $f^1(i) \equiv 0$ is a Nash equilibrium but not a perfect equilibrium.

The claim above shows that perfect equilibrium is a strict refinement of Nash equilibrium in large games.

It is a straightforward observation that any strong perfect equilibrium is a weak perfect equilibrium, based on the fact that the topology on $\mathcal{M}(A)$ induced by the strong metric is finer than the topology induced by the weak metric. Furthermore, the notions of strong perfect equilibrium and weak perfect equilibrium coincide when the action set A is finite. However, there are essential differences between these two concepts in large games with infinitely many actions. The following example presents such a large game, which illustrates that the set of strong perfect equilibria is a proper subset of weak perfect equilibria.[15]

Example 2. Let the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ be the player space, and the set $A = \{-1\} \cup [0,1]$ the common action space. The payoff functions are defined as follows.

For each player $i \in [0, \frac{1}{2})$, given her own action $a \in A$ and a societal aggregate $\xi \in \mathcal{M}(A)$, her payoff function is $u_1(a,\xi) = \int_A v(a, a_2)\, \mathrm{d}\xi(a_2)$, where

$$v(a,a_2) = \begin{cases} \frac{1}{8}a_2, & \text{if } a = -1,\ a_2 \in [0,1], \\ a, & \text{if } a \in [0, \frac{a_2}{2}),\ a_2 \in [0,1], \\ a_2 - a, & \text{if } a \in [\frac{a_2}{2}, 1],\ a_2 \in [0,1], \\ 0, & \text{if } a \in A,\ a_2 = -1. \end{cases}$$

For each player $i \in [\frac{1}{2}, 1]$, given her own action $a \in A$ and a societal aggregate $\xi \in \mathcal{M}(A)$, her payoff function is $u_2(a,\xi) = -a$ if $a \in [0,1]$ and $u_2(-1,\xi) = -2$.

Claim 2. For the game in Example 2, we have the following results.

(1) There is a unique strong perfect equilibrium f^1 (modulo negligible subsets of players), where $f^1(i) = -1$ for $i \in [0, \frac{1}{2})$ and $f^1(i) = 0$ for $i \in [\frac{1}{2}, 1]$.

(2) Besides f^1, there is another weak perfect equilibrium $f^2(i) \equiv 0$.

The claim above shows that the set of strong perfect equilibria is a possibly proper subset of the set of weak perfect equilibria in large games with infinitely many actions.

2.3 The existence of perfect equilibria

Khan *et al.* (1997) constructed a large game (with infinitely many actions), which does not have a pure strategy Nash equilibrium; see Section 2 therein. Since a pure strategy perfect

[15]This example is motivated by Example 2.3 in Simon and Stinchcombe (1995).

equilibrium is always a pure strategy Nash equilibrium as shown in Proposition 1, a pure strategy prefect equilibrium in that game does not exist either. In this section, we systematically study the existence issue of pure strategy perfect equilibria, and prove that the nowhere equivalence condition, introduced in He *et al.* (2016), is sufficient and necessary to guarantee the existence.

In our formulation, the player space is endowed with two σ-algebras $\mathcal{F}$ and $\mathcal{G}$, where $\mathcal{G}$ is a countably-generated sub-σ-algebra of $\mathcal{F}$. We assume that large games and action correspondences are $\mathcal{G}$-measurable. The two σ-algebras $\mathcal{F}$ and $\mathcal{G}$ are called the *universal σ-algebra* and the *characteristics type σ-algebra*, respectively, and the probability space $(I, \mathcal{G}, \lambda)$ is called the *characteristic type space*. Here $\mathcal{G}$ can be viewed as the σ-algebra generated by the mapping $(G, \mathcal{A})$ specifying the individual payoff functions and action sets.

For any non-negligible subset $D \in \mathcal{F}$, *i.e.*, $\lambda(D) > 0$, the restricted probability space $(D, \mathcal{G}^D, \lambda^D)$ is defined as follows: $\mathcal{G}^D$ is the σ-algebra $\{D \cap D' \colon D' \in \mathcal{G}\}$ and λ^D is the probability measure re-scaled from the restriction of λ to $\mathcal{G}^D$; the restricted probability space $(D, \mathcal{F}^D, \lambda^D)$ is defined similarly. We state the nowhere equivalence condition introduced in He *et al.* (2016) as follows.

Definition 5 (Nowhere equivalence condition)**.** The σ-algebra $\mathcal{F}$ is said to be nowhere equivalent to the sub-σ-algebra $\mathcal{G}$, if for every non-negligible subset $D \in \mathcal{F}$, there exists an $\mathcal{F}$-measurable subset D_0 of D such that $\lambda(D_0 \triangle D_1) > 0$ for any $D_1 \in \mathcal{G}^D$, where $\triangle$ denotes the symmetric difference operator. That is, $D_0 \triangle D_1 = (D_0 \setminus D_1) \cup (D_1 \setminus D_0)$.

The condition of nowhere equivalence requires that given any non-trivial collection of players, when the player space and the characteristic type space are restricted to such a collection, the former contains the latter strictly in terms of measure spaces.

By distinguishing the player space from the characteristic type space, the condition of nowhere equivalence allows the heterogeneity that different players with the same payoff to select different optimal (pure) actions, which in turn guarantees the existence of pure strategy equilibria; see He *et al.* (2016). However, in the example constructed by Khan *et al.* (1997), both the spaces of players and their characteristic types are the Lebesgue unit interval, which fails to have such a heterogeneity. In particular, a continuum version of the classical sequential replica model as in Debreu and Scarf (1963) satisfies the nowhere equivalence condition.

To view an analogous situation in a finite-player setting, let k and n be positive integers, $I = \{1, 2, \ldots, k\,n\}$ be the space of players, and $\mathcal{F}$ the power set of I. Let $\mathcal{G}$ be the σ-algebra generated by a partition $\{I_1, I_2, \ldots, I_k\}$ of I, where each I_ℓ has n players sharing the same characteristics. In some asymptotic sense, $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ when n goes to infinity.

Under the nowhere equivalence condition, we have the following existence result.

Theorem 1. *Every $\mathcal{G}$-measurable large game G has an $\mathcal{F}$-measurable pure strategy strong perfect equilibrium if and only if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.*

Remark 1. The nowhere equivalence condition is needed for the purification of behavioral strategy perfect equilibria so to allow us obtaining a pure strategy equilibrium. In general, in a large game, such a purification method does not work; see He *et al.* (2016) and its references. In a game where some players with a given characteristics have multiple optimal (pure) actions, to purify a behavioral strategy equilibrium, one has to allows the heterogeneity that different players with the same characteristics to select different optimal (pure) actions; see the examples in He *et al.* (2016) and their discussions.

2.4 Limit admissible perfect equilibria

In games with finite players and finite actions, it is well known that a perfect equilibrium is always admissible, that is, it can not be in weakly dominated strategies. However, this result does not necessarily hold in general. For finite-player games with infinitely many actions, an admissible perfect equilibrium may fail to exist; see Example 2.1 in Simon and Stinchcombe (1995). Furthermore, Rath (1998) constructed a large game with finite actions, in which there is a perfect equilibrium that is not admissible; see Section 5 therein. In the following, we construct a large game with infinitely many actions, motivated by Example 2.1 in Simon and Stinchcombe (1995), in which the unique strong perfect equilibrium is not admissible. This suggests that the existence and the admissibility may not be compatible in large games with infinitely many actions.

Example 3. Let the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ be the player space and $A = [0, \frac{1}{2}]$ the common action space. The common payoff function for each player i is $u(a_i, \xi) = \int_0^{\frac{1}{2}} v(a_i, y)\, \mathrm{d}\xi(y)$ where a_i denotes player i's action, ξ is a societal summary, and $v(\cdot, \cdot)$ is a continuous function on $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$ given by:

$$v(x, y) = \begin{cases} x, & \text{if } x \leq \frac{1}{2}y, \\ \frac{y(1-x)}{2-y}, & \text{if } \frac{1}{2}y < x. \end{cases}$$

Claim 3. The unique strong perfect equilibrium for this game is f, where $f(i) \equiv 0$ (modulo non-negligible subsets of players). However, the strategy 0 is weakly dominated for each player.

This claim shows that an admissible perfect equilibrium may fail to exist for large games with infinitely many actions. Therefore, an admissible perfect equilibrium may not exist even when the players have a common payoff function.[16]

In finite-player games, as suggested by Simon and Stinchcombe (1995), the limit admissibility could be an appropriate alternative: a strategy is limit admissible if it puts no mass on the interior of the set of weakly dominated strategies. Thus, in the framework of large games, one may turn to the *limit admissibility* to resolve this incompatibility.

Definition 6 (Limit admissibility)**.** A pure strategy $a_i \in A_i$ is said to be a *weakly dominated strategy* for player i if there exists a behavioral strategy $\mu_i \in \mathcal{M}(A_i)$ such that $u_i(a_i, \mu) \leq u_i(\mu_i, \mu)$ for every $\mu \in \mathcal{M}(A)$ and $u_i(a_i, \bar{\mu}) < u_i(\mu_i, \bar{\mu})$ for some $\bar{\mu} \in \mathcal{M}(A)$, where $u_i(\mu_i, \mu) = \int_{A_i} u_i(a_i, \mu)\, \mathrm{d}\mu_i(a)$. For each $i \in I$, let Θ_i be the set of weakly dominated strategies for player i.

- A strategy profile f is said to be *admissible* if for λ-almost all $i \in I$, $f(i) \in \Theta_i^c$, where Θ_i^c is the complement of the set Θ_i relative to A_i.
- A strategy profile f is said to be *limit admissible* if for λ-almost all $i \in I$, $f(i) \in (\Theta_i^o)^c$, where Θ_i^o is the interior of the set Θ_i and $(\Theta_i^o)^c$ is the complement of the set Θ_i^o relative to A_i.

[16] The nowhere equivalence condition is trivially satisfied.

Furthermore, two pure strategies a_i and $\bar{a}_i$ in A_i are said to be *equivalent* if $u_i(a_i, \mu) = u_i(\bar{a}_i, \mu)$ for all $\mu \in \mathcal{M}(A)$.

Remark 2. In a large game, a "limit admissible strategy" is the same as a "admissible strategy".

Under the nowhere equivalence condition, the following result shows that the existence and limit admissibility of perfect equilibria are compatible in large games with infinitely many actions.

Theorem 2. *Every $\mathcal{G}$-measurable large game has an $\mathcal{F}$-measurable limit admissible strong perfect equilibrium in pure strategies if and only if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.*

2.5 Proper equilibria of large games

In this section,[17] we consider the notion of proper equilibrium, which further refines the notion of trembling hand perfect equilibrium by assuming that more costly trembles are made with significantly smaller probability than less costly ones. To define a proper equilibrium of a large game, we follow the approach of finite partitions adopted by Simon and Stinchcombe (1995).

Given a finite Borel measurable partition $\mathcal{P} = \{\mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^m\}$ of the action space A, let $\mathcal{P}_i^j = \mathcal{P}^j \cap A_i$ for each $i \in I$ and $j = 1, 2, \ldots, m$. Denote $\mathcal{P}_i = \{\mathcal{P}_i^j \mid j = 1, 2, \ldots, m \text{ and } \mathcal{P}_i^j \neq \emptyset\}$. We first define the ϵ-proper equilibrium relative to a finite partition $\mathcal{P}$, which requires the relative weight condition (Condition (2) in Definition 7) to hold for the finite partition $\mathcal{P}$.

Definition 7 (ϵ-proper equilibrium relative to a finite partition $\mathcal{P}$)**.** Let $\epsilon > 0$ and $\mathcal{P} = \{\mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^m\}$ be a finite measurable partition of the action space A. A (behavioral strategy) strong (resp. weak) ϵ-proper equilibrium relative to $\mathcal{P}$, $g = g^\epsilon(\mathcal{P})$, is an $\mathcal{F}$-measurable (behavioral) strategy profile with full support such that

(1) g is a strong (resp. weak) ϵ-perfect equilibrium,

[17]We are grateful to the anonymous referee for suggesting us to study proper equilibria in large games with infinitely many actions.

(2) for λ-almost all $i \in I$, if $u_i\big(\mathcal{P}_i^\ell, \int_I g(i)\,\mathrm{d}\lambda(i)\big) \ll u_i\big(\mathcal{P}_i^k, \int_I g(i)\,\mathrm{d}\lambda(i)\big)$ for $\mathcal{P}_i^\ell$ and $\mathcal{P}_i^k$ in $\mathcal{P}_i$, then $g(i;\mathcal{P}_i^\ell) \le \epsilon \cdot g(i;\mathcal{P}_i^k)$, where $u_i\big(\mathcal{P}_i^\ell, \int_I g(i)\,\mathrm{d}\lambda(i)\big) \ll u_i\big(\mathcal{P}_i^k, \int_I g(i)\,\mathrm{d}\lambda(i)\big)$ means

$$\sup_{a'\in\mathcal{P}_i^\ell} u_i\big(a', \textstyle\int_I g(i)\,\mathrm{d}\lambda(i)\big) < \inf_{a''\in\mathcal{P}_i^k} u_i\big(a'', \textstyle\int_I g(i)\,\mathrm{d}\lambda(i)\big).$$

Let $\mathrm{Pro}^s(\epsilon, \mathcal{P})$ (resp. $\mathrm{Pro}^w(\epsilon, \mathcal{P})$) denote the set of all $\mathcal{F}$-measurable behavioral strategy strong (resp. weak) ϵ-proper equilibria relative to $\mathcal{P}$.

Definition 8 (Proper equilibrium relative to a finite partition $\mathcal{P}$)**.** Let $\mathcal{P} = \{\mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^m\}$ be a finite measurable partition of the action space A. A pure (resp. behavioral) strategy strong proper equilibrium relative to $\mathcal{P}$, $g = g(\mathcal{P})$, is an $\mathcal{F}$-measurable pure (resp. behavioral) strategy profile such that there exists a sequence $\{g^n\}_{n\in\mathbb{Z}_+}$ where

(3) each g^n is a strong ϵ_n-proper equilibrium relative to $\mathcal{P}$ with $\epsilon_n \to 0$ as n approaches infinity,

(4) for λ-almost all $i \in I$, $g(i) \in \operatorname{supp} \operatorname{Ls} g^n(i)$ (resp. $\operatorname{Ls} g^n(i)$),

(5) $\lim_{n\to\infty} \int_I g^n(i)\,\mathrm{d}\lambda(i) = \lambda g^{-1}$ (resp. $\int_I g(i)\,\mathrm{d}\lambda(i)$).

Let $\mathrm{Pro}^s(\mathcal{P})$ denote the set of all $\mathcal{F}$-measurable behavioral strategy strong proper equilibria relative to $\mathcal{P}$.

The definition of weak proper equilibria relative to $\mathcal{P}$ is similar, and the set of all $\mathcal{F}$-measurable behavioral strategy weak proper equilibria relative to $\mathcal{P}$ is denoted by $\mathrm{Pro}^w(\mathcal{P})$.

The following definition requires that the set of proper equilibria does not depend on any particular finite partition by "anchoring" the finite partitions. Given a finite measurable partition $\mathcal{P} = \{\mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^m\}$ of the action space A, the diameter of $\mathcal{P}$ is defined to be the maximum of the diameter of its elements, *i.e.*, $\mathrm{diam}(\mathcal{P}) = \max_{j=1,2,\ldots,m} \mathrm{diam}(\mathcal{P}^j) = \max_{j=1,2,\ldots,m} \sup_{a,b\in\mathcal{P}^j} d(a, b)$ where $d(\cdot,\cdot)$ is a metric on A. Let $\beta > 0$. We further denote

$$\mathrm{Pro}^s(\beta) = \cup_{\{\mathcal{P}:\ \mathrm{diam}(\mathcal{P})=\beta\}} \mathrm{Pro}^s(\mathcal{P}) \text{ and } \mathrm{Pro}^w(\beta) = \cup_{\{\mathcal{P}:\ \mathrm{diam}(\mathcal{P})=\beta\}} \mathrm{Pro}^w(\mathcal{P}).$$

Definition 9 (Proper equilibrium)**.** A pure (resp. behavioral) strategy strong proper equilibrium g is an $\mathcal{F}$-measurable pure (resp. behavioral) strategy profile such that there exists a sequence $\{g^n\}_{n\in\mathbb{Z}_+}$ where

(6) each g^n is a strong proper equilibrium relative to a finite measurable partition with the diameter β_n, where $\beta_n \to 0$ as n approaches infinity,

(7) for λ-almost all $i \in I$, $g(i) \in \operatorname{supp} \operatorname{Ls} g^n(i)$ (resp. $\operatorname{Ls} g^n(i)$),

(8) $\lim\limits_{n\to\infty} \int_I g^n(i)\,\mathrm{d}\lambda(i) = \lambda g^{-1}$ (resp. $\int_I g(i)\,\mathrm{d}\lambda(i)$).

The definition of weak proper equilibria is similar.

Remark 3. Simon and Stinchcombe (1995) required a strong (resp. weak) proper equilibrium g to be in $\cap_{\mathcal{P}} \operatorname{Pro}^s(\mathcal{P})$ (resp. $\cap_{\mathcal{P}} \operatorname{Pro}^w(\mathcal{P})$), where the intersection is taken over all possible finite measurable partitions. However, $\operatorname{Pro}^s(\mathcal{P})$ may not be a closed set in our setting, and hence $\cap_{\mathcal{P}} \operatorname{Pro}^s(\mathcal{P})$ may not be well defined.

Under the nowhere equivalence condition, we have the following existence result.

Theorem 3. *Every $\mathcal{G}$-measurable large game G has an $\mathcal{F}$-measurable pure strategy strong proper equilibrium if and only if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.*

2.6 Condition of best response robustness

In this section, we will consider the condition of best response robustness, which is motivated by the boundary condition in Rath (1998). We follow the notation in the previous sections and restrict our attention to games with a common action space and strong perfect equilibria unless otherwise stated explicitly.

Definition 10 (Best response robustness)**.** A large game G is said to *have robust best response with respect to a given societal aggregate $\mu \in \mathcal{M}(A)$* if there exists an open neighborhood V_μ of μ such that $a \in \operatorname{Br}_i(\mu)$ implies $a \in \operatorname{Br}_i(\tau)$ for all $\tau \in V_\mu$ and for λ-almost all $i \in I$.

When the best response with respect to μ is robust, it means that every best response of μ is a best response with respect to any societal aggregate with small perturbation of μ. The following proposition shows that the best response robustness condition is sufficient for a Nash equilibrium to be a perfect equilibrium.

Proposition 2. *Let f be a pure strategy Nash equilibrium in a large game $G\colon (I, \mathcal{G}, \lambda) \to \mathcal{U}_A$. Suppose that either (1) the societal aggregate $\lambda f^{-1} \in \mathcal{M}(A)^{fs}$ or (2) the societal aggregate $\lambda f^{-1} \notin \mathcal{M}(A)^{fs}$ and the game G has robust response with respect to λf^{-1}. Then f is a perfect equilibrium. Moreover, there exists a sequence of ϵ_n-perfect equilibria which converges almost everyone to f (ϵ_n converges to zero as n approaches infinity).*

Now we revisit Example 2. The strategy profile $f^2 \equiv 0$, which induces the Dirac measure on the point 0, is a pure strategy Nash equilibrium. For any $\mu \in \mathcal{M}(A)$ with small perturbation of δ_0 (the Dirac probability measure at the point 0), as shown in Example 2, the action -1 is the unique best response for player $i \in [\frac{1}{2}, 1]$. That is, the condition of best response robustness is violated there.

In the definition of perfect equilibria, we do not require that the perfect equilibrium is a limit of a sequence of ϵ-perfect equilibria. However, the following result, which is an extension of Theorem 1(b) in Rath (1998), shows that every perfect equilibrium can be a limit of a sequence of ϵ-perfect equilibria under the condition of best response robustness.

Proposition 3. *Suppose that for any societal aggregate $\mu \in \mathcal{M}(A) \setminus \mathcal{M}(A)^{fs}$, the large game G has robust best response with respect to μ. Then every pure strategy perfect equilibrium of G is the almost everyone limit of a sequence of ϵ-perfect equilibria. That is, for every pure strategy perfect equilibrium g^*, there is a sequence $\{g^n\}_{n \in \mathbb{Z}_+}$ such that (1) each g^n is an ϵ_n-perfect equilibrium with $\epsilon_n \to 0$ as n approaches infinity; (2) g^n converges almost everyone to g^*; (3) $\lim\limits_{n \to \infty} \int_I g^n(i)\, \mathrm{d}\lambda(i) = \lambda(g^*)^{-1}$.*

Rath (1998) also studied a stronger condition that the order of actions with respect to a given societal aggregate is robust: for a societal aggregate μ, there exists an open neighborhood V_μ of μ such that $G(i)(a, \mu) \geq G(i)(b, \mu)$ implies $G(i)(a, \tau) \geq G(i)(b, \tau)$ for each $\tau \in V_\mu$ and for λ-almost all player $i \in I$. It is a mild boundary condition when the action space is finite. However, such a condition is quite strong when the action space is infinite. The following proposition illustrates the restrictiveness of this condition under infinite action space.

Proposition 4. *Suppose that for any societal aggregate $\mu \in \mathcal{M}(A) \setminus \mathcal{M}(A)^{fs}$, there exists an open neighborhood V_μ of μ such that $G(i)(a, \mu) \geq G(i)(b, \mu)$ implies $G(i)(a, \tau) \geq G(i)(b, \tau)$ for each $\tau \in V_\mu$ and for λ-almost all player $i \in I$. Suppose that A is an infinite compact metric space. Then for each i and any two actions a and b of player i, then one and only one of the following statements holds: a weakly dominates b, b weakly dominates a, or a and b are equivalent.*

Notice that the proposition above may fail if there are only finite actions; see the following simple example.

Example 4. Let the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ be the player space, and the set $A = \{0, 1\}$ the common action space. Thus, a societal aggregate can be described as a two-dimensional vector $\mu = (\mu_0, \mu_1)$, where μ_a is the proportion of players choosing action a ($a = 0, 1$). The common symmetric payoff function is defined as follows:

$$u(a, \mu) = \begin{cases} \mu_0, & \text{if } a \in [0, \frac{1}{2}], \\ 1 - \mu_0, & \text{if } a \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, either 0 or 1 will be dominated by the other under some particular societal aggregates.

Although the best response robustness condition can guarantee a Nash equilibrium to be a perfect equilibrium, the equilibrium strategy could be weakly dominated. This point has been illustrated in (Rath, 1998, Section 4). Here we provide a simple example.[18]

Example 5. Let the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ be the player space, and the set $A = [0, 1]$ the common action space. The symmetric payoff function is $u_i(a, \mu) = a \cdot \rho^w(\mu, \eta)$, where η is the uniform distribution (*i.e.*, the Lebesgue measure) on $[0, 1]$.

[18]The authors thank Lei Qiao for suggesting the idea of this example.

Let us consider the following two Nash equilibria: (1) everyone plays the dominant strategy 1, (2) for each i, player i chooses the strategy i. Clearly, this large game satisfies the best response robustness condition, and hence these two Nash equilibria are perfect equilibria. However, the strategy adopted by player i in the second equilibrium is a weakly dominated strategy.

2.7 Proofs

2.7.1 Regular conditional distribution

In this subsection, we state the definitions of the transition probability and the regular conditional distribution here for the convenience of readers.

Recall that $(I, \mathcal{F}, \lambda)$ is an atomless probability space and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. Given a measurable subset E, $\lambda|_E$ denotes the restriction of λ to E. Let X be a Polish space (complete separable metrizable topological space), $\mathcal{B}(X)$ the Borel σ-algebra of X, and $\mathcal{M}(X)$ the space of all Borel probability measures on X with the topology of weak convergence. We recall that $\mathcal{M}(X)$ is again a Polish space, and if X is compact then so is $\mathcal{M}(X)$. We use $C_b(X)$ to denote the set of all bounded continuous function from X to $\mathbb{R}$.

Definition 11. A $\mathcal{G}$-measurable transition probability from I to X is a mapping $\phi\colon I \to \mathcal{M}(X)$ such that for every $B \in \mathcal{B}(X)$ the mapping

$$\phi(\cdot; B)\colon i \mapsto \phi(i; B)$$

is $\mathcal{G}$-measurable, where $\phi(i; B)$ is the value of the probability measure $\phi(i)$ on the Borel subset $B \subseteq A$.

We use $\mathcal{R}^{\mathcal{G}}(X)$, or $\mathcal{R}^{\mathcal{G}}$ when it is clear, to denote the set of all $\mathcal{G}$-measurable transition probabilities from I to X. Let $\mathbb{Z}_+$ denote the set of positive integers.

Definition 12. A sequence $\{\phi_n\}_{n\in\mathbb{Z}_+}$ in $\mathcal{R}^{\mathcal{G}}$ is said to converge weakly to ϕ in $\mathcal{R}^{\mathcal{G}}$, denoted by

$\phi_n \Rightarrow \phi$, if for every bounded Carathéodory function $c\colon I \times X \to \mathbb{R}$,[19]

$$\lim_{n\to\infty} \int_I \left[\int_X c(i,x)\phi_n(i;\mathrm{d}x)\right] \mathrm{d}\lambda(i) = \int_I \left[\int_X c(i,x)\phi(i;\mathrm{d}x)\right] \mathrm{d}\lambda(i).$$

The weak topology on $\mathcal{R}^{\mathcal{G}}$ is defined as the weakest topology for which the functional

$$\phi \mapsto \int_I \left[\int_X c(i,x)\phi(i;\mathrm{d}x)\right] \mathrm{d}\lambda(i)$$

is continuous for every bounded Carathéodory function $c\colon I \times X \to \mathbb{R}$.

We next review the regular conditional distribution. Let f be an $\mathcal{F}$-measurable mapping from I to X. A mapping $\mu^{f|\mathcal{G}}\colon I \times \mathcal{B}(X) \to [0,1]$ is said to be a regular conditional distribution for f given $\mathcal{G}$, if

1. for λ-almost all $i \in I$, $\mu^{f|\mathcal{G}}(i,\cdot)$ is a probability measure on X,
2. for each Borel subset $B \subseteq X$, $\mu^{f|\mathcal{G}}(i,B) = \mathbf{E}[1_B(f) \mid \mathcal{G}](i)$ for λ-almost all $i \in I$, where $\mathbf{E}[1_B(f) \mid \mathcal{G}]$ is the conditional expectation of the indicator function $1_B(f)$ given $\mathcal{G}$.

Since X is assumed to be a Polish space endowed with the Borel σ-algebra, the regular conditional distribution for f given $\mathcal{G}$ always exists; see Theorem 5.1.9 in Durrett (2010).

Let F be an $\mathcal{F}$-measurable correspondence from I to X. We use

$$\mathcal{R}_F^{(\mathcal{F},\mathcal{G})} = \left\{\mu^{f|\mathcal{G}} \;\middle|\; f \text{ is an } \mathcal{F}\text{-measurable selection of } F\right\}$$

to denote the set of regular conditional distributions induced by $\mathcal{F}$-measurable selections of F conditional on $\mathcal{G}$.

The following results on regular conditional distributions of correspondences are stated as a lemma here for the convenience of readers.

Lemma 1.

[19]Given a probability space $(I,\mathcal{G},\lambda)$ and a Polish space X, a function $c\colon I \times X \to \mathbb{R}$ is a Carathéodory function if $c(\cdot,x)$ is $\mathcal{G}$-measurable for each $x \in X$ and $c(i,\cdot)$ is continuous for each $i \in I$.

1. *(Theorem 3(C5) in He and Sun (2013)) Suppose that $(I, \mathcal{F}, \lambda)$ is atomless and $\mathcal{G}$ is a countably generated sub-σ-algebra of $\mathcal{F}$. If $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$, then for any $g \in \mathcal{R}^{\mathcal{G}}$, there exists an $\mathcal{F}$-measurable mapping g^* such that $\mu^{g^*|\mathcal{G}} = g$.*

2. *(Corollary 1 in He and Sun (2013)) Given $g \in \mathcal{R}^{\mathcal{G}}$ and an $\mathcal{F}$-measurable mapping g^*, if $\mu^{g^*|\mathcal{G}} = g$, then for λ-almost all $i \in I$, $g^*(i) \in \operatorname{supp} g(i)$.*

3. *(Corollary 8(B8) in He (2014, p.67)) Let Z be a separable Banach space endowed with the norm $\|\cdot\|$. Suppose that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ and G is a p-integrably bounded,[20] weak* compact valued correspondence from $(I, \mathcal{G}, \lambda)$ to Z^* with $1 \leq p < \infty$. Let F be a weak* closed valued correspondence from $I \times Y$ to Z^* where Y is a metric space. If*

   - *$F(i, y) \subseteq G(i)$ for λ-almost all $i \in I$;*
   - *for each $y \in Y$, $F(\cdot, y)$ is $\mathcal{G}$-measurable;*
   - *for each $i \in I$, $F(i, \cdot)$ is weak* upper-hemicontinuous;*

   then $H(y) = I^{\mathcal{F}}_{F(\cdot,y)}$ is weak upper-hemicontinuous, where*

   $$I^{\mathcal{F}}_{F(\cdot,y)} = \left\{ \int_I f \,\mathrm{d}\lambda(i) \mid f \text{ is an } \mathcal{F}\text{-measurable Gelfand integrable selection of } F(\cdot, y) \right\}.$$

2.7.2 Proofs of results in Section 2.2

Proof of Proposition 1. We show that a pure strategy weak perfect equilibrium is a pure strategy Nash equilibrium at first.

Let g be an $\mathcal{F}$-measurable pure strategy weak perfect equilibrium, and $\{g^n\}_{n \in \mathbb{Z}_+}$ a sequence of ϵ_n-perfect equilibria satisfying the three corresponding conditions in Definition 3. Thus, for λ-almost all i, there is a probability measure μ_i such that $\mu_i \in \operatorname{Ls} g^n(i)$ and $g(i) \in \operatorname{supp} \mu_i$. Hence, there is a subsequence of $\{g^n(i)\}_{n \in \mathbb{Z}_+}$, say $\{g^n(i)\}_{n \in \mathbb{Z}_+}$ itself for simplicity, such that $\mu_i = \lim_{n \to \infty} g^n(i)$.

[20] A correspondence G is said to be p-integrably bounded if there is a real-valued function h such that h^p is integrable and $\sup\{\|x\| \mid x \in G(i)\} \leq h(t)$ for almost all i.

For each $n \in \mathbb{Z}_+$, since g^n is a weak ϵ_n-perfect equilibrium, we have

$$\rho^w\big(g^n(i), \mathcal{M}(\mathrm{Br}_i(g^n))\big) < \epsilon_n,$$

which implies that $g^n(i)\big(\mathrm{Br}_i(g^n)^{\epsilon_n}\big) \geq 1 - \epsilon_n$ for λ-almost all i, where $\mathrm{Br}_i(g^n)^{\epsilon_n}$ is the ϵ_n-neighborhood of the Borel measurable subset $\mathrm{Br}_i(g^n)$.

Hence for any n and N with $N \leq n$, we have

$$g^n(i)\left(\cap_{k=1}^{N}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)^{\epsilon_j}}\right) \geq 1 - \epsilon_n.$$

Since $\mu_i = \lim_{n\to\infty} g^n(i)$, letting n go to infinity, we get

$$\mu_i\left(\cap_{k=1}^{N}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)^{\epsilon_j}}\right) \geq 1 \text{ for all } N.$$

Then letting N go to infinity, we have

$$\mu_i\left(\cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)^{\epsilon_j}}\right) \geq 1. \tag{2.1}$$

We claim that:

(i) $\cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)^{\epsilon_j}} = \cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)}$ if ϵ_n goes to 0 as n approaches infinity.

(ii) $\cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)} \subseteq \mathrm{Br}_i(g)$.

Based on these two claims and Equation (2.1), we have

$$\mu_i(\mathrm{Br}_i(g)) = 1,$$

where $\mathrm{Br}_i(g)$ is a compact set. Thus, $\mathrm{supp}\,\mu_i$ is a subset of $\mathrm{Br}_i(g)$. Moreover, since $g(i) \in \mathrm{supp}\,\mu_i$, $g(i) \in \mathrm{Br}_i(g)$ for λ-almost all i. Therefore, g is a pure strategy Nash equilibrium.

For a behavioral strategy weak perfect equilibrium g, following the above proof by replacing μ_i with $g(i)$, we have $g(i)\big(\mathrm{Br}_i(g)\big) = 1$ for λ-almost all i. Thus, g is a behavioral strategy Nash equilibrium.

Now we need to prove the validity of Claims (i) and (ii).

Proof of Claim (i): It suffices to show $\cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)^{\epsilon_j}} \subseteq \cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)}$ if $\epsilon_n \to 0$ as $n \to \infty$. For any $x \in \cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)^{\epsilon_j}}$ and for any $\epsilon > 0$, we have

$$B(x,\epsilon) \cap \left(\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)^{\epsilon_j}\right) \neq \emptyset \text{ for each } k \in \mathbb{Z}_+,$$

where $B(x,\epsilon)$ is the ϵ-neighborhood of x. Since ϵ_n goes to zero as n approaches infinity, there exists an positive integer N_1 such that $\epsilon_n < \epsilon$ whenever $n > N_1$. For each $N_2 \in \mathbb{Z}_+$, it is clear that

$$B(x,\epsilon) \cap \left(\cup_{j=\max\{N_1,N_2\}}^{\infty}\mathrm{Br}_i(g^j)^{\epsilon_j}\right) \neq \emptyset,$$

which implies that there exists an positive integer $\ell \geq \max\{N_1, N_2\}$ such that $B(x,\epsilon) \cap \mathrm{Br}_i(g^\ell)^{\epsilon_\ell} \neq \emptyset$. By the choice of N_1, we know that $B(x, 2\epsilon) \cap \mathrm{Br}_i(g^\ell) \neq \emptyset$. Therefore, $x \in \mathrm{Ls}\,\mathrm{Br}_i(g^n) = \cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)}$.

Proof of Claim (ii): Since A_i is compact, there exists a countable dense set $\{\bar{a}_m\}_{m\in\mathbb{Z}_+} \subseteq A_i$. For any $a \in \cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)}$, let $a = \lim_{n\to\infty} a_n$ where $a_n \in \mathrm{Br}_i(g^n)$. Hence,

$$u_i\big(a, \lambda g^{-1}\big) = \lim_{n\to\infty} u_i\big(a_n, \textstyle\int_I g^n \,\mathrm{d}\lambda\big) \geq \lim_{n\to\infty} u_i\big(\bar{a}_m, \textstyle\int_I g^n \,\mathrm{d}\lambda\big) = u_i\big(\bar{a}_m, \lambda g^{-1}\big),$$

for all $\bar{a}_m$ and λ-almost all i. Thus, $a \in \mathrm{Br}_i(g)$ and $\cap_{k=1}^{\infty}\overline{\cup_{j=k}^{\infty}\mathrm{Br}_i(g^j)} \subseteq \mathrm{Br}_i(g)$. □

Proof of Claim 1. If $\xi(\{1\}) = 0$, then $u_i(1,\xi) = u_i(0,\xi)$ and if $\xi(\{1\}) > 0$, then $u_i(1,\xi) > u_i(0,\xi)$. Thus, for each player, 1 weakly dominates 0. It is easy to see that there are two Nash equilibria: (1) $f^1(i) = 0$ for all i and (2) $f^2(i) = 1$ for all i.

For each $\epsilon \in (0, \frac{1}{2})$ and each player $i \in I$, let $g^\epsilon(i) = (1-\epsilon, \epsilon)$ be a behavioral strategy of player i, that is, she has probabilities $1-\epsilon$ and ϵ to choose actions 1 and 0 respectively. Then it is clear that g^ϵ is an ϵ-perfect equilibrium. Since g^ϵ converges weakly to f^2 as ϵ goes to zero, f^2 is a perfect equilibrium.

However, we can not find a sequence of ϵ-perfect equilibria converging to f^1. To show it, we suppose that $h^\epsilon = (h_1^\epsilon, h_2^\epsilon)$ is a sequence of ϵ-perfect equilibria converging to f^1, where h_1^ϵ and h_2^ϵ specify the probabilities choosing actions 1 and 0 respectively. Then for η-almost all i, $h_1^\epsilon(i) \neq 0$ should be close to 0. However, for each player i, the action 1 is the best choice given h^ϵ, which leads to a contradiction. Thus, f^1 is not a perfect equilibrium. □

Proof of Claim 2. For each player $i \in [\frac{1}{2}, 1]$, for any societal response ξ, her unique best response is 0. Thus, the societal response ξ induced by a Nash equilibrium should be in the form of $\frac{1}{2}\delta_0 + \frac{1}{2}\mu$ for some $\mu \in \mathcal{M}(A)$, where δ_0 is the Dirac probability measure at 0.

For each player $i \in [0, \frac{1}{2})$, we have $u_1(0, \xi) = 0$ and

$$u_1(-1, \xi) = \tfrac{1}{2} \int_0^1 v(-1, a_2)\, \mathrm{d}\mu(a_2) + \tfrac{1}{2} v(-1, 0) = \tfrac{1}{16} \int_0^1 a_2\, \mathrm{d}\mu(a_2),$$

which is nonnegative.

For $0 < a < \frac{1}{2}$, we have

$$\begin{aligned} u_1(a, \xi) &= \textstyle\int_A v(a, a_2)\, \mathrm{d}\xi(a_2) = \tfrac{1}{2} v(a, 0) + \tfrac{1}{2} \int_A v(a, a_2)\, \mathrm{d}\mu(a_2) \\ &= \tfrac{1}{2}(0 - a) + \tfrac{1}{2}\mu(-1) v(a, -1) + \tfrac{1}{2} \textstyle\int_0^{2a} v(a, a_2)\, \mathrm{d}\mu(a_2) + \tfrac{1}{2} \int_{2a}^1 v(a, a_2)\, \mathrm{d}\mu(a_2) \\ &= -\tfrac{a}{2} \left(\mu(-1) + \textstyle\int_0^1 \mathrm{d}\mu(a_2)\right) + \tfrac{1}{2} \left(\textstyle\int_0^{2a} (a_2 - a)\, \mathrm{d}\mu(a_2) + \int_{2a}^1 a\, \mathrm{d}\mu(a_2)\right) \\ &= -\tfrac{a}{2}\mu(-1) + \tfrac{1}{2} \textstyle\int_0^{2a} (a_2 - 2a)\, \mathrm{d}\mu(a_2) \le 0, \end{aligned}$$

and the equality holds if and only if $\mu = \delta_{2a}$.

For $\frac{1}{2} \le a \le 1$, we have

$$\begin{aligned} u_1(a, \xi) &= \textstyle\int_A v(a, a_2)\, \mathrm{d}\xi(a_2) = \tfrac{1}{2} v(a, 0) + \tfrac{1}{2} \int_A v(a, a_2)\, \mathrm{d}\mu(a_2) \\ &= \tfrac{1}{2}(0 - a) + \tfrac{1}{2}\mu(-1) v(a, -1) + \tfrac{1}{2} \textstyle\int_0^1 v(a, a_2)\, \mathrm{d}\mu(a_2) \\ &= -\tfrac{a}{2} \left(\mu(-1) + \textstyle\int_0^1 \mathrm{d}\mu(a_2)\right) + \tfrac{1}{2} \textstyle\int_0^1 (a_2 - a)\, \mathrm{d}\mu(a_2) \\ &= -\tfrac{a}{2}\mu(-1) + \tfrac{1}{2} \textstyle\int_0^1 (a_2 - 2a)\, \mathrm{d}\mu(a_2) \le \tfrac{1-2a}{2}, \end{aligned}$$

and the equality holds if and only if $\mu = \delta_1$.

When $\mu = \delta_{2a}$ for some $a \in (0, \frac{1}{2})$, we have $u_1(-1, \xi) = \frac{1}{2}v(-1,0) + \frac{1}{2}v(-1,2a) = \frac{a}{8} > 0$, and when $\mu = \delta_1$ (the Dirac probability measure at 1), we have $u_1 = (-1, \xi) = \frac{1}{2}v(-1,0) + \frac{1}{2}v(-1,1) = \frac{1}{16} > 0$. Thus, for player $i \in [0, \frac{1}{2})$, any action $a \in [0,1)$ can not be a best response. That is, the possible best responses of player $i \in [0, \frac{1}{2})$ are -1 and 0, which implies that $\operatorname{supp}\mu \subseteq \{-1, 0\}$. Hence, $u_1(-1,\xi) = \frac{1}{16}\int_0^1 a_2 \, \mathrm{d}\mu(a_2) = 0 = u_1(0,\xi)$. The actions -1 and 0 are the two best responses of each player $i \in [0, \frac{1}{2})$.

(1) To show that f^1 is the unique strong perfect equilibrium, we shall prove that in any strong ϵ-perfect equilibrium, the action -1 is the unique best response for each player $i \in [0, \frac{1}{2})$ given the other players' equilibrium strategies.

For each $\epsilon > 0$, let g^ϵ be a strong ϵ-perfect equilibrium. Denote $\frac{g_1^\epsilon}{2} = \int_0^{1/2} g^\epsilon(i) \, \mathrm{d}\eta(i)$ and $\frac{g_2^\epsilon}{2} = \int_{1/2}^1 g^\epsilon(i) \, \mathrm{d}\eta(i)$. Obviously, g_1^ϵ and g_2^ϵ are two probability measures on A with full support. As discussed above, for each $i \in [\frac{1}{2}, 1]$, 0 is the unique best action of player i. Due to the definition of the strong perfect equilibrium, we have $g^\epsilon(i)(\{0\}) \geq 1 - \epsilon$ for $i \in [\frac{1}{2}, 1]$.

For each player $i \in [0, \frac{1}{2})$, we consider the following three cases. First, suppose that $a \in [0, \frac{1}{2})$. Since $g_2^\epsilon(\{0\}) \geq 1 - \epsilon$, we have

$$\int_A v(a, a_2)\frac{g_2^\epsilon}{2}(\mathrm{d}a_2) \leq \frac{1}{2}g_2^\epsilon(\{0\}) \cdot v(a, 0) + \frac{\epsilon}{2}a \leq \frac{1}{2}(-a)(1-\epsilon) + \frac{1}{2}a\epsilon = -\frac{a}{2} + a\epsilon.$$

Furthermore, we have

$$\begin{aligned} u_1(a, \textstyle\int g^\epsilon) &= \int_A v(a, a_2)\frac{g_1^\epsilon}{2}(\mathrm{d}a_2) + \int_A v(a, a_2)\frac{g_2^\epsilon}{2}(\mathrm{d}a_2) \\ &\leq \int_A v(a, a_2)\frac{g_1^\epsilon}{2}(\mathrm{d}a_2) - \frac{a}{2} + a\epsilon \\ &= \int_0^{2a} (a_2 - a)\frac{g_1^\epsilon}{2}(\mathrm{d}a_2) + \int_{2a}^1 a\frac{g_1^\epsilon}{2}(\mathrm{d}a_2) - \frac{a}{2} + a\epsilon \\ &= \int_0^{2a} (a_2 - 2a)\frac{g_1^\epsilon}{2}(\mathrm{d}a_2) - a\frac{g_1^\epsilon}{2}(\{-1\}) + a\epsilon, \end{aligned}$$

which is in turn less than zero when ϵ is sufficiently small.

Second, suppose that $a \in [\frac{1}{2}, 1]$. Then we have

$$\int_A v(a, a_2) \frac{g_2^\epsilon}{2}(\mathrm{d}a_2) \leq \frac{1}{2} g_2^\epsilon(\{0\}) \cdot v(a, 0) + \frac{\epsilon}{2}(1-a) \leq \frac{1}{2}(1-\epsilon) \cdot (-a) + \frac{\epsilon}{2}(1-a) = \frac{\epsilon - a}{2},$$

and hence

$$u_1(a, \int g^\epsilon) \leq \int_0^1 (a_2 - 2a) \frac{g_1^\epsilon}{2}(\mathrm{d}a_2) - \frac{a}{2} g_1^\epsilon(\{-1\}) + \frac{\epsilon}{2} < 0,$$

when ϵ is sufficiently small.

Lastly, suppose that $a = -1$. Since g_1^ϵ and g_2^ϵ have full support on A, we have

$$u_1(-1, \int g^\epsilon) = \int_0^1 \frac{1}{8} a_2 \frac{g_1^\epsilon}{2}(\mathrm{d}a_2) + \int_0^1 \frac{1}{8} a_2 \frac{g_2^\epsilon}{2}(\mathrm{d}a_2),$$

which is strictly larger than 0.

Therefore, for $i \in [0, \frac{1}{2})$, the action -1 is player i's unique best action, and hence $g^\epsilon(i)(\{-1\}) > 1 - \epsilon$. Let ϵ go to zero. Then we get the unique strong perfect equilibrium f^1, where $f^1(i) = -1$ for each $i \in [0, \frac{1}{2})$ and $f^1(i) = 0$ for each $i \in [\frac{1}{2}, 1]$.

(2) In the following, we shall show that $f^2(i) \equiv 0$ is a weak perfect equilibrium. Let ν be a probability measure on A with full support. For each $\epsilon > 0$, consider the strategy profile g^ϵ: for each player $i \in [0, 1]$,

$$g^\epsilon(i) = \mu := (1-\epsilon)\delta_\epsilon + \epsilon\nu,$$

where δ_ϵ is the Dirac probability measure on A at the point ϵ. It suffices to show that the distance between the societal response μ and the set of best responses of player $i \in [0, \frac{1}{2})$ is less than $k\epsilon$ for some $k \in \mathbb{Z}_+$.

For each player $i \in [0, \frac{1}{2})$, if she chooses the action -1, then her payoff is

$$u_1(-1, \int g^\epsilon) = \int_A \frac{1}{8} a_2 \mu(\mathrm{d}a_2) = (1-\epsilon)\frac{\epsilon}{8} + \epsilon \int_A \frac{a_2}{8} \nu(\mathrm{d}a_2) < \frac{\epsilon}{4},$$

when ϵ is sufficiently small. If she chooses $\frac{\epsilon}{2}$, then her payoff is

$$\begin{aligned} u_1(\tfrac{\epsilon}{2}, \textstyle\int g^\epsilon) &= \textstyle\int_A v(\frac{\epsilon}{2}, a_2)\mu(\mathrm{d}a_2) = \int_0^\epsilon v(\frac{\epsilon}{2}, a_2)\mu(\mathrm{d}a_2) + \int_\epsilon^1 v(\frac{\epsilon}{2}, a_2)\mu(\mathrm{d}a_2) \\ &> (1-\epsilon)\tfrac{\epsilon}{2} > \tfrac{\epsilon}{4}, \end{aligned}$$

when ϵ is sufficiently small. If she chooses an action $a \geq 2\epsilon$, then her payoff is

$$\begin{aligned} u_1(a, \textstyle\int g^\epsilon) &= \textstyle\int_A v(a, a_2)\mu(\mathrm{d}a_2) = (1-\epsilon)\int_0^1 v(a, a_2)\delta_\epsilon(\mathrm{d}a_2) + \epsilon\int_0^1 v(a, a_2)\nu(\mathrm{d}a_2) \\ &= (1-\epsilon)(\epsilon - a) + \epsilon\textstyle\int_0^{2a}(a_2 - a)\nu(\mathrm{d}a_2) + \epsilon\int_{2a}^1 a\nu(\mathrm{d}a_2) \\ &= (1-\epsilon)(\epsilon - a) + \epsilon\textstyle\int_0^{2a}(a_2 - 2a)\nu(\mathrm{d}a_2) + \epsilon a - \epsilon a\nu(\{-1\}) \\ &< \epsilon - a - \epsilon^2 + 2a\epsilon < -\epsilon + 3\epsilon^2 < 0, \end{aligned}$$

when ϵ is sufficiently small. Therefore, the best response of player $i \in [0, \frac{1}{2})$ should be in $[0, 2\epsilon]$ given the societal response μ. It follows that the distance between the societal response μ and the set of best responses is at most 3ϵ under the weak metric. Let ϵ go to zero. Then the limit $f^2(i) \equiv 0$ is a weak perfect equilibrium. □

2.7.3 Proof of Theorem 1

Before proving Theorem 1, we show the existence of $\mathcal{G}$-measurable strong ϵ-perfect equilibria for $\mathcal{G}$-measurable large games.

Lemma 2. For any $\epsilon \in (0, 1)$, every $\mathcal{G}$-measurable large game G has a $\mathcal{G}$-measurable strong ϵ-perfect equilibrium.

Proof. The set of $\mathcal{G}$-measurable behavioral strategy profiles, denoted by $\Psi^{\mathcal{G}}$, is the set of transition probabilities ϕ from $(I, \mathcal{G}, \lambda)$ to the set A such that $\phi(i; A_i) = 1$ for λ-almost all $i \in I$, where $\phi(i; A_i)$ is the value of the probability measure $\phi(i)$ on the subset $A_i \subseteq A$. One can show that the set of all these transition probabilities is convex and weakly compact; see Balder (1995) for example.

For any $\epsilon \in (0, 1)$, randomly pick a $\mathcal{G}$-measurable behavioral strategy profile with full support $g_0 \in \Psi^{fs}$.[21] Let

$$N(g_0) = \big\{g \in \Psi^{\mathcal{G}} \mid \text{for } \lambda\text{-almost all } i, g(i;B) \geq \tfrac{\epsilon}{3} g_0(i;B) \text{ for any } B \in \mathcal{B}(A_i)\big\},$$

and $S(g_0) = \big\{ \int_I g(i;\cdot)\, \mathrm{d}\lambda(i) \in \mathcal{M}(A) \mid g \in N(g_0)\big\}$. It is clear that $S(g_0)$ is a nonempty convex subset of $\mathcal{M}(A)$.

In the following, we will show that $S(g_0)$ is weakly closed. Let $\{\int_I g^n(i;\cdot)\, \mathrm{d}\lambda(i)\}_{n\in\mathbb{Z}_+}$ be a sequence in $S(g_0)$, which converges weakly to $\mu \in \mathcal{M}(A)$. By the definition of $S(g_0)$, we have $g^n \in N(g_0)$ for each $n \in \mathbb{Z}_+$. That is, for λ-almost all $i \in I$, $g^n(i;B) \geq \frac{\epsilon}{3} g_0(i;B)$ for any $B \in \mathcal{B}(A_i)$. Thus, $\left\{\frac{g^n(i;\cdot) - \frac{\epsilon}{3} g_0(i;\cdot)}{1-\frac{\epsilon}{3}}\right\}_{n\in\mathbb{Z}_+}$ is a sequence in $\Psi^{\mathcal{G}}$. By the weak compactness of $\Psi^{\mathcal{G}}$, there exists a $\mathcal{G}$-measurable $g' \in \Psi^{\mathcal{G}}$ such that a subsequence of $\left\{\frac{g^n(i;\cdot) - \frac{\epsilon}{3} g_0(i;\cdot)}{1-\frac{\epsilon}{3}}\right\}_{n\in\mathbb{Z}_+}$, without loss of generality say itself, converges weakly to $g'(i;\cdot)$. For each $i \in I$, let $g(i;\cdot) = (1 - \frac{\epsilon}{3}) g'(i;\cdot) + \frac{\epsilon}{3} g_0(i;\cdot)$. Thus, we have that $\{g^n\}_{n\in\mathbb{Z}_+}$ converges weakly to g. By the definition of the weak convergence of transition probabilities, $\int_I g^n(i)\, \mathrm{d}\lambda(i)$ converges weakly to $\int_I g(i)\, \mathrm{d}\lambda(i)$, which leads to $\mu = \int_I g(i)\, \mathrm{d}\lambda(i)$. This implies that $\mu \in S(g_0)$. Since $S(g_0)$ is weakly closed in the weakly compact set $\mathcal{M}(A)$, $S(g_0)$ is also weakly compact in $\mathcal{M}(A)$.

For each $i \in I$ and $\tau \in \mathcal{M}(A)$, let

$$\hat{N}(i, g_0) = \big\{\nu \in \mathcal{M}(A_i) \mid \nu(B) \geq \tfrac{\epsilon}{3} g_0(i;B) \text{ for any } B \in \mathcal{B}(A_i)\big\},$$

and

$$D_{g_0}(i,\tau) = \operatorname*{arg\,max}_{\mu \in \hat{N}(i,g_0)} \int_A u_i(a,\tau)\, \mathrm{d}\mu(a).$$

Notice that $D_{g_0}(i,\tau)$ is the set of best responses restricted to the set of behavioral strategies $\hat{N}(i, g_0)$, given the societal response τ.

Since $\hat{N}(i, g_0)$ is a weakly closed set in $\mathcal{M}(A_i)$, it is also weakly compact. Then, Berge Maximum Theorem (see Theorem 17.31 in Aliprantis and Border (2006)) implies that the correspon-

[21] Such a $\mathcal{G}$-measurable behavioral strategy profile with full support exists due to Casting Representation Theorem; see Corollary 18.14 in Aliprantis and Border (2006). In particular, there exists a sequence $\{f_n\}_{n\in\mathbb{Z}_+}$ of $\mathcal{G}$-measurable selections from the action correspondence $\mathcal{A}$ satisfying $A_i = \overline{\{f_1(i), f_2(i), \ldots\}}$ for each $i \in I$. Let $g_0(i;B) = \sum_{n\in\mathbb{Z}_+} \frac{1}{2^n} 1_B\big(f_n(i)\big)$ for each $i \in I$ and $B \in \mathcal{B}(A_i)$. Then g_0 is a $\mathcal{G}$-measurable behavioral strategy profile with full support.

dence $D_{g_0}(i,\cdot)\colon \mathcal{M}(A) \twoheadrightarrow \hat{N}(i,g_0)$ is nonempty, compact-valued and upper-hemicontinuous.

Define a correspondence $B_{g_0}\colon S(g_0) \twoheadrightarrow \mathcal{M}(A)$ as follows: for any $\tau \in S(g_0)$, let

$$B_{g_0}(\tau) = \int_I D_{g_0}(i,\tau)\,\mathrm{d}\lambda(i),$$

which is the set of integrals of all $\mathcal{G}$-measurable selections of $D_{g_0}(i,\tau)$. Then it is easy to see that the correspondence B_{g_0} is indeed from $S(g_0)$ to $S(g_0)$. Clearly, B_{g_0} is also convex and upper-hemicontinuous. Hence, by Kakutani-Fan-Glicksberg fixed-point theorem, B_{g_0} has a fixed-point, say τ^*. That is, there is a $g^* \in N(g_0)$ such that

$$\tau^* = \int_I g^*(i)\,\mathrm{d}\lambda(i), \text{ and } g^*(i) \in D_{g_0}(i,\tau^*) \text{ for } \lambda\text{-almost all } i \in I.$$

Finally, we claim that for λ-almost all $i \in I$,

$$g^*\big(i;\mathrm{Br}_i(g^*)\big) \geq 1 - \tfrac{\epsilon}{3} g_0\big(i;A_i \setminus \mathrm{Br}_i(g^*)\big) \geq 1 - \tfrac{\epsilon}{3} g_0(i;A_i) = 1 - \tfrac{\epsilon}{3}.$$

Thus, $\rho^s\big(g^*(i),\mathcal{M}(\mathrm{Br}_i(g^*))\big) < \epsilon$ for λ-almost all $i \in I$. Therefore, g^* is a $\mathcal{G}$-measurable strong ϵ-perfect equilibrium.

We prove the remaining claim by contradiction. Assume that there exists a non-negligible subset $E \in \mathcal{G}$ such that for each $i \in E$,

$$g^*\big(i;\mathrm{Br}_i(g^*)\big) < 1 - \tfrac{\epsilon}{3} g_0\big(i;A_i \setminus \mathrm{Br}_i(g^*)\big).$$

Fix a player $i \in E$, and pick an action $a_0 \in \mathrm{Br}_i(g^*)$. Denote $u_i\big(a_0, \int_I g^*(i)\,\mathrm{d}\lambda(i)\big)$ by c for simplicity, which is player i's maximum payoff given other players' strategy profiles g^*_{-i}. Without loss of generality, we assume that c is positive.[22]

[22]It suffices if some big enough constants are added to the payoff functions.

We define a new strategy for player i as follows:

$$\xi = \Big(g^*(i) + \Big(g^*\big(i; A_i \setminus \mathrm{Br}_i(g^*)\big) - \tfrac{\epsilon}{3} g_0\big(i; A_i \setminus \mathrm{Br}_i(g^*)\big)\Big) \cdot \delta_{a_0}\Big)\Big|_{\mathrm{Br}_i(g^*)} + \tfrac{\epsilon}{3} g_0(i)\Big|_{A_i \setminus \mathrm{Br}_i(g^*)}.$$

It is not difficult to see that $\xi \in \mathcal{M}(A_i)$ and $\xi(B) \geq \frac{\epsilon}{3} g_0(i; B)$ for any $B \in \mathcal{B}(A_i)$.

In the following we shall show that player i will be better off when she deviates from the strategy $g^*(i)$ to the strategy ξ defined above.

$$\begin{aligned}
&\int_{A_i} u_i\left(a, \int_I g^*(j)\,\mathrm{d}\lambda(j)\right) g^*(i; \mathrm{d}a) \\
&= \int_{\mathrm{Br}_i(g^*)} u_i\left(a, \int_I g^*\,\mathrm{d}\lambda\right) g^*(i; \mathrm{d}a) + \int_{A_i \setminus \mathrm{Br}_i(g^*)} u_i\left(a, \int_I g^*\,\mathrm{d}\lambda\right) g^*(i; \mathrm{d}a) \\
&= c \cdot g^*\big(i; \mathrm{Br}_i(g^*)\big) + \int_{A_i \setminus \mathrm{Br}_i(g^*)} u_i\left(a, \int_I g^*\,\mathrm{d}\lambda\right) g^*(i; \mathrm{d}a) \\
&< c \cdot g^*\big(i; \mathrm{Br}_i(g^*)\big) + c \cdot g^*\big(i; A_i \setminus \mathrm{Br}_i(g^*)\big) - c \cdot \tfrac{\epsilon}{3} g_0\big(i; A_i \setminus \mathrm{Br}_i(g^*)\big) \\
&\quad + \int_{A_i \setminus \mathrm{Br}_i(g^*)} u_i\left(a, \int_I g^*\,\mathrm{d}\lambda\right) \tfrac{\epsilon}{3} g_0(i; \mathrm{d}a) \\
&= \int_{\mathrm{Br}_i(g^*)} u_i\left(a, \int_I g^*\,\mathrm{d}\lambda\right) \mathrm{d}\xi(a) + \int_{A_i \setminus \mathrm{Br}_i(g^*)} u_i\left(a, \int_I g^*\,\mathrm{d}\lambda\right) \mathrm{d}\xi(a) \\
&= \int_{A_i} u_i\left(a, \int_I g^*(j)\,\mathrm{d}\lambda(j)\right) \mathrm{d}\xi(a).
\end{aligned}$$

By the definition of $D_{g_0}(i, \tau^*)$, we have $g^*(i) \notin D_{g_0}(i, \tau^*)$, which holds for every player i in E. Thus, $g^*(i)$ is not a measurable selection of the correspondence $D_{g_0}(i, \tau^*)$, which leads to a contradiction. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We first prove the sufficiency part.

Let $\{\epsilon_n\}_{n \in \mathbb{Z}_+}$ be a sequence such that $0 < \epsilon_n < 1$ and ϵ_n goes to zero as n approaches infinity. By Lemma 2, for each ϵ_n, there exists $g^n \in \Psi^{fs}$ such that g^n is a $\mathcal{G}$-measurable strong ϵ_n-perfect equilibrium. Since $\{\int_I g^n(i; \cdot)\,\mathrm{d}\lambda(i)\}_{n \in \mathbb{Z}_+}$ is a sequence in the compact metric space $\mathcal{M}(A)$, there is a subsequence, without loss of generality say itself, converges weakly to some measure $\mu \in \mathcal{M}(A)$.

By Proposition 1 in He *et al.* (2016), there exists a σ-algebra $\mathcal{H}$ such that $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$, $\mathcal{H}$ is nowhere equivalence to $\mathcal{G}$ and $\mathcal{F}$ nowhere equivalence to $\mathcal{H}$.

Let $Y = \{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, \dots, 0\}$. It is clear that Y is a metric space (equipped with the Euclidean metric on $\mathbb{R}$). Define a weakly closed correspondence $F \colon I \times Y \twoheadrightarrow \mathcal{M}(A)$ as follows: for each $i \in I$,

$$F(i, \tfrac{1}{k}) = \cap_{\ell=1}^{k} \overline{\cup_{n=\ell}^{\infty} \{g^n(i;\cdot)\}} \text{ for } k = 1, 2, \dots,$$
$$F(i, 0) = \cap_{\ell=1}^{\infty} \overline{\cup_{n=\ell}^{\infty} \{g^n(i;\cdot)\}} = \operatorname{Ls} g^n(i;\cdot).$$

It can be checked that for each $y \in Y$, $F(\cdot, y)$ is $\mathcal{G}$-measurable, and for each $i \in I$, $F(i, \cdot)$ is weak* upper-hemicontinuous.

Since $\mathcal{H}$ is nowhere equivalent to $\mathcal{G}$, by Corollary 8(B8) in He (2014) (see also Lemma 1 or He and Sun Y (2014)), we have $I^{\mathcal{H}}_{F(\cdot,y)}$ is weak* upper-hemicontinuous, where

$$I^{\mathcal{H}}_{F(\cdot,y)} = \left\{ \int_I f \,\mathrm{d}\lambda(i) \mid f \text{ is an } \mathcal{H}\text{-measurable Gelfand integrable selection of } F(\cdot, y) \right\}.$$

Since $\int_I g^n(i;\cdot)\,\mathrm{d}\lambda(i) \in I^{\mathcal{H}}_{F(\cdot,\frac{1}{n})}$, and as n approaches infinity $\int_I g^n(i;\cdot)\,\mathrm{d}\lambda(i)$ converges weakly to μ and $\frac{1}{n} \to 0$, by the upper-hemicontinuity, we have $\mu \in I^{\mathcal{H}}_{F(\cdot,0)}$. Hence, there exists an $\mathcal{H}$-measurable selection g of $F(\cdot, 0)$ such that $\mu = \int_I g(i;\cdot)\,\mathrm{d}\lambda(i)$.[23] It is not difficult to see that g is an $\mathcal{H}$-measurable transition probabilities from I to A.

Since $\mathcal{F}$ is nowhere equivalent to $\mathcal{H}$, Theorem 3(C5) in He and Sun (2013) (see also Lemma 1 or He and Sun Y (2014)) implies the existence of an $\mathcal{F}$-measurable mapping $g^* \colon I \to A$ such that $\mu^{g^*|\mathcal{H}} = g$, which leads to $\lambda(g^*)^{-1} = \int_I g(i;\cdot)\,\mathrm{d}\lambda(i)$.[24] Moreover, by Corollary 1 in He and Sun (2013) (see also Lemma 1 or He and Sun Y (2014)), for λ-almost all $i \in I$, $g^*(i) \in \operatorname{supp} g(i)$, where $g(i) \in F(i, 0) = \operatorname{Ls} g^n(i)$ for λ-almost all $i \in I$. Thus, $\int_I g^n(i, \cdot)\,\mathrm{d}\lambda(i)$ converges weakly

[23] We use the nowhere equivalent condition to prove the existence of behavioral strategy equilibria for simplicity. Note that an alternative proof can be done through Fatou lemma directly. More specifically, Fatou lemma for Gelfand integrals implies the existence of a $\mathcal{G}$-measurable strategy profile g such that $\int_I g(i;\cdot)\,\mathrm{d}\lambda(i) = \mu = \lim_{n\to\infty} \int_I g^n(i;\cdot)\,\mathrm{d}\lambda(i)$ and $g(i) \in \operatorname{supp} \operatorname{Ls} g^n(i)$ for λ-almost all $i \in I$. In this way, such a strategy profile is a behavioral strategy perfect equilibrium and its existence does not rely on the nowhere equivalence condition.

[24] For each Borel subset $B \in \mathcal{B}(A)$, we have $\lambda(g^*)^{-1}(B) = \mathbf{E}[1_B(g^*)] = \mathbf{E}[\mathbf{E}[1_B(g^*) \mid \mathcal{H}]] = \int_I g^*(i; B)\,\mathrm{d}\lambda(i)$.

to $\lambda(g^*)^{-1}$ and $g^*(i) \in \operatorname{supp} \operatorname{Ls} g^n(i)$ for λ-almost all $i \in I$. For λ-almost all $i \in I$, since $\operatorname{supp} \operatorname{Ls} g^n(i) \subseteq A_i$, we have $g^*(i) \in A_i$. It is clear that g^* is an $\mathcal{F}$-measurable pure strategy strong perfect equilibrium.

The necessity follows directly from Theorem 2 in He *et al.* (2016) and Proposition 1 above. □

2.7.4 Proofs of results in Section 2.4

Proof of Claim 3. Clearly, for any x and y, $v(x, y) \geq 0$, and the equality holds if and only if $x = 0$ or $y = 0$. When $y > 0$, the function $v(\cdot, y)$ strictly increases on $[0, \frac{y}{2}]$, and strictly decreases on $[\frac{y}{2}, \frac{1}{2}]$. That is, the unique maximum is achieved at the point $\frac{y}{2}$.

When player i chooses the action 0, then she will always get the payoff 0 regardless of other players' actions. However, when player i chooses the action $\frac{1}{2}$, then she will obtain a strictly positive payoff except λ-almost all players choose the action 0. Therefore, the action 0 is weakly dominated for each player.

In the following, we shall first show that f is the unique Nash equilibrium (modulo non-negligible subsets of players), where $f(i) = 0$ for each $i \in I$. Suppose that ξ is the social response induced by a Nash equilibrium.[25] Let $s = \inf\{0 \leq t \leq 1/2 \mid \xi([0, t]) = 1\}$. For each $y \in [0, s]$, player i's payoff $v(\cdot, y)$ is strictly decreasing on $[\frac{s}{2}, \frac{1}{2}]$. This implies that player i's best response to ξ must assign total probability on the set $[0, \frac{s}{2}]$. The symmetry implies that $\xi([0, \frac{s}{2}]) = 1$. By the definition of s, we have $s \leq \frac{s}{2}$, and hence $s = 0$. Therefore, f is the unique Nash equilibrium.

Next, we will find a sequence of ϵ-perfect equilibria which converges weakly to f. We first consider the two-player game in Example 2.1 of Simon and Stinchcombe (1995). The common

[25] It is clear that this large game has a Nash equilibrium $f \equiv 0$.

action is $[0, \frac{1}{2}]$, and the symmetric payoff function is given by

$$v(x,y) = \begin{cases} x, & \text{if } x \leq \frac{1}{2}y, \\ \frac{y(1-x)}{2-y}, & \text{if } \frac{1}{2}y < x. \end{cases}$$

As shown in Simon and Stinchcombe (1995), $(0,0)$ is the unique strong perfect equilibrium. That is, there exists a sequence $\{\epsilon_n\}_{n\in\mathbb{Z}_+}$ converging to 0 and a sequence of ϵ_n-perfect equilibria (ξ_n, ξ_n) such that $\{(\xi_n, \xi_n)\}_{n\in\mathbb{Z}_+}$ converges weakly to $(0,0)$ as n goes to infinity. Now move to the game in our example. Let f_n be the constant transition probability, mapping each $i \in I$ to ξ_n. Since ξ_n converges weakly to δ_0 (the Dirac probability measure at the point 0), we have $f_n(i)$ converges weakly to $f(i)$ for each i. For each $i \in I$, we have $\mathrm{Br}_i(f_n) = \mathrm{Br}(\xi_n)$, and hence $f_n(i)(\mathrm{Br}_i(f_n)) = \xi_n(\mathrm{Br}(\xi_n)) \geq 1 - \epsilon_n$. Therefore, f_n is a strong ϵ_n-perfect equilibrium, which leads to the conclusion that f is the unique strong perfect equilibrium. □

Before we offer a proof of Theorem 2, we shall need two additional lemmas. The following lemma shows that for each player i, the complement of the interior of the set of weakly undominated strategies is nonempty.

Lemma 3. In every large game, the set $(\Theta_i^o)^c$ is nonempty for each player $i \in I$.

Proof. Fix an arbitrary player i. We consider the following two-player game: the two players are i and $-i$; the action sets for player i and player $-i$ are A_i and $\mathcal{M}(A)$, respectively; the payoff functions for player i and player $-i$ are $u_i(a_i, \mu)$ and $-u_i(a_i, \mu)$ respectively, for each $a_i \in A_i$ and $\mu \in \mathcal{M}(A)$.

By Theorem 2.1 in Simon and Stinchcombe (1995), for this two-player game, the set of behavioral strategy strong perfect equilibria is a closed, nonempty subset of the set of behavioral strategy weak perfect equilibria which is a closed subset of behavioral strategy limit admissible Nash equilibria. Thus, there is a behavioral strategy strong perfect equilibrium (μ_i, μ_{-i}) which is limit admissible. That is, $\mu_i(\Theta_i^o) = 0$, which in turn implies that $(\Theta_i^o)^c$ is nonempty. □

Let H^c be the correspondence from I to A given by $H^c(i) = (\Theta_i^o)^c$ for each i, and $\tilde{H}^c$ the correspondence from I to $\mathcal{M}(A)$ given by $\tilde{H}^c(i) = \mathcal{M}\big((\Theta_i^o)^c\big)$, where $(\Theta_i^o)^c$ is the complement of the interior of the set of weakly dominated strategies for player i.

Lemma 4. *Both H^c and $\tilde{H}^c$ are two $\mathcal{G}$-measurable correspondences.*

Proof. If H^c is a measurable correspondence, then Lemma 18.4 in Aliprantis and Border (2006) implies that the correspondence $\mathrm{co}\, H^c$ is a measurable correspondence as well, where $\mathrm{co}\, H^c(i)$ is the convex hull of $(\Theta_i^o)^c$ for each i. Since A is a compact metric space, Density Theorem (Theorem 15.10 in Aliprantis and Border (2006)) implies that the set of probability measures on $(\Theta_i^o)^c$ with finite support is dense in $\mathcal{M}\big((\Theta_i^o)^c\big)$. By the embedding mapping $a \mapsto \delta_a$, the set of probability measures on $(\Theta_i^o)^c$ with finite support can be identified with $\mathrm{co}(\Theta_i^o)^c = \mathrm{co}\, H^c(i)$. By Lemmas 18.2 and 18.3 in Aliprantis and Border (2006), the measurability of $\mathrm{co}\, H^c$ implies the measurability of the correspondence $\overline{\mathrm{co}\, H^c}$ which is $\tilde{H}^c$. Therefore, we only need to show the measurability of H^c.

Let H be the correspondence from I to A given by $H(i) = \Theta_i^o$. Clearly, the correspondence H^c is measurable if and only if H is measurable. By Lemmas 18.2 and 18.3 in Aliprantis and Border (2006), the correspondence H is measurable if and only if $\bar{H}$ is measurable, where $\bar{H}$ is a correspondence from I to A given by $\bar{H}(i) = \overline{\Theta_i^o}$.

Since each $\mathcal{M}(A_i)$ and $\mathcal{M}(A)$ are compact, let $\{\mu_i^n\}_{n\in\mathbb{Z}_+}$ and $\{\mu^n\}_{n\in\mathbb{Z}_+}$ be the dense sets of $\mathcal{M}(A_i)$ and $\mathcal{M}(A)$, respectively. For each m and n in $\mathbb{Z}_+$, define the correspondence H_n^m from I to A as follows: for each i, $H_n^m(i)$ is

$$\bigcup_{j=1}^{\infty}\Big(\big[\cap_{k=1}^{n}\{a \mid u_i(a,\mu^k) \le u_i(\mu_i^j,\mu^k) + \tfrac{1}{n}\}\big] \cap \big[\cup_{k=1}^{\infty}\{a \mid u_i(a,\mu^k) + \tfrac{1}{m} \le u_i(\mu_i^j,\mu^k)\}\big]\Big).$$

By Lemma 18.4 in Aliprantis and Border (2006), H_n^m is $\mathcal{G}$-weakly measurable. We claim that $\cup_{m=1}^{\infty}\cap_{n=1}^{\infty} H_n^m(i)$ is the set of weakly dominated strategies for player i for each i.

For each $a \in \cup_{m=1}^{\infty}\cap_{n=1}^{\infty} H_n^m(i)$, there exists an integer m such that $a \in \cap_{n=1}^{\infty} H_n^m(i)$. Thus, for each n, there exists $\xi_i^n \in \{\mu_i^n\}_{n\in\mathbb{Z}_+}$ such that $u_i(a,\mu^k) \le u_i(\xi_i^n,\mu^k) + \frac{1}{n}$ for $k = 1, 2, \ldots, n$. And there also exists $\eta_i^n \in \{\mu^n\}_{n\in\mathbb{Z}_+}$ such that $u_i(a,\eta_i^n) + \frac{1}{m} \le u_i(\xi_i^n,\eta_i^n)$. Since $\mathcal{M}(A_i)$ and $\mathcal{M}(A)$ are compact, there exist convergent subsequences of $\{\xi_i^n\}_{n\in\mathbb{Z}_+}$ and $\{\eta_i^n\}_{n\in\mathbb{Z}_+}$, say themselves respectively. Suppose that the limits of $\{\xi_i^n\}$ and $\{\eta_i^n\}$ are ξ and η respectively. For each $j \in \mathbb{Z}_+$, $u_i(a, \mu^j) \leq u_i(\xi_i^n, \mu^j) + \frac{1}{n}$ holds for any $n \geq j$. Let n go to infinity. Then we have $u_i(a, \mu^j) \leq u_i(\xi, \mu^j)$ for each $j \in \mathbb{Z}_+$. Similarly, we have $u_i(a, \eta) + \frac{1}{m} \leq u_i(\xi, \eta)$. Therefore, a is a weakly dominated strategy of player i.

Next, suppose that a is a weakly dominated strategy for player i. Then there exists a strategy ξ such that $u_i(a, \mu^k) \leq u_i(\xi, \mu^k)$ for each $k \in \mathbb{Z}_+$ and $u_i(a, \mu^{k_0}) < u_i(\xi, \mu^{k_0})$ for some $k_0 \in \mathbb{Z}_+$. Thus, there exists a positive integer m such that $u_i(a, \mu^{k_0}) + \frac{1}{m} < u_i(\xi, \mu^{k_0})$. Fix this integer m. For each $n \in \mathbb{Z}_+$, the denseness of $\{\mu_i^n\}_{n\in\mathbb{Z}_+}$ and the continuity of u_i imply the existence of $\xi_i^n \in \{\mu_i^n\}_{n\in\mathbb{Z}_+}$ such that $u_i(a, \mu^j) \leq u_i(\xi_i^n, \mu^j)$ for each $j \in \mathbb{Z}_+$. Thus, $a \in H_n^m(i)$ for the fixed integer m and for each $n \in \mathbb{Z}_+$. Therefore, $a \in \cup_{m=1}^{\infty} \cap_{n=1}^{\infty} H_n^m(i)$.

By definition, we have $\bar{H} = \overline{\cup_{m=1}^{\infty} \cap_{n=1}^{\infty} H_n^m}$, and hence $\bar{H}$ is $\mathcal{G}$-measurable. □

Proof of Theorem 2. We first prove the sufficiency part.

For any $\epsilon \in (0, 1)$, randomly pick a $\mathcal{G}$-measurable behavioral strategy profile with full support $g_0 \in \Psi^{fs}$. Let

$$N(g_0) = \left\{g \in \Psi^{\mathcal{G}} \mid \text{for } \lambda\text{-almost all } i, g(i; B) \geq \tfrac{\epsilon}{3} g_0(i; B) \text{ for any } B \in \mathcal{B}(A_i)\right\},$$

and $S(g_0) = \left\{ \int_I g(i; \cdot)\, \mathrm{d}\lambda(i) \in \mathcal{M}(A) \mid g \in N(g_0)\right\}$. By the similar arguments in Lemma 2, $S(g_0)$ is a nonempty, convex and weakly closed subset of $\mathcal{M}(A)$.

By Lemma 3, $(\Theta_i^o)^c$ is a nonempty compact set in A_i. For each $i \in I$ and $\tau \in \mathcal{M}(A)$, define

$$\hat{N}(i, g_0) = \{\nu \in \mathcal{M}(A_i) \mid \nu = (1 - \tfrac{\epsilon}{3})\kappa + \tfrac{\epsilon}{3} g_0(i), \kappa \in \mathcal{M}((\Theta_i^o)^c)\},$$

and

$$D_{g_0}(i, \tau) = \operatorname*{arg\,max}_{\mu \in \hat{N}(i, g_0)} \int_A u_i(a, \tau)\, \mathrm{d}\mu(a).$$

For each i, since $\hat{N}(i, g_0)$ is a weakly closed set in $\mathcal{M}(A)$, it is also weakly compact. Then Berge Maximum Theorem (see Theorem 17.31 in Aliprantis and Border (2006)) implies that the correspondence $D_{g_0}(i, \cdot)\colon \mathcal{M}(A) \twoheadrightarrow \hat{N}(i, g_0)$ is nonempty, compact and upper-hemicontinuous.

Define a correspondence $B_{g_0}\colon S(g_0) \twoheadrightarrow \mathcal{M}(A)$ as follows: for any $\tau \in S(g_0)$, let

$$B_{g_0}(\tau) = \int_I D_{g_0}(i,\tau)\,\mathrm{d}\lambda(i),$$

which is the set of integrals of all $\mathcal{G}$-measurable selections of $D_{g_0}(i,\tau)$. Then it is easy to see that the correspondence B_{g_0} is indeed from $S(g_0)$ to $S(g_0)$. Clearly, B_{g_0} is also convex and upper-hemicontinuous. Hence, by Kakutani-Fan-Glicksberg fixed-point theorem, B_{g_0} has a fixed-point, say τ^*. That is, there is a $g^* \in N(g_0)$ such that

$$\tau^* = \int_I g^*(i)\,\mathrm{d}\lambda(i), \text{ and } g^*(i) \in D_{g_0}(i,\tau^*) \text{ for } \lambda\text{-almost all } i \in I.$$

We claim that for λ-almost all $i \in I$,

$$g^*\big(i;\mathrm{Br}_i(g^*)\big) \geq 1 - \tfrac{\epsilon}{3} g_0\big(i; A_i \setminus \mathrm{Br}_i(g^*)\big) \geq 1 - \tfrac{\epsilon}{3} g_0(i;A_i) = 1 - \tfrac{\epsilon}{3}.$$

Thus, $\rho^s\big(g^*(i), \mathcal{M}(\mathrm{Br}_i(g^*))\big) < \epsilon$ for λ-almost all $i \in I$, that is, g^* is a strong ϵ-perfect equilibrium.

We prove the claim by contradiction. Assume that there exists a non-negligible subset $E \in \mathcal{G}$ such that for each $i \in E$,

$$g^*\big(i;\mathrm{Br}_i(g^*)\big) < 1 - \tfrac{\epsilon}{3} g_0\big(i; A_i \setminus \mathrm{Br}_i(g^*)\big).$$

Fix a player $i \in E$. Lemma 3 implies that there exists an undominated strategy $a_0 \in (\Theta_i^o)^c$ which is a best response to g^*. Denote $u_i\big(a_0, \int_I g^*(i)\,\mathrm{d}\lambda(i)\big)$ by c for simplicity, which is player i's maximum payoff given other players' strategy profiles g^*_{-i}. Without loss of generality, we assume that c is positive.

We define a new strategy for player i as follows:

$$\xi = \Big(g^*(i) + \Big(g^*\big(i;A_i \setminus \mathrm{Br}_i(g^*)\big) - \tfrac{\epsilon}{3} g_0\big(i;A_i \setminus \mathrm{Br}_i(g^*)\big)\Big)\cdot \delta_{a_0}\Big)\Big|_{\mathrm{Br}_i(g^*)} + \tfrac{\epsilon}{3} g_0(i)\Big|_{A_i\setminus \mathrm{Br}_i(g^*)},$$

where δ_{a_0} is the Dirac measure on A at the point a_0. It is easy to see that $\xi \in (1-\frac{\epsilon}{3})\mathcal{M}((\Theta_i^o)^c) + \frac{\epsilon}{3}g_0(i)$. By the similar arguments in the proof of Lemma 2, player i is better off when she deviates from the strategy $g^*(i)$ to the strategy ξ defined above.

Take a sequence of $\{\epsilon_n\}_{n\in\mathbb{Z}_+}$ such that ϵ_n goes to zero when n approaches infinity. For each $n \in \mathbb{Z}_+$, the proof above implies that there exists a $\mathcal{G}$-measurable ϵ_n-perfect equilibrium h^k such that $h^k(i) \in (1-\frac{\epsilon_n}{3})\mathcal{M}((\Theta_i^o)^c) + \frac{\epsilon_n}{3}g_0(i)$. By the similar arguments in the proof of Theorem 1, there exists an $\mathcal{F}$-measurable pure strategy profile h^* such that $\int_I h^n(i;\cdot)\,\mathrm{d}\lambda(i)$ converges weakly to $\lambda(h^*)^{-1}$ and $h^*(i) \in \operatorname{supp}\operatorname{Ls} h^n(i)$ for λ-almost all $i \in I$. For each $n \in \mathbb{Z}_+$, let $\tilde{h}^n(i) = \frac{h^n(i)-\frac{\epsilon_n}{3}g_0(i)}{1-\frac{\epsilon_n}{3}}$. Clearly, we have $\lim_{n\to\infty}\left|\int_I h^n(i;\cdot)\,\mathrm{d}\lambda(i) - \int_I \tilde{h}^n(i;\cdot)\,\mathrm{d}\lambda(i)\right| = 0$ and $\operatorname{Ls} h^n(i) = \operatorname{Ls}\tilde{h}^n(i)$ for all $i \in I$. Since $\operatorname{supp}\tilde{h}^n(i) \subseteq H^c(i)$ for all $i \in I$, we have $\operatorname{supp}\operatorname{Ls}\tilde{h}^n(i) \subseteq H^c(i)$ for all $i \in I$. Thus, $h^*(i) \in \operatorname{supp}\operatorname{Ls} h^n(i) \subseteq H^c(i) = (\Theta_i^o)^c$ for all $i \in I$. Therefore, h^* is an $\mathcal{F}$-measurable limit admissible perfect equilibrium in pure strategies.

The necessity follows directly from Theorem 2 in He *et al.* (2016) and Proposition 1 above. $\square$

2.7.5 Proof of Theorem 3

To prove Theorem 3, we need the following two lemmas.

Lemma 5. *For any fixed finite partition $\mathcal{P} = \{\mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^m\}$ and any $\epsilon \in (0, \frac{1}{\sqrt{m}})$, $\mathrm{Pro}^s(\epsilon, \mathcal{P})$ is nonempty and contains a $\mathcal{G}$-measurable ϵ-proper equilibria relative to $\mathcal{P}$.*

Proof. For any $i \in I$, let $\mathcal{J}_i = \{j = 1,2,\ldots,m \mid \mathcal{P}_i^j \neq \emptyset\}$ and $\Omega^j = \{i \in I \mid \mathcal{P}_i^j \neq \emptyset\}$. Without loss of generality, we assume $\Omega^j \neq \emptyset$ for each $j = 1,2,\ldots,m$. For each $j = 1,2,\ldots,m$, we consider the correspondence $\mathcal{A}^j \colon \Omega^j \twoheadrightarrow \mathcal{P}^j$ such that $\mathcal{A}^j(i) = \mathcal{P}_i^j = A_i \cap \mathcal{P}^j$ for each $i \in \Omega^j$. It is clear that this correspondence has $\mathcal{G}\otimes\mathcal{B}(A)^{\mathcal{P}^j}$-measurable graph and nonempty values. Then the standard measurable selection theorem (see Theorem 18.26 in Aliprantis and Border (2006)) implies that $\mathcal{A}^j$ admits a $\mathcal{G}$-measurable selection f^j. For each $r_i^m = (x_i^1, x_i^2, \ldots, x_i^m) \in [0,1]^m$ with $\sum_{j=1}^m x_i^j = 1$ and $x_i^j = 0$ for each $j \notin \mathcal{J}_i$, we define a probability measure $\sigma(r_i^m)$ on $(A_i, \mathcal{B}(A_i))$ as follows:

$$\sigma(r_i^m) = \sum_{j \in \mathcal{J}_i} x_i^j \cdot \delta_{f^j(i)}.$$

For each $i \in I$, we then define a correspondence $T(i, \cdot) \colon \mathcal{M}(A) \twoheadrightarrow \mathcal{M}(A_i)$ as follows:

$$T(i, \mu) = \left\{ \sigma(r_i^m) \;\middle|\; u_i(\mathcal{P}_i^\ell, \mu) \ll u_i(\mathcal{P}_i^k, \mu) \text{ implies } x_i^\ell \leq \epsilon^2 x_i^k, \text{ and } x_i^j \geq \epsilon^{2m} \text{ for } j \in \mathcal{J}_i \right\}.$$

It is easy to check that the correspondence $T(i, \cdot)$ is convex, compact and upper-hemicontinuous.

In the following we shall prove that $T(i, \cdot)$ has nonempty values. Given μ, it suffices to show that there exists a vector $r_i^m = (x_i^1, x_i^2, \ldots, x_i^m)$ such that $x_i^j \geq \epsilon^{2m}$ for $j \in \mathcal{J}_i$ and $x_i^\ell \leq \epsilon^2 x_i^k$ if $u_i(\mathcal{P}_i^\ell, \mu) \ll u_i(\mathcal{P}_i^k, \mu)$ for $\mathcal{P}_i^\ell$ and $\mathcal{P}_i^k$ in $\mathcal{P}_i$. For each $i \in I$, $\mu \in \mathcal{M}(A)$ and $\mathcal{P}_i^j \in \mathcal{P}_i$, let $V(i, \mu, \mathcal{P}_i^j) = \{\mathcal{P}_i^k \in \mathcal{P}_i \mid u_i(\mathcal{P}_i^j, \mu) \ll u_i(\mathcal{P}_i^k, \mu)\}$, and $W(i, \mu) = \{\mathcal{P}_i^j \in \mathcal{P}_i \mid V(i, \mu, \mathcal{P}_i^j) = \emptyset\}$. It is easy to see that $W(i, \mu)$ is nonempty. Define $\rho_\ell = 1 + |V(i, \mu, \mathcal{P}_i^\ell)|$ for $\ell \in \mathcal{J}_i$. For the fixed $\mathcal{P}_i^j$, define $x_i^\ell = \epsilon^{2\rho_\ell}$ for $\ell \in \mathcal{J}_i \setminus \{j\}$, $x_i^\ell = 0$ for $\ell \notin \mathcal{J}_i$ and $x_i^j = 1 - \sum_{\ell \neq j} x_i^\ell$. Consider the vector $r = (x_i^1, x_i^2, \ldots, x_i^m)$. By the definition of ρ_ℓ, $\rho_\ell \leq m$ for $\ell \in \mathcal{J}_i$, and hence $x_i^\ell = \epsilon^{2\rho_\ell} \geq \epsilon^{2m}$ for $\ell \in \mathcal{J}_i \setminus \{j\}$. Moreover, $x_i^j = 1 - \sum_{\ell \neq j} x_i^\ell > 1 - (m-1)\epsilon^2 > \epsilon^2$ since $\epsilon^2 < \frac{1}{m}$. Suppose that $u_i(\mathcal{P}_i^\ell, \mu) \ll u_i(\mathcal{P}_i^k, \mu)$ for some $\mathcal{P}_i^\ell$ and $\mathcal{P}_i^k$ in $\mathcal{P}_i$. If $\mathcal{P}_i^k = \mathcal{P}_i^j$, then $x_i^k = x_i^j > \epsilon^2$ and $x_i^\ell \leq \epsilon^4$ (since $V(i, \mu, \mathcal{P}_i^\ell)$ contains at least one element $\mathcal{P}_i^k$). If $\mathcal{P}_i^k \neq \mathcal{P}_i^j$, then $\rho_\ell \geq \rho_k + 1$, which implies $x_i^\ell \leq \epsilon^2 x_i^k$. Therefore, the correspondence $T(i, \cdot)$ has nonempty values.

Randomly pick a $\mathcal{G}$-measurable behavioral strategy profile with full support $g_0 \in \Psi^{fs}$. Let

$$N(g_0) = \left\{ g \in \Psi^{\mathcal{G}} \mid \text{for } \lambda\text{-almost all } i \in I,\, g(i; B) \geq \epsilon^N g_0(i; B) \text{ for any } B \in \mathcal{B}(A_i) \right\},$$

and $S(g_0) = \left\{ \int_I g(i; \cdot)\, \mathrm{d}\lambda(i) \in \mathcal{M}(A) \mid g \in N(g_0) \right\}$, where N is an positive integer which is sufficiently large. For any $i \in I$ and $\tau \in \mathcal{M}(A)$, we define

$$\hat{N}(i, g_0) = \left\{ \nu \in \mathcal{M}(A) \mid \nu(B) \geq \epsilon^N g_0(i; B) \text{ for any } B \in \mathcal{B}(A_i) \right\},$$

and

$$D_{g_0}(i,\tau) = (1-\epsilon)\operatorname*{arg\,max}_{\mu\in\mathcal{M}(A_i)} \int_{A_i} u_i(a,\tau)\,\mathrm{d}\mu(a) + (\epsilon-\epsilon^N)T(i,\tau) + \epsilon^N g_0(i).$$

Then, Berge Maximum Theorem (see Theorem 17.31 in Aliprantis and Border (2006)) implies that the correspondence $D_{g_0}(i,\cdot)\colon \mathcal{M}(A) \twoheadrightarrow \hat{N}(i,g_0)$ is nonempty, compact-valued and upper-hemicontinuous.

Define a correspondence $B_{g_0}\colon S(g_0) \twoheadrightarrow \mathcal{M}(A)$ as follows: for any $\tau \in S(g_0)$, let

$$B_{g_0}(\tau) = \int_I D_{g_0}(i,\tau)\,\mathrm{d}\lambda(i),$$

which is the set of integrals of all $\mathcal{G}$-measurable selections of $D_{g_0}(i,\tau)$. Then it is easy to see that the correspondence B_{g_0} is indeed from $S(g_0)$ to $S(g_0)$. Clearly, B_{g_0} is also convex and upper-hemicontinuous. Hence, by Kakutani-Fan-Glicksberg fixed-point theorem, B_{g_0} has a fixed-point, say τ^*. That is, there is a $g^* \in N(g_0)$ such that

$$\tau^* = \int_I g^*(i)\,\mathrm{d}\lambda(i), \text{ and } g^*(i) \in D_{g_0}(i,\tau^*) \text{ for } \lambda\text{-almost all } i \in I.$$

Denote $g^* = (1-\epsilon)g_1^* + (\epsilon-\epsilon^N)g_2^* + \epsilon^N g_0$ where $g_1^*(i) \in \operatorname{arg\,max}_{\mu\in\mathcal{M}(A_i)} \int_{A_i} u_i(a,\tau^*)\,\mathrm{d}\mu(a)$ and $g_2^*(i) \in T(i,\tau^*)$ for λ-almost all $i \in I$.

Finally, by definition, we have for λ-almost all $i \in I$,

$$g^*\big(i;\mathrm{Br}_i(g^*)\big) = (1-\epsilon)g_1^*\big(i;\mathrm{Br}_i(g^*)\big) + (\epsilon-\epsilon^N)g_2^*\big(i;\mathrm{Br}_i(g^*)\big) + \epsilon^N g_0\big(i;\mathrm{Br}_i(g^*)\big) \geq 1-\epsilon.$$

Thus, $\rho^s\big(g^*(i),\mathcal{M}(\mathrm{Br}_i(g^*))\big) < \epsilon$ for λ-almost all $i \in I$. Therefore, g^* is a $\mathcal{G}$-measurable strong ϵ-perfect equilibrium.

For λ-almost all $i \in I$, if $u_i(\mathcal{P}_i^\ell,\tau^*) \ll u_i(\mathcal{P}_i^k,\tau^*)$ for $\mathcal{P}_i^\ell$ and $\mathcal{P}_i^k$ in $\mathcal{P}_i$, then $\mathcal{P}_i^\ell \cap \mathrm{Br}_i(g^*) = \emptyset$. Since $\operatorname{supp}\big(\operatorname{arg\,max}_{\mu\in\mathcal{M}(A_i)} \int_{A_i} u_i(a,\tau^*)\,\mathrm{d}\mu(a)\big) \subseteq \mathrm{Br}_i(g^*)$, then $g_1^*(i;\mathcal{P}_i^\ell) = 0$. Hence, $g^*(i;\mathcal{P}_i^\ell) = (\epsilon-\epsilon^N)g_2^*(i;\mathcal{P}_i^\ell) + \epsilon^N g_0(i;\mathcal{P}_i^\ell) = (\epsilon-\epsilon^N)x_i^\ell + \epsilon^N g_0(i;\mathcal{P}_i^\ell) \leq \epsilon[(\epsilon-\epsilon^N)x_i^k + \epsilon^N g_0(i;\mathcal{P}_i^k)] \leq \epsilon g^*(i;\mathcal{P}_i^k)$ for some vector $r_i^m = (x_i^1, x_i^2, \ldots, x_i^m)$ and the sufficiently large integer N. Since $\mathcal{P}$ is a finite partition, such an integer N exists. Therefore, we conclude that g^*

is a $\mathcal{G}$-measurable strong ϵ-proper equilibrium respect to the finite partition $\mathcal{P}$. □

Lemma 6. *For any fixed finite partition $\mathcal{P} = \{\mathcal{P}^1, \mathcal{P}^2, \ldots, \mathcal{P}^m\}$, the set of proper equilibrium respect to $\mathcal{P}$, $\mathrm{Pro}^s(\mathcal{P})$, is nonempty and contains an $\mathcal{H}_1$-measurable pure strategy strong proper equilibria relative to $\mathcal{P}$, where $\mathcal{H}_1$ is a σ-algebra on I such that $\mathcal{G} \subseteq \mathcal{H}_1 \subseteq \mathcal{F}$, $\mathcal{F}$ is nowhere equivalent to $\mathcal{H}_1$ and $\mathcal{H}_1$ is nowhere equivalent to $\mathcal{G}$.*

Proof. Let $\{\epsilon_n\}_{n\in\mathbb{Z}_+}$ be a sequence such that $0 < \epsilon_n < \frac{1}{\sqrt{m}}$ and ϵ_n goes to zero as n approaches infinity. By Lemma 5, for each ϵ_n, there exists a $\mathcal{G}$-measurable strong ϵ_n-proper equilibrium relative to $\mathcal{P}$, denoted by g^n. By the similar arguments in the proof of Theorem 1, there exists an $\mathcal{H}_1$-measurable pure strategy profile g^* such that $\int_I g^n(i;\cdot)\,\mathrm{d}\lambda(i)$ converges weakly to $\lambda(g^*)^{-1}$ and $g^*(i) \in \mathrm{supp}\,\mathrm{Ls}\,g^n(i)$ for λ-almost all $i \in I$. That is, g^* is an $\mathcal{H}_1$-measurable pure strategy strong proper equilibrium relative to $\mathcal{P}$. □

Proof of Theorem 3. We first prove the sufficiency part.

Let $\{\beta_n\}_{n\in\mathbb{Z}_+}$ be a sequence such that $0 < \beta_n < \min\{1, \mathrm{diam}(A)\}$ and β_n goes to zero as n approaches infinity. By Lemma 6, for each β_n, there exists an $\mathcal{H}_1$-measurable strong proper equilibrium relative to a finite partition with diameter β_n, denoted by g^n. Since $\{\int_I g^n(i;\cdot)\,\mathrm{d}\lambda(i)\}_{n\in\mathbb{Z}_+}$ is a sequence in the compact metric space $\mathcal{M}(A)$, there is a subsequence, without loss of generality say itself, converges weakly to some measure $\mu \in \mathcal{M}(A)$. By Proposition 1 in He *et al.* (2016), there exists a σ-algebra $\hat{\mathcal{H}}$ such that $\mathcal{H}_1 \subseteq \hat{\mathcal{H}} \subseteq \mathcal{F}$, $\mathcal{F}$ nowhere equivalence to $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}$ is nowhere equivalence to $\mathcal{H}_1$.

Let $Y = \{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}, \ldots, 0\}$. It is clear that Y is a metric space (equipped with the Euclidean metric on $\mathbb{R}$). Define a weakly closed correspondence $F\colon I \times Y \twoheadrightarrow \mathcal{M}(A)$ as follows: for each $i \in I$,

$$F(i, \tfrac{1}{k}) = \cap_{\ell=1}^{k} \overline{\cup_{n=\ell}^{\infty}\{g^n(i;\cdot)\}} \text{ for } k = 1, 2, \ldots,$$
$$F(i, 0) = \cap_{\ell=1}^{\infty} \overline{\cup_{n=\ell}^{\infty}\{g^n(i;\cdot)\}} = \mathrm{Ls}\, g^n(i;\cdot).$$

It can be checked that for each $y \in Y$, $F(\cdot, y)$ is $\mathcal{H}_1$-measurable, and for each $i \in I$, $F(i,\cdot)$ is weak* upper-hemicontinuous.

Since $\hat{\mathcal{H}}$ is nowhere equivalent to $\mathcal{H}_1$, by Corollary 8(B8) in He (2014) (see also Lemma 1 or He and Sun Y (2014)), we have $I^{\hat{\mathcal{H}}}_{F(\cdot,y)}$ is weak* upper-hemicontinuous, where

$$I^{\hat{\mathcal{H}}}_{F(\cdot,y)} = \left\{ \int_I f \,\mathrm{d}\lambda(i) \mid f \text{ is an } \hat{\mathcal{H}}\text{-measurable Gelfand integrable selection of } F(\cdot, y) \right\}.$$

Since $\int_I g^n(i;\cdot)\,\mathrm{d}\lambda(i) \in I^{\hat{\mathcal{H}}}_{F(\cdot,\frac{1}{n})}$, and $\int_I g^n(i;\cdot)\,\mathrm{d}\lambda(i)$ converges weakly to μ and $\frac{1}{n} \to 0$ as n approaches infinity, by the upper-hemicontinuity, we have $\mu \in I^{\hat{\mathcal{H}}}_{F(\cdot,0)}$. Hence, there exists an $\hat{\mathcal{H}}$-measurable selection g of $F(\cdot, 0)$ such that $\mu = \int_I g(i;\cdot)\,\mathrm{d}\lambda(i)$. It is not difficult to see that g is an $\hat{\mathcal{H}}$-measurable transition probabilities from I to A.

Since $\mathcal{F}$ is nowhere equivalent to $\hat{\mathcal{H}}$, Theorem 3(C5) in He and Sun (2013) (see also Lemma 1 or He and Sun Y (2014)) implies the existence of an $\mathcal{F}$-measurable mapping $g^*\colon I \to A$ such that $\mu^{g^*|\mathcal{H}} = g$, which leads to $\lambda(g^*)^{-1} = \int_I g(i;\cdot)\,\mathrm{d}\lambda(i)$. Moreover, by Corollary 1 in He and Sun (2013) (see also Lemma 1 or He and Sun Y (2014)), for λ-almost all $i \in I$, $g^*(i) \in \operatorname{supp} g(i)$, where $g(i) \in F(i, 0) = \operatorname{Ls} g^n(i)$ for λ-almost all $i \in I$. Thus, $\int_I g^n(i,\cdot)\,\mathrm{d}\lambda(i)$ converges weakly to $\lambda(g^*)^{-1}$ and $g^*(i) \in \operatorname{supp} \operatorname{Ls} g^n(i)$ for λ-almost all $i \in I$. It is clear that g^* is an $\mathcal{F}$-measurable pure strategy strong proper equilibrium.

The necessity follows directly from Theorem 2 in He *et al.* (2016) and Proposition 1 above. □

2.7.6 Proofs of results in Section 2.6

Proof of Proposition 2. It suffices to show that, under the assumption, f is the almost everywhere limit of a sequence of ϵ_n-perfect equilibria, where ϵ_n goes to zero as n approaches infinity. Let $\eta = \lambda f^{-1}$, and V_η be an open neighborhood of η. The rest of the proof is divided into two parts: (i) when $\eta \notin \mathcal{M}(A)^{fs}$ and (ii) when $\eta \in \mathcal{M}(A)^{fs}$.

(i) First suppose that $\eta \notin \mathcal{M}(A)^{fs}$. Since f is a Nash equilibrium, we have $f(i) \in \operatorname{Br}_i(\lambda f^{-1})$ for λ-almost all $i \in I$, where $\operatorname{Br}_i(\lambda f^{-1}) = \arg\max_{a\in A} u_i(a, \lambda f^{-1})$. Randomly pick a probability measure $\mu \in \mathcal{M}(A)^{fs}$ and a sequence $\{\epsilon_n\}_{n\in\mathbb{Z}_+}$ such that $\epsilon_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{Z}_+$, let

$$f^n(i) = (1 - \tfrac{\epsilon_n}{2}) f(i) + \tfrac{\epsilon_n}{2} \mu.$$

It is easy to see that $f^n(i) \in \mathcal{M}(A)^{fs}$ and $f^n(i)$ converges weakly to $f(i)$ for λ-almost all $i \in I$ as $n \to \infty$. For the open neighborhood V_η, there exists a sufficiently large integer N such that $\int_I f^n(i)\, \mathrm{d}\lambda(i) \in V_\eta$ if $n \geq N$. Thus, by the assumption, $\mathrm{Br}_i(\lambda f^{-1}) \subseteq \mathrm{Br}_i\left(\int_I f^n(i)\, \mathrm{d}\lambda(i)\right)$ for λ-almost all $i \in I$ and $n \geq N$. Hence, for $n \geq N$, we have

$$\begin{aligned} f^n(i)\left(\mathrm{Br}_i\left(\textstyle\int_I f^n(i)\, \mathrm{d}\lambda(i)\right)\right) &\geq f^n(i)\left(\mathrm{Br}_i(\lambda f^{-1})\right) \\ &= (1 - \tfrac{\epsilon_n}{2}) f(i)\left(\mathrm{Br}_i(\lambda f^{-1})\right) + \tfrac{\epsilon_n}{2} \mu\left(\mathrm{Br}_i(\lambda f^{-1})\right) \\ &\geq 1 - \tfrac{\epsilon_n}{2} - \tfrac{\epsilon_n}{2} = 1 - \epsilon_n \end{aligned}$$

Thus, f^n is an ϵ_n-perfect equilibrium for each $n \in \mathbb{Z}_+$, $\int_I f^n(i)\, \mathrm{d}\lambda(i)$ converges weakly to λf^{-1} and f^n converges almost everywhere to f.

(ii) Now suppose that $\eta \in \mathcal{M}(A)^{fs}$. As in part (i), randomly pick a sequence $\{\epsilon_n\}_{n \in \mathbb{Z}_+}$ such that $\epsilon_n \to 0$ as $n \to \infty$. For each $n \in \mathbb{Z}_+$, let

$$f^n(i) = (1 - \tfrac{\epsilon_n}{2}) f(i) + \tfrac{\epsilon_n}{2} \lambda f^{-1}.$$

It is clear that $f^n(i) \in \mathcal{M}(A)^{fs}$ and $\int_I f^n(i)\, \mathrm{d}\lambda(i) = \lambda f^{-1}$ for each $n \in \mathbb{Z}_+$. Thus, $\mathrm{Br}_i(\lambda f^{-1}) = \mathrm{Br}_i\left(\int_I f^k(i)\, \mathrm{d}\lambda(i)\right)$ for all $i \in I$. Moreover, for each $n \in \mathbb{Z}_+$, we have

$$\begin{aligned} f^n(i)\left(\mathrm{Br}_i\left(\textstyle\int_I f^n(i)\, \mathrm{d}\lambda(i)\right)\right) &= f^n(i)\left(\mathrm{Br}_i(\lambda f^{-1})\right) \\ &= (1 - \tfrac{\epsilon_n}{2}) f(i)\left(\mathrm{Br}_i(\lambda f^{-1})\right) + \tfrac{\epsilon_n}{2} \lambda f^{-1}\left(\mathrm{Br}_i(\lambda f^{-1})\right) \\ &\geq 1 - \tfrac{\epsilon_n}{2} - \tfrac{\epsilon_n}{2} = 1 - \epsilon_n \end{aligned}$$

Thus, again, we have the conclusion that f^n is an ϵ_n-perfect equilibrium for each $n \in \mathbb{Z}_+$, $\int_I f^n(i)\, \mathrm{d}\lambda(i)$ converges weakly to λf^{-1} and f^n converges almost everywhere to f. □

Proof of Proposition 3. It suffices to show the existence of a perfect equilibrium which is an almost everyone limit of a sequence of ϵ_n-perfect equilibria. By Theorem 1, there is a pure strategy perfect equilibrium g^*. Let $\eta^* = \lambda(g^*)^{-1}$, and V_{η^*} be an open neighborhood of η^* which satisfies the assumption in the statement of this theorem. By the similar arguments in the proof of Proposition 1, it is easy to see the existence of a sequence of ϵ_n-perfect equilibria whose almost everyone limit is g^*. $\square$

Proof of Proposition 4. Suppose that there exists a player i and two actions a and b such that $u_i(a, \mu_1) > u_i(b, \mu_1)$ and $u_i(a, \mu_2) < u_i(b, \mu_2)$ for some societal aggregates μ_1 and μ_2. We will show that there is a contradiction. Let $E = \{\mu \in \mathcal{M}(A) \mid u_i(a, \mu) = u_i(b, \mu)\}$. The continuity of the payoff function implies that E is a closed set. By the Density Theorem (Theorem 15.10 in Aliprantis and Border (2006)), the set of probability measures on A with finite support is dense in $\mathcal{M}(A)$. Thus, there exist two finite support measures ν_1 and ν_2 such that $u_i(a, \nu_1) > u_i(b, \nu_1)$ and $u_i(a, \nu_2) < u_i(b, \nu_2)$. Let $\nu_\beta = (1 - \beta)\nu_1 + \beta\nu_2$ for $\beta \in [0, 1]$. By the intermediate value theorem, there exists β' such that the finite support measure $\nu_{\beta'} = (1 - \beta')\nu_1 + \beta'\nu_2$ satisfies $u_i(a, \nu_{\beta'}) = u_i(b, \nu_{\beta'})$. Thus E is nonempty and contains a measure $\nu_{\beta'}$ with finite support.

For any $\nu \in E$, if $\nu \notin \mathcal{M}(A)^{fs}$, then there exists a open neighborhood V_ν such that for any $\tau \in V_\nu$, $\tau \in E$. That is, every measure in E with finite support is an interior point of E. Let $\mu_\gamma = (1 - \gamma)\nu_1 + \gamma\nu_{\beta'}$ for $\gamma \in [0, 1]$. Since E is a closed set in $\mathcal{M}(A)$, there exists γ' such that $\gamma' = \min\{\gamma \in [0, 1] \mid (1 - \gamma)\nu_1 + \gamma\nu_{\beta'} \in E\}$. It is easy to see that the finite support measure $\mu_{\gamma'}$ is on the boundary of E, which leads a contradiction. $\square$

Chapter 3

Pure-strategy Equilibrium in General Bayesian Games

3.1 Introduction

Since Harsanyi (1967–68), Bayesian games, where players have incomplete information on some important parameters (such as payoff functions, the information other players have, *etc.*), have been widely studied over the years, and many contributions have been made to the underlying theory of information. One of the most fundamental questions is known to be the existence of pure-strategy Bayesian Nash equilibria. To guarantee such existence, a condition imposing absolute continuity on the information structures is essential. Indeed, without an absolutely continuous information structure, Simon (2003) constructed a Bayesian game that has no pure strategy equilibria. His idea was further developed by Hellman (2014), who constructed a simpler example to show that without such absolute continuity condition, a Bayesian game may not even have measurable approximate pure-strategy equilibria.

In this paper, we consider general Bayesian games which allow players' actions to be infinite, types to be correlated, and payoffs to be interdependent. We introduce a condition of "coarser density weighted payoff-relevant information", a version of the absolute continuity condition in a general setting, to distinguish the informational influence of a player on her own strategies and the informational influence on the payoffs of all the players. With this condition, we establish the existence of pure strategy equilibria for general Bayesian games; see Theorem 4. More importantly, such condition is also shown to be minimal in the sense that if every general Bayesain game of an identical information structure with the same set of players and actions has a pure-strategy equilibrium, then every player has coarser density weighted payoff-relevant information; see Proposition 5.

To obtain the existence of pure-strategy equilibria in Bayesian games with general action spaces, several other assumptions on the payoff functions have been proposed. Vives (1990) showed that the supermodularity in strategies is a sufficient condition for the existence of pure strategy equilibria.[1] By assuming Spence-Mirrlees single crossing property, Athey (2001) proved that a monotone pure-strategy equilibrium exists, and McAdams (2003) generalized this result to setting in which spaces of type and action are multidimensional and only partially ordered. Reny (2011) further generalized Athey (2001)'s and McAdams (2003)'s results to the setting which allows action spaces to be compact locally complete metric semilattices and type spaces to be partially ordered probability spaces.

Note that all these existence results mentioned above rely on the assumptions of independence and diffuseness of information. To allow players' types to be correlated and payoffs to be interdependent, He and Sun (2014a) proposed the condition of "coarser inter-player information", and showed that this condition is not only sufficient but also necessary for the existence of pure-strategy equilibria in finite-action Bayesian games.[2] However, Khan *et al.* (1999) pointed out that a pure-strategy equilibrium may not exist in Bayesian games with general action spaces, even the private information are independent and diffuse. This nonexistence result leads us to find a suitable condition to retain the existence of pure-strategy equilibria for general Bayesian games.

Besides assuming specific structures on the payoff structures, another approach has been

[1] A game is supermodular in the strategies if one player's strategy increases pointwise, the best response strategies of all opponents must increase pointwise.

[2] In Bayesian games with finitely many actions, the existence of behavioral/distributional strategy equilibria was well established; see, for example, Milgrom and Weber (1985). Based on Dvoretsky, Wald and Wolfowitz (1951)'s purification result, it is straightforward to obtain the existence of pure strategy equilibria; see also Radner and Rosenthal (1982) and Khan *et al.* (2006). Khan and Sun (1995) extended the existence result to Bayesian games with countably many actions.

proposed to guarantee the existence of pure-strategy equilibria, which imposes rich structures on the information spaces. Khan and Sun (1999) modeled the players' information spaces by atomless Loeb spaces,[3] and showed that a pure-strategy equilibrium exists. Such existence result also holds when each player's information space is modeled by a saturated probability space; see Keisler and Sun (2009) and Khan and Zhang (2014).[4] Based on the condition of "nowhere equivalence" proposed in He *et al.* (2016), He and Sun X (2014) introduced the "relative diffuseness" assumption to characterize the differences between payoff-relevant and strategy-relevant diffuseness of information, which leads to the existence of pure-strategy equilibria as well.

3.2 General Bayesian games

A Bayesian game can be described as follows. The set of players is $I = \{1, \ldots, n\}$. Each player $i \in I$ observes a private signal/type t_i, whose value lies in some measurable space $(T_i, \mathcal{T}_i)$. After observing the type, player i then chooses an action from some nonempty compact metric space. Each player's payoff may depend on the actions chosen by all the players, and on the type profiles as well. Let λ be an information structure for the game, which is a joint probability distribution on $(\prod_{1\leq i\leq n} T_i, \otimes_{1\leq i\leq n}\mathcal{T}_i)$. We allow players' types to be correlated and payoffs to be interdependent.

Formally, a Bayesian game Γ consists of the following five elements.

- The set of players: $I = \{1, 2, \ldots, n\}$.
- The set of actions available to each player: $\{A_i\}_{i\in I}$. Each A_i is a nonempty compact metric space endowed with the Borel σ-algebra $\mathcal{B}(A_i)$. Denote $A = \prod_{i=1}^n A_i$.
- The (private) information space for each player: $\{T_i\}_{i\in I}$. Each T_i is endowed with a σ-algebra $\mathcal{T}_i$. Denote $T = \prod_{i=1}^n T_i$, and $\mathcal{T} = \otimes_{i=1}^n \mathcal{T}_i$.

[3] Such probability spaces were introduced by Loeb (1975). For the construction, see Loeb and Wolff (2015).

[4] Khan and Zhang (2014, 2016) also showed that the saturation property is necessary for the existence of pure strategy equilibria. That is, if the information space of only one player is not saturated, a pure strategy equilibrium in Bayesian games with general action space may not exist.

- The prior: λ, a probability measure on the measurable space $(T, \mathcal{T})$.

- The payoff functions: $\{u_i\}_{i \in I}$. Each u_i is a mapping from $A \times T$ to $\mathbb{R}$ such that

  1. $u_i(a, \cdot)$ is measurable for each $a \in A$ and $u_i(\cdot, t)$ is continuous for each $t \in T$,

  2. u_i is integrably bounded in the sense that there is a real-valued integrable function h_i on $(T, \mathcal{T}, \lambda)$ with $|u_i(a,t)| \leq h_i(t)$ for all $(a,t) \in A \times T$.

For each $i \in I$, let λ_i be the marginal probability of λ on $(T_i, \mathcal{T}_i)$. Throughout this paper, we assume that λ is absolutely continuous with respect to $\otimes_{i=1}^n \lambda_i$, with the corresponding Radon-Nikodym derivative q. As usual, the notation $-i$ denotes all the players except player i, and $\lambda_{-i} = \otimes_{j \neq i} \lambda_j$.

For each $i \in I$, a behavioral strategy (resp. pure strategy) for player i is a $\mathcal{T}_i$-measurable function from T_i to $\mathcal{M}(A_i)$ (resp. A_i), where $\mathcal{M}(A_i)$ denotes the space of Borel probability measures on A_i endowed with the topology of weak convergence. Let $L_i^{\mathcal{T}_i}$ (resp. $L_{0,i}^{\mathcal{T}_i}$) be the set of all the behavioral strategies (resp. pure strategies) for player i, and $L^{\mathcal{T}} = \prod_{i=1}^n L_i^{\mathcal{T}_i}$.[5] A pure strategy can be viewed as a behavioral strategy by taking it as a Dirac measure for all $t_i \in T_i$.

Given a strategy profile $f = (f_1, f_2, \ldots, f_n)$, player i's expected payoff is

$$\begin{aligned} U_i(f) &= \int_T \int_A u_i(a,t) \cdot \prod_{j \in I} f_j(t_j; \mathrm{d}a_j)\, \lambda(\mathrm{d}t) \\ &= \int_T \int_A u_i(a,t) \cdot \prod_{j \in I} f_j(t_j; \mathrm{d}a_j) \cdot q(t) \otimes_{j \in I} \lambda_j(\mathrm{d}t_j), \end{aligned}$$

where $f_j(t_j; B)$ denotes the value of $f_j(t_j)$ on the Borel subset B of A_j.

A behavioral (resp. pure) strategy equilibrium is a behavioral (resp. pure) strategy profile $f^* = (f_1^*, f_2^*, \ldots, f_n^*)$ such that f_i^* maximizes $U_i(f_i, f_{-i}^*)$ in $L_i^{\mathcal{T}_i}$ (resp. $L_{0,i}^{\mathcal{T}_i}$) for each $i \in I$.

[5] A behavioral strategy of player i is a transition probability from $(T_i, \mathcal{T}_i)$ to $\big(A_i, \mathcal{B}(A_i)\big)$. The definition and several properties of transition probability are stated in Section 3.5.1.

3.3 Main results

In this section, we will prove the existence of pure strategy equilibria in general Bayesian games under the condition "coarser density weighted payoff-relevant information." More importantly, we show that this condition is also necessary for the existence result. Furthermore, we consider the related purification issue as well.

3.3.1 Coarser density weighted payoff-relevant information

For each player $i \in I$, we consider the **density weighted payoff**

$$w_i(a,t) = u_i(a,t) \cdot q(t)$$

for each $a \in A$ and $t \in T$. Let $\mathcal{F}_i$ be the σ-algebra on T_i generated by the collection of mappings

$$\left\{w_j(a,\cdot,t_{-i}) \mid j \in I, a \in A, t_{-i} \in T_{-i}\right\}.$$

That is, the σ-algebra $\mathcal{F}_i \subseteq \mathcal{T}_i$ represents player i's **density weighted payoff-relevant information**. To be more precise, $\mathcal{F}_i$ is player i's information flow to the density weighted payoffs of all players, which describes the influence of player i's private information in all players' density weighted payoffs. Throughout this paper, we assume that $(T_i, \mathcal{F}_i, \lambda_i)$ is an atomless probability space for each $i \in I$. We will characterize the relationship between $\mathcal{T}_i$ and $\mathcal{F}_i$ in the following.

For each $i \in I$, recall that $(T_i, \mathcal{T}_i, \lambda_i)$ is an atomless probability space. For any nonnegligible subset $D \in \mathcal{T}_i$, the restricted probability space $(D, \mathcal{F}_i^D, \lambda_i^D)$ is defined as follows: $\mathcal{F}_i^D$ is the σ-algebra $\{D \cap D' \colon D' \in \mathcal{F}_i\}$ and λ_i^D the probability measure re-scaled from the restriction of λ_i on $\mathcal{F}_i^D$. Furthermore, $(D, \mathcal{T}_i^D, \lambda_i^D)$ can be defined similarly.

Below, we state the definition of nowhere equivalence (see He *et al.* (2016); He and Sun Y (2014)).

Definition 13. A σ-algebra $\mathcal{T}_i$ is said to be **nowhere equivalent** to its sub-σ-algebra $\mathcal{F}_i$ if for every nonnegligible subset $D \in \mathcal{T}_i$, there exists a $\mathcal{T}_i$-measurable subset D_0 of D such that

$\lambda_i(D_0 \triangle D_1) > 0$ for any $D_1 \in \mathcal{F}_i^D$, where $D_0 \triangle D_1$ is the symmetric difference $(D_0 \setminus D_1) \cup (D_1 \setminus D_0)$.

We shall introduce the definition of coarser density weighted payoff-relevant information as follows.

Definition 14. Player i is said to have **coarser density weighted payoff-relevant information** if $\mathcal{T}_i$ is nowhere equivalent to $\mathcal{F}_i$ under λ_i.

A Bayesian game is said to have coarser density weighted payoff-relevant information if each player has coarser density weighted payoff-relevant information.

This assumption implies that on any nonnegligible subset $D \subseteq T_i$, $\mathcal{T}_i^D$ is always essentially larger than $\mathcal{F}_i^D$, which means that the player i's information influencing her own strategies is "much" richer than the information influencing the payoffs of all the players.

3.3.2 Existence of pure strategy equilibria

The theorem on the existence of pure strategy equilibria is presented below, and its proof is given in Section 3.5.2.

Theorem 4. *Every Bayesian game with coarser density weighted payoff-relevant information has a pure strategy equilibrium.*

Remark 4. For Bayesian games with coarser density weighted payoff-relevant information, players' payoffs may be interdependent and types could be correlated. Indeed, no matter whether types are independent or correlated, the Radon-Nikodym derivative q can always absorb them into density weighted payoffs.

If types are independent and each player's payoff only depends on her own type, the condition of "coarser density weighted payoff-relevant information" becomes the condition of "relative diffuseness" in He and Sun X (2014). In such Bayesian games, He and Sun X (2014) showed that the relative diffuseness condition can guarantee the existence of pure strategy equilibria.

In Theorem 4, we show that the condition of "coarser density weighted payoff-relevant information" is sufficient for the existence of pure strategy equilibria. More importantly, this condition is necessary as well.

Fix a finite set of players $I = \{1, 2, \ldots, n\}$ with $n \geq 2$ and a compact metric space A with infinitely many elements. Each player $i \in I$ has a private information space $(T_i, \mathcal{T}_i, \lambda_i)$ and atomless density weighted payoff-relevant information $\mathcal{F}_i$. Let H_n be the collection of all Bayesian games with the player space I, the common action set A and the above private information spaces $\{(T_i, \mathcal{T}_i/\mathcal{F}_i, \lambda_i)\}_{i\in I}$. The following result is a generalization of Theorem 5 in He and Sun (2014a).

Proposition 5. *If every Bayesian game in H_n with type-irrelevant payoffs has a pure strategy equilibrium,*[6] *then every player i has coarser density weighted payoff-relevant information.*

3.4 Purification

In this section, we shall consider the notion of strong purification, and prove its existence based on the condition of coarser density weighted payoff-relevant information.[7]

Definition 15. Let $f = (f_1, f_2, \ldots, f_n)$ and $g = (g_1, g_2, \ldots, g_n)$ be two behavioral strategy profiles.

1. The strategy profiles f and g are said to be payoff equivalent if for each player $i \in I$, $U_i(f) = U_i(g)$.

2. The strategy profiles f and g are said to be strongly payoff equivalent if

   (a) they are payoff equivalent;

   (b) for each player $i \in I$ and any given behavioral strategy h_i, the two strategy profiles (h_i, f_{-i}) and (h_i, g_{-i}) are payoff equivalent.

[6] A Bayesian game is said to have type-irrelevant payoffs if the payoff function of each player does not depend on the type profile t.

[7] The notion of strong purification was proposed in Khan *et al.* (2006). Its existence has been proved therein for the case that the action space is finite for every player.

3. The strategy profiles f and g are said to be distribution equivalent if for each player $i \in I$ and Borel subset $B \subseteq A_i$,

$$\int_{T_i} f_i(t_i)(B)\lambda_i(\mathrm{d}t_i) = \int_{T_i} g_i(t_i)(B)\lambda_i(\mathrm{d}t_i).$$ [8]

4. Suppose that f is a pure strategy profile. For player i, f_i is said to be belief consistent with g_i if $f_i(t_i) \in \operatorname{supp} g_i(t_i)$ for λ_i-almost all $t_i \in T_i$, where $\operatorname{supp} g_i(t_i)$ is the support of the measure $g_i(t_i)$. Moreover, f is said to be belief consistent with g if they are belief consistent for each player $i \in I$.

Now we are ready to state the definition of strong purification.

Definition 16. Suppose that f is a pure strategy profile and g is a behavioral strategy profile. Then f is said to be a strong purification of g if they are strongly payoff equivalent, distribution equivalent and belief consistent.

In the following, we shall establish the existence result of strong purification based on the condition of coarser density weighted payoff-relevant information.

Proposition 6. *In a Bayesian game with coarser density weighted payoff-relevant information, every behavioral strategy profile g possesses a strong purification f.*

Remark 5. *It is easy to see that if the given behavioral strategy profile g is an equilibrium, then its strong purification f is a pure strategy equilibrium.*

Recall that H_n is the collection of all Bayesian games with the player space I and the private information spaces $\{(T_i, \mathcal{T}_i/\mathcal{F}_i, \lambda_i)\}_{i \in I}$. The following result shows that the condition of coarser density weighted payoff-relevant information is not only sufficient, but also necessary for the existence of a strong purification.

Proposition 7. *If for every Bayesian game in H_n with private values, every behavioral strategy profile has a strong purification, then every player i has coarser density weighted payoff-relevant information.*

[8] When f_i is a pure strategy, $\int_{T_i} f_i(t_i)(B)\,\mathrm{d}\lambda_i(t_i) = \lambda_i\big(f_i^{-1}(B)\big)$ for any Borel subset B in A_i.

3.5 Proofs

3.5.1 Preliminary results

Asymptotic independent supplement

Below, we shall state the definition of "asymptotic independent supplement".

Definition 17. *The σ-algebra $\mathcal{F}$ admits an* ***asymptotic independent supplement*** *in $\mathcal{T}$ if for some strictly increasing sequence $\{n_k\}$ and each $k \geq 1$, there exists a $\mathcal{T}$-measurable partition $\{E_1, E_2, \ldots, E_{n_k}\}$ of T with $\lambda(E_j) = \frac{1}{n_k}$ and E_j being independent of $\mathcal{F}$ for $j = 1, 2, \ldots, n_k$.*

This condition will be used for deriving the necessity results. The following lemma is from He and Sun (2013) (see also He and Sun Y (2014)). It shows that this condition is indeed equivalent to the condition of nowhere equivalence.

Lemma 7. *The following statements are equivalent.*

(i) $\mathcal{T}$ is nowhere equivalent to $\mathcal{F}$.

(ii) $\mathcal{F}$ admits an asymptotic independent supplement in $\mathcal{T}$.

Transition probability and regular conditional distribution

Let $(T, \mathcal{T}, \lambda)$ be an atomless complete countably-additive probability space and $\mathcal{F}$ a sub-σ-algebra of $\mathcal{T}$. Let X be a Polish space (complete metrizable topological space), $\mathcal{B}(X)$ the Borel σ-algebra, and $\mathcal{M}(X)$ the space of all Borel probability measures on X endowed with the topology of weak convergence. We use $C_b(X)$ to denote the set of all bounded continuous function from X to $\mathbb{R}$.

Definition 18. An $\mathcal{F}$-measurable transition probability from T to X is a mapping $\phi\colon T \to \mathcal{M}(X)$ such that for every $B \in \mathcal{B}(X)$, the mapping

$$\phi(\cdot; B)\colon t \mapsto \phi(t; B)$$

is $\mathcal{F}$-measurable, where $\phi(i; B)$ is the value of the probability measure $\phi(i)$ on the Borel subset B of A.

We use $\mathcal{R}^{\mathcal{F}}(X)$, or $\mathcal{R}^{\mathcal{F}}$ when it is clear, to denote the set of all $\mathcal{F}$-measurable transition probabilities from T to X.

Definition 19. A sequence $\{\phi_n\}_{n\in\mathbb{Z}_+}$ in $\mathcal{R}^{\mathcal{F}}$ is said to weakly converge to ϕ in $\mathcal{R}^{\mathcal{F}}$, denoted by $\phi_n \implies \phi$, if for every bounded Carathéodory function $c\colon T \times X \to \mathbb{R}$,[9]

$$\lim_{n\to\infty} \int_T \left[\int_X c(t,x)\phi_n(t; \mathrm{d}x) \right] \mathrm{d}\lambda(t) = \int_T \left[\int_X c(t,x)\phi(t; \mathrm{d}x) \right] \mathrm{d}\lambda(t).$$

The weak topology on $\mathcal{R}^{\mathcal{F}}$ is defined as the weakest topology for which the functional

$$\phi \mapsto \int_T \left[\int_X c(t,x)\phi(t; \mathrm{d}x) \right] \mathrm{d}\lambda(t)$$

is continuous for every bounded Carathéodory function $c\colon T \times X \to \mathbb{R}$.

We next review the regular conditional distribution. Let f be a $\mathcal{T}$-measurable mapping from T to X. A mapping $\mu^{f|\mathcal{F}}\colon T \times \mathcal{B}(X) \to [0,1]$ is said to be a regular conditional distribution for f given $\mathcal{F}$, if

1. for λ-almost all $t \in T$, $\mu^{f|\mathcal{F}}(t, \cdot)$ is a probability measure on X;

2. for each Borel subset $B \subseteq X$, $\mu^{f|\mathcal{F}}(\cdot, B)$ is a version of $\mathbf{E}[1_B(f) \mid \mathcal{F}]$, where $\mathbf{E}[1_B(f) \mid \mathcal{F}]$ is the conditional expectation of the indicator function $1_B(f)$ given $\mathcal{F}$.

Since X is assumed to be a Polish space endowed with the Borel σ-algebra, the regular conditional distribution for f given $\mathcal{F}$ exists; see Theorem 5.1.9 in Durrett (2010).

Let F be a $\mathcal{T}$-correspondence from T to X. We use

$$\mathcal{R}_F^{(\mathcal{T},\mathcal{F})} = \left\{ \mu^{f|\mathcal{F}} \mid f \text{ is a } \mathcal{T}\text{-measurable selection of } F \right\}$$

[9] Given a probability space $(T, \mathcal{F}, \lambda)$ and a Polish space X, a function $c\colon T \times X \to \mathbb{R}$ is a Carathéodory function if $c(\cdot, x)$ is $\mathcal{F}$-measurable for each $x \in X$ and $c(t, \cdot)$ is continuous for each $t \in T$.

to denote the set of all regular conditional distributions induced by $\mathcal{T}$-measurable selections of F conditional on $\mathcal{F}$.

The following result on regular conditional distributions of correspondences is stated as a lemma here for the convenience of readers, which is Theorem 3 in He and Sun (2013) (see also He and Sun Y (2014)).

Lemma 8. *Suppose that $(T, \mathcal{T}, \lambda)$ is atomless and $\mathcal{F}$ is a countably-generated sub-σ-algebra of $\mathcal{T}$. If $\mathcal{T}$ is nowhere equivalent to $\mathcal{F}$, then for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, we have the following results.*

(C1) For any closed valued $\mathcal{F}$-measurable correspondence F from T to X, $\mathcal{R}_F^{(\mathcal{T},\mathcal{G})}$ is convex.

(C2) For any closed valued $\mathcal{F}$-measurable correspondence F from T to X, $\mathcal{R}_F^{(\mathcal{T},\mathcal{G})}$ is weakly closed.

(C3) For any compact valued $\mathcal{F}$-measurable correspondence F from T to X, $\mathcal{R}_F^{(\mathcal{T},\mathcal{G})}$ is weakly compact.

(C4) Let F be a compact valued $\mathcal{F}$-measurable correspondence from T to X, Z a metric space and G a closed valued correspondence from $T \times Z$ to X such that

- *for each $(t, z) \in T \times Z$, $G(t, z) \subseteq F(t)$,*
- *for each $z \in Z$, $G(\cdot, z)$ (denoted as G_z) is $\mathcal{F}$-measurable from T to X,*
- *for each $t \in T$, $G(t, \cdot)$ (denoted as G_t) is upper-hemicontinuous from Z to X.*

Then $H(z) = \mathcal{R}_{G_z}^{(\mathcal{T},\mathcal{G})}$ is upper-hemicontinuous from T to $\mathcal{R}^{\mathcal{G}}$.

(C5) For any $G \in \mathcal{R}^{\mathcal{F}}$, there exists a $\mathcal{T}$-measurable mapping g such that $\mu^{g|\mathcal{F}} = G$.

3.5.2 Proof of Theorem 4

Lemma 9. *Let $(T, \mathcal{T}, \lambda)$ be a probability space, $\mathcal{F}$ a sub-σ-algebra of $\mathcal{T}$, h a $\mathcal{T}$-measurable mapping from T to a Polish space X with the Borel σ-algebra $\mathcal{B}$, and $\mu^{h|\mathcal{F}}$ a regular conditional distribution of h given $\mathcal{F}$. Let g be a $\mathcal{B} \otimes \mathcal{F}$-measurable function from $X \times T$ to $\mathbb{R}$ such that*

$|g(x,t)| \leq \phi(t)$ for all $(x,t) \in X \times T$, where ϕ is integrable on $(T, \mathcal{T}, \lambda)$. Define a function ψ from T to $\mathbb{R}$ by letting $\psi(t) = g(h(t),t)$ for any $t \in T$. Then, for λ-almost all $t \in T$,

$$E[\psi|\mathcal{F}](t) = \int_X g(x,t)\mu^{h|\mathcal{F}}(t; \mathrm{d}x). \tag{3.1}$$

Proof. Let $\mathcal{H}$ be the class of non-negative $\mathcal{B} \otimes \mathcal{F}$-measurable function g from $X \times T$ to $\mathbb{R}$ such that equation (3.1) holds.

For any measurable sets $B \in \mathcal{B}$ and $C \in \mathcal{F}$, let $g = 1_{B \times C}$. Since $\mu^{h|\mathcal{F}}$ is a regular conditional distribution of h given $\mathcal{F}$, we have $E[1_B(h)|\mathcal{F}] = \mu^{h|\mathcal{F}}(B) = \int_X 1_B(x)\mu^{h|\mathcal{F}}(\mathrm{d}x)$. Since C is $\mathcal{F}$-measurable, we obtain that

$$E[1_B(h)1_C|\mathcal{F}] = 1_C E[1_B(h)|\mathcal{F}] = \int_X 1_B(x) 1_C \mu^{h|\mathcal{F}}(\mathrm{d}x),$$

which implies that equation (3.1) holds for $g = 1_{B \times C}$. Hence $1_{B \times C} \in \mathcal{H}$

By the properties of conditional expectation, it is obvious that $\mathcal{H}$ is a λ-system in the sense that (1) the constant function 1 is in $\mathcal{H}$; (2) for any non-negative real numbers c_1, c_2, any $g_1, g_2 \in \mathcal{H}$, $c_1 g_1 + c_2 g_2 \in \mathcal{H}$; (3) for any increasing sequence of functions $g_n, n \geq 1$ in $\mathcal{H}$ with a limit function g, one has $g \in \mathcal{H}$. Since the class $\mathcal{D}$ of measurable rectangles $B \times C$ with $B \in \mathcal{B}$ and $C \in \mathcal{F}$ is a π-class (i.e., closed under the operation of finite intersections), the usual $\pi - \lambda$ Theorem[10] implies that equation (3.1) holds for every non-negative $\mathcal{B} \otimes \mathcal{F}$-measurable function g from $X \times T$ to $\mathbb{R}$.

For a $\mathcal{B} \otimes \mathcal{F}$-measurable function from $X \times T$ to $\mathbb{R}$ satisfying the conditions in the statement of the lemma, one can consider the positive and negative parts of g. The rest is clear. □

Proof of Theorem 4:

For each $i \in I$, let $\mathcal{R}^{\mathcal{F}_i}$ be the set of $\mathcal{F}_i$-measurable transition probabilities from T_i to A_i. Clearly, $\mathcal{R}^{\mathcal{F}_i}$ is nonempty, convex and weakly compact (under the topology of weak convergence). Let $\mathcal{R}^{\mathcal{F}} = \prod_{i \in I} \mathcal{R}^{\mathcal{F}_i}$, which is endowed with the product topology.

[10] See, for example, Theorem 1.4.3 of Chow and Teicher (1997, p. 16).

Given a pure strategy profile $h = (h_1, h_2, \dots, h_n)$, for any two distinct players i and j, any actions $a_{-j} \in A_{-j}$, any types $t_{-j} \in T_{-j}$, the Fubini property implies that the function $w_i(a_{-j}, a_j, t_{-j}, t_j)$ in terms of (a_j, t_j) is $\mathcal{B}(A_j) \otimes \mathcal{F}_j$-measurable, where $\mathcal{B}(A_j)$ is the Borel σ-algebra on A_j. Lemma 9 implies that for λ_j-almost all $t_j \in T_j$,

$$\mathbf{E}^{\lambda_j}\left[w_i(a_{-j}, h_j(t_j), t_{-j}, t_j) \;\middle|\; \mathcal{F}_j\right] = \int_{A_j} w_i(a_{-j}, a_j, t_{-j}, t_j) \cdot \mu^{h_j|\mathcal{F}_j}(t_j; \mathrm{d}a_j). \tag{3.2}$$

Fix player 1. For any $t_1 \in T_1$ and any $a_1 \in A_1$, we have

$$\begin{aligned}
&\int_{T_{-1}} u_1(a_1, h_{-1}(t_{-1}), t_1, t_{-1}) \cdot q(t_1, t_{-1})\, \lambda_{-1}(\mathrm{d}t_{-1}) \\
&= \int_{T_{-1}} w_1(a_1, h_{-1}(t_{-1}), t_1, t_{-1})\, \lambda_{-1}(\mathrm{d}t_{-1}) \\
&= \int_{T_{-(1,2)}} \int_{T_2} w_1\big(a_1, h_2(t_2), h_{-(1,2)}(t_{-(1,2)}), t_{-2}, t_2\big)\, \lambda_2(\mathrm{d}t_2)\, \lambda_{-(1,2)}(\mathrm{d}t_{-(1,2)}) \\
&= \int_{T_{-(1,2)}} \int_{T_2} \mathbf{E}^{\lambda_2}\left[w_1\big(a_1, h_2(t_2), h_{-(1,2)}(t_{-(1,2)}), t_{-2}, t_2\big) \;\middle|\; \mathcal{F}_2\right] \lambda_2(\mathrm{d}t_2)\, \lambda_{-(1,2)}(\mathrm{d}t_{-(1,2)}) \\
&= \int_{T_{-(1,2)}} \int_{T_2} \int_{A_2} w_1\big(a_1, a_2, h_{-(1,2)}(t_{-(1,2)}), t_{-2}, t_2\big) \cdot \mu^{h_2|\mathcal{F}_2}(t_2; \mathrm{d}a_2)\, \lambda_2(\mathrm{d}t_2)\, \lambda_{-(1,2)}(\mathrm{d}t_{-(1,2)}) \\
&= \cdots \\
&= \int_{T_{-1}} \int_{A_{-1}} w_1(a_1, a_{-1}, t_1, t_{-1}) \cdot \prod_{j \neq 1} \mu^{h_j|\mathcal{F}_j}(t_j; \mathrm{d}a_j)\, \lambda_{-1}(\mathrm{d}t_{-1}),
\end{aligned}$$

where the subscript $-(1,2)$ denotes all the players except players 1 and 2. The first equality is due to the definition of density weighted payoff. The second equality is due to the Fubini property. The third equality holds by taking the conditional expectation. The fourth equality is implied by Equation (3.2). Then the previous four equalities are repeated for $n-2$ times (from T_3 to T_n). This procedure is omitted in the fifth equality, and finally leads to the last equality. One can repeat the argument and show that for any $i \in I$, $a_i \in A_i$ and $t_i \in T_i$,

$$\begin{aligned}
&\int_{T_{-i}} u_i(a_i, h_{-i}(t_{-i}), t_i, t_{-i}) \cdot q(t_i, t_{-i})\, \lambda_{-i}(\mathrm{d}t_{-i}) \\
&= \int_{T_{-i}} \int_{A_{-i}} w_i(a_i, a_{-i}, t_i, t_{-i}) \cdot \prod_{j \neq i} \mu^{h_j|\mathcal{F}_j}(t_j; \mathrm{d}a_j)\, \lambda_{-i}(\mathrm{d}t_{-i}).
\end{aligned} \tag{3.3}$$

For each $i \in I$, let F_i be a mapping from $T_i \times A_i \times \mathcal{R}^{\mathcal{F}}$ to $\mathbb{R}$ defined as follows:

$$F_i(t_i, a_i, g_1, \ldots, g_n) = \int_{T_{-i}} \int_{A_{-i}} w_i(a_i, a_{-i}, t_i, t_{-i}) \cdot \prod_{j \neq i} g_j(t_j; \mathrm{d}a_j)\ \lambda_{-i}(\mathrm{d}t_{-i}).$$

It is clear that F_i is $\mathcal{T}_i$-measurable on T_i and continuous on A_i. For each $i \in I$ and each $(g_1, g_2, \ldots, g_n)$, consider the best response correspondence G_i from T_i to A_i:

$$G_i(t_i, g_1, \ldots, g_n) = \arg\max_{a_i \in A_i} F_i(t_i, a_i, g_1, \ldots, g_n).$$

Then Berge maximum theorem and measurable maximum theorem imply that this correspondence is nonempty, compact-valued and upper-hemicontinuous, and admits a $\mathcal{T}_i$-measurable selection; see Theorems 17.31 and 18.19 in Aliprantis and Border (2006). Thus, $\mathcal{R}^{(\mathcal{T}_i, \mathcal{F}_i)}_{G_i(\cdot, g_1, \ldots, g_n)}$ is nonempty. Since $\mathcal{T}_i$ is nowhere equivalent to $\mathcal{F}_i$, Theorem 3 in He and Sun (2013) implies that $\mathcal{R}^{(\mathcal{T}_i, \mathcal{F}_i)}_{G_i(\cdot, g_1, \ldots, g_n)}$ is convex, weakly compact and upper-hemicontinuous on $\mathcal{R}^{\mathcal{F}} = \prod_{i=1}^{n} \mathcal{R}^{\mathcal{F}_i}$ (each $\mathcal{R}^{\mathcal{F}_i}$ is endowed with the topology of weak convergence).

Consider the correspondence Ψ from $\mathcal{R}^{\mathcal{F}}$ to itself as follows:

$$\Psi(g_1, g_2, \ldots, g_n) = \prod_{i=1}^{n} \mathcal{R}^{(\mathcal{T}_i, \mathcal{F}_i)}_{G_i(\cdot, g_1, \ldots, g_n)}.$$

It is clear that Ψ is nonempty, convex, weakly compact and upper-hemicontinuous on $\mathcal{R}^{\mathcal{F}}$. By Fan-Glicksberg's fixed-point theorem, there exists a fixed point $(g_1^*, g_2^*, \ldots, g_n^*)$ of Ψ. That is, for each $i \in I$, there exists a $\mathcal{T}_i$-measurable selection f_i^* of $G_i(\cdot, g_1^*, \ldots, g_n^*)$ such that $g_i^* = \mu^{f_i^* | \mathcal{F}_i}$.

Under the strategy profile $(f_1^*, f_2^* \ldots, f_n^*)$, the payoff of player i is

$$\begin{aligned}
U_i(f^*) &= \int_T w_i(f_i^*(t_i), f_{-i}^*(t_{-i}), t_i, t_{-i})\ \lambda(\mathrm{d}t) \\
&= \int_{T_i} \int_{T_{-i}} w_i(f_i^*(t_i), f_{-i}^*(t_{-i}), t_i, t_{-i})\ \lambda_{-i}(\mathrm{d}t_{-i})\ \lambda_i(\mathrm{d}t_i) \\
&= \int_{T_i} \int_{T_{-i}} \int_{A_{-i}} w_i(f_i^*(t_i), a_{-i}, t_i, t_{-i}) \cdot \prod_{j \neq i} g_j^*(t_j; \mathrm{d}a_j)\ \lambda_{-i}(\mathrm{d}t_{-i})\ \lambda_i(\mathrm{d}t_i).
\end{aligned}$$

The first equality holds due to the definition of U_i. The second equality holds based on the

Fubini property, and the third equality relies on Equation (3.3). By the choices of $(g_1^*, g_2^* \ldots, g_n^*)$ and $(f_1^*, f_2^*, \ldots, f_n^*)$, we have that $(f_1^*, f_2^* \ldots, f_n^*)$ is a pure strategy equilibrium.

3.5.3 Proof of Proposition 5

To prove Proposition 5, we first consider a sequence of auxiliary games.

Example 6. Let ρ be a metric on the given compact metric space A. For each fixed integer $m \geq 2$, pick m distinct elements in A, denoted as $a_1, a_2, \ldots, a_m$. For each $i = 1, 2, \ldots, m$, we choose a positive real number $r_m < 1$ such that closed balls $\bar{B}(a_i, r_m) \colon = \{a \in A \mid \rho(a_i, a) \leq r_m\}$ $(i = 1, 2, \ldots, m)$ are disjoint. By Urysohn's Lemma and the property that every closed set is a G_δ set in metric spaces, there exist continuous functions $\{f_i^m\}_{i=1,2,\ldots,m}$ and h_m satisfying the following requirements:

- $f_i^m \colon A \to [0, 1]$ such that $f_i^m(a) = 1$ for all $a \in \bar{B}(a_i, \frac{r_m}{2})$, $f_i^m(a) = 0$ for $a \in B(a_i, r_m)^c$ and $f_i^m(a) \in (0, 1)$ for $a \in B(a_i, r_m) \cap \bar{B}(a_i, \frac{r_m}{2})^c$;
- $h_m \colon A \to [0, 1]$ such that $h_m(a) = 0$ for $a \in \cup_{i=1}^m \bar{B}(a_i, \frac{r_m}{2})$, $h_m(a) = -5$ for $(\cup_{i=1}^m B(a_i, r_m))^c$.

Now we construct a 2-player Bayesian games G_m. The two players have a common action space A. The information structure is the same as the Bayesian game in Example 1 of He and Sun (2014a). The payoff functions are given as follows. Given the private information $(t_1, t_2) \in T_1 \times T_2$, when player 1 chooses action s_1 and player 2 chooses action s_2, their payoffs are:

$$
\begin{aligned}
u_1^m(s_1, s_2, t_1, t_2) = & \sum_{i=1}^m f_i^m(s_1) \cdot f_i^m(s_2) \cdot \big(3 - \rho(s_1, a_i)\big) + \sum_{i=1}^m f_i^m(s_1) \cdot f_{i+1}^m(s_2) \cdot \big(1 - \rho(s_1, a_i)\big) \\
& + \sum_{i,j \colon i \neq j,\ i+1 \neq j} f_i^m(s_1) \cdot f_j^m(s_2) \cdot \big(2 - \rho(s_1, a_i)\big) + h_m(s_1) - 2, \\
u_2^m(s_1, s_2, t_1, t_2) = & \sum_{i=1}^m f_i^m(s_1) \cdot f_{i+1}^m(s_2) \cdot \big(3 - \rho(s_2, a_{i+1})\big) + \sum_{i=1}^m f_i^m(s_1) \cdot f_i^m(s_2) \cdot \big(1 - \rho(s_2, a_i)\big) \\
& + \sum_{i,j \colon i \neq j,\ i+1 \neq j} f_i^m(s_1) \cdot f_j^m(s_2) \cdot \big(2 - \rho(s_2, a_i)\big) + h_m(s_2) - 2,
\end{aligned}
$$

where we denote $f_{m+1}^m = f_1^m$ and $a_{m+1} = a_1$.

In the following we will show that only $\{a_1, a_2, \dots, a_m\}$ survive in the iterated elimination of strictly dominated strategies for both players.

Round 1: For any $(t_1, t_2) \in T_1 \times T_2$ and any $s_2 \in A$, player 1's payoff is $u_1^m(x, s_2, t_1, t_2) = h_m(x) - 2 = -7$ if she chooses an action $x \in \left(\cup_{i=1}^m B(a_i, r_m)\right)^c$, while her payoff will be

$$u_1^m(a_1, s_2, t_1, t_2) = \begin{cases} -2, & \text{if } s_2 \in \left(\cup_{i=1}^m B(a_i, r_m)\right)^c, \\ 3 \cdot f_1^m(s_2) - 2, & \text{if } s_2 \in B(a_1, r_m), \\ f_2^m(s_2) - 2, & \text{if } s_2 \in B(a_2, r_m), \\ 2 \cdot f_j^m(s_2) - 2, & \text{if } s_2 \in B(a_j, r_m) \text{ for some } j \neq 1, 2, \end{cases}$$

if she chooses the action a_1.

Since for each $i = 1, 2, \dots, m$ $0 \leq f_i^m(a) \leq 1$ for any $a \in A$, the action x is strictly dominated by a_1 for player 1. Therefore, every action in $\left(\cup_{i=1}^m B(a_i, r_m)\right)^c$ is strictly dominated and will be eliminated from player 1's action space. That is, $A_1' = \cup_{i=1}^m B(a_i, r_m)$ survives after this round.

Round 2: For any $(t_1, t_2) \in T_1 \times T_2$ and any $s_1 \in A_1'$, player 2's payoff is $u_2^m(s_1, y, t_1, t_2) = h_m(y) - 2 = -7$ if she chooses an action $y \in \left(\cup_{i=1}^m B(a_i, r_m)\right)^c$, while her payoff will be

$$u_2^m(s_1, a_1, t_1, t_2) = \begin{cases} 3 \cdot f_m^m(s_1) - 2, & \text{if } s_1 \in B(a_m, r_m), \\ f_1^m(s_1) - 2, & \text{if } s_1 \in B(a_1, r_m), \\ 2 \cdot f_j^m(s_1) - 2, & \text{if } s_1 \in B(a_j, r_m) \text{ for some } j \neq 1, m, \end{cases}$$

if she chooses the action a_1.

Since for each $i = 1, 2, \dots, m$, $0 \leq f_i^m(a) \leq 1$ for any $a \in A$, the action y is strictly dominated by a_1 for player 2. Therefore, every action in $\left(\cup_{i=1}^m B(a_i, r_m)\right)^c$ is strictly dominated and will be eliminated from player 2's action space. That is, $A_2' = \cup_{i=1}^m B(a_i, r_m)$ survives after this round.

Round 3: For any $(t_1, t_2) \in T_1 \times T_2$ and any $s_2 \in A_2'$, when player 1 chooses an action $x_1' \in B(a_1, r_m) \setminus \{a_1\}$, her payoff is

$$\begin{cases} f_1^m(x_1') \cdot f_1^m(s_2) \cdot \big(3 - \rho(x_1', a_1)\big) + h_m(x_1') - 2, & \text{if } s_2 \in B(a_1, r_m), \\ f_1^m(x_1') \cdot f_2^m(s_2) \cdot \big(1 - \rho(x_1', a_1)\big) + h_m(x_1') - 2, & \text{if } s_2 \in B(a_2, r_m), \\ f_1^m(x_1') \cdot f_j^m(s_2) \cdot \big(2 - \rho(x_1', a_1)\big) + h_m(x_1') - 2, & \text{if } s_2 \in B(a_j, r_m) \text{ for some } j \neq 1, 2. \end{cases}$$

On the other hand, if player 1 chooses the action a_1, her payoff will be

$$\begin{cases} 3 \cdot f_1^m(s_2) - 2, & \text{if } s_2 \in B(a_1, r_m), \\ f_2^m(s_2) - 2, & \text{if } s_2 \in B(a_2, r_m), \\ 2 \cdot f_j^m(s_2) - 2, & \text{if } s_2 \in B(a_j, r_m) \text{ for some } j \neq 1, 2. \end{cases}$$

Since for each $i = 1, 2, \ldots, m$, $0 \leq f_i^m(a) \leq 1$ for any $a \in A$, the action x_1' is strictly dominated by a_1 for player 1. Therefore, every action in $B(a_1, r_m) \setminus \{a_1\}$ is strictly dominated and will be eliminated from player 1's action space. For $i = 2, 3, \ldots, m$, similar arguments show that every action in $B(a_i, r_m) \setminus \{a_i\}$ is strictly dominated by a_i for player 1. Therefore, $A_1'' = \{a_1, a_2, \ldots, a_m\}$ survives after this round.

Round 4: For any $(t_1, t_2) \in T_1 \times T_2$ and any $s_1 \in A_1''$, if player 2 chooses an action $y_1' \in B(a_1, r_m) \setminus \{a_1\}$, her payoff is

$$\begin{cases} f_1^m(y_1') \cdot \big(3 - \rho(y_1', a_1)\big) + h_m(y_1') - 2, & \text{if } s_1 = a_m, \\ f_1^m(y_1') \cdot \big(1 - \rho(y_1', a_1)\big) + h_m(y_1') - 2, & \text{if } s_1 = a_1, \\ f_1^m(y_1') \cdot \big(2 - \rho(y_1', a_1)\big) + h_m(y_1') - 2, & \text{if } s_1 = a_j \text{ for some } j \neq 1, m. \end{cases}$$

On the other hand, if player 2 chooses the action a_1, her payoff will be

$$\begin{cases} 1, & \text{if } s_1 = a_m, \\ -1, & \text{if } s_1 = a_1, \\ 0, & \text{if } s_1 = a_j \text{ for some } j \neq 1, m. \end{cases}$$

Since for each $i = 1, 2, \ldots, m$, $0 \leq f_i^m(a) \leq 1$ for any $a \in A$, the action y_1' is strictly dominated by a_1 for player 2. Therefore, every action in $B(a_1, r_m) \setminus \{a_1\}$ is strictly dominated and will be eliminated from player 2's action space. For $i = 2, 3, \ldots, m$, similar arguments show that every action in $B(a_i, r_m) \setminus \{a_i\}$ is strictly dominated by a_i for player 2. Therefore, $A_2'' = \{a_1, a_2, \ldots, a_m\}$ survives after this round.

Clearly, given any behavioral strategy, a rational player will choose a strictly dominated action with probability zero. Thus, we can allow these two players to focus on the actions in $A'' = \{a_1, a_2, \ldots, a_m\}$. The payoff matrix restricted on the action set A'' is as follows. Notice

		Player 2				
		a_1	a_2	a_3	$\cdots$	a_m
Player 1	a_1	$1, -1$	$-1, 1$	$0, 0$	$\cdots$	$0, 0$
	a_2	$0, 0$	$1, -1$	$-1, 1$	$\cdots$	$0, 0$
	a_3	$0, 0$	$0, 0$	$1, -1$	$\cdots$	$0, 0$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	a_m	$-1, 1$	$0, 0$	$\cdots$	$0, 0$	$1, -1$

that it is the same as the one in Example 1 of He and Sun (2014a).

Since m is an arbitrary fixed integer, the arguments above works for each integer $m \geq 2$.

Claim 4. *Suppose that (s_1, s_2) is a pure strategy equilibrium of the game G_m. Then for λ_i-almost all $t_i \in T_i$, $s_i(t_i) \in A'' = \{a_1, a_2, \ldots, a_m\}$, where $i = 1, 2$.*

Proof of Claim. Firstly we want to show $s_1(t_1) \in \cup_{j=1}^m B(a_j, r_m)$ for λ_1-almost all $t_1 \in T_1$. Suppose that there exists a measurable subset $T_1' \subseteq T_1$ such that $\lambda_1(T_1') > 0$ and $s_1(t_1) \in$

$(\cup_{j=1}^m B(a_j, r_m))^c$ for each $t_1 \in T_1'$. Since s_1 solves

$$\max_{s_1' \in L_{0,1}^{\mathcal{T}_1}} U_1(s_1', s_2) = \int_{T_2} \left[\int_{T_1} u_i(s_1'(t_1), s_2(t_2), t_1, t_2) \cdot q(t_1, t_2) \cdot \lambda_1(t_1) \right] \lambda_2(t_2).$$

Based on the simple calculation, we have

$$q(t_1, t_2) = \begin{cases} \frac{1}{2(1-h_1(t_1))h_2(t_2)}, & \text{if } 0 < h_1(t_1) \leq h_2(t_2) < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where h_i is a measure preserving mapping from $(T_i, \mathcal{F}_i, \lambda_i)$ to $([0,1], \mathcal{B}, \tau_i)$ such that for any $E \in \mathcal{F}_i$ there exists a set $E' \in \mathcal{B}$ such that $\lambda_i(E \Delta h_i^{-1}(E')) = 0$, and τ_i is the marginal distribution of the uniform distribution on the triangle of the unit square $\{(l_1, l_2) \mid 0 \leq l_1 \leq l_2 \leq 0\}$. It can be checked that there exists $T_1'' \subseteq T_1'$ and $T_2' \subseteq T_2$ such that $\lambda_1(T_1'') > 0$, $\lambda_2(T_2') > 0$ and $0 < h_1(t_1) \leq h_2(t_2) < 1$ for each $t_1 \in T_1''$ and $t_2 \in T_2'$. Let s_1^* be the strategy defined by

$$s_1^*(t_1) = \begin{cases} a_1, & \text{if } t_1 \in T_1'', \\ s_1(t_1), & \text{otherwise.} \end{cases}$$

Then by the same arguments in Round 1 and the simple calculation, we have $U_1(s_1, s_2) < U_1(s_1^*, s_2)$. which leads to a contradiction. Therefore, $s_1(t_1) \in \cup_{j=1}^m B(a_j, r_m)$ for λ_1-almost all $t_1 \in T_1$. Similarly, we have $s_2(t_2) \in A''$ for λ_2-almost all $t_2 \in T_2$.

Secondly we want to show $s_1(t_1) \in A''$ for λ_1-almost all $t_1 \in T_1$ in this case. Suppose that there exists a measurable subset $T_1' \subseteq T_1$ such that $\lambda_1(T_1') > 0$ and $s_1(t_1) \in (\cup_{j=1}^m B(a_j, r_m))^c \setminus A''$ for each $t_1 \in T_1'$. Let $T_{1,j}' = T_1' \cap B(a_j, r_m)$ for $j = 1, 2, \ldots, m$. Then there exists $j \in \{1, \ldots, m\}$ such that $\lambda_1(T_{1,j}') > 0$. Since s_1 solves

$$\max_{s_1' \in L_{0,1}^{\mathcal{T}_1}} U_1(s_1', s_2) = \int_{T_2} \left[\int_{T_1} u_i(s_1'(t_1), s_2(t_2), t_1, t_2) \cdot q(t_1, t_2) \cdot \lambda_1(t_1) \right] \lambda_2(t_2).$$

It can be checked that there exists $T_{1,j}'' \subseteq T_{1,j}'$ and $T_2' \subseteq T_2$ such that $\lambda_1(T_{1,j}'') > 0$, $\lambda_2(T_2') > 0$

and $0 < h_1(t_1) \leq h_2(t_2) < 1$ for each $t_1 \in T''_{1,j}$ and $t_2 \in T'_2$. Let s_1^* be the strategy defined by

$$s_1^{**}(t_1) = \begin{cases} a_j, & \text{if } t_1 \in T''_{1,j}, \\ s_1(t_1), & \text{otherwise.} \end{cases}$$

Then by the same arguments in Round 3 and the simple calculation, we have $U_1(s_1, s_2) < U_1(s_1^{**}, s_2)$. which leads to a contradiction. Therefore, $s_1(t_1) \in A''$ for λ_1-almost all $t_1 \in T_1$. Similarly, we have $s_2(t_2) \in A''$ for λ_2-almost all $t_2 \in T_2$. □

Proof of Proposition 5. First we consider a sequence of 2-player games $\{G_m\}_{m\geq 2}$ in Example 6, and then extend it to a sequence of n-player games. Based on the claim above and Theorem 5 in He and Sun (2014a), we thus have $\mathcal{T}_1$ has no $\mathcal{F}_1$-atom under λ_1, and $\mathcal{T}_2$ has no $\mathcal{F}_2$-atom under λ_2 either.

By adding dummy players as in the proof of Theorem 5 in He and Sun (2014a), we can get a sequence of n-player games $\{G_{m,i}\}_{m\geq 2, i=1,2,\ldots,m}$, where for each $m \geq 2$, players i and $i+1$ (denote $m+1 = 1$) are the only active players in the game $G_{m,i}$. Adopting the same arguments as in the proof of Theorem 5 in He and Sun (2014a), all the players have coarser density weighted payoff-relevant information. □

3.5.4 Proof of Proposition 6

Let g be an $\mathcal{F}$-measurable behavioral strategy profile. By Theorem 3 in He and Sun (2013), there exists a pure strategy profile f such that for each $i \in I$, $g_i = \mu^{f_i|\mathcal{F}_i}$. It is obvious that f and g are distribution equivalent. Hence, we have

$$\begin{aligned} U_i(g) &= \int_T \int_A u_i(a,t) \cdot \prod_{j\in I} g_j(t_j; \mathrm{d}a_j)\ \lambda(\mathrm{d}t) \\ &= \int_T \int_A w_i(a,t) \cdot \prod_{j\in I} g_j(t_j; \mathrm{d}a_j)\ \otimes_{i\in I} \lambda_i(\mathrm{d}t_i) \\ &= \int_T \int_A w_i(a,t) \cdot \prod_{j\in I} \mu^{f_j|\mathcal{F}_j}(t_j; \mathrm{d}a_j)\ \otimes_{i\in I} \lambda_i(\mathrm{d}t_i) \\ &= \int_T \int_A w_i(f_i(t_i), f_{-i}(t_{-i}), t_i, t_{-i})\ \otimes_{i\in I} \lambda_i(\mathrm{d}t_i) \end{aligned}$$

$$= U_i(f).$$

The first three equalities are due to the definitions of U_i, w_i, and f_i respectively. The fourth equality follows from repeated applications of Lemma 9 as in the proof of Equation (3.3).

Next, we verify strong payoff equivalence. For any player $i \in I$ and any given behavioral strategy h_i, we can obtain

$$\begin{aligned}
U_i(h_i, g_{-i}) &= \int_T \int_{A_i} \int_{A_{-i}} w_i(a_i, a_{-i}, t) \cdot \prod_{j \neq i} g_j(t_j; \mathrm{d}a_j) \cdot h_i(t_i; \mathrm{d}a_i) \; \otimes_{i \in I} \lambda_i(\mathrm{d}t_i) \\
&= \int_{T_i} \int_{A_i} \int_{T_{-i}} \int_{A_{-i}} w_i(a_i, a_{-i}, t_i, t_{-i}) \cdot \prod_{j \neq i} g_j(t_j; \mathrm{d}a_j) \; \otimes_{j \neq i} \lambda_j(\mathrm{d}t_j) h_i(t_i; \mathrm{d}a_i) \lambda_i(\mathrm{d}t_i) \\
&= \int_{T_i} \int_{A_i} \int_{T_{-i}} \int_{A_{-i}} w_i(a_i, a_{-i}, t_i, t_{-i}) \cdot \prod_{j \neq i} \mu^{f_j | \mathcal{F}_j}(t_j; \mathrm{d}a_j) \; \otimes_{j \neq i} \lambda_j(\mathrm{d}t_j) h_i(t_i; \mathrm{d}a_i) \lambda_i(\mathrm{d}t_i) \\
&= \int_{T_i} \int_{A_i} \int_{T_{-i}} \int_{A_{-i}} w_i(a_i, f_{-i}(t_{-i}), t_i, t_{-i}) \; \otimes_{j \neq i} \lambda_j(\mathrm{d}t_j) \, h_i(t_i; \mathrm{d}a_i) \; \lambda_i(\mathrm{d}t_i) \\
&= U_i(h_i, f_{-i}),
\end{aligned}$$

where the fourth identity follows from Equation (3.3) and the rest are clear. Thus, f and g are strongly payoff equivalent.[11]

Finally, we shall show that f and g are belief consistent. Fix $i \in I$. Define a nonnegative valued function c_i from $A_i \times T_i$ such that for each $a_i \in A_i$ and $t_i \in T_i$, $c_i(a_i, t_i) = d(a_i, \operatorname{supp} g_i(t_i))$, where $d(a_i, \operatorname{supp} g_i(t_i))$ is the distance of a_i to the support of the probability measure $g_i(t_i)$. It is obvious that for each $t_i \in T_i$, $\int_{A_i} c_i(a_i, t_i) g_i(t_i; \mathrm{d}a_i) = 0$. Since g_i is $\mathcal{F}_i$-measurable, c_i is $\mathcal{B}(A_i) \otimes \mathcal{F}_i$-measurable. Lemma 9 implies that

$$\begin{aligned}
\int_{T_i} c_i\big(f_i(t_i), t_i\big) \; \lambda_i(\mathrm{d}t_i) &= \int_{T_i} E\left[c_i\big(f_i(t_i), t_i\big) \mid \mathcal{F}_i\right] \; \lambda_i(\mathrm{d}t_i) \\
&= \int_{T_i} \int_{A_i} c_i(a_i, t_i) g_i(t_i; \mathrm{d}a_i) \; \lambda_i(\mathrm{d}t_i) = 0,
\end{aligned}$$

which implies that $c\big(t_i, f_i(t_i)\big) = 0$ for λ_i-almost all $t_i \in T_i$. Hence, $f_i(t_i) \in \operatorname{supp} g_i(t_i)$ for λ_i-almost all $t_i \in T_i$, and f_i and g_i are belief consistent. Thus, f and g are belief consistent.

[11] It is also obvious that strongly payoff equivalence preserves the equilibrium property. Thus, if g is a Bayesian-Nash equilibrium in behavioral strategy, then f is a Bayesian-Nash equilibrium in pure strategy.

Therefore, f is a strong purification of g.

3.5.5 Proof of Proposition 7

Let $([0,1], \mathcal{B}, \eta)$ be the Lebesgue unit interval, where $\mathcal{B}$ is the Borel σ-algebra on $[0,1]$ and η is the Lebesgue measure. As is well known, there is a measure preserving mapping ϕ_i from $(T_i, \mathcal{F}_i, \lambda_i)$ to $([0,1], \mathcal{B}, \eta)$ such that for any $E \in \mathcal{F}_i$, there exists a set $E' \in \mathcal{B}$ with $\lambda_i(E \triangle \phi_i^{-1}(E')) = 0$ (see, for example, Lemma 6 in He *et al.* (2016)).

Fix a positive integer $m \geq 2$. We consider the following game.

The set of players is I. Player i has the private information space $(T_i, \mathcal{T}_i, \lambda_i)$. The common prior is $\lambda = \otimes_{i \in I} \lambda_i$. The common action space is $A = [0, m]$. For the action profile $(a_1, \dots, a_n) \in \prod_{i \in I} A_i$ and state profile $(t_1, \dots, t_n) \in \prod_{i \in I} T_i$, the payoff of player i is

$$u_i(a_1, \dots, a_n, t_1, \dots, t_n) = - \prod_{0 \leq j \leq m-1} (a_i - \phi_i(t_i) - j)^2.$$

Notice that the payoff of player i does not depend on the action and the state of her opponents.

Define a behavioral strategy g_1 for player 1 as

$$g_1(t_1) = \frac{1}{m} \sum_{0 \leq j \leq m-1} \delta_{\phi_1(t_1)+j},$$

where $\delta_{\phi_1(t_1)+j}$ is the Dirac measure on A at the points $\phi_1(t_1) + j$. Let $\hat{g}_1$ be a function on the unit interval $L_1 = [0,1]$ as

$$\hat{g}_1(l_1) = \frac{1}{m} \sum_{0 \leq j \leq m-1} \delta_{l_1+j}.$$

Then $g_1 = \hat{g}_1 \circ \phi_1$ and is $\mathcal{F}_1$-measurable. Let τ be a probability measure on A such that for any Borel subset B in A,

$$\tau(B) = \int_{T_1} g_1(t_1)(B) \lambda_1(\mathrm{d}t_1),$$

which implies that

$$\tau(B) = \int_{T_1} \hat{g}_1(\phi_1(t_1))(B)\lambda_1(\mathrm{d}t_1) = \int_{[0,1]} \hat{g}_1(l_1)(B)\eta(\mathrm{d}l_1).$$

It is clear that τ is the uniform distribution on A.

Due to the condition of Proposition 7, the behavioral strategy g_1 has a strong purification f_1. By the definition, f_1 and g_1 are distribution equivalent. As a result, $\lambda_1 f_1^{-1} = \tau$. Based on the belief consistency condition, $f_1(t_1) \in \{\phi_1(t_1), \phi_1(t_1)+1, \ldots, \phi_1(t_1)+m-1\}$ for λ_1-almost all $t_1 \in T_1$. Thus, $f_1(t_1) = \phi_1(t_1) + j$ on a $\mathcal{T}_1$-measurable set $C_j \subseteq T_1$ for $j = 0, \ldots, m-1$.

Recall that for any $E \in \mathcal{F}_1$, there exists a set $E' \in \mathcal{B}$ with $\lambda_1(E \triangle \phi_1^{-1}(E')) = 0$. Then for any $j = 0, \ldots, m-1$, we have

$$\begin{aligned} \lambda_1(C_j \cap E) &= \lambda_1(C_j \cap \phi_1^{-1}(E')) = \lambda_1(f_1 \in E' + j) \\ &= \tau(E' + j) = \frac{1}{m}\eta(E') = \frac{1}{m}\lambda_1(\phi_1^{-1}(E')) = \frac{1}{m}\lambda_1(E), \end{aligned}$$

and hence $\lambda_1(C_j) = \frac{1}{m}$. Therefore, $\{C_0, \ldots, C_{m-1}\}$ is a $\mathcal{T}_1$-measurable partition of T_1 such that C_j is independent of $\mathcal{F}_1$ for $j = 0, \ldots, m-1$, which implies that $\mathcal{F}_1$ admits an asymptotic independent supplement in $\mathcal{T}_1$ under λ_1. By Lemma 7, $\mathcal{T}_1$ is nowhere equivalent to $\mathcal{F}_1$ under λ_1. Based on an analogous argument, $\mathcal{T}_i$ is nowhere equivalent to $\mathcal{F}_i$ under λ_i for every $i \in I$.

Chapter 4

Stationary Markov Perfect Equilibria in Large Stochastic Games

4.1 Introduction

Dynamic strategic interactions are common phenomena in economics. To consider such interactions in the setting of many agents, this paper studies stochastic games with a continuum of agents. The focus will be on stationary Markov perfect equilibria.

A large stochastic game is played by a continuum of players in discrete time with evolving publicly observable states, where the state evolution is governed by a transition probability determined by the state and the players' action distribution. In each stage, the current state induces a large game in which players simultaneously choose feasible actions and realize their state-dependent stage payoffs. A natural starting point for strategic analysis in this setting is to consider equilibria that reflect the stationary structure of the environment, which leads us to study stationary Markov perfect equilibria.

There is an extensive literature on the existence of behavioral-strategy stationary Markov perfect equilibria in finite-player stochastic games.[1] However, Levy (2013) and Levy and McLennan (2015) constructed examples on the non-existence of such equilibria for the case of

[1]See, for example, the book Neyman and Sorin (2003) and the survey Jaśkiewicz and Nowak (2016).

general state spaces. Based on the condition of "(decomposable) coarser transition kernels", He and Sun (2015) showed the existence of stationary Markov perfect equilibria. This paper demonstrates the existence of behavioral-strategy stationary Markov perfect equilibria in large stochastic games without any special conditions.

Given that large games often have pure-strategy Nash equilibria, a natural question is whether pure-strategy stationary Markov perfect equilibria exist in large stochastic games. We point out that pure-strategy stationary Markov perfect equilibria may not exist in general by presenting a counterexample in Example 7. To circumvent this nonexistence issue, we follow He, Sun and Sun (2016) by using the "nowhere equivalence" condition on the agent space to characterize the existence of pure-strategy stationary Markov perfect equilibria in large stochastic games (Theorem 6). With the same condition, we show in Theorem 7 that for every behavioral-strategy stationary Markov perfect equilibrium, there is an equivalent pure-strategy stationary Markov perfect equilibrium with the same total discounted payoffs for almost all the agents.

Next, we consider the fundamental issue of whether the model of stochastic games with a continuum of agents is in general a good proxy to large finite-agent stochastic games; namely, whether the closed graph property for the correspondence of stationary Markov perfect equilibria holds. Example 8 indicates that such a property may fail in general. By imposing uniform norm continuity on the transition probabilities as in Duffie *et al.* (1994), we show the closed graph property for the behavioral-strategy stationary Markov perfect equilibria in Theorem 8. We also use the nowhere equivalence condition on the agent space to characterize the closed graph property for pure-strategy stationary Markov perfect equilibria.

The rest of this chapter is organized as follows. Section 2 presents the large stochastic game model. Section 3 provides the existence theorem and the purification theorem. Section 4 discusses the upper hemi-continuity and the closed graph property. The appendix contains all the proofs of our results.

4.2 Markov perfect equilibria in large stochastic games

In this section, we shall specify the formulation of large stochastic games with general action and state spaces. The player space is modeled by an atomless probability space $(I, \mathcal{G}, \lambda)$.[2] All players observe the realized public state s, whose value lies in a Polish space $(S, \mathcal{S}, \mu)$. They then simultaneously choose actions from a common feasible action set A, which is a compact metric space. The set of Borel probability measures on A, denoted by $\mathcal{M}(A)$, will serve as the space of societal summaries. Note that $\mathcal{M}(A)$ is also a compact metrizable space endowed with the topology of weak convergence of measures and the resulting Borel σ-algebra $\sigma(\mathcal{M}(A))$.

Given the realized public state, each player's payoff continuously depends on her/his own action as well as on a societal summary that addresses the interdependence of everyone's actions.[3] The space of stage payoffs $\mathcal{U}_A$ is then defined as the space of all bounded continuous functions on the product space $A \times \mathcal{M}(A)$ with its sup-norm topology and the resulting Borel σ-algebra.

Time is infinite and discrete, *i.e.*, $t \in \mathbb{N}$. The public state evolves in a Markov fashion with a (stationary) transition probability P, which assigns probabilities on future states depending on the current state and the current societal summary. Technically, The transition probability P is a $\mathcal{S} \otimes \sigma(\mathcal{M}(A))$-measurable mapping from $S \times \mathcal{M}(A)$ to $\mathcal{M}(S)$, where $\mathcal{M}(A)$ and $\mathcal{M}(S)$ are endowed with the Borel σ-algebras $\sigma(\mathcal{M}(A))$ and $\sigma(\mathcal{M}(S))$ respectively. The existence of such transition probability is a standard result of regular conditional distributions in probability theory (see, for example, Theorem 1.6, Chapter 4 of Durrett (2005)). Note that all players discount the future by the same rate β where $\beta \in [0, 1)$.

Now, we are ready to present the definition of large stochastic games.

Definition 20 (Large stochastic game)**.** *A discrete time (stationary)* large stochastic game G with the player space $(I, \mathcal{G}, \lambda)$, public state space $(S, \mathcal{S}, \mu)$ and the common action space A consists of a $\mathcal{G} \otimes \mathcal{S}$-measurable mapping u from $I \times S$ to the space of stage payoffs $\mathcal{U}_A$, a transition probability P and a common discounted factor $\beta \in [0, 1)$.

[2] A probability space $(I, \mathcal{G}, \lambda)$ (or its σ-algebra) is atomless if for any non-negligible subset $E \in \mathcal{G}$, there is a $\mathcal{G}$-measurable subset E' of E such that $0 < \lambda(E') < \lambda(E)$.

[3] In fact, the payoff function for each player under a realised state is a bounded continuous function on $A \times \mathcal{M}(A)$.

Remark 6. *This model fits into a wide range of applications. For example, a homogeneous goods industry with a continuum of indentical firms. Each firm has negligible influence on the whole market and hence a price taker. Based on the observation of the current price (the public state), it will choose its own production level. And the aggregate production in return will affect the market price in the next stage.*

In general, the strategy of a player is a complete set of plans that describes the choices a player will make, given every possible realization of initial histories for any time. In the relevant literature, attention is typically focused on Markov strategies, which based on current states only (Marskin and Tirole (2001)). One may, therefore, define a Markov behavioral strategy profile in a large stochastic game as a sequence of $\mathcal{G} \otimes \mathcal{S}$ measurable functions $\{f^t\}_{t\in\mathbb{N}}$ such that $f^t : I \times S \to \mathcal{M}(A)$ with each player i plays behavioral strategy $\{f^t(i,\cdot)\}_{t\in\mathbb{N}}$.

Without loss of generality, player i's stage payoff function in large stochastic game G under public state s is denoted as $u(i, s, \cdot, \cdot)$ with a uniform upper bound B_u for all i, s.[4] Given a Markov behavioral strategy profile $\{f^t\}_{t\in\mathbb{N}}$, the continuation payoff of player i from a certain sequence of realizations of the public states $\{s^t\}_{t\in\mathbb{N}}$ is thus given by

$$\sum_{t=0}^{\infty} \beta^t \int_A u(i, s^t, a, \int_I f^t(i, s^t)\mu(\mathrm{d}i)) f^t(i, s^t, \mathrm{d}a),$$

where the Gelfand integral $\int_I f^t(i, s^t)\mu(\mathrm{d}i)$ is the societal summery at time t under the Markov behavioral strategy profile f^t with the realized public state s^t.

As the public state evolves with the transition probability P, we can further define the sequence of continuation payoffs $\{V^t(i, s^t, \{f^{\tilde{t}}\}_{\tilde{t}\in\mathbb{N}})\}_{t\in\mathbb{N}}$, where V^t is player i's discounted expected continuation payoffs started from time t given Markov strategy $\{f^{\tilde{t}}\}_{\tilde{t}\in\mathbb{N}}$ and initial state s^t,

$$V^t(i, s^t, \{f^{\tilde{t}}\}_{\tilde{t}\in\mathbb{N}}) = \int_A u(i, s^t, a, \int_I f^t(i, s^t)\mu(\mathrm{d}i)) f^t(i, s^t, \mathrm{d}a)$$

[4] It is well-known that $u(\cdot,\cdot,\cdot,\cdot)$ is a Carathéodory function, see for example, Theorem 4.55 in Aliprantis and Border (2006). If u is not uniformly bounded, we can replace each $u \in \mathcal{U}_A$ by composing a continuous increasing function $arctan \circ u$.

$$+\sum_{\tilde{t}=t+1}^{\infty}\beta^{\tilde{t}-t}\int_S\dots\int_S\int_A u(i,s^{\tilde{t}},a,\int_I f^{\tilde{t}}(i,s^{\tilde{t}})\mu(\mathrm{d}i))f^{\tilde{t}}(i,s^{\tilde{t}},\mathrm{d}a)$$
$$\times P(\mathrm{d}s^{\tilde{t}}|s^{\tilde{t}-1},\int_I f^{\tilde{t}-1}(i,s^{\tilde{t}-1})\mu(\mathrm{d}i))\dots P(\mathrm{d}s^{t+1}|s^t,\int_I f^t(i,s^t)\mu(\mathrm{d}i)).$$

From the literature of dynamic programming, we know the sequence $\{V^t\}_{t\in\mathbb{N}}$ is well-defined, $\mathcal{G}\otimes\mathcal{S}$-measurable for any given Markov strategy $\{f^{\tilde{t}}\}_{\tilde{t}\in\mathbb{N}}$. We now formally define a behavioral-strategy Markov prefect equilibrium.[5] For other related discussions of this equilibrium concept one may refer to Fudenberg and Tirole (1991) and Marskin and Tirole (2001). The definition here is thus an adaptation of the original concept to large stochastic games, synthesizing equilibrium concepts used in finite player stochastic games and large games.

Definition 21 (Markov perfect equilibrium)**.** A *behavioral-strategy Markov perfect equilibrium* $f^*=\{f^{*t}\}_{t\in\mathbb{N}}$ in a large stochastic game is a Markov behavioral strategy profile such that at any t and μ-almost all s,

$$V^t(i,s,f^*)\geq V^t(i,s,(f^*_{-i},f_i)),$$

for any other strategy f_i of player i for λ-almost all $i\in I$.

Notice that in the definition above, players are choosing optimally at every time for every possible realization of the states given that they will continue to choose optimally in the future. Though Markov strategy only allows players to condition their choices on the current state instead of the entire realized past history, a Markov perfect equilibrium is indeed a subgame perfect equilibrium. One may be able to see this by observing that, it is optimal for a player to use a Markov strategy when all other players are using this special kind of subgame strategy. Therefore, a Markov perfect equilibrium strategy is actually a subgame perfect equilibrium strategy.

The rest of this chapter is concerned with stationary Markov perfect equilibrium, which is a $\mathcal{G}\otimes\mathcal{S}$-measurable function f^* from $I\times S$ to $\mathcal{M}(A)$ such that for every $t\in\mathbb{N}$, $f^t=f^*$ and $\{f^t\}_{t\in\mathbb{N}}$ constitutes a Markov perfect equilibrium.

To study the stationary Markov perfect equilibrium, one may, therefore, seek to understand

[5] Pure strategy Markov equilibrium can be defined analogously, with a slight modification on the societal summaries and the stage payoffs.

the stationary structure of the large stochastic game. Recall that a large game is defined as a $\mathcal{G}$-measurable mapping from the player space I to the payoff space $\mathcal{U}_A$. Hence a $\mathcal{G} \otimes \mathcal{S}$-measurable mapping $u \colon I \times S \to \mathcal{U}_A$ in large stochastic game G, which assigns each player a specific stage payoff function under any state, defines a state-contingent large game. This game captures all the static elements of large stochastic game G, and thus represents the stationary structure of G. It is thus called *the auxiliary game* of large stochastic game G and denoted as G^a,

A state-contingent large game $u \colon I \times S \to \mathcal{U}_A$ is in fact a class of static large games parametrized by state s. A behavioral strategy profile f, which is $\mathcal{G} \otimes \mathcal{S}$-measurable from $I \times S$ to $\mathcal{M}(A)$, is called a state-contingent Nash equilibrium in the state-contingent large game u, if $f(\cdot, s)$ stands as a Nash equilibrium for large game $u(\cdot, s)$ under almost all state s.

Definition 22 (State-contingent Nash equilibrium)**.** A behavioral strategy profile f constitutes a *behavioral-strategy state-contingent Nash equilibrium* for a state-contingent large game u, if for μ-almost all fixed $s \in S$,

$$\int_A u(i, s, a', \int_I f(i', s)\lambda(\mathrm{d}i'))g(i, s, \mathrm{d}a') \geq u(i, s, a, \int_I f(i', s)\lambda(\mathrm{d}i')),$$

for all $a \in A$ and λ-almost all $i \in I$, where $\int_I f(i', s)\lambda(\mathrm{d}i')$ denotes the Ge′lfand integral of $f(\cdot, s)$.

4.3 The existence of stationary Markov perfect equilibria

4.3.1 Behavioral-strategy stationary Markov perfect equilibria

Rath, Sun and Yamashige (1995) constructed a large game with infinitely many actions, which does not have a pure-strategy Nash equilibrium. Motivated by their example, we construct a large stochastic game in the following which does not have any pure-strategy stationary Markov perfect equilibrium.

Example 7. *Consider a large stochastic game G with the unit interval $[0, 1]$ as the player space I, which is endowed with Lebesgue measure λ and the related σ-algebra. The action space*

A is the interval $[-1, 1]$, and $\mathcal{M}(A)$ denotes the set of social summaries. Let $\mathcal{U}_A$ be the set of continuous function on $A \times \mathcal{M}(A)$, which denotes the utilities. Define the public space S to be the finite set $\{1, 2, 3, ..., 10\}$. The large stochastic game G then maps from $I \times S$ to $\mathcal{U}_A$, whose payoff function of this large game is specified as follows:

$$u(i, s, a, \mu) : = g(a, 0.5d(\lambda^*, \mu)), -|i - |a|| + s,$$

where λ^* is the uniform distribution on $[-1, 1]$, $d(\lambda^*, \mu)$ is a Prohorov distance between λ^* and μ, and for any $a \in A$ and $l \in (0, 1]$,

$$g(a, l) = \begin{cases} \frac{a}{2} & 0 \leq a \leq \frac{l}{2}; \\ \frac{l-a}{2} & \frac{l}{2} \leq a \leq l; \\ -g(a-l, l) & l \leq a \leq 2l. \end{cases}$$

Claim 5. *The large stochastic game G has a behavioral-strategy stationary Markov perfect equilibrium, however, there is no pure-strategy stationary Markov perfect equilibrium.*

This example sheds some light on existence results. One may, therefore, explore the existence of behavioral-strategy stationary equilibria as well as the validity of purifications. In this section, we systematically study the existence issue of stationary Markov perfect equilibria. First the existence result for behavioral-strategy stationary Markov perfect equilibria is presented.

Theorem 5. *Every large stochastic game has a behavioral-strategy stationary Markov perfect equilibrium.*

We now briefly explain the idea behind the proof. The problem of the equilibrium existence, by Lemma 12, boils down to finding a state-contingent Nash equilibria in its auxiliary game. The fact that an auxiliary game is indeed a class of large games parametrized by states and every large game possesses a behavioral-strategy Nash equilibrium, leads to the study of a specific kind of measurable selection, which selects measurable functions in a measurable way that the resulted function is product measurable. To obtain such selection, we need to extend the main theorem Mertens (2003) to infinite dimensional space.

The general existence of behavioral-strategy stationary Markov perfect equilibria sheds light on the existence of pure stationary Markov perfect equilibria under a special circumstance, namely the large stochastic game with finite-dimensional action spaces. Indeed, with a slight modification of setting societal summaries to be action integrations instead of distributions, we derive the following proposition.

Proposition 8. *Under the setting that the common action set is compact in a finite-dimensional real topological vector space and societal summaries are action-integrations, every large stochastic game has a pure-strategy stationary Markov perfect equilibrium.*

Notice that in a large stochastic game with n actions, its societal summaries, or action distributions, lies in the unit ball in $\mathbb{R}^n$, which hence can also be interpreted as action integrations. Therefore, the proposition above can be applied directly in (i) every large stochastic game with finitely many actions. Due to the fact that every isomorphism of $\mathbb{R}^n$ onto an n-dimensional (real) topological vector space is a homeomorphism, another application is to the case of (ii) large stochastic games of action-integrated societal summaries, with action space to be a compact set in the n-dimensional Euclidean space.We summarise the above observations into the following corollary.

Corollary 1.

(i) Every large stochastic game with finite actions admits a pure-strategy stationary Markov perfect equilibrium.

(ii) Every large stochastic game, with compact feasible action sets in $\mathbb{R}^n$ and action-integrated societal summaries, has a pure-strategy stationary Markov perfect equilibrium.

All details of the proofs in this subsection are placed in Section 4.5.2.

4.3.2 Pure-strategy stationary Markov perfect equilibria

In the previous section we have studied the existence of behavioral-strategy stationary Markov perfect equilibria and explored the equilibrium existence in pure strategy under several special cases. Now we turn to the study of the existence of pure-strategy stationary Markov perfect equilibria in general. We show that the nowhere equivalence condition, introduced in He *et al.* (2016), is sufficient and necessary to guarantee such existence in general. A further purification result follows under this condition.

In our current formulation, two σ-algebras $\mathcal{F}$ and $\mathcal{G}$ are introduced to the player space such that $(I, \mathcal{F}, \lambda)$ is the probability space for players with $\mathcal{G}$ being a sub σ-algebra of $\mathcal{F}$. These two σ-algebras $\mathcal{F}$ and $\mathcal{G}$ are called the *universal σ-algebra* and the *characteristics type σ-algebra*, respectively. Here $\mathcal{G}$ can be viewed as the σ-algebra induced by stage payoffs under all possible states[6]. That is to say, $\mathcal{F}$ consists of every event that may happen among the whole population, while $\mathcal{G}$ only consists of those perceivable ones. Hence, the information processed by $\mathcal{F}$ is essentially richer than that by $\mathcal{G}$.

Nowhere equivalence condition requires certain properties on every probability space restricted to a non-negligible set in $\mathcal{F}$. To be exact, for any non-negligible subset $D \in \mathcal{F}$, *i.e.*, $\lambda(D) > 0$, the restricted probability space $(D, \mathcal{G}^D, \lambda^D)$ is defined as follows: $\mathcal{G}^D$ is the σ-algebra $\{D \cap D' \colon D' \in \mathcal{G}\}$ and λ^D is the probability measure re-scaled from the restriction of λ to $\mathcal{G}^D$; the restricted probability space $(D, \mathcal{F}^D, \lambda^D)$ is defined similarly. We state the nowhere equivalence condition introduced in He *et al.* (2016) as follows.

Definition 23 (Nowhere equivalence condition)**.** The σ-algebra $\mathcal{F}$ is said to be nowhere equivalent to the sub-σ-algebra $\mathcal{G}$, if for every non-negligible subset $D \in \mathcal{F}$, there exists an $\mathcal{F}$-measurable subset D_0 of D such that $\lambda(D_0 \triangle D_1) > 0$ for any $D_1 \in \mathcal{G}^D$, where $\triangle$ denotes the symmetric difference operator. That is, $D_0 \triangle D_1 = (D_0 \setminus D_1) \cup (D_1 \setminus D_0)$.

By relaxing the measurability of strategy profiles to the enlarger σ-algebra $\mathcal{F} \otimes \mathcal{S}$, we are able to establish the general existence of pure stationary Markov perfect equilibria. Furthermore, this nowhere equivalence is shown to be a complete characterization of such a general existence result. We are now ready to present the existence theorem.

Theorem 6. *Every $\mathcal{G} \otimes \mathcal{S}$-measurable*[7] *large stochastic game has an $\mathcal{F} \otimes \mathcal{S}$-measurable pure-*

[6]That is to say, $\mathcal{G}$ is the σ-algebra generated by all $u(\cdot, s\cdot, \cdot)$ for all possible s, where $u(\cdot, \cdot, \cdot, \cdot)$ is a stage payoff mapping from $I \times S$ to $\mathcal{U}_A$.

[7]We slightly mix the notation here just for illustrative simplicity, to be accurate one should say "every large

strategy stationary Markov perfect equilibrium if and only if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.

It is known that behavioral-strategy stationary Markov perfect equilibria exist without any condition in large stochastic games (Theorem 5). Theorem 6 further implies that, under nowhere equivalence condition, it may be possible to purify a behavioral-strategy stationary equilibrium to a pure-strategy stationary one in general. We then proceed to the details of this purification result. Normally, a pure strategy profile is a purification of a behavioral strategy profile if they both induce the same societal summary and are consistent with individual strategies. With a slight modification regarding the term "state contingent", the formal definition of purification that suits our setting follows.

Definition 24 (Purification)**.** *A pure-strategy stationary Markov perfect equilibrium (resp. state-contingent Nash equilibrium) f is a purification of a behavioral-strategy stationary Markov perfect equilibrium (resp. state-contingent Nash equilibrium) $\tilde{f}$ if for μ-almost all s,*

(i) (societal summary equivalence) the distribution of $f(\cdot, s)$ is equivalent to that of $\tilde{f}$,

$$\lambda f^{-1}(\cdot, s) = \int_I \tilde{f}(i', s)\lambda(\mathrm{d}i').$$

(ii) (strategy consistency) each player i's action in $f(i, s)$ is selected from the support of the corresponding behavioral strategy $\tilde{f}(i, s)$; that is,

$$f(\cdot, s) \in \operatorname{supp} \tilde{f}(\cdot, s) \text{ for } \lambda\text{-almost all } i.$$

We are now ready to show the purification under the nowhere equivalence condition. It proves the sufficiency of this particular condition for the pure-strategy equilibrium existence.

Theorem 7. *For every $\mathcal{G} \otimes \mathcal{S}$-measurable large stochastic game, every $\mathcal{G} \otimes \mathcal{S}$-measurable behavioral-strategy stationary Markov perfect equilibrium processes a $\mathcal{F} \otimes \mathcal{S}$-measurable purification if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.*

All details of the proofs in this section are placed in Section 4.5.2.

stochastic game with with $\mathcal{G} \otimes \mathcal{S}$-measurable stage payoff mapping u from $I \times S$ to $\mathcal{U}_A$".

4.4 Approximation and the closed-graph property

A significant difference between finite-player stochastic games and large stochastic games is that the former has strategic behaviors but the later does not. Nevertheless, connections of the behaviors between these two environment can be established by approximation.

From now on, we will study another representation of large stochastic games, which is in terms of choosing a sequence of finite-player stochastic games and look at the stationary Markov perfect equilibrium in the limit game. The linking between these two kinds of representations is the "closed graph property".

Preliminary: Let $(I^n, \mathcal{I}^n, \lambda^n)$ be an n-element probability space with $n \in \mathbb{N}$, where $\mathcal{I}^n$ denotes its power set [8] and λ^n is its probability measure. In particular, λ^n assigns equal weight to each element. $(I^n, \mathcal{I}^n, \lambda^n)$ then serves as the space of n players.

The public state space $(S, \mathcal{S}, \mu)$ is a polish space. The common action space A is compact and metrizable, and its probability distribution space is $\mathcal{M}(A)$, denotes the whole collection of societal summaries. Let $\mathcal{U}_A$ be the Banach space of continuous functions on $A \times \mathcal{M}(A)$ endowed with the sup-norm topology and the resulting Borel σ-algebra. β is the common discount factor. The transition probability P^n is a $\mathcal{S} \otimes \sigma(\mathcal{M}(A))$-measurable mapping from $S \times \mathcal{M}(A)$ to $\mathcal{M}(S)$.

A large finite stochastic game G^n consists of a $\mathcal{I}^n \otimes \mathcal{S}$ measurable mapping $u^n(i^n, s, \cdot, \cdot)$ from $I^n \times S$ into $\mathcal{U}_A$, together with a $\mathcal{S} \otimes \sigma(\mathcal{M}(A))$-measurable transition probability P^n and a common discounted factor β. As a well-known fact in the literature, a behavioral-strategy stationary Markov perfect equilibrium in the respective large finite stochastic game G^n can be therefore defined in the following recursive form, see He and Sun (2015) for example.

Definition 25. *(Stationary Markov perfect equilibria in large finite stochastic games) For any $n \in \mathbb{N}$, a behavioral strategy profile $f^n \colon I^n \times S \to \mathcal{M}(A)$ is a behavioral-strategy stationary Markov perfect equilibrium in large finite stochastic game G^n, if there exists a continuation*

[8] The power set $\mathcal{I}^n$ is the collection of all subsets of I^n.

utility function $v^n(i,s) \in L_\infty^{\mathcal{I}^n \otimes \mathcal{S}}(I^n \times S)$[9] *which satisfies for* μ*-almost all fixed* $s \in S$,

$$v^n(i,s) = (1-\beta)\int_A u^n(i,s,a,\int_{I^n} f^n(i',s)\lambda^n(\mathrm{d}i'))f^n(i,s,\mathrm{d}a) + \beta\int_S v^n(i,s')P^n(ds'|s,\int_{I^n} f^n(i',s)\lambda^n(\mathrm{d}i')$$
$$= \max_{a\in A}(1-\beta)u^n(i,s,a,\int_{I^n}(a,f^n_{-i}(i',s))\lambda^n(\mathrm{d}i')) + \beta\int_S v^n(i,s')P^n(ds'|s,\int_{I^n}(a,f^n_{-i}(i',s))\lambda^n(\mathrm{d}i'))$$

for all $i \in I^n$*, and* $\int_{I^n}(a,f^n_{-i}(i',s))\lambda^n(\mathrm{d}i')$ *denotes a strategy profile that player* i *plays* a *and his opponent player* j *plays* $f^n(j,s)$ *for* $j \neq i$ *under state* s.

Note that a proper approximation to a large stochastic game requires each player's influence in the sequence of large finite stochastic games decreases to zero as the number of players tends to infinity. In our setting, this is satisfied since $\sup_{i\in I_n}\lambda_n(i) = \frac{1}{n} \to 0$ as $n \to \infty$. We further impose uniform continuity on the convergent class of transition probabilities under norm topology. This condition smoothes the flow of information in the evolution of states, which we will explain in details later. The closed graph property, therefore, can be described formally as below:

Definition 26 (The closed graph property)**.** *A behavioral-strategy (resp. pure-strategy) stationary Markov perfect equilibrium correspondence of a large stochastic game* G^0 *has the closed graph property, if for any sequence of large finite stochastic games* $\{G^n\}_{n>0}$[10] *satisfies that for* μ*-almost all* s,

(i) the sequence of stage utilities $\{u^n(\cdot,s,\cdot,\cdot)\}_{n>0}$ *converges weakly to* $u^0(\cdot,s,\cdot,\cdot)$ *as* $n\to\infty$,

(ii) the sequence of behavioral-strategy (resp. pure-strategy) stationary Markov perfect equilibria f^n *of* G^n *for each* $n>0$ *is such that* $\{\int_{I^n} f^n(i,s)\lambda^n(di)\}_{n>0}$ *converges weakly to some* $\nu(s)\in\mathcal{M}(A)$ *as* $n\to\infty$,

(iii) $\{P^n(s'|s,\tau)\}_{n\geq 0}$ *is uniformly continuous in any* $\tau\in\mathcal{M}(A)$ *where* $\mathcal{M}(A)$ *and* $\mathcal{M}(S)$ *are endowed with the prohorov metric and total variation norm respectively,*

9 $L_\infty^{\mathcal{I}^n \otimes \mathcal{S}}(I^n \times S)$ denotes the set of $\mathcal{I}^n \otimes \mathcal{S}$ measurable function whose sup-norm is finite, *i.e.*, $||h||_\infty < \infty$ for $h \in L_\infty^{\mathcal{I}^n \otimes \mathcal{S}}(I^n \times S)$.

10 Equal weight condition can be further relaxed to the condition $\sup_{i\in I^n}\lambda^n(i)\to 0$.

there exists a behavioral-strategy (resp. pure-strategy) stationary Markov perfect equilibrium f^0 *of* G^0 *such that* $\int_{I^0} f^0(i,s)\lambda^0(di) = \nu(s)$ *for* μ*-almost all* s.

4.4.1 On the norm continuity of the transition

In general, the closed graph property is not necessarily true in a large stochastic game, for small information conveying in the discrete environment may lead to strategic behaviors surviving in the sequence of large finite stochastic games but not in the limit continuum game. To avoid such failure, strategy behaviors should decline with increasing number of players, which in fact narrows the difference between the discrete and continuum environments.

A Common approach, followed in the literature on stochastic games, is to specify certain level of prerequisite continuity on the transition probability in societal summaries, ranging from (1) weak continuous (e.g., Federgruen (1978) and Dutta and Sundaram (1998)); (2) norm-continuous (e.g., Solan (1998) and Mertens and Parthasarathy (2003)) (3) norm-continuous and absolutely continuous with respect to a fixed, non-atomic probability measure (e.g, Duffie *et al.* (1994) and Duggan (2012)); etc. We adopt the norm-continuous condition herein, which is a proper choice due to the failure of the closed graph property under weaker continuity conditions (Example 8).

Definition 27 (Norm continuity of the transition)**.**

- *The total variation norm* $\|\cdot\|$ *of any element* ρ *in a space of signed measures on an arbitrary measurable space* $(S, \mathcal{S})$ *is defined by*

$$||\rho||_{TV}: \; = \sup \sum_{i=1}^{k} |\rho(A_i)|,$$

where the supremum is over all finite partitions of S *into disjoint measurable subsets. For any two probability measures* μ_1 *and* $\mu_2 \in \mathcal{M}(S)$*, the distance between them specified by the total variation norm has an equivalent definition (Lemma 11.5, Stokey, Lucas and Prescott (1989)):*

$$||\mu_1 - \mu_2||_{TV}: \; = 2 \cdot \sup_{A \in \mathcal{S}} |\mu_1(A) - \mu_2(A)|.$$

- *We call a transition probability P is norm-continuous in $\mathcal{M}(A)$ if the transition probability $P\colon S \times \mathcal{M}(A) \to \mathcal{M}(S)$ is a $\mathcal{S} \otimes \sigma(\mathcal{M}(A))$-measurable mapping and for μ-almost all s,*

$$P(s'|s, \lambda) \text{ is a continuous mapping from } \mathcal{M}(A) \text{ to } \mathcal{M}(S),$$

where $\mathcal{M}(A)$ is endowed with weak topology and $\mathcal{M}(S)$ is endowed with total variation norm topology.[11]

In the following, we will discuss the appropriateness of the norm-continuous condition by two examples. We first present a counnterexample in which the transition probability is weak continuous in but not norm continuous in $\mathcal{M}(A)$, and the closed graph property for stationary Markov perfect equilibria correspondence fails.

Example 8. Let a sequence of large finite games $\{G^n\}_{n\in\mathbb{N}}$ with probability space $(I^n, \mathcal{I}^n, \lambda^n)$ to be the player space of G^n, where $I^n = \{\frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\}$ with equally distributed weights on each player, i.e, $\lambda^n(\frac{k}{n}) = \frac{1}{n}$ for $k = 1, 2, ..., n$, and the Lebesgue unite interval $(I, \mathcal{I}, \lambda)$ be the continuum player space of G in the limit, where $I = [0, 1]$, $\mathcal{I}$ is the Lebesgue σ-algebra, and λ is the Lebesgue measure. The rest elements are the same for both the sequence of large finite stochastic games $\{G^n\}_{n\in\mathbb{N}}$ and the limit large stochastic game G: the two-action space $A = \{C, D\}$ and the type space $(S, \mathcal{S}, \mu)$ where $S = [0, 1]$, $\mathcal{S}$ is the Lebesgue σ-algebra and μ is the Lebesgue measure. All games $\{G^n\}_{n\in\mathbb{N}}$ and G share the same transition probability P, which is defined as:

$$P(s'|s, \lambda) = \delta_{\frac{a_1}{2}}, \text{ and } \delta_s \text{ denotes the Dirac measure on state } s.$$

where $\lambda = (a_1, 1 - a_1)$ is a societal summary with a_1 being the proportion of players choosing "C". The payoff function for each player is defined in the following: if the realized state s satisfies $0 \le s < \frac{1}{2}$, then each player receives -3 with action "C" and -2 with action "D"; on the other hand, if $\frac{1}{2} \le s \le 1$, then each player receives -1 with "C" and 0 with "D". And the discount factor $\delta > \frac{1}{2}$.

[11]Though the measurability statement seems the same, it is actually strengthened compared to the initial setting due to the strengthen of the image space's topology.

Claim 6. For the games described in Example 8, we have the following results:

(i) For all $s \in S$, the transition probability $P\colon \mathcal{M}(A) \to \mathcal{M}(S)$ is weak continuous in $\mathcal{M}(A)$ where both $\mathcal{M}(A)$ and $\mathcal{M}(S)$ are equipped with weak topologies. But P is not norm-continuous in $\mathcal{M}(A)$.

(ii) The strategic behavior that every player choose "C" if state $\frac{1}{2} \leq s \leq 1$, and "D" otherwise, *i.e,* $f_n(\cdot, s) \equiv C$ for $\frac{1}{2} \leq s \leq 1$ and $f_n(\cdot, s) \equiv D$ for $0 \leq s < \frac{1}{2}$, is a stationary Markov perfect equilibrium in every finite player game G^n.

(iii) The unique stationary Markov perfect equilibrium in the limit large stochastic game G is that every player chooses "D" ($f \equiv D$) .

The large stochastic game we constructed in the example above fails to satisfy the closed graph property[12] under (uniformly) weak continuity, which implies stronger continuity is required. In the following we revisit Example 8, and regain the closed graph property by modifying the transition probability such that it is uniformly norm-continuous.[13] A further discussion shows that the closed graph property is valid under such modification.

Example 9. *Use the identical setting in Example 8, except that the transition probability P is redefined as follows:*

$$P(s'|s, \lambda) = a_1\mu_1 + (1 - a_1)\mu_0, \text{ where } \mu_0 \text{ and } \mu_1 \text{ are two distinguished probability measures on } [0, 1].$$

Then we claim that

(i) The transition probability is norm-continuous;

(ii) There exists a large number N, such that for all $n \geq N$, all players play "D" in all states, i.e., $f^n \equiv D$, is the unique stationary Markov perfect equilibrium in those large finite stochastic game G^n. Therefore the stationary Markov perfect equilibrium correspondence in the large stochastic game G has the closed graph property.

[12] That is, the sequence of stationary Markov perfect equilibria f^n in the sequence of large finite stochastic games clearly does not converge weakly to f for μ-almost all state s.

[13] Note that existence, hence nowhere equivalence, is not a key issue here, since Corollary 1 can guarantee the equilibrium existence in finite-action environment.

The intuition for Example 9 is the following. Although μ_0 and μ_1 can be quite different (e.g. $\mu_i = \delta_i$ for $i \in \{0, 1\}$)[14], the state evolution rule gives defectors a chance to get away with a probability proportional to the size of cooperators. Since a single player's influence is almost negligible in a large population, an unilateral deviation will become profitable to any individual player as the ppopulation size goes beyond certain level. Hence always playing "D" is the only stationary Markov perfect equilibrium in the game G^n when the population is significantly large. The proofs in this section are given in Section 4.5.3.

4.4.2 The closed graph theorem

In this section, we consider the closed graph theorem. As suggested in Lemma 12, this asymptotic implementation requires that, as a sequence of finite-player stochastic games converges towards the limit, the set of stationary Markov perfect equilibria in each finite-player stochastic game should approximate the set of state-contingent Nash equilibria in its auxiliary game. Uniform continuity in the state evolution process (if valid) will help to control the evolution such that it becomes smooth and tractable, since no sharp drifts occur from time to time. Hence it is a suitable prerequisite for such approximation.

Throughout this section, we are working with a general framework, except for adopting such condition. Although this condition itself is quite strong, the proof of Lemma 14 suggests that such continuity is actually reasonable for considering the closed graph property in general large stochastic games, due to the arbitrary choices of continuation payoffs. Also previous example (Example 8) shows that with weaker continuity such property may fail.

The validity of the closed graph theorem hinges on two key factors: one is the issue of equilibrium existence, and the other is a smooth information flow, *i.e.*, no revelation of unilateral deviations when the population size is large enough. For behavioral-strategy stationary Markov perfect equilibria, the former is guaranteed by Theorem 5, and the later is supported by the uniform norm-continuity in the transition probability.

Theorem 8 (The closed graph theorem)**.** *The behavioral-strategy stationary Markov perfect*

[14]Here δ_a denotes the Dirac measure which places mass 1 on the point a.

equilibrium correspondence of any large stochastic game has the closed graph property.

When restricted to pure strategies, the nowhere equivalence condition completely characterizes the equilibrium existence (Theorem 6). Therefore, the two conditions together: the nowhere equivalence and uniform norm-continuity, are sufficient for the closed graph theorem to hold for the pure-strategy stationary Markov perfect equilibrium correspondence of any large stochastic game.

A further scrutiny gives that nowhere equivalence is also necessary for the closed graph theorem of pure-strategy stationary Markov perfect equilibria. Indeed, this is a straightforward application of the closed graph property in He *et al.* (2016). They show that nowhere equivalence is the sufficient and necessary condition for the pure Nash equilibrium correspondence of every large game to satisfy the closed-graph property. The large repeated game bridges their asymptotic implementation and the current one. While the closed-graph property in He *et al.* (2016) is static, the current one involves the stochastic evolutions of states. It is, therefore, conceivable that the nowhere equivalence is also a necessary condition for the closed graph theorem herein.

We then summerize the above discussions in the following proposition, which completely characterizes the closed graph property for pure-strategy stationary Markov perfect equilibrium correspondences of large stochastic games.

Proposition 9. *The $\mathcal{F} \otimes \mathcal{S}$-measurable pure-strategy stationary Markov perfect equilibrium correspondence of any $\mathcal{G} \otimes \mathcal{S}$-measurable large stochastic game has the closed graph property if and only if $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$.*

In the following we will give an example illustrating the necessity of nowhere equivalence. Motivated by Example 4 in He *et al.* (2016), we construct a counterexample using large repeated games to explain that the lack of the nowhere equivalence condition leads to the failure of the (pure) closed graph theorem.

Example 10. *Define a large repeated game G such that the player space is the Lebesgue unit interval $(I, \mathcal{F}, \lambda)$ where $I = [0, 1]$, $\mathcal{F}$ is the Lebesgue σ-algebra, and λ is the Lebesgue measure. The common action space is $A = [-1, 1]$. The stage payoffs for game G are defined as follows.*

For player $i \in [0,1]$, action $a \in A$ and societal action distribution $v \in \mathcal{M}(A)$,

$$G(i, a, v) = -(a+i)^2 \cdot (a-i)^2.$$

Now we define a sequence of finite-player repeated games $\{G_k\}_{k\in\mathbb{N}}$. For each $k \in \mathbb{N}$, the probability space $(I_k, \mathcal{F}_k, \lambda_k)$ is the agent space of the game G_k, where $I_k = \{1, 2, ..., 2k\}$, $\mathcal{F}_k$ is the power set of I_k, and λ_k is the counting probability over $\mathcal{F}_k$. The stage payoff function for player $j \in I_k$ in the game G_k is

$$G_k(j, a, v) = -(a + \frac{j}{2k})^2 \cdot (a - \frac{j}{2k})^2,$$

for her own action $a \in A$ and societal action distribution $v \in \mathcal{M}(A)$. For each $k \in \mathbb{N}$ and $j \in I_k$, let

$$f_k(j) = (-1)^j \frac{j}{2k}.$$

All games $\{G_k\}_{k\in\mathbb{N}}$ and G are endowed with the same discount factor β and trivial transition probability. Consider all stationary Markov perfect equilibria in the large repeated game G are measurable in the Lebesgue σ-algebra $\mathcal{F}$, which is clearly not satisfied the nowhere equivalence condition. Then we have the following claim:

Claim 7.

(i) The sequence of finite-player repeated games $\{G_k\}_{k\in\mathbb{N}}$ converges weakly to G, i.e., $P_k G_k^{-1} \Rightarrow PG^{-1}$ as $k \to \infty$.

(ii) f_k is a pure-strategy stationary Markov perfect equilibrium of finite-player repeated game G_k for each $k \in \mathbb{N}$, and the sequence of $\{P_k f_k^{-1}\}_{k\in\mathbb{N}}$ converges weakly to the uniform distribution on A.

(iii) The uniform distribution on A is not a distribution of any $\mathcal{F}$-measurable stationary Markov perfect equilibrium in large repeated game G.

All proofs in this section are given in Section 4.5.3.

4.5 Proofs

We begin this appendix by collecting recent results on the regular conditional distributions of correspondences.

4.5.1 Regular conditional distribution

In this subsection, we state the definitions of the transition probability and the regular conditional distribution here for the convenience of readers.

Recall that $(I, \mathcal{F}, \lambda)$ is an atomless probability space and $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. Given a measurable subset E, $\lambda|_E$ denotes the restriction of λ to E. Let X be a Polish space (complete separable metric topological space), $\mathcal{B}(X)$ the Borel σ-algebra of X, and $\mathcal{M}(X)$ the space of all Borel probability measures on X with the topology of weak convergence. We recall that $\mathcal{M}(X)$ is again a Polish space, and if X is compact then so is $\mathcal{M}(X)$. We use $C_b(X)$ to denote the set of all bounded continuous function from X to $\mathbb{R}$.

Definition 28. A $\mathcal{G}$-measurable transition probability from I to X is a mapping $\phi\colon I \to \mathcal{M}(X)$ such that for every $B \in \mathcal{B}(X)$ the mapping

$$\phi(\cdot; B)\colon i \mapsto \phi(i; B)$$

is $\mathcal{G}$-measurable, where $\phi(i; B)$ is the value of the probability measure $\phi(i)$ on the Borel subset $B \subseteq A$.

We use $\mathcal{R}^{\mathcal{G}}(X)$, or $\mathcal{R}^{\mathcal{G}}$ when it is clear, to denote the set of all $\mathcal{G}$-measurable transition probabilities from I to X. Let $\mathbb{Z}_+$ denote the set of positive integers.

Definition 29. A sequence $\{\phi_n\}_{n \in \mathbb{Z}_+}$ in $\mathcal{R}^{\mathcal{G}}$ is said to weakly converge to ϕ in $\mathcal{R}^{\mathcal{G}}$, denoted by $\phi_n \Rightarrow \phi$, if for every bounded Carathéodory function $c\colon I \times X \to \mathbb{R}$,[15]

$$\lim_{n\to\infty} \int_I \left[\int_X c(i,x)\phi_n(i; \mathrm{d}x) \right] \mathrm{d}\lambda(i) = \int_I \left[\int_X c(i,x)\phi(i; \mathrm{d}x) \right] \mathrm{d}\lambda(i).$$

[15] Given a probability space $(I, \mathcal{G}, \lambda)$ and a Polish space X, a function $c\colon I \times X \to \mathbb{R}$ is a Carathéodory function if $c(\cdot, x)$ is $\mathcal{G}$-measurable for each $x \in X$ and $c(i, \cdot)$ is continuous for each $i \in I$.

The weak topology on $\mathcal{R}^{\mathcal{G}}$ is defined as the weakest topology for which the functional

$$\phi \mapsto \int_I \left[\int_X c(i,x)\phi(i;\mathrm{d}x)\right] \mathrm{d}\lambda(i)$$

is continuous for every bounded Carathéodory function $c\colon I \times X \to \mathbb{R}$.

We next review the regular conditional distribution. Let f be an $\mathcal{F}$-measurable mapping from I to X. A mapping $\mu^{f|\mathcal{G}}\colon I \times \mathcal{B}(X) \to [0,1]$ is said to be a regular conditional distribution for f given $\mathcal{G}$, if

1. for λ-almost all $i \in I$, $\mu^{f|\mathcal{G}}(i,\cdot)$ is a probability measure on X;
2. for each Borel subset $B \subseteq X$, $\mu^{f|\mathcal{G}}(i,B) = \mathbf{E}[1_B(f) \mid \mathcal{G}](i)$ for λ-almost all $i \in I$, where $\mathbf{E}[1_B(f) \mid \mathcal{G}]$ is the conditional expectation of the indicator function $1_B(f)$ given $\mathcal{G}$.

Since X is assumed to be a Polish space endowed with the Borel σ-algebra, the regular conditional distribution for f given $\mathcal{G}$ always exists; see Theorem 5.1.9 in Durrett (2005).

Let F be an $\mathcal{F}$-measurable correspondence from I to X. We use

$$\mathcal{R}_F^{(\mathcal{F},\mathcal{G})} = \left\{\mu^{f|\mathcal{G}} \;\middle|\; f \text{ is an } \mathcal{F}\text{-measurable selection of } F\right\}$$

to denote the set of regular conditional distributions induced by $\mathcal{F}$-measurable selections of F conditional on $\mathcal{G}$.

We present results on regular conditional distributions of correspondences culled from Theorem 3, Corollary 1 and Lemma 3 in He and Sun (2013) (see also He and Sun Y (2014)).

Lemma 10. *Suppose that $(T,\mathcal{T},\lambda)$ is atomless and $\mathcal{F}$ is a countably generated sub-σ-algebra of $\mathcal{T}$. If $\mathcal{T}$ is nowhere equivalent to $\mathcal{F}$, then for any sub-σ-algebra $\mathcal{G}$ of $\mathcal{F}$, we have the following results.*

C1 *For any closed valued $\mathcal{F}$-measurable correspondence F from T to X, $\mathcal{R}_F^{(\mathcal{T},\mathcal{G})}$ is convex.*

C2 *For any closed valued $\mathcal{F}$-measurable correspondence F from T to X, $\mathcal{R}_F^{(\mathcal{T},\mathcal{G})}$ is weakly closed.*

C3 *For any compact valued $\mathcal{F}$-measurable correspondence F from T to X, $\mathcal{R}_F^{(\mathcal{T},\mathcal{G})}$ is weakly compact.*

C4 *Let F be a compact valued $\mathcal{F}$-measurable correspondence from T to X, Z is a metric space, G is a closed valued correspondence from $T \times Z$ to X, such that*

a $\forall (t,z) \in T \times Z$, $G(t,z) \subseteq F(t)$;

b $\forall z \in Z$, $G(\cdot, z)$ *(denoted as G_z) is $\mathcal{F}$-measurable from T to X;*

c $\forall t \in T$, $G(t, \cdot)$ *(denoted as G_t) is upper-hemicontinuous from Z to X;*

Then $H(z) = \mathcal{R}_{G_z}^{(\mathcal{T},\mathcal{G})}$ is upper-hemicontinuous from Z to $\mathcal{R}^{\mathcal{G}}$.

C5 *For any $G \in \mathcal{R}^{\mathcal{F}}$, there exists a $\mathcal{T}$ measurable mapping g such that $\mu^{g|\mathcal{F}} = G$.*

C6 *Given $G \in \mathcal{R}^{\mathcal{F}}$ and a $\mathcal{T}$-measurable mapping g. If $\mu^{g|\mathcal{F}} = G$, then for λ-a.e. $t \in T$, $g(t) \in \operatorname{supp} G(t)$.*

C7 *Let F be a compact valued correspondence from T to X, Y is a metric space, G is a closed valued correspondence from $T \times Y$ to X, such that*

a $\forall (t,y) \in T \times Y$, $G(t,y) \subseteq F(t)$;

b $\forall y \in Y$, $G(,y)$*(denoted as G_y) is $\mathcal{F}$-measurable from T to X;*

c $\forall t \in T$, $G(t,)$ *(denoted as G_t) is upper-hemicontinuous from Y to X;*

Then $H(y) = D_{G_y}^{\mathcal{T}}$ is upper-hemicontinuous from Y to $\mathcal{M}(X)$.

Lemma 11. *For any σ-algebra $\mathcal{G}$, $\phi_n, \phi \in \mathcal{R}^{\mathcal{G}}$ for $n \in \mathbb{N}$. Then $\phi_n \Rightarrow \phi$ if and only if*

$$\lim_{n\to\infty} \int_E \int_X c(x)\phi_n(t, \mathrm{d}x)] \,\mathrm{d}\lambda(t) = \int_E \int_X c(x)\phi(t, dx)] \,\mathrm{d}\lambda(t)$$

for every $E \in \mathcal{G}$ and $c \in C_b(X)$.

4.5.2 Proofs of results in Section 4.3

Notice that for each player, to find a subgame perfect equilibrium strategy is actually solving the one-player optimal policy problem in a stochastic programming given other players' strategies. Holding this view, a stationary Markov perfect equilibrium, therefore, can be defined in a recursive form, which is also known as the "Bellman equation" (see, for example, Bellman (1957)). We give the formal definition of a stationary Markov perfect equilibrium in the following, which is equivalent to the definition in Section 4.2.

Definition 30 (Stationary Markov perfect equilibrium)**.** A *behavioral-strategy stationary Markov perfect equilibrium* in a large stochastic game f is a $\mathcal{G} \otimes \mathcal{S}$ measurable strategy profile from $I \times S$ to $\mathcal{M}(A)$ such that there exists a continuation utility function $v(i,s) \in L_\infty^{\mathcal{G}\otimes\mathcal{S}}(I \times S)$[16], which satisfies for μ-almost all fixed $s \in S$,

$$\begin{aligned} v(i,s) &= (1-\beta)\int_A u(i,s,a',\int_I f(i',s)\lambda(\mathrm{d}i'))f(i,s,\mathrm{d}a') + \beta\int_S v(i,s')P(ds'|s,\int_I f(i',s)\lambda(\mathrm{d}i')) \\ &= \max_{a\in A}(1-\beta)u(i,s,a,\int_I f(i',s)\lambda(\mathrm{d}i')) + \beta\int_S v(i,s')P(ds'|s,\int_I f(i',s)\lambda(\mathrm{d}i')) \end{aligned}$$

for λ-almost all $i \in I$.

We first need to present a technical lemma, which is essential for reducing problems in a stochastic environment to those in a static environment with a continuum of players.

Lemma 12 (Equivalence theorem)**.** *In any large stochastic game, the set of pure-strategy/behavioral-strategy stationary Markov perfect equilibria coincides with that of the pure-strategy/behavioral-strategy state-contingent Nash equilibrium in its auxiliary game.*

Proof of Lemma 12. Recall that in Section 4.2, the space of payoffs $\mathcal{U}_A$ is defined as the space of all continuous functions on the product space $A \times \mathcal{M}(A)$ uniformly bounded by B_u with its sup-norm topology and the resulting Borel σ-algebra.

We show that a behavioral-strategy stationary Markov perfect equilibrium is a behavioral-strategy state-contingent Nash equilibrium at first.

[16] $L_\infty^{\mathcal{G}\otimes\mathcal{S}}(I \times S)$ denotes the set of $\mathcal{G} \otimes \mathcal{S}$ measurable function whose sup-norm is finite, *i.e.*, $||h||_\infty < \infty$ for $h \in L_\infty^{\mathcal{G}\otimes\mathcal{S}}(I \times S)$.

Let $f\colon I \times S \to \mathcal{M}(A)$ be an $\mathcal{G} \otimes \mathcal{S}$-measurable stationary Markov perfect equilibrium satisfying the corresponding conditions in Definition 30: there exists an $\mathcal{G} \otimes \mathcal{S}$-measurable function $v(i,s) \in L_\infty(I \times S)$ such that for μ-almost all fixed $s \in S$,

$$\begin{aligned} v(i,s) &= (1-\beta)\int_A u(i,s,a',\int_I f(i',s)\lambda(\mathrm{d}i'))f(i,s,\mathrm{d}a') + \beta\int_S v(i,s')P(ds'|s,\int_I f(i',s)\lambda(\mathrm{d}i')) \\ &= \max_{a\in A}(1-\beta)u(i,s,a,\int_I f(i',s)\lambda(\mathrm{d}i')) + \beta\int_S v(i,s')P(ds'|s,\int_I f(i',s)\lambda(\mathrm{d}i')) \end{aligned}$$

for λ-almost all $i \in I$, which implies for μ-almost all fixed $s \in S$,

$$\int_A u(i,s,a',\int_I f(i',s)\lambda(\mathrm{d}i'))f(i,s,\mathrm{d}a') = \max_{a\in A} u(i,s,a,\int_I f(i',s)\lambda(\mathrm{d}i'))$$

for λ-almost all $i \in I$. Therefore, f is a state-contingent Nash equilibrium in its auxiliary game by Definition 22.

Now we need to prove that in any large stochastic game, a behavioral-strategy state-contingent Nash equilibrium in its auxiliary game is also a behavioral-strategy stationary Markov perfect equilibrium.

Let $g\colon I \times S \to \mathcal{M}(A)$ be an $\mathcal{G} \otimes \mathcal{S}$-measurable behavioral-strategy state-contingent Nash equilibrium satisfying the Definition 22, and $H\colon L_\infty(I \times S) \to L_\infty(I \times S)$ be a mapping such that for any $v \in L_\infty(I \times S)$,

$$H(v) = (1-\beta)\int_A u(i,s,a',\int_I g(i',s)\lambda(\mathrm{d}i'))g(i,s,\mathrm{d}a') + \beta\int_S v(i,s')P(ds'|s,\int_I g(i',s)\lambda(\mathrm{d}i')).$$

Given $\beta \in [0,1)$, it is easy to verify that H is a contraction mapping, which is, for any $v_1, v_2 \in L_\infty(I \times S)$,

$$\begin{aligned} ||H(v_1) - H(v_2)||_\infty &= ||\beta\int_S v_1(i,s') - v_2(i,s')P(ds'|s,\int_I g(i',s)\lambda(\mathrm{d}i'))||_\infty \\ &\le \beta\int_S ||v_1 - v_2||_\infty P(ds'|s,\int_I g(i',s)\lambda(\mathrm{d}i')) \\ &= \beta||v_1 - v_2||_\infty. \end{aligned}$$

Therefore, Banach fixed point theorem shows there is a unique $v^* \in L_\infty(I \times S)$ such that $H(v^*) = v^*$, which combined with the fact that g is a state-contingent Nash equilibrium in the auxiliary game derives the following equalities:

$$\begin{aligned} H(v^*) = v^* &= (1-\beta)u(i,s,a',\int_I g(i',s)\lambda(\mathrm{d}i'))g(i,s,\mathrm{d}a') + \beta \int_S v^*(i,s')P(ds'|s,\int_I g(i',s)\lambda(\mathrm{d}i')) \\ &= \max_{a\in A}(1-\beta)u(i,s,a',\int_I g(i',s)\lambda(\mathrm{d}i'))g(i,s,\mathrm{d}a') + \beta \int_S v^*(i,s')P(ds'|s,\int_I g(i',s)\lambda(\mathrm{d}i')). \end{aligned}$$

Therefore, g is also a behavioral-strategy stationary Markov equilibrium in the specific large stochastic game. Note that pure-strategy case can be shown similarly, and hence we omit its proof here. □

Remark 7. *The theorem above can be further generated to the case of non-stationary Markov perfect equilibria in large stochastic games with time-varying auxiliary games, i.e., in a non-stationary large stochastic game, a Markov strategy $\{f^t\}_{t\in\mathbb{N}}$ is a pure-strategy/behavioral-strategy Markov perfect equilibrium if and only if for every t, f^t is a pure-strategy/behavioral-strategy state-contingent Nash equilibrium in the corresponding auxiliary game G^t.*[17]

Proof of Claim 5. Since a stationary Markov perfect equilibrium in a large stochastic game is always a state-contingent Nash equilibrium in its auxiliary game (Lemma 12). Fixed each s, a standard fixed point result shows the large game $G(\cdot, s)$ has a behavioral-strategy Nash equilibrium. A collection of all such equilibria for all s results in a behavioral-strategy stationary Markov perfect equilibrium. By Rath *et al.* (1995), there is no pure-strategy Nash equilibrium in the large game $G(\cdot, s)$ for each s. It is thus easy to conclude that no pure stationary Markov perfect equilibrium ever exists in the constructed large stochastic game G either. □

Before proving Theorem 5, we present the main theorem in Mertens (2003) for readers' conveniences.

Lemma 13 (Theorem 3 in Mertens (2003))**.** *Let $P(\mathrm{d}\omega|e)$ be a bounded $\mathbb{R}^k$-valued kernel*

[17] Similar case was studied by Chakrabarti (2003). However, his proof still has a gap regarding the product measurability issue. By Remark 7, our method in fact can be used to establish the existence of non-stationary Markov perfect equilibrium in his setting. Moreover, his assumption of the continuity on transition probability actually is not needed.

from $(E, \mathcal{E})$ *to* $(\Omega, \mathcal{A})$*, two measurable spaces (i.e.,* $\forall e \in E$*,* $P(\cdot|e)$ *is a bounded* $\mathbb{R}^k$*-valued measure on* $(\Omega, \mathcal{A})$*, and* $\forall A \in \mathcal{A}$*,* $P(A|\cdot)$ *is* $\mathcal{E}$ *-measurable). Let* N *be a measurable map from* $(E \times \Omega, \mathcal{E} \otimes \mathcal{A})$ *to* $K^*(\mathbb{R}^l)$*, which is* P*-integrable in the sense that for any measurable selection* f *from* N *,* $f(\omega, e)$ *is* $P(\mathrm{d}\omega|e)$*-absolutely integrable for any* $e \in E$*. Define* $\int N \,\mathrm{d}P$ *as the map from* E *to subsets of* $R^{k\cdot l}$ *(the tensor product of* R^k *and* R^l*) defined by* $\int N \,\mathrm{d}P$ *:* $e \to \{f(\omega, e)\,\mathrm{d}P(d\,\mathrm{d}\omega|e)|f$ *is an* $\mathcal{E} \otimes \mathcal{A}$*-measurable selection from* $N\}$*, and denote its graph by* $(F, \mathcal{F}) \subseteq (E \times R^{k\cdot l}, \mathcal{E} \otimes$ *Borel sets). Assume that* $\mathcal{A}$ *is separable,*

B1: $(\int N \,\mathrm{d}P)$ *is an* $\mathcal{E}$*-measurable map to* $K^*(R^{kl})$*, and* F *is measurable in* $E \times R^{kl}$*.*

B2: there exists a measurable, R^l*-valued function* f *on* $(F \times \Omega, \mathcal{F} \otimes \mathcal{A})$ *such that* $f(e, x, \omega) \in N(e, \omega)$ *and* $x = \int f(e, x, \omega)P(\mathrm{d}\omega|e)$*.*

Now we are ready to prove Theorem 5.

Proof of Theorem 5. Recall that the player space is $(I, \mathcal{G}, \lambda)$, and the state space is $(S, \mathcal{S}, \mu)$. The action space and the societal summary space are A and $\mathcal{M}(A)$ respectively. Denote the large stochastic game and its auxiliary game as G and G^a respectively[18]. The stage payoff function can be rewritten as $u(i, s, a, \nu)$, which is $\mathcal{G} \otimes \mathcal{S}$-measurable on $I \times S$ and continuous on $A \times \mathcal{M}(A)$.

By the equivalence theorem, to prove the existence of behavioral-strategy stationary Markov perfect equilibria, it is equivalent to prove the existence of behavioral-strategy state-contingent Nash equilibria in its auxiliary game.

For μ-almost all fixed s, $G^a(\cdot, s)$ is a large game and it is well-known that there exists an $\mathcal{G}$-measurable behavioral-strategy Nash equilibrium. Define a correspondence N mapping each state $s \in S$ into the set of societal summaries of behavioral-strategy Nash equilibria in the auxiliary game $G^a(\cdot, s)$; that is,

$$N(s) = \{\int_I h(i', s)\lambda(\mathrm{d}i') \colon h \text{ is a } \mathcal{G}\text{-measurable behavioral-strategy Nash equilibrium of the auxiliary game } G$$

Now we claims the followings to be true, and will show their validity at the end of this proof:

[18]These notations are inherited from the standard setting in Section 4.2.

Claim 8.

(i) N *is an* $\mathcal{S}$*-measurable, nonempty-valued, compact-valued correspondence.*

(ii) $\tilde{G}$ *is a nonempty, convex and compact-valued,* $\mathcal{G} \otimes \mathcal{S}$*-measurable correspondence from* $I \times S$ *to* $\mathcal{M}(A)$*, where* $\mathcal{G}$ *is countable generated and* $\mathcal{M}(A)$ *is the space of probability measures on a Polish space* A*. Let a correspondence* Φ *from* S *to the set of probability measure on action space* $\mathcal{M}(A)$ *be:*

$$\Phi(s) = \{\int_I g(i')\lambda(\mathrm{d}i')\colon g \text{ is an } \mathcal{G}\text{-measurable selection of } \tilde{G}(\cdot, s)\}.$$

Then for any $\mathcal{S}$*-measurable selection* $\tilde{\phi}$ *of* Φ*, there exists an* $\mathcal{G} \otimes \mathcal{S}$*-measurable mapping* ψ *from* $I \times S$ *to* $\mathcal{M}(A)$ *such that* $\psi(i, s) \in \tilde{G}(i, s)$ *for* $\lambda \times \mu$*-almost all* (i, s) *and* $\phi(s) = \int_I \psi(i', s)\lambda(\mathrm{d}i')$ *for* μ*-almost all* $s \in S$*.*

By Claim 8 (i), there exists an $\mathcal{S}$-measurable selection n of N. Then define a correspondence B such that

$$B(i, s)\colon = \{\int_I g(i')\lambda(\mathrm{d}i')\colon g \text{ is an } \mathcal{G}\text{-measurable selection of } \arg\max_{\nu \in \mathcal{M}(A)} \int_A u(i, s, a', n(s))\nu(\mathrm{d}a')\}.$$

Then Claim 8 (ii) gives the existence of a behavioral-strategy state-contingent Nash equilibrium.

Now we prove Claim 8 (i) and (ii).

Proof of Claim 8 (i): Define a correspondence H mapping from $S \times \mathcal{M}(A)$ to $\mathcal{M}(A)$ as follows:

$$H(s, \tau) = \{\int_I \phi(i')\lambda(\mathrm{d}i')\colon \phi(\cdot) \text{ is a } \mathcal{G}\text{-measurable selection of } \operatorname{argmax}_{\nu \in \mathcal{M}(A)} \int_A u(i, s, a', \tau)\nu(\mathrm{d}a')\}$$

for any $s \in S$ and $\tau \in \mathcal{M}(A)$. We now show that for any fixed τ, $H(\cdot, \tau)$ is an $\mathcal{S}$-measurable correspondence. Since A is a compact set, the set of bounded continuous function $C_b(A)$ is equivalent to $C(A)$, which is a separable Banach lattice. Denote $\{f_n\}_{n \in \mathbb{N}}$ as the set of countable bounded continuous function which is dense in $C(A)$. Since $C_b(A)$ separate points in $\mathcal{M}(A)$, with the dense countable set $\{f_n\}_{n\in\mathbb{N}}$ the probability measure set $\mathcal{M}(A)$ can be identified as a subset of $\mathbb{R}^{\mathbb{N}}$ for mapping every $\mu \in \mathcal{M}(A)$ to $(\int_A f_1(a)\mu(\mathrm{d}a), \int_A f_2(a)\mu(\mathrm{d}a), ...) \in \mathbb{R}^{\mathbb{N}}$. Then the original measurability issue can be transformed to the measurability issue in the correspondence $\tilde{H}$ mapping from $S \times \mathcal{M}(A)$ to $\mathbb{R}^{\mathbb{N}}$ where

$$\tilde{H}(s,\tau)\colon = \{(\int_A f_1(a) \int_I h(i,\mathrm{d}a)\lambda(\mathrm{d}i), \int_A f_2(a) \int_I h(i,\mathrm{d}a)\lambda(\mathrm{d}i), ...)\colon \int_I h(i,\cdot)\lambda(\mathrm{d}i) \in H(s,\tau)\}.$$

For fixed τ, consider every $n \in \mathbb{N}$. Since $\int_A f_n(a) \int_I h(i,\mathrm{d}a)\lambda(\mathrm{d}i) = \int_I \int_A f_1(a)h(i,\mathrm{d}a)\lambda(\mathrm{d}i)$, the correspondence $\int_I \int_A f_n(a)B(i,s,\tau)\lambda(\mathrm{d}i)$, where $B(i,s,\tau) = \underset{\nu\in\mathcal{M}(A)}{\operatorname{argmax}} \int_A u(i,s,a',\tau)\nu(\mathrm{d}a')$, is $\mathcal{S}$-measurable by Theorem 2 in Mertens (2003) (see also Lemma 13, B1). Lemma 18.4 (2) in Aliprantis and Border (2006) implies that countable product correspondence whose components are weakly measurable functions is weakly measurable, and Lemma 18.2 (2) shows that weakly measurability and measurability are the same when dealing with compact-valued correspondences. Based on those results, we are able to establish that $H(\cdot,\tau)$ is $\mathcal{S}$-measurable with fixed $\tau \in \mathcal{M}(A)$.

Now we need to show for any fixed s, $H(s,\cdot)$ is upper hemi-continuous on τ. Since by the Berge's maximum theorem $\underset{\nu\in\mathcal{M}(A)}{\operatorname{argmax}} \int_A u(i,s,a',\tau)\nu(\mathrm{d}a')$ is upper hemi-continuous on τ, the integration of this compact-valued and convex-valued correspondence is also upper hemicontinuous. Again let $d(\cdot,\cdot)$ be the Prohorov metric on $M(A)$, and a function D on $S \times \mathcal{M}(A)$ be $D(s,\tau) = -d(H(s,\tau),\tau)$, then for any fixed s by Lemma 17.23 in Aliprantis and Border (2006), $D(s,\cdot)$ is upper semi-continuous, and for any fixed τ, and $D(\cdot,\tau)$ is $\mathcal{S}$-measurable.

To show N is $\mathcal{S}$-measurable, we first show for any open set $O \subseteq \mathbb{R}$, ϕ_O is a $\mathcal{S}$-measurable correspondence, whose definition is in the following.

$$\phi_O(s) = \{\nu \in \mathcal{M}(A)\colon D(s,\nu) \in O\}.$$

To prove such measurability we use definition directly. For any closed set $F \subseteq \mathcal{M}(A)$, we can find a countable dense subset $\{\mu_1, \mu_2 ...\}$ of F due to the compactness of $\mathcal{M}(A)$. Denote ϕ_O's lower inverse image of F as $\phi^l_O(F)$, then

$$\begin{aligned}\phi^l_O(F) &= \{s \in S \colon \phi_O(s) \cap F \neq \emptyset\} \\ &= \{s \in S \colon D(s,\nu) \in O \text{ for some } \nu \in F\} \\ &= \{s \in S \colon D(s,\mu_n) \in O \text{ for some } n\} \\ &= \cup_{n=1}^{\infty}\{s \in S \colon D(s,\mu_n) \in O\},\end{aligned}$$

where the third equality follows from the openness of O and the upper semicontinuity of D in $\mathcal{M}(A)$. Since the function D is measurable in s for each $\nu \in \mathcal{M}(A)$, each of the sets $\{s \in S \colon D(s,\mu_n) \in O\}$ belongs to the σ-algebra $\mathcal{S}$, so ϕ_O is a $\mathcal{S}$-measurable correspondence.

Finally, notice that the correspondence N whose definition is given previously:

$$N(s) = \{\int_I h(i',s)\lambda(\mathrm{d}i') \colon h \text{ is a } \mathcal{G}\text{-measurable behavioral-strategy Nash equilibrium of the auxiliary game } G$$

has an alternative but equivalent definition by utilizing the newly defined function D:

$$N(s) = \{\tau \in \mathcal{M}(A) \colon D(s,\tau) = 0\}.$$

We now show that N is an $\mathcal{S}$-measurable compact-valued correspondence and conclude our proof. Define $N_n \colon S \to \mathcal{M}(A)$ by $N_n(s) = \{\nu \in \mathcal{M}(A) \colon |D(s,\nu)| < \frac{1}{n}\}$. Then by previous argument that ϕ_O is $\mathcal{S}$-measurable, each correspondence N_n is measurable, which combined with Lemma 18.3 in Aliprantis and Border (2006), deduces its closure correspondence $\overline{N_n}$ is also measurable. And $N(s) \subseteq \overline{N_n}(s) = \{\nu \in \mathcal{M}(A) \colon |D(s,\nu)| \leq \frac{1}{n}\}$, so $N(s) = \cap_{n=1}^{\infty}\overline{N_n}(s)$, which is obviously compact-valued. By Lemma 18.4(3) in Aliprantis and Border (2006), the intersection N is measurable.

Proof of Claim 8 (ii): Through out this proof, we will deal with compact metric space A instead of the Polish space A as we claimed in the beginning. But this is in fact without loss of generality here, since otherwise we can always embed the Polish space A into a larger compact metric space $\tilde{A}$ such that the topology in $\mathcal{M}(\tilde{A})$ restricted on $\mathcal{M}(A)$ is exactly the same as the one in $\mathcal{M}(A)$.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a set of countable real valued bounded continuous functions on A which is dense in $C(A)$ and hence separates points in $\mathcal{M}(A)$. Given $\{f_n\}_{n\in\mathbb{N}}$, define a sequence of operators $\{\phi_n\}_{n\in\mathbb{N}}$ such that $\phi_n \colon \mathcal{M}(A) \to \mathbb{R}$ and $\phi_n(v) = \int_A f_n \,\mathrm{d}\nu$, for any $\nu \in \mathcal{M}(A)$ and $n \in \mathbb{N}$. And define another sequence of operators $\{\phi^n\}_{n\in\mathbb{N}}$ such that $\phi^n \colon \mathcal{M}(A) \to \mathbb{R}^n$,

$$\phi^n(\nu) = (\phi_1(\nu), \phi_2(\nu), ..., \phi_n(\nu)) = (\int_A f_1 \,\mathrm{d}\nu, \int_A f_2 \,\mathrm{d}\nu, ..., \int_A f_n \,\mathrm{d}\nu),$$

for any $\nu \in \mathcal{M}(A)$ and $n \in \mathbb{N}$.

Hence, for the given $\tilde{G}$, define a sequence of correspondence $\{G^n\}_{n\in\mathbb{N}}$ based on the sequence of oprators $\{\phi^n\}_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$:

$$G^n(i,s) \colon = \phi^n \circ \tilde{G}(i,s) = \{(\int_A f_1\nu, \cdots, \int_A f_n\nu) \colon \nu \in \tilde{G}(i,s)\}.$$

Then G^n is a nonempty, compact-valued, $\mathcal{G} \otimes \mathcal{S}$-measurable correspondence from $I \times S$ to $\mathbb{R}^n$. Given any mapping g from $I \times S$ to $\mathcal{M}(A)$, denote the composite function $g_n \colon = \phi_n(g(i,s))$ which maps from $I \times S$ to $\mathbb{R}$ for any $n \in \mathbb{N}$ and $g^n \colon = \phi^n \circ g = (g_1, \cdots, g_n)$ which maps from $I \times S$ to $\mathbb{R}^n$.

Given $\Phi(s) = (\int \tilde{G} \,\mathrm{d}\lambda)(s) = \{\int_I p(i,s)\lambda(\mathrm{d}i) \colon p$ is an $\mathcal{G}$-measurable selection of $\tilde{G}(\cdot, s)\}$ for each $s \in S$. Then for any $n \in \mathbb{N}$, let

$$\Phi^n(s) = (\int G^n \,\mathrm{d}\lambda)(s) = \{\int_I \tilde{p}(i,s)\lambda(\mathrm{d}i) \colon \tilde{p} \text{ is an } \mathcal{G}\text{-measurable selection of } G^n(\cdot, s)\}.$$

Denote the graph of Φ^n by H^n with the corresponding σ-algebra $\mathcal{H}^n \colon = \mathcal{S} \otimes B(\mathbb{R}^n)$[19].

Suppose that $\tilde{\phi}$ is an $\mathcal{S}$-measurable selection of Φ, which by definition satisfies that: for any s, there exists a function g_s such that g_s is an $\mathcal{G}$-measurable selection of $\tilde{G}(\cdot, s)$ and $\tilde{\phi}(s) = \int_I g_s(i')\lambda(\mathrm{d}i')$. Then for each fixed $s \in S$, note that $\tilde{\phi}(s)$ is a probability measure on A.

[19] $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-algebra on the n-dimensional Euclidian space $\mathbb{R}^n$

For any $n \in \mathbb{N}$, let

$$\alpha_n(s) = \phi_n(\tilde{\phi}(s, \cdot)) \text{ and } \alpha^n(s) = \phi^n(\tilde{\phi}(s, \cdot)) = (\alpha_1(s), ...\alpha_n(s)). \tag{4.1}$$

Obviously $\alpha^n(s)$ is an $\mathcal{S}$-measurable function, since ϕ^n is a continuous function on $\mathcal{M}(A)$. Note that $\alpha^n(s) \in \Phi^n(s)$ for each $s \in S$, and hence α^n is a $\mathcal{S}$-measurable selection of Φ^n.

Due to Theorem 3 in Mertens (2003) (see also Lemma 13, B2), there exists a measurable, $\mathbb{R}^n$-valued function h^n on $(I \times H^n, \mathcal{G} \times \mathcal{H}^n)$ such that $h^n(i, s, y) \in G^n(i, s)$ for any $(i, s, y) \in I \times H^n$ and $y = \int_I h^n(i, s, y)\lambda(\mathrm{d}i)$. Let $l^n(i, s) = h^n(i, s, \alpha^n(s))$. Then l^n is a $\mathcal{G} \otimes \mathcal{S}$-measurable selection of G^n such that

$$\alpha^n(s) = \int_I l^n(i, s)\lambda(\mathrm{d}i) \text{ where } \alpha_m(s) = \int_I l^n_m(i, s)\lambda(\mathrm{d}i), 1 \leq m \leq n \tag{4.2}$$

for any $s \in S$. Since $l^n(i, s) = h^n(i, s, \alpha^n(s)) \in G^n(i, s)$, then by definition, for each (i, s), there exists an $\nu_{(i,s)} \in \tilde{G}(i, s)$ such that $l^n(i, s) = (\int_A f_1(a)\nu_{(i,s)}(\mathrm{d}a), ..., \int_A f_n(a)\nu_{(i,s)}(\mathrm{d}a))$. Therefore, $(\phi^n)^{-1}l^n(i, s) \cap \tilde{G}(i, s) \neq \emptyset$.

For any $n \in \mathbb{N}$, define the sequence of correspondence G^n as $\tilde{G}^n(i, s) := (\phi^n)^{-1}(l^n(i, s)) \cap \tilde{G}(i, s) \neq \emptyset$, which is a sub-correspondence of $\tilde{G}$ with $\tilde{G}^n \subseteq \tilde{G}$. For each $n \in \mathbb{N}$, pick a $\mathcal{G} \otimes \mathcal{S}$-measurable selection ψ^n of $\tilde{G}^n$. Then ψ^n is also a $\mathcal{G} \otimes S$-measurable selection of $\tilde{G}$. Since $\tilde{G}$ has compact valued, by the weak compactness of young measure, there exists a subsequence of $\{\psi^n\}_{n\in\mathbb{N}}$ such that it converges to a limit, denoted as ψ^∞. Let the convergence subsequence of $\{\psi^n\}_{n\in\mathbb{N}}$ to be $\{\psi^n\}_{n\in\mathbb{N}}$ itself for notation simplicity.

We then claim that (iii) $\psi^\infty \in \tilde{G}$ for $\lambda \times \mu$-almost all (i, s), and (vi) $\phi(s) = \int_I \psi^\infty(i', s)\lambda(\mathrm{d}i')$ for μ-almost all $s \in S$.

To prove claim (iii), suppose that ψ^∞ is not a selection of $\tilde{G}$. Then there exists some measurable set $E \in \mathcal{G} \otimes S$ with $\lambda \times \mu(E) > 0$ such that $\psi^\infty(i, s) \notin \tilde{G}(i, s)$ for any $(i, s) \in E$. Since $\{f_n\}_{n\in\mathbb{N}}$ separates points in $\mathcal{M}(A)$ and $\tilde{G}$ is convex and compact valued, there exists some positive integer n' and a measurable subset $E' \subseteq E$ with positive measure such that for any $(i,s) \in E'$,

$$\int_A f_{n'}(a)\psi^\infty(\mathrm{d}a|(i,s)) \notin \{\int_A f_{n'}(a)P(\mathrm{d}a)\colon P \in \tilde{G}(\omega)\}.$$

Without loss of generality, we can assume that there exists some measurable subset $\tilde{E} \subseteq E'$ with $\lambda \times \mu(\tilde{E}) > 0$ and a sufficiently small $\epsilon > 0$ such that for any $(i,s) \in \tilde{E}$,

$$\int_A f_{n'}(a)\psi^\infty(\mathrm{d}a|(i,s)) > \{\int_A f_{n'}(a)P(\mathrm{d}a) + \epsilon\colon P \in \tilde{G}(\omega)\}.$$

Then

$$\int_{\tilde{E}}\int_A f_{n'}(a)\psi^\infty(\mathrm{d}a|\omega)P(\mathrm{d}\omega) > \{\int_{\tilde{E}}\int_A f_{n'}(a)P(\mathrm{d}a) + \epsilon\colon P \in \tilde{G}(\omega)\}.,$$

which contradicts with the assumption that the sequence $\{\psi^n\}_{n\in\mathbb{N}}$ weakly converges to ψ^∞.

Now we turn to prove the claim (iv). Denote $\overline{\psi^n}(s)$ be the Gel′fand integral of $\psi^n(\cdot,s)$ for each s; that is, $\overline{\psi^n}(s) = \int_I \psi^n(i,s,\cdot)\lambda(\mathrm{d}i) \in \mathcal{M}(A)$ for all $n \in \mathbb{N}\cup\{\infty\}$. Then given the set of countable bounded continuous function $\{f_m\}_{m\in\mathbb{N}}$ on A which is dense in $C_b(A)$.

For the $\mathcal{G}\otimes\mathcal{S}$-measurable selection ψ^n of $\tilde{G}^n$ where $\tilde{G}^n(i,s)\colon = (\phi^n)^{-1}(l^n(i,s))\cap\tilde{G}(i,s) \neq \emptyset$, $\forall n \in \mathbb{N}$, it is easy to see that $l^n_m = f_m \circ \psi^n$ for $1 \le m \le n$, which is the m-th component of n-tuple vector $l^n = (l^n_1, ..., l^n_n)$. Fix $m \in \mathbb{N}$, for any measurable set $\tilde{E} \subseteq \mathcal{S}$,

$$\begin{aligned}
\int_{\tilde{E}}\int_A f_m(a)\overline{\psi^\infty}(s,\mathrm{d}a)\mu(\mathrm{d}s) &= \int_{\tilde{E}}\int_I\int_A f_m(a)\psi^\infty(i,s,\mathrm{d}a)\lambda(\mathrm{d}i)\mu(\mathrm{d}s)\\
&= \lim_{n\to\infty}\int_{\tilde{E}}\int_I\int_A f_m(a)\psi^n(i,s,\mathrm{d}a)\lambda(\mathrm{d}i)\mu(\mathrm{d}s)\\
&= \lim_{n\to\infty}\int_{\tilde{E}}\int_I l^n_m(i,s)\lambda(\mathrm{d}i)\mu(\mathrm{d}s)\\
&= \int_{\tilde{E}} \alpha_m(s)\mu(\mathrm{d}s)\\
&= \int_{\tilde{E}}\int_A f_m(a)\phi(s,\mathrm{d}a)\mu(\mathrm{d}s).
\end{aligned}$$

The second equality is by Theorem 2.1.3 of Castaing, Fitte and Valadier (2004) (see also Lemma 11). The forth equality is valid because of Equation (4.2), whereas the last equality holds for Equation (4.1). Therefore,

$$\int_A f_m(a)\overline{\psi^\infty}(s,\mathrm{d}a) = \int_A f_m(a)\phi(s,\mathrm{d}a)$$

for each $m \in \mathbb{N}$ for μ-lamost all s. As a result, $\overline{\psi^\infty}(s) = \phi(s)$ for μ-almost all $s \in S$. This concludes our proof. □

Proof of Proposition 8. It is easy to see that, if there exists an $\mathcal{S}$-measurable function mapping each s to a social summary of Nash equilibria in large game $G^a(\cdot, s)$, with the measurable selection theorem in Mertens (2003) (see also Lemma 13) we can directly have the conclusion. Therefore, we just need to show the correspondence from the state space S to the set of social summaries of Nash equilibria in large game $G(\cdot, s)$, is $\mathcal{S}$-measurable and compact-valued, i.e.,

$$N(s) = \{\int_I g(i')\lambda(i') \colon g \text{ is a Nash equilibrium of the auxiliary game } G^a(\cdot, s)\},$$

is $\mathcal{S}$-measurable, and compact valued, which will implies the existence of a measurable selection.

This is an analogue to the previous proof. Without loss of generality, we will show the case in $\mathbb{R}^n$ instead of finite-dimensional vector space for simplicity. Define a correspondence H mapping from $S \times \mathbb{R}^n$ to $\mathbb{R}^n$ as follows:

$$H(s,\tau) = \{\int_I \phi(i')\lambda(\mathrm{d}i') \colon \phi(\cdot) \text{ is a } \mathcal{G}\text{-measurable selection of } \operatorname*{argmax}_{a' \in A} u(i,s,a',\tau)\}$$

for any $s \in S$ and $\tau \in \mathbb{R}^n$. Since A is a compact set, the correspondence $\operatorname*{argmax}_{a' \in A} u(i,s,a',\tau)\}$ is $\mathcal{S}$-measurable for any fixed τ is directly from the measurable maximum theorem, and hence we can further deduce the correspondence $H(\cdot, \tau)$ is $\mathcal{S}$-measurable with fixed $\tau \in \mathbb{R}^n$ by Theorem 2 in Mertens (2003) (see also Lemma 13, B1).

Since by the Berge's maximum theorem, for any fixed s, $\operatorname*{argmax}_{a' \in A} u(i,s,a',\tau)$ is upper hemi-continuous on τ, the integration $H(s,\cdot)$ is also upper hemicontinuous due to Proposition 8 in Hildernbrand (1974) that integration preserves upper hemicontinuity. Again let $d(\cdot,\cdot)$ be the Prohorov metric on $M(A)$, and a function D on $S \times \mathbb{R}^n$ be $D(s,\tau) = -d(H(s,\tau),\tau)$, then for any fixed s by Lemma 17.23 in Aliprantis and Border (2006), $D(s, \cdot)$ is upper semi-continuous, and for any fixed τ, and $D(\cdot, \tau)$ is $\mathcal{S}$-measurable.

To show N is $\mathcal{S}$-measurable, we first show for any open set $O \subseteq \mathbb{R}$, ϕ_O is a $\mathcal{S}$-measurable correspondence, whose definition is in the following.

$$\phi_O(s) = \{a \in \mathbb{R}^n \colon D(s, a) \in O\}.$$

To prove such measurability we use definition directly. For any closed set $F \subseteq \mathbb{R}^n$, we can find a countable dense subset $\{a_1, a_2 ...\}$ of F due to the separability of $\mathbb{R}^n$. Denote ϕ_O's lower inverse image of F as $\phi_O^l(F)$, then

$$\begin{aligned}
\phi_O^l(F) &= \{s \in S \colon \phi_O(s) \cap F \neq \emptyset\} \\
&= \{s \in S \colon D(s, \nu) \in O \text{ for some } \nu \in F\} \\
&= \{s \in S \colon D(s, a_n) \in O \text{ for some } n\} \\
&= \cup_{n=1}^{\infty} \{s \in S \colon D(s, a_n) \in O\},
\end{aligned}$$

where the third equality follows from the openness of O and the upper semicontinuity of D in $\mathbb{R}^n$. Since the function D is measurable in s for each $\tau \in \mathbb{R}^n$, each of the sets $\{s \in S \colon D(s, a_n) \in O\}$ belongs to the σ-algebra $\mathcal{S}$, so ϕ_O is a $\mathcal{S}$-measurable correspondence.

Finally, notice that the correspondence N whose definition is given previously:

$$N(s) = \{\int_I g(i')\lambda(i') \colon g \text{ is a Nash equilibrium of the auxiliary game } G^a(\cdot, s)\},$$

has an alternative but equivalent definition by utilizing the newly defined function D:

$$N(s) = \{\tau \in \mathcal{M}(A) \colon D(s, \tau) = 0\}.$$

We now show that N is an $\mathcal{S}$-measurable compact-valued correspondence and conclude our proof. Define $N_n \colon S \to \mathbb{R}^n$ by $N_n(s) = \{a \in \mathbb{R}^n \colon |D(s, \nu)| < \frac{1}{n}\}$. Then by previous argument that ϕ_O is $\mathcal{S}$-measurable, each correspondence N_n is measurable, which combined with Lemma 18.3 in Aliprantis and Border (2006), deduces its closure correspondence $\overline{N_n}$ is also measurable. And $N(s) \subseteq \overline{N_n}(s) = \{\nu \in \mathcal{M}(A) \colon |D(s,\nu)| \leq \frac{1}{n}\}$, so $N(s) = \cap_{n=1}^{\infty} \overline{N_n}(s)$, which is obviously compact-valued. By Lemma 18.4(3) in Aliprantis and Border (2006), the intersection N is measurable. □

Under the nowhere equivalence condition, we extend Theorem 1, Mertens (2003) to the case of general probability measure space in the following, which will serve to establish the existence result as well as for independent interests.

Theorem 9 (Extension of Mertens' theorem)**.** *Let $(I, \mathcal{F}, \lambda)$ be an atomless probability space with a countablely-generated sub σ-algebra $\mathcal{G}$, satisfying that $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$. $(S, \mathcal{S}, \mu)$ is a probability space with the Borel σ-algebra $\mathcal{S}$. $\tilde{G}$ is a nonempty, compact-valued, $\mathcal{G} \otimes \mathcal{S}$-measurable correspondence from $I \times S$ to a compact metric space A. Let a correspondence Φ from S to the set of probability measure on action space $\mathcal{M}(A)$ be:*

$$\Phi(s) = \{\lambda \circ g(\cdot)^{-1} \colon g \text{ is an } \mathcal{F}\text{-measurable selection of } \tilde{G}(\cdot, s)\}.$$

Then for any $\mathcal{S}$-measurable selection ϕ of Φ, there exists an $\mathcal{F} \otimes \mathcal{S}$-measurable mapping ψ from $I \times S$ to A such that $\psi(i,s) \in \tilde{G}(i,s)$ for $\lambda \times \mu$-almost all (i,s) and $\phi(s) = \lambda \circ \psi(\cdot, s)^{-1}$ for μ-almost all $s \in S$.

Proof of Theorem 9. Recall that A is a compact metric space, whereas $C_b(A)$ denotes the set of all bounded continuous real functions on A, which coincides with the set of continuous functions on A, $C(A)$, due to compactness of A. Since $C(A)$ separates points in the set of probability measures on A, *i.e.*, $\mathcal{M}(A)$ (Theorem 15.1 in Aliprantis and Border (2006)) and $C(A)$ is a separable Banach lattice (Theorem 9.14 in Aliprantis and Border (2006)), there exists a set of countable dense continuous functions separating points in $\mathcal{M}(A)$.

Let $\{f_n\}_{n \in \mathbb{N}}$ be a set of countable real valued bounded continuous functions on A which is dense in $C(A)$ and hence separates points in $\mathcal{M}(A)$. Given $\tilde{G}$, define a sequence of correspondence $\{G^n\}_{n \in \mathbb{N}}$ based on the countable continuous functions $\{f_n\}_{n \in \mathbb{N}}$ such that for each

$n \in \mathbb{N}$:

$$G^n(i,s)\colon = (f_1, \cdots, f_n) \circ \tilde{G}(i,s) = \{(f_1(a), \cdots, f_n(a))\colon a \in \tilde{G}(i,s)\}.$$

Then G^n is a nonempty, compact-valued, $\mathcal{G} \otimes \mathcal{S}$-measurable correspondence from $I \times S$ to $\mathbb{R}^n$. For notation simplicity, given any mapping g from $I \times S$ to A, denote the composite function $g_n\colon = f_n \circ g$ which maps from $I \times S$ to $\mathbb{R}$ for any $n \in \mathbb{N}$ and hence $g^n\colon = (f_1, \cdots, f_n) \circ g = (g_1, \cdots, g_n)$ which maps from $I \times S$ to $\mathbb{R}^n$.

Given $\Phi(s) = (\int \tilde{G}\, \mathrm{d}\lambda)(s) = \{\int_I p(i,s)\lambda(\mathrm{d}i)\colon p$ is an $\mathcal{F}$-measurable selection of $\tilde{G}(\cdot, s)\}$ for each $s \in S$. Then for any $n \in \mathbb{N}$, let

$$\Phi^n(s) = (\int G^n\, \mathrm{d}\lambda)(s) = \{\int_I \tilde{p}^n(i,s)\lambda(\mathrm{d}i)\colon \tilde{p}^n \text{ is an } \mathcal{F}\text{-measurable selection of } G^n(\cdot, s)\}.$$

Denote the graph of Φ^n by H^n with the corresponding σ-algebra $\mathcal{H}^n\colon = \mathcal{S} \otimes B(\mathbb{R}^n)$[20]. Notice that the Borel σ-algebra $\mathcal{S}$ on the polish space S is generated by the set of all its open sets, which has a countable basis, and therefore $\mathcal{S}$ is countable generated (or separable).

Suppose that ϕ is an $\mathcal{S}$-measurable selection of Φ, which by definition satisfies that: for any s, there exists a function g_s such that g_s is an $\mathcal{F}$-measurable selection of $\tilde{G}(\cdot, s)$ and $\phi(s) = \lambda \circ g_s(\cdot)^{-1}$. Then for each fixed $s \in S$, note that $\phi(s)$ is a probability measure on A. For any $n \in \mathbb{N}$, let

$$\alpha_n(s) = \int_A f_n(a)\phi(s, \mathrm{d}a) = \int_I f_n\left(g_s(i)\right)\lambda(\mathrm{d}i), \tag{4.3}$$

and $\alpha^n = (\alpha_1, \ldots, \alpha_n)$. Obviously $\alpha^n(s) = \iint_{A^n}(f_1, ..., f_n)\phi(s, \mathrm{d}a)$ is an $\mathcal{S}$-measurable function, since $\iint_{A^n}(f_1, ..., f_n)(\cdot)(\mathrm{d}a)$ is a continuous function on $\mathcal{M}(A)$. Note that $\alpha^n(s) \in \Phi^n(s)$ for each $s \in S$, and hence α^n is a $\mathcal{S}$-measurable selection of Φ^n.

Due to Theorem 3 in Mertens (2003) (see also Lemma 13, B2), there exists a measurable, $\mathbb{R}^n$-valued function h^n on $(I \times H^n, \mathcal{G} \otimes \mathcal{H}^n)$ such that $h^n(i,s,y) \in G^n(i,s)$ for any $(i,s,y) \in I \times H^n$ and $y = \int_I h^n(i,s,y)\lambda(\mathrm{d}i)$. Let $g^n(i,s) = h^n(i,s,\alpha^n(s))$. Then g^n is a $\mathcal{G} \otimes \mathcal{S}$-measurable

[20] $\mathcal{B}(\mathbb{R}^n)$ denotes the Borel σ-algebra on the n-dimensional Euclidian space $\mathbb{R}^n$

selection of G^n such that

$$\alpha^n(s) = \int_I g^n(i,s)\lambda(\mathrm{d}i) \text{ where } \alpha_m(s) = \int_I g^n_m(i,s)\lambda(\mathrm{d}i), 1 \leq m \leq n \tag{4.4}$$

for any $s \in S$. Since $g^n(i,s) = h^n(i,s,\alpha^n(s)) \in G^n(i,s)$, then by definition, for each (i,s), there exists an $a_{(i,s)} \in \tilde{G}(i,s)$ such that $g^n(i,s) = (f_1(a_{(i,s)}), ..., f_n(a_{(i,s)}))$. Therefore, $(f_1, ..., f_n)^{-1} g^n(i,s) \cap \tilde{G}(i,s) \neq \emptyset$.

For any $n \in \mathbb{N}$, define the sequence of correspondence G^n as $\tilde{G}^n(i,s) :\ = (f^n)^{-1}(g^n(i,s)) \cap \tilde{G}(i,s) \neq \emptyset$, which is a sub-correspondence of $\tilde{G}$ with $\tilde{G}^n \subseteq \tilde{G}$. For each $n \in \mathbb{N}$, pick a $\mathcal{G} \otimes \mathcal{S}$-measurable selection ψ^n of $\tilde{G}^n$. Then ψ^n is also a $\mathcal{G} \otimes S$-measurable selection of $\tilde{G}$.

For each $n \in \mathbb{N}$, let $\varphi^n(i,s,\cdot)$ be the regular conditional distribution of ψ^n conditional on $\{\emptyset, I\} \otimes \mathcal{S}$[21] under the product probability measure $\lambda \times \mu$; that is, $\varphi^n = \mu^{\psi^n | \{\emptyset, I\} \otimes \mathcal{S}}$ which is a mapping from $I \times S \times \mathcal{B}(X)$ to $[0,1]$. Therefore, $\{\varphi^n\}_{n \in \mathbb{N}}$ is a sequence of regular conditional probabilities regarding to a sequence of measurable selections of $\mathcal{G} \otimes \mathcal{S}$-measurable selections of $\tilde{G}$, *i.e.*, $\{\varphi^n\}_{n \in \mathbb{N}} \subseteq \mathcal{R}_G^{\mathcal{F} \otimes \mathcal{S}, \{\emptyset, I\} \otimes \mathcal{S}}$.

Due to Theorem 3 in He and Sun (2013) (see also C2 and C3, Lemma 10), $\mathcal{R}_{\tilde{G}}^{\mathcal{F} \otimes \mathcal{S}, \{\emptyset, I\} \otimes \mathcal{S}}$ is weakly closed and weakly compact. Therefore, a subsequence of $\{\varphi^n\}_{n \in \mathbb{N}}$ will weakly converge to some φ^∞, and there exists an $\mathcal{F} \otimes \mathcal{S}$-measurable selection of $\tilde{G}$, ψ^∞, such that $\varphi^\infty = \mu^{\psi^\infty | \{\emptyset, I\} \otimes \mathcal{S}}$. Since φ^n is $\{\emptyset, I\} \otimes \mathcal{S}$-measurable for each $n \in \mathbb{N}$, it is obvious that $\varphi^n(i,s,\cdot)$ is constant element in the probability measure space $\mathcal{M}(A)$ for μ-almost all s. Hence, denote $\overline{\varphi^n}(s)$ be the Gelfand integral of $\varphi^n(\cdot, s)$ for each s; that is, $\overline{\varphi^n}(s) = \int_I \varphi^n(i,s,\cdot)\lambda(\mathrm{d}i) \in \mathcal{M}(A)$. Then given the set of countable bounded continuous function $\{f_m\}_{m \in \mathbb{N}}$ on A which is dense in $C_b(A)$, for fixed $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$,

$$\begin{aligned}\int_E \int_A f_m(a)\overline{\varphi^n}(s,\mathrm{d}a)\mu(\mathrm{d}s) &= \int_E \int_I \int_A f_m(a)\varphi^n(i,s,\mathrm{d}a)\lambda(\mathrm{d}i)\mu(\mathrm{d}s) \\ &= \int_E \int_I f_m(\psi^n(i,s))\lambda(\mathrm{d}i)\mu(\mathrm{d}s).\end{aligned}$$

holds for each measurable set $E \subseteq \mathcal{S}$. Hence, $\int_A f_m(a)\overline{\varphi^n}(s,\mathrm{d}a) = \int_I f_m(\psi^n(i,s))\lambda(\mathrm{d}i)$ for

[21] The defnition of regular conditional distribution is given in Section 4.5.1.

μ-almost all s. Since there are countable pairs of (m,n) where $m,n \in \mathbb{N}$, we can conclude that $\int_A f_m(a)\overline{\varphi^n}(s,\mathrm{d}a) = \int_I f_m(\psi^n(i,s))\lambda(\mathrm{d}i)$ for μ-almost all s holds for each $m,n \in \mathbb{N}$. Fix $n \in \mathbb{N}$ and vary m, $\int_A f_m(a)\overline{\varphi^n}(s,\mathrm{d}a) = \int_I f_m(\psi^n(i,s))\lambda(\mathrm{d}i)$ is valid for μ-almost all s. This result with the fact that $\{f_m\}_{m\in\mathbb{N}}$ is dense in $C_b(X)$ gives that, for μ-almost all s,

$$\overline{\varphi^n}(s) = \lambda \circ \psi^n(s,\cdot)^{-1} \text{ for all } n \in \mathbb{N}\cup\{\infty\}. \tag{4.5}$$

Let the convergence subsequence of $\{\varphi^n\}_{n\in\mathbb{N}}$ to be $\{\varphi^n\}_{n\in\mathbb{N}}$ itself for notation simplicity. Recall that $\{f_m\}_{m\in\mathbb{N}}$ is the set of countable bounded continuous function dense in $C_b(A)$. For the $\mathcal{G}\otimes\mathcal{S}$-measurable selection ψ^n of G^n where $G^n(i,s) \colon = (f^n)^{-1}(g^n(i,s)) \cap \tilde{G}(i,s) \neq \emptyset$, $\forall n \in \mathbb{N}$, it is easy to see that $g^n_m = f_m \circ \psi^n$ for $1 \leq m \leq n$, which is the m-th component of n-tuple vector $g^n = (g^n_1, ..., g^n_n)$. Fix $m \in \mathbb{N}$, for any measurable set $\tilde{E} \subseteq \mathcal{S}$,

$$\begin{aligned}
\int_{\tilde{E}}\int_A f_m(a)\overline{\varphi^\infty}(s,\mathrm{d}a)\mu(\mathrm{d}s) &= \int_{\tilde{E}}\int_I\int_A f_m(a)\varphi^\infty(i,s,\mathrm{d}a)\lambda(\mathrm{d}i)\mu(\mathrm{d}s)\\
&= \lim_{n\to\infty}\int_{\tilde{E}}\int_I\int_A f_m(a)\varphi^n(i,s,\mathrm{d}a)\lambda(\mathrm{d}i)\mu(\mathrm{d}s)\\
&= \lim_{n\to\infty}\int_{\tilde{E}}\int_I f_m(\psi^n(i,s))\lambda(\mathrm{d}i)\mu(\mathrm{d}s)\\
&= \lim_{n\to\infty}\int_{\tilde{E}}\int_I g^n_m(i,s)\lambda(\mathrm{d}i)\mu(\mathrm{d}s)\\
&= \int_{\tilde{E}}\alpha_m(s)\mu(\mathrm{d}s)\\
&= \int_{\tilde{E}}\int_A f_m(a)\phi(s,\mathrm{d}a)\mu(\mathrm{d}s).
\end{aligned}$$

The fifth equation is valid because of Equation (4.4), whereas the last equation holds for Equation (4.3). Therefore,

$$\int_A f_m(a)\overline{\varphi^\infty}(s,\mathrm{d}a) = \int_A f_m(a)\phi(s,\mathrm{d}a)$$

for each $m \in \mathbb{N}$ for μ-lamost all s. As a result, $\overline{\varphi^\infty}(s) = \phi(s)$ for μ-almost all $s \in S$. Combined with Equation (4.5), we have $\lambda \circ \psi^\infty(\cdot,s)^{-1} = \phi(s)$ for μ-almost all $s \in S$. □

Theorem 9 implies that by the nowhere equivalence condition, we can select one Nash equilibrium corresponding to one state in a measurable way among the whole state space such that the joint mapping is product measurable. It effectively tackles the measurability issue where the existence result hinges on.

Proof of Theorem 6. First we assume $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$ and prove the existence of stationary Markov equilibria in a large stochastic game. Notice that by the equivalence theorem (Theorem 12) it is sufficient to prove the existence of state-contingent Nash equilibria in its auxiliary game. Denote the large stochastic game and its auxiliary game as G and G^a respectively[22]. Denote the stage payoff function as $u(i, s\cdot, \cdot)$ for player i at state s.

Define Φ such that

$$\Phi(s) = \{\lambda \circ g(\cdot)^{-1} \colon g \text{ is an } \mathcal{F}\text{-measurable selection of } \arg\max_{a \in A} G(i, s, \cdot, \int_I \tilde{f}(i', s)\lambda(\mathrm{d}i'))\},$$

and Φ is nonempty-valued. Note that $\int_I \tilde{f}(i', \cdot)\lambda(\mathrm{d}i')$ is a $\mathcal{S}$-measurable selection of Φ. By Theorem 9, the result follows.

For the converse, assume every large stochastic game has a pure strategy stationary Markov perfect equilibrium. Restrict to the degenerated case which has only one state, and again by the equivalence theorem, the assumption implies that every large game has one pure strategy Nash equilibrium, which by Theorem 2 in He *et al.* (2016), is a necessary and sufficient condition for $\mathcal{F}$ being nowhere equivalent to $\mathcal{G}$. Then we reach our conclusion. □

Proof of Theorem 7. It is the same as the sufficient part of Theorem 6. □

4.5.3 Proofs of results in Section 4.4

Proofs of results in Section 4.4.1

Proof of Claim 6. (i) Clearly, for any action distribution $(a_0, 1 - a_0) \in \mathcal{M}(A)$ where $a_0 \in [0, 1]$ and a sequence of action distributions $\{(a_n, 1 - a_n)\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} a_n \to a_0$, $P(s'|a_n, s) = \delta_{\frac{a_n}{2}}(s')$ converges weakly to $\delta_{\frac{a_0}{2}}(s')$ for μ-almost all s, which implies that P is continuous when $\mathcal{M}(A)$ and $\mathcal{M}(S)$ are both endowed with weak topologies. However,

[22] These notations are inherited from the standard setting in Section 4.2.

such convergence cannot be strengthened into total variation norm sense in $\mathcal{M}(S)$, since the total variation norm between two distinguish Dirac measure is 2, *i.e,*

$$\|\delta_a - \delta_{a'}\|_{TV} = 2 \cdot \sup_{A \in \mathcal{S}} |\delta_a(A) - \delta_{a'}(A)| = 2 \text{ for } a \neq a',$$

which implies that for $\lim_{n\to\infty} a_n \to a_0$, $P(s'|a_n, s)$ keep distance 2 with $P(s'|a_0, s)$ in total variation norm, and hence does not converge to $P(s'|a_0, s)$ in total variation norm sense.

(ii) For every $n \in \mathbb{N}$, we need to prove that the strategy profile f^n of large finite stochastic game G^n is a stationary Markov perfect equilibrium:

$$f^n(i, s) = \begin{cases} D, & 0 \leq s < \frac{1}{2}, i \in I^n, \\ C, & \frac{1}{2} \leq s \leq 1, i \in I^n. \end{cases}$$

The case that starting game G^n with initial state s_0 where $0 \leq s_0 < \frac{1}{2}$ is trivially ture, *i.e,* no player will ever deviate from playing "D" and the transition probability is δ_0 in all stages, which means everyone will remain forever in state 0 from second stage on.

On the other hand, consider starting game G^n with initial state s_0 where $\frac{1}{2} \leq s_0 \leq 1$. Given every player's strategy as above, player i knows that his opponents are going to choose "C" in this stage. If he plays "C", the transition probability is $\delta_{\frac{1}{2}}$, which means the state s_1 in the next stage is $\frac{1}{2}$ and everyone still cooperates, then by keeping choosing "C" his expected payoff is at least $\frac{-1}{1-\delta}$. But if he deviate to "D" in this stage, the transition probability for next stage is $\delta_{\frac{n-1}{2n}}$, and the state in the next stage will be $\frac{n-1}{2n} < \frac{1}{2}$ which means everyone will start to choose "D" in the next stage and with this trend, keep "D" forever, which gives him the expected payoff $\frac{-2\delta}{1-\delta}$. Given that $\delta > \frac{1}{2}$, "C" strictly dominates "D" and player i will always choose "C" whenever state s satisfies $\frac{1}{2} \leq s \leq 1$. Such analysis goes through in reaching each stage (in the beginning of each subgame), which means f_n is actually a stationary Markov perfect equilibrium.

(iii) By the equivalence theorem (Theorem 12), the stationary Markov perfect equilibria in a large stochastic game are the state-contingent Nash equilibria in its auxiliary game. Therefore, we just need to find out all the state-contingent Nash equilibria in the component game of the large stochastic game we constructed. Notice that for every state $s \in S$, "D" strictly dominates "C" and hence the unique state-contingent Nash equilibrium is $f \equiv D$, which is also the unique stationary Markov perfect equilibrium in the large stochastic game G in Example 8.

□

Proof of the claims in Example 9. (i) Denote the Borel σ-algebra of state space $[0, 1]$ as $\mathcal{B}[0, 1]$. If $a_n \to a$, then

$$\|a_n\mu_1 - a\mu_1\|_{TV} = 2 \cdot \sup_{B \in \mathcal{B}[0,1]} |a_n\mu_1(B) - a\mu_1(B)| \leq 2|a_n - a| \to 0, \text{ and}$$

$$\|(1 - a_n)\mu_0 - (1 - a)\mu_0\|_{TV} = 2 \cdot \sup_{B \in \mathcal{B}[0,1]} |(1 - a_n)\mu_0(B) - (1 - a)\mu_0(B)| \leq 2|a_n - a| \to 0,$$

which shows the norm continuity in transition probability.

(ii) We prove this claim by contradictions. Assume in the constructed sequence of large finite stochastic game $\{G^n\}_{n \in \mathbb{N}}$, there exists a sequence of stationary Markov perfect equilibria f^n and a positive measure set S^n such that for any $s \in S^n$, there exists an $\epsilon(s) > 0$, the distributions under state s $\lambda^n(f^n)^{-1}(\cdot, s) := (a_n(s), 1 - a_n(s))$ satisfies $a_n(s) \geq \epsilon(s)$ for all $n \in \mathbb{N}$.

Fixed $n \in \mathbb{N}$ for the game G^n, and consider the case under state $s_0 \in S^n \bigcap [\frac{1}{2}, 1]$. Assume there is a player i using "C" in state s_0(*i.e.*, $f^n(i, s_0) = C$) in the equilibrium f^n. For simplicity, denote his expected payoff to be $V_s^n(a_i)$ if the current state is s and he plays action a_i while others are all following actions specified by the equilibrium f^n. When the current state is s_0, his expected payoff of choosing the equilibrium action "C" is $V_{s_0}^n(C) = -1 + \delta \left(a^n(s) \int_S V_{s'}^n(f(i, s'))\mu_1(ds') + (1 - a^n(s)) \int_S V_{s'}^n(f(i, s'))\mu_0(ds')\right)$ where $a^n(s)$ is the equilibrium actions distribution under state s. If he chooses to deviate to "D" one-time in current state s_0 after which he goes back to his original equilibrium actions, then his expected payoff turns out to be

$$V_{s_0}(D) = \delta\left((a^n(s) - \frac{1}{n})\int_S V^n_{s'}(f(i,s'))\mu_1(ds') + (1 - a^n(s) + \frac{1}{n})\int_S V^n_{s'}(f(i,s'))\mu_0(ds')\right),$$

When n is large enough, $V^n_{s_0}(C) < V^n_{s_0}(D)$. Since $\sup_{s\in S, a\in\{C,D\}, n\in\mathbb{N}} |V^n_s(a)| \leq \frac{3}{1-\delta}$, we can find an integer N (e.g., $N > \frac{6\delta}{1-\delta}$) such that if $n > N$ then $V^n_{s_0}(C) < V^n_{s_0}(D)$ for all state $s_0 \in S^n \bigcap [\frac{1}{2}, 1]$. Things will be the same when the next stage turns out to be $s_0 \in S^n \bigcap [\frac{1}{2}, 1]$, and the following stages as well. Hence in large finite stochastic games G^n with $n > \frac{6\delta}{1-\delta}$, no player will choose "C" in state $s_0 \in S^n \bigcap [\frac{1}{2}, 1]$, which contracts with the assumption that $a^n(s_0) \geq \epsilon(s_0) > 0$ for all $n \in \mathbb{N}$ and all $s_0 \in S^n \bigcap [\frac{1}{2}, 1]$.

Now turn to the current state $s'_0 \in S^n \bigcap [0, \frac{1}{2})$, similar analysis will gives that there exists an integer N', if the number of players n excesses N, $V^n_{s'_0}(C) < V^n_{s'_0}(D)$ for all state $s'_0 \in S^n \bigcap [0, \frac{1}{2})$, which derives similar contradictions. Therefore, we conclude that when the number of players is large enough, all players will choose to play "D" is the only stationary Markov perfect equilibrium, which implies the validity of limit principle in the modified large game G. □

Proofs of results in Section 4.4.2

We first define the concept of ϵ-approximate state-contingent Nash equilibrium.

Recall that G^n is a large finite stochastic game with the player space $(I^n, \mathcal{I}^n, \lambda^n)$ (a finite-element probability measurable space), the state space $(S, \mathcal{S}, \mu)$ (a polish space), the action space A (a compact metric space) and the societal summary space $\mathcal{M}(A)$ (a compact metric space) such that u^n assigns each player and state with a state payoff function, i.e., u^n is a measurable mapping from $I^n \times S$ to $\mathcal{U}_A$, where the payoff space $\mathcal{U}_A$ is the (uniformly bounded) Banach space of continuous functions on $A \times \mathcal{M}(A)$ endowed with the sup-norm topology and the resulting Borel σ-algebra.

Definition 31 (State-contingent ϵ-approximate Nash equilibrium)**.** A behavioral strategy profile $f^n\colon I^n \times S \to \mathcal{M}(A)$ is a behavioral-strategy state-contingent ϵ[23]-approximate Nash equilibrium

[23] ϵ is a $\mathcal{S}$-measurable mapping from S to $\mathbb{R}^+$.

of the state-contingent large game u^n, if for μ-almost all state s, $\epsilon(s) \geq 0$ such that

$$\int_A u^n(i,s,a,\int_{I^n} f^n(i,s)\lambda^n(i))f^n(i,s)(\mathrm{d}a) \geq \max_{a\in A} u^n(i,s,a,\int_{I^n}(a,f^n_{-i}(i,s))\lambda^n(i)) - \epsilon(s),$$

for all $i \in I_n^{\epsilon(s)}$ where $I_n^{\epsilon(s)} \subseteq I_n$ with $\lambda_n(I_n^{\epsilon(s)}) \geq 1-\epsilon(s)$, and $(a, f^n_{-i}(i,s))$ denotes a strategy profile that player i plays a and his opponent player j plays $f^n(j,s)$ for $j \neq i$ under state s.

Notice that pure strategy State-contingent ϵ-approximate Nash equilibrium can be defined analogously. Now we are ready to present the lemma.

Lemma 14 (Apporximation)**.** For every sequence of large finite stochastic games $\{G^n\}_{n\in\mathbb{N}}$ with $\sup_{i\in I^n} \lambda^n(i) \to 0$, and the sequence of pure-strategy/behavioral-strategy stationary Markov perfect equilibria f^n of G^n for each $n \in \mathbb{N}$, if the sequence of transition probabilities $\{P^n\}_{n\in\mathbb{N}}$ is uniformly continuous in norm, then there exists a sequence of $\mathcal{S}$-measurable mapping $\epsilon^n \colon S \to \mathbb{R}^+$ such that f^n is a pure-strategy/behavioral-strategy state-contingent ϵ^n-approximate Nash equilibrium of the auxiliary game G^n_a, and $\lim_{n\to\infty} \epsilon^n(s) = 0$ for μ-almost all s.

Proof of Lemma 14. We only prove the case for pure stationary Markov perfect equilibria. The behavioral-strategy equilibrium case can be proven similarly. For any player i, we need to quantify the difference between his equilibrium stage payoff $G^n_a(i,s)(f^n(i,s),\lambda^n(f^n(\cdot,s))^{-1})$ and his possible maximal stage payoff $\max_{a\in A} G^n_a(i,s)(a,\lambda^n(a,f^n_{-i}(\cdot,s))^{-1})$. By definition, there exists a continuation utility function $v^n(i,s) \in L_\infty^{\mathcal{I}^n\times\mathcal{S}}(I^n\times S)$[24] which satisfies for μ-almost all fixed $s \in S$,

$$\begin{aligned} v^n(i,s) &= (1-\beta)G^n_a(i,s)(f^n(i,s),\lambda^n(f^n(\cdot,s))^{-1}) + \beta\int_S v^n(i,s')P^n(ds'|s,\lambda^n(f^n(\cdot,s))^{-1}) \\ &= \max_{a\in A}(1-\beta)G^n_a(i,s)(a,\lambda^n(a,f^n_{-i}(\cdot,s))^{-1}) + \beta\int_S v^n(i,s')P^n(ds'|s,\lambda^n(a,f^n_{-i}(\cdot,s))^{-1}) \end{aligned}$$

for λ^n-almost all $i \in I^n$, and $(a, f^n_{-i}(\cdot,s))$ denotes a strategy profile that player i plays a and his opponent player j plays $f^n(j,s)$ for $j \neq i$ under state s. Therefore, we can achieve our goal by

[24] $L_\infty^{\mathcal{I}^n\times\mathcal{S}}(I^n\times S)$ denotes the set of $\mathcal{I}^n\times\mathcal{S}$ measurable function whose sup-norm is finite, *i.e.*, $||h||_\infty < \infty$ for $h \in L_\infty^{\mathcal{I}^n\times\mathcal{S}}(I^n\times S)$.

looking into the absolute difference between these two terms in the following:

$$|\max_{a\in A} G_a^n(i,s)(a,\lambda^n(a,f_{-i}^n(\cdot,s))^{-1}) - G_a^n(i,s)(f^n(i,s),\lambda^n(f^n(\cdot,s))^{-1})| \tag{4.6}$$
$$= \frac{\beta}{1-\beta}\left(\max_{a\in A}\int_S v^n(i,s')P^n(ds'|s,\lambda^n(a,f_{-i}^n(\cdot,s))^{-1}) - \int_S v^n(i,s')P^n(s'|s,\lambda^n(f^n(\cdot,s))^{-1})\right)$$

Let d be the prohorov metric on $M(A)$, by the assumption that $\sup_{i\in I^n}\lambda^n(i)\to 0$, for any a, $d(\lambda^n(f^n(\cdot,s))^{-1},\lambda^n(a,f_{-i}^n(\cdot,s))^{-1}) \le 2\sup_{i\in I^n}\lambda^n(i)\to 0$ as $n\to+\infty$. Therefore $\max_{a\in A} d(\lambda^n(f^n(\cdot,s))^{-1},\lambda^n(a,f_{-i}^n(\cdot,s))^{-1}) \le 2\sup_{i\in I^n}\lambda^n(i)\to 0$ as $n\to+\infty$, which by the uniformly norm continuity in the transition deduces, for any $\epsilon>0$, there exists an integer N large enough such that for all $n\ge N$,

$$\max_{a\in A}\|P^n(\cdot|s,\lambda^n(a,f_{-i}^n(\cdot,s))^{-1}) - P^n(\cdot|s,\lambda^n(f^n(\cdot,s))^{-1})\|_{TV}\le\epsilon \tag{4.7}$$

By Theorem 11.7, Stokey, Lucas and Prescott (1989) the above equation equals to

$$\max_{a\in A}\left(\sup_{f\in B_\infty^{\mathcal{S}}(S)}|\int_S f(s')P(ds'|s,\lambda^n(a,f_{-i}^n(\cdot,s))^{-1}) - P(s'|s,\lambda^n(f^n(\cdot,s))^{-1})\right)\to 0, \tag{4.8}$$

where $B_\infty^{\mathcal{S}}(S)$ is the space of bounded measurable functions endowed with sup norm: $\|f\|:=\sup_{s\in S}|f(s)|$.

Since v^n is the (discounted) continuation utility of the players with uniformly bounded stage payoffs [25], therefore (modify $v_n(\cdot,s)$ on s in a μ-zero-measure set if necessary) v_n is also uniformly bounded by $\frac{B_u}{1-\beta}$ for all $s\in S$ and $n\in\mathbb{N}$. By Equation (4.8) the following equation :

$$\max_{a\in A}|\int_S v^n(i,s')P^n(ds'|s,\lambda^n(a,f_{-i}^n(\cdot,s))^{-1}) - \int_S v^n(i,s')P^n(ds'|s,\lambda^n(f^n(\cdot,s))^{-1})|$$
$$\le \max_{a\in A}\|P^n(\cdot|s,\lambda^n(a,f_{-i}^n(\cdot,s))^{-1}) - P^n(\cdot|s,\lambda^n(f^n(\cdot,s))^{-1})\|_{TV}\to 0 \tag{4.9}$$

holds uniformly for λ^n-almost all $i\in I^n$. For all $s\in S$, let $\epsilon^n(s)$ be as follows:

$$\epsilon^n(s):=\frac{\beta}{1-\beta}\max_{a\in A}\|P^n(\cdot|s,\lambda^n(a,f_{-i}^n(\cdot,s))^{-1}) - P^n(\cdot|s,\lambda^n(f^n(\cdot,s))^{-1})\|_{TV},$$

[25] $\mathcal{U}_A$ is bounded by B_u in the basic setting.

for all finite sets I^n where $n \in \mathbb{N}$. That ϵ^n is $\mathcal{S}$-measurable is implied by the norm continuity of the transition probability P^n. Combine Equations (4.6), (4.7) and (4.9), for μ-almost all s and λ^n-almost all $i \in I^n$,

$$
\begin{aligned}
&|\max_{a\in A} G^n_a(i,s)(a,\lambda^n(a,f^n_{-i}(\cdot,s))^{-1}) - G^n_a(i,s)(f^n(i,s),\lambda^n(f^n(\cdot,s))^{-1})| \\
&\leq \frac{\beta}{1-\beta}\max_{a\in A} |\left(\int_S v^n(i,s')P^n(ds'|s,\lambda^n(a,f^n_{-i}(\cdot,s))^{-1}) - \int_S v^n(i,s')P^n(ds'|s,\lambda^n(f^n(\cdot,s))^{-1})\right)| \\
&\leq \epsilon^n(s) \to 0,
\end{aligned}
$$

which concludes our lemma. □

Proof of Theorem 8. Recall that a behavioral-strategy stationary Markov perfect equilibrium correspondence of a large stochastic game G^0 has the closed graph property, if for any sequence of large finite stochastic games $\{G^n\}_{n>0}$[26] satisfies that for μ-almost all s,

(i) the stage utilities of this sequence $u^n(\cdot,s,\cdot,\cdot)$ converge weakly to $u^0(\cdot,s,\cdot,\cdot)$ as $n \to \infty$, where u^n is a measurable mapping from $I^n \times S$ to $\mathcal{U}_A$ for $n \geq 0$;

(ii) the sequence of behavioral-strategy stationary Markov perfect equilibria f^n of G^n for each $n > 0$ is such that $\{\int_{I^n} f^n(\cdot,s)\lambda^n\}_{n>0}$ converges weakly to some $\nu(s) \in \mathcal{M}(A)$ as $n \to \infty$;

(iii) $\{P^n(s'|s,\tau)\}_{n\geq 0}$ is uniformly continuous in any $\tau \in \mathcal{M}(A)$ where $\mathcal{M}(A)$ and $\mathcal{M}(S)$ are endowed with the prohorov metric and norm topology respectively;

there exists a behavioral-strategy (resp. pure-strategy) stationary Markov perfect equilibrium f^0 of G^0 such that $\int_{I^0} f^0(\cdot,s)\lambda^0 = \nu(s)$ (resp. $\lambda^0(f^0(\cdot,s))^{-1} = \nu(s)$) for μ-almost all s.

By Lemma 14, f^n is a behavioral-strategy state-contingent ϵ^n-approximate state-contingent Nash equilibrium of G^n with $\epsilon \to 0$ for μ-almost all s. Then for μ-almost all s, fixed such s, $f^n(\cdot,s)$ is a behavioral-strategy $\epsilon^n(s)$-approximating Nash equilibrium for large game $G^n_a(\cdot,s)$, whose payoff function is $u^n(\cdot,s,\cdot,\cdot)$. Such pair $(G^n_a(\cdot,s),f^n(\cdot,s))$ corresponds to a $\epsilon^n(s)$-

[26] Equal weight condition can be further relaxed to the condition $\sup_{i\in I^n}\lambda^n(i) \to 0$.

approximating Nash equilibrium distribution[27] $\tau^n(s) :\ = (\int_I G_a^n(i,s)\lambda^n(i), \int_I f^n(i,s)\lambda^n(i))$, which is a Borel probability measure on $\mathcal{U}_A \times A$, and its marginal distributions satisfy that $\tau^n(s)_{\mathcal{U}_A} = \lambda^n G_a^n(\cdot,s)^{-1} = \lambda^n u^n(\cdot,s,\cdot,\cdot)^{-1}$ and $\tau^n(s)_A = \int_{I^n} f^n(i',s)\lambda^n(\mathrm{d}i')$. Since $\{\tau^n(s)\}_{n>0}$ is relatively compact, there exists an limit point, i.e., there exists a subsequence of $\{\tau^n(s)\}_{n>0}$ weakly converges to a probability measure, denoted as τ_s^0. Without loss of generality, we denote the subsequence as the sequence itself. Since $u^n(\cdot,s,\cdot,\cdot) \Rightarrow u^0(\cdot,s,\cdot,\cdot)$ and $\int_{I^n} f^n(i',s)\lambda^n(\mathrm{d}i') \Rightarrow \nu(s) \in \mathcal{M}(A)$, hence $(\tau_s^0)_{\mathcal{U}_A} = \lambda^0 u^0(\cdot,s,\cdot,\cdot)^{-1}$ and $(\tau_s^0)_A = \nu(s)$.

We now show that τ_s^0 is again a Nash equilibrium distribution of $\lambda^0 G_a^0(\cdot,s)^{-1}$, which therefore by Theorem 1 in Sun *et al.* (2016), there exists a Nash equilibrium f_s^0 of large game $G_a^0(\cdot,s)$, which induces τ_s^0, i.e., $(\tau_s^0)_A = \int_{I^0} f_s^0(i')\lambda^0(\mathrm{d}i') = \nu(s)$.

Define $\mathrm{Br}^{\epsilon^n(s)}(\tau^n(s)) :\ = \{(v,a) \in \mathcal{U}_A \times A | v(a,\tau^n) \geq \max_{x\in A} v(x,\tau^n) - \epsilon^n(s)\}$ and $\mathrm{Br}(\tau_s^0) :\ = \{(v,a) \in \mathcal{U}_A \times A | v(a,\tau_s^0) \geq \max_{x\in A} v(x,\tau_s^0)\}$. By the definition of $\epsilon^n(s)$-approximate Nash equilibrium distribution, for $n \geq N$,

$$\tau^n(s)\left(\cap_{k=1}^N \overline{\cup_{j=k}^\infty \mathrm{Br}^{\epsilon^n(s)}(\tau^n(s))}\right) \geq 1 - \epsilon^n(s).$$

Then letting n go to infinity, we have

$$\tau_s^0\left(\cap_{k=1}^N \overline{\cup_{j=k}^\infty \mathrm{Br}^{\epsilon^n(s)}(\tau^n(s))}\right) \geq 1.$$

We then let N goes to infinity,

$$\tau_s^0\left(\cap_{k=1}^\infty \overline{\cup_{j=k}^\infty \mathrm{Br}^{\epsilon^n(s)}(\tau^n(s))}\right) \geq 1. \tag{4.10}$$

It is easy to see that $\cap_{k=1}^\infty \overline{\cup_{j=k}^\infty \mathrm{Br}^{\epsilon^n(s)}(\tau^n(s))} \subseteq \mathrm{Br}(\tau_s^0)$ so we conclude the claim.

Since for G_a^0, its payoff $u^0(i,s,\cdot,\nu(s))$ is $\mathcal{G} \otimes \mathcal{S}$-measurable on $I \times S$ and continuous on A.

[27] A similar proof of Theorem 1 in Sun, Sun and Yu (2016)

Define a new correspondence $\tilde{G}_a^0$ from $I \times S$ to $\mathcal{M}(A)$ such that

$$\tilde{G}_a^0(i,s) := \underset{v \in \mathcal{M}(A)}{\arg\max} \int_A u^0(i,s,a',\nu(s))v(\mathrm{d}a').$$

By Measurable Maximum theorem (Theorem 18.19 in Aliprantis and Border (2006)), $\tilde{G}_a^0$ is a $\mathcal{G} \otimes \mathcal{S}$-measurable, nonempty, convex and compact-valued correspondence. And let $\Phi(s)$ is a correspondence such that

$$\Phi(s) := \{\int_I g(i')\lambda(i') : g \text{ is an } \mathcal{G}\text{-measurable selection of } \tilde{G}_a^0(\cdot,s)\}.$$

Since $\nu(s) = \int_{I^0} f_s^0(i')\lambda^0(\mathrm{d}i')$ where f_s^0 is a Nash equilibrium in the component game $G_a^0(\cdot,s)$ for μ-almost all s, $\nu(s)$ is a $\mathcal{S}$-measurable selection of Φ. Therefore, by Claim 8 in the proof of Theorem 5, there exists an $\mathcal{G} \otimes \mathcal{S}$-measurable mapping f^0 from $I \times S$ to $\mathcal{M}(A)$ such that $f^0(i,s) \in \tilde{G}_a^0(i,s)$ and $\int_{I^0} f^0(i',s)\lambda^0(\mathrm{d}i') = \nu(s)$ for μ-almost all s, which concludes that the behavioral-strategy stationary Markov perfect equilibrium correspondence of large stochastic game G^0 has the closed graph property. □

Proof of Proposition 9. We first prove the sufficient part. Given the sequence of converging large finite stochastic games G^n and its pure stationary Markov perfect equilibrium f^n, By Theorem 8 there exists a behavioral-strategy stationary Markov perfect equilibrium f^0 in the limit large stochastic game G^0 such that for μ-almost all s, $\int_{I^0} f^0(\cdot,s)\lambda^0 = \nu(s)$, which therefore by the purification theorem (Theorem 7) shows the validity of pure stationary Markov perfect equilibrium correspondence of large stochastic game G^0 has the closed graph property.

To prove the necessity of nowhere equivalence condition, consider the reduced large stochastic games with only one state, in other words, the large repeated games. Regardless, all players repeatedly play the strategies specified in one of their static Nash equilibrium is always a (trivial) stationary Markov perfect equilibrium of the repeated game (See, for example, Theorem 1 in Ratliff (1990)). Therefore, if restrict the limit principle to a special case that, for any large repeated game, and any sequence of large finite-player repeated games weakly converging to it with the sequence of converging trivial stationary Markov perfect equilibria[28], the limit principle property is satisfied. By the equivalence theorem, such reduction can be interpreted as every $\mathcal{G}$-measurable large game satisfies the closed-graph property for $\mathcal{F}$-measurable Nash equilibria. Then by Theorem 2 (iv) in He *et al.* (2016), $\mathcal{F}$ is nowhere equivalent to $\mathcal{G}$. □

Proof of Claim 7. (i) It is easy to verify that the sequence of finite-player repeated games $\{P_k G_k^{-1}\}_{k\in\mathbb{N}}$ converges weakly to PG^{-1}, and hence such verification is omitted here.

(ii) The discussion in Example 4, He *et al.* (2016) shows that f_k is a pure-strategy Nash equilibrium in the component game of G_k for $k \in \mathbb{N}$, whose distribution $P_k f_k^{-1}$ converges weakly to the uniform distribution on A. It is a classical result that any Nash equilibrium in the component game of a finite-player repeated game is a history-independent subgame perfect equilibrium (See, for example, Theorem 1 in Ratliff (1990)). Hence, $\{f_k\}_{k\in\mathbb{N}}$ is a sequence of stationary Markov perfect equilibria converging weakly to the uniform distribution on A.

(iii) By the equivalence theorem (Theorem 12), the set of stationary Markov perfect equilibrium in a large stochastic game is the same as the set of Nash equilibrium in its component game. By Claim 3, He *et al.* (2016), the uniform distribution is not a distribution of any Nash equilibrium of the component game of the large repeated game G, which implies that it cannot be a distribution of a stationary Markov perfect equilibrium as well. Therefore, the closed-graph property fails in this case.

□

[28] The "trivial stationary Markov perfect equilibria in a large finite-player repeated game" here means Nash equilibria in its component game.

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