OPTIMAL ORDERING POLICIES FOR BROWNIAN INVENTORY MODELS WITH GENERAL SETUP COSTS

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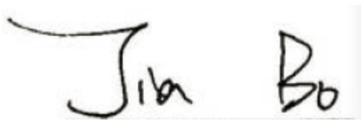
Declaration

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.



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Contents

Summary

We consider continuous-review inventory models with general quantity-dependent setup costs. The demand processes of the inventory models are modeled as Brownian motions with a positive drift. The inventory level can be adjusted by a positive amount at any time and the lead time of each order is zero. Each order incurs a proportional cost and a setup cost that is a step function of the order quantity. We further assume that the holding cost is a general convex function of the inventory level. By a lower bound approach, we obtain optimal ordering policies for three continuous-review inventory models: (a) an inventory model without backlogs under the long-run average cost criterion; (b) an inventory model without backlogs under the discounted cost criterion; (c) an inventory model with backlogs under the discounted cost criterion. Since the smooth pasting technique does not apply when the setup cost is quantity-dependent, we propose a four-step procedure to obtain optimal policy parameters for the inventory models. To cope with the quantity-dependent setup cost and upward adjustments, we provide a comparison theorem under the discounted cost criterion. With this comparison theorem, we can prove the global optimality within a tractable subset of admissible policies.

Keywords: continuous-review inventory models; inventory control; impulse control; quantity-dependent setup cost; free boundary problems; (s, S) policy.

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Chapter 1

Introduction

This thesis explores optimal ordering policies for continuous-review single-product Brownian inventory models with quantity-dependent setup costs. In this chapter, we introduce existing inventory models with quantity-dependent setup cost and show the limitations of these works in Section 1.1. Based on the limitations discussed in Section 1.1, the objectives and contributions of this thesis are specified in Section 1.2. Section 1.3 outlines the organization of this thesis.

1.1 Motivation

Because of the prosperity of commercial society, most enterprises involve in the management of inventories in the face of a diverse collection of factors, e.g., cost patterns, modes of shipment and methods of delivery to customers. Due to this phenomenon, inventory management has received considerable attention in the literature. Silver (1981) states that inventory management attempts to cope with three main questions: (i) How often should the inventory status be determined? (ii) When should an order be placed? (iii) How large should the replenishment order be? Inventory models can be divided into periodic-review models and continuous-review models according to frequency of inventory inspection. In periodic-review inventory models, managers can adjust the inventory level only at discrete times. In contrast, the inventory level can be adjusted at any time in continuous-review inventory models. In this thesis, we consider continuous-review inventory models.

The objectives of inventory management include cost minimization, profit maximization, maximization of rate of return on stock investment, maximization of chance of survival, ensuring flexibility of operation and determination of a feasible solution (Silver, 1981). The majority of the inventory management literature focuses on investigating the optimal control policies that minimize the cost of inventory systems. The optimal policies specify when to order and how large each order should be. During the procedure of constructing the optimal policy that minimizes the cost of inventory systems, managers face a trade-off between different types of costs. There are two types of costs that are related to inventory decision making procedure: the replenishment cost and the holding and shortage cost. The replenishment cost consists of the setup cost and the proportional cost for each order. The holding and shortage cost refers to the holding cost, when there are products in stock and the shortage cost, when there are backlogs. In the existing literature, inventory models, including both periodic-review models and continuous-review models, assume a constant setup cost or a monotone piecewise constant setup cost. The optimal inventory control policies have not been completely characterized for the inventory models with the monotone piecewise constant setup cost (Lippman, 1969; Chao and Zipkin, 2008; Zhou et al., 2009). Furthermore, the setup cost for each order in the real world arises from different activities and thus may not be a monotone function of the order quantity. A general step function of the ordering quantity is more appropriate to model the setup cost. Next, we will examine some existing periodic-review inventory models and continuous-review inventory models with different setup cost structures.

Classical inventory models usually assume a fixed setup cost when placing an order or starting a production run to replenish the inventory. For periodic-review single-product inventory models, Scarf (1960) and Iglehart (1963) prove the well known result that an ordering policy of the (s, S) type attains the optimality when the setup cost is constant for any order quantity.

The constant setup cost assumption may not be practical. To generalize this assumption, one may model the setup cost as a step function. In some cases, the setup cost increases as the ordering quantity grows. For example, if the setup cost consists of transportation fees only, the number of trucks needed grows as the shipping amount increases, thus resulting in the rise of the setup cost. Assuming that the capacity of a vehicle is Q and the fixed cost for each vehicle is K_0, then the ordering setup cost is a non-decreasing step function of the order quantity ξ which is specified by

$$K(\xi) = K_0 \cdot \lceil \frac{\xi}{Q} \rceil. \tag{1.1}$$

The study of stochastic inventory models with such a setup cost can be traced back to Lippman (1969). In this paper, Lippman considers a periodic-review single-product stochastic inventory model with a nondecreasing and subadditive ordering cost function. The author proves the existence of optimal ordering policies for both finite-horizon problem and infinite-horizon problem. However, the optimal ordering policy that minimizes the discounted cost is only partially characterized for both the multi-period problem and the infinite-horizon problem: it is pointed out that at the beginning of each period, the optimal policy is to replenish the inventory if the inventory level drops below a certain level and not to order if the inventory exceeds another level. However, the optimal policy is not specified when the inventory is between these two levels. With the cost structure in (1.1), Iwaniec (1979) figures out several conditions under which the optimal policy minimizing the discounted cost is a full-batch ordering policy. In addition to full-batch ordering policies, partial-batch ordering policies are studied in Alp et al. (2013) and the optimal ordering policy that minimizes the long-run average cost is partially characterized.

Chao and Zipkin (2008) consider a periodic-review single-product stochastic inventory model with another type of monotone quantity-dependent setup cost function

$$K(\xi) = K_0 \cdot 1_{\{\xi \in (C,\infty)\}}. \tag{1.2}$$

The authors state that the setup cost in (1.2) is specified in supply contracts and could be interpreted as the cost of disruption for the suppliers, such as finding more trucks, arranging extra processing capacity, persuading other customers to postpone orders and so on. This setup cost structure is imposed by the supplier in the supply contract in order to prevent irregular larger orders. Under such a supply contract, no setup cost is incurred if the ordering quantity does not exceed the contract volume C. However, the buyer is charged a setup cost K_0 if the order quantity exceeds the contract volume C. The authors partially characterize the optimal ordering policy for their periodic-review system and propose an effective heuristic policy under both discounted and average cost criterion. The optimal order policy is not specified when the starting inventory level was in a certain interval. Caliskan-Demirag et al. (2012) consider a stochastic periodic-review single-product inventory model with several cost structures derived from a nondecreasing quantity-dependent setup cost function

$$K(\xi) = \sum_{i=1}^{n} K_i \cdot 1_{\{\xi \in (C_{i-1}, C_i]\}}, \tag{1.3}$$

where ξ is the order quantity, $K_i \leq K_{i+1}$ and $C_0 = 0$. The setup cost structure (1.3) includes the cost structure (1.1) and (1.2) as special cases. However, they only provide partial characterization of the optimal ordering policy for the finite-horizon problem. The optimal policy is not specified for a certain interval.

Contrary to the increasing setup cost structure, setup cost may also decline as the order size increases. To explore the economies of scale, suppliers often encourage buyers to order more by providing shipping discounts or free shipping for large orders. Such promotions indeed increase additional sales. In Lewis et al. (2006), the authors also point out that customers are sensitive to shipping charges and promotions for large orders are effective to generate more profits. Zhou et al. (2009) consider a periodic-review single-product stochastic inventory system with a setup cost function

$$K(\xi) = K_0 \cdot 1_{\{\xi \in (0, C)\}}. \tag{1.4}$$

Assuming a linear holding and shortage cost function, the authors provide the optimal inventory control policy and its structural properties for the single-period model and propose a heuristic policy for the multi-period inventory system. Zhou et al. (2009) provide numerical results, but the authors do not provide complete characterization of the optimal policy for the multi-period inventory system.

In existing inventory models with quantity-dependent setup cost, such as (1.1)-(1.4), the optimal policies are not completely characterized (Lippman, 1969; Chao and Zipkin, 2008; Zhou et al., 2009; Caliskan-Demirag et al., 2012; Alp et al., 2013). Furthermore, the existing setup cost structures in (1.1)-(1.4) are all monotone with respect to the order quantity. However, the setup cost may not be a monotone function of the order quantity. In practical inventory systems, setup cost arises from different activities, such as transportation, loading and unloading, equipment installation and so on. As a result, the setup cost may not be monotone with respect to the order quantity. For example, when the inventory manager orders more, quantity discounts are offered for the transportation fees (Zhou et al., 2009) and labor fees, however, are charged more (Caliskan-Demirag et al., 2012). To the best of our knowledge, He et al. (2015) is the first and only study that completely characterizes the optimal ordering policy for an inventory system with quantity-dependent setup cost. Furthermore, the authors consider a general step setup cost structure in the continuous-review single-product Brownian inventory system. However, He et al. (2015) only consider an inventory model with complete backlogs under the long-run average cost criterion. There are no studies about what type the optimal order policies will be under the no-backlog scenario or under the discounted cost criterion. Since backlogs are not acceptable in some industries (Ormeci et al., 2008), it is necessary to investigate the the optimal order policies for the inventory system with quantity-dependent setup cost under the no-backlog scenario. Moreover, when the interest rate is considered, it is more appropriate to apply the discounted cost criterion rather than the long-run average cost criterion.

In this thesis, we consider continuous-review inventory models due to the technical advantage of continuous-review models over periodic-review models. Since the ordering cost function may be neither convex nor concave, it is difficult to identify the cost structures that can be preserved through dynamic programming. However under the continuous-review model and Brownian demand process, we can obtain the optimal policy by solving a Brownian control problem, which is more tractable than solving a dynamic programming for the periodic-review inventory problem under a quantity-dependent setup cost.

1.2 Objectives and Contributions

This thesis considers a continuous-review single-product inventory model with a general step setup cost function

$$K(\xi) = \sum_{i=1}^{M} K_i \cdot 1_{\{\xi \in (Q_i, Q_{i+1})\}} + \sum_{i=1}^{M-1} (K_i \wedge K_{i+1}) \cdot 1_{\{\xi = Q_{i+1}\}}, \tag{1.5}$$

where

$$0 = Q_1 < Q_2 < \cdots < Q_{M-1} < Q_M < Q_{M+1} = \infty,$$
$$K_i > 0 \quad \text{for } i = 1, 2, ..., M,$$
$$K_i \neq K_{i+1} \quad \text{for } i = 1, 2, ..., M-1.$$

See Figure 1.1 for an example of the general setup cost function (1.5) with $M = 5$.

In addition to the quantity-dependent setup cost, we further assume a general convex holding and shortage cost and a proportional cost. Our objective is to extend the results of He et al. (2015) to no-backlog cases and discounted cost criterion cases. The specific objectives of this research are to:

(a) Investigate the optimal inventory control policy for the continuous-review inventory system with the general setup cost and without backlogs under the long-run average cost criterion.

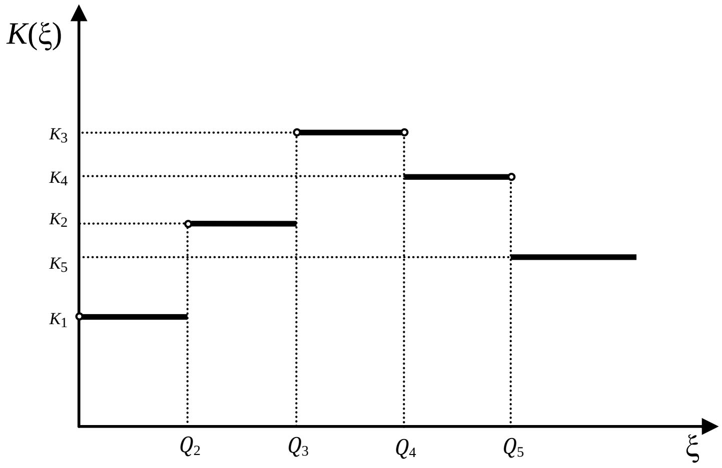

Figure 1.1: An example of the general setup cost $K(\xi)$ in (1.5) with $M = 5$.

(b) Investigate the optimal inventory control policy for the continuous-review inventory system with the general setup cost and without backlogs over the infinite horizon under the discounted cost criterion.

(c) Investigate the optimal inventory control policy for the continuous-review inventory system with the general setup cost and with backlogs allowed over the infinite horizon under the discounted cost criterion.

The main difference between He et al. (2015) and our thesis is that we discuss two discounted cost models. The technical proof under the discounted cost model is quite different from the proof under the average cost model adopted by He et al. (2015).

In this thesis, we prove the optimality of an (s, S) policy with $s = 0$ for the average-cost inventory model and the discounted-cost inventory model without backlogs by the lower-bound approach. A similar no-backlog assumption was made in Harrison et al. (1983) and Ormeci et al. (2008). In addition, we prove that the optimal policy for the discounted-cost inventory model with backlogs is of the (s, S) type if the initial inventory level is non-negative by the lower-bound approach.

In this thesis, we follow the lower-bound approach which was first proposed in Harrison et al. (1983), while there are new issues arising from our models. The lower-bound approach is a two-step self-contained method. In the first step, we establish a lower bound theorem, also known as a verification theorem, for all admissible policies. The lower bound theorem under the long-run average cost criterion, Theorem 3.2, shows that if a function f and a constant γ jointly satisfy several conditions, the constant γ will be a lower bound for the cost incurred by any admissible policy. The lower bound theorems under the discounted cost criterion, Theorems 4.2 and 5.2, state that if a function f satisfies several conditions, $f(x)$ will be the lower bound for the cost incurred by an arbitrary admissible policy with initial inventory level x. The lower bound under the long-run average cost criterion does not depend on the initial inventory level while the lower bound under the discounted cost criterion does. This is because the cost of the initial order will not influence the long-run average cost, but it will affect the discounted cost. We derive the lower bound theorems by Itô's formula as in Harrison et al. (1983), Ormeci et al. (2008) and Dai et al. (2013a,b). The Brownian inventory models in these papers allow both downward and upward adjustments, and thus the optimal policies are expected to be control band policies. Under control band policies, the controlled inventory level is restricted to a finite interval and the value function (or the relative value function under long-run average cost criterion) is Lipschitz continuous. With this fact, they can assume f to be Lipschitz continuous. This assumption allows them to prove the lower bound theorem by solely relying on Itô's formula. However only upward adjustments are allowed in our models. The optimal policies of our models are expected to be (s, S) policies under which the value function is not Lipschitz continuous. Without the Lipschitz assumption, Wu and Chao (2013) and Yao et al. (2015) establish the lower bound theorem for a subset, rather than for all admissible policies. Thus the proposed (s, S) policies in these papers are optimal among the same subset of admissible policies. To tackle this issue, He et al. (2015) propose a comparison theorem under the long-run average cost criterion. In this thesis, we prove the comparison theorem under the discounted cost criterion. Theorems 4.3 and 5.3 state that for any admissible policy, we can find an admissible policy with a finite order-up-to bound whose discounted cost is either less than or arbitrarily close to the discounted cost under the given policy. Namely, if a policy is optimal among policies with order-up-to bound, it must be optimal among all admissible policies. With an order-up-to bound, we do not require f to be Lipschitz continuous to prove the lower bound theorem by Itô's formula.

In the second step, we seek a policy whose discounted cost attains the lower bound. The main difference between the lower bound approach for inventory models with constant setup cost (Ormeci et al., 2008; Dai et al., 2013a,b) and the lower bound approach for inventory models with quantity-dependent cost in this thesis lies in this step. For Brownian inventory models with constant setup cost, the optimal (s, S) policies are obtained by imposing smoothness conditions at the the reorder level and the order-up-to level, known as smooth pasting conditions (Bather, 1966; Sulem, 1986; Bar-Ilan and Sulem, 1995). However for the our Brownian models with quantity-dependent setup cost, the difference between the reorder level and the order-up-to level is confined to an interval from Q_i to Q_{i+1} when the setup cost is K_i. With this constraint, the smooth pasting conditions may not hold at the optimal reorder level and the optimal order-up-to level. Instead of imposing the smooth pasting conditions, we construct a procedure to find the optimal policy parameters by examining the monotonicity of discounted cost, which is illustrated in Sections 3.5, 4.5 and 5.5 respectively.

The contributions of this paper are in two aspects. Firstly, we obtain optimal policies for the continuous-review inventory models with quantity-dependent setup cost structure under no-backlog cases and discounted cost criterion cases. We provide an explicit four-step procedure for calculating the optimal parameters. Although the obtained optimal policies are for continuous-review inventory models, these policies can serve as near-optimal solutions for the periodic-review models. Secondly, we provide comparison theorems under the discounted cost criterion. With this comparison theorem, we prove the optimality of (s, S) policies within a tractable subset of admissible policies instread of all admissible policies. The proof of the comparison theorem under the discounted cost criterion is different from the proof under the average cost criterion in He et al. (2015) because the time points of incurred costs do not affect the long-run average cost but the time points may affect the discounted cost. To prove the comparison theorem under the discounted cost criterion, we establish a new comparison scheme and let the cost difference go to zero by taking the up-to-bound to infinity.

1.3 Thesis Organization

This thesis is organized as follows. In Chapter 2, we will provide a review of the related literature. We explore an optimal inventory control policy that minimizes the long-run average cost under no-backlog scenario in Chapter 3, an optimal inventory control policy that minimizes the discounted cost under no-backlog scenario in Chapter 4 and an optimal inventory control policy that minimizes the discounted cost with backlogs and non-negative initial inventory level in Chapter 5.

Chapter 2

Literature Review

In this chapter, we provide a literature review of optimal policies for different inventory models. In Section 2.1, we review optimal policies for periodic-review inventory models with a constant setup cost. Section 2.2 provides a review of optimal policies for periodic-review models with a piecewise constant setup cost. Finally, we look into optimal policies for continuous-review inventory models with a constant setup cost in Section 2.3.

2.1 Periodic-Review Inventory Models with a Constant Setup Cost

The earliest studies of inventory policies for dynamic periodic-review inventory models can be traced back to Arrow et al. (1951), Arrow (1958) and Scarf (1960). Scarf (1960) is the first to prove that the optimal ordering policy for an n-period dynamic inventory system can be specified by n pairs $\{(s_i, S_i) : i = 1, ..., n\}$. Scarf (1960) assumes that an ordering cost consists of a constant setup cost and a proportional cost and that the holding and shortage cost is convex. The author introduces the concept of K-convexity to tackle dynamic inventory problems. This study is regarded as a milestone in the theory of inventory control. Based on the study of Scarf (1960), Iglehart (1963) proves the optimality of (s, S) policies for the infinite-horizon problem and Veinott (1966) solves

Scarf's problem under a weaker assumption about the one-period expected cost. Bensoussan et al. (1983) provide a rigorous formulation of the problem with non-stationary but stochastic independent demand.

Instead of assuming a linear ordering cost, Porteus (1971) examines a concave increasing ordering cost function in a dynamic periodic-review inventory model. Assuming that the demand of each period is an i.i.d. random variable that has a one-sided Polya-density, the author proves that a generalized (s, S) policy is optimal for the n-period problem by generalizing K-convex and quasi-convex functions to quasi-K-convex functions. Porteus (1972) extends the optimality of a generalized (s, S) policy to a dynamic periodic-review inventory model when the probability densities of demand are uniform or convolutions of a finite number of uniform and\or one-sided Polya densities. Lu and Song (2014) characterize the optimal policy for a periodic-review inventory model with an ordering cost, which consists of a fixed cost and a piecewise linear convex variable cost.

To tackle demand fluctuations caused by external factors such as business cycles and new substitutes, Sethi and Cheng (1997) consider an inventory model in which the distribution of demand is determined by a Markov chain. The authors consider a fixed ordering setup cost and demonstrate that (s, S) policies are optimal for this generalized model. Sethi and Cheng further incorporate some realistic constraints, such as no ordering periods and storage and service level constraints, and prove that the optimal policies for the extended models are of the (s, S) type. The Markovian demand and the constraints incorporated in the constrained inventory models bridge gaps between theoretical inventory models and practical inventory systems.

Instead of assuming a single class of demand, Veinott (1965) studies a multi-period inventory model in which there are several classes of demand for a single product in each period. The author proposes an optimal rationing policy with critical inventory levels that minimizes the discounted cost. Topkis (1968) considers a single-product inventory model with lost sales and n classes of stochastic demand of different importance. Topkis characterizes the optimal rationing policy that minimizes the expected costs, including holding costs, ordering costs, penalty costs for unsatisfied demand and salvage value. Evans (1968) extends the analysis to an inventory model with two different classes of demand. Kaplan (1969) and Frank et al. (2003) examine the rationing policy for periodic-review inventory models with two priority demand classes and backlogs. Hung et al. (2012) extend the analysis to inventory systems with multiple demand classes and backlogs.

In the classical inventory model in Scarf (1960), the author provides the optimal inventory policies when the lead time is constant. The constant lead time assumption is not always practical in the real world. Hadley and Whitin (1963) point out that it is of great difficulty to handle the stochastic lead time. A fundamental problem of the dynamic programming formulation for periodic-review inventory model with stochastic lead time is that a multi-dimensional state vector is required in order to keep track of outstanding orders. The curse of dimensionality is a serious issue when the state vector is in high dimensions. Instead of using a stochastic process to represent the random lead times (Arrow, 1958; Agin, 1966), Kaplan (1970) builds a probabilistic model for arrivals of outstanding orders. The author demonstrates how a multi-dimensional minimization problem can be reduced to an one-dimensional problem in a finite-horizon periodic-review inventory model. Ehrhardt (1984) extends a finite-horizon problem to an infinite-horizon problem and proves the optimality of (s, S) policies under the assumption that the distribution of lead time for a given order is independent of the number and sizes of outstanding orders. Furthermore, the author provides an efficient algorithm for calculating the parameters of the optimal (s, S) policy. Song and Zipkin (1996) incorporate a Markovian model of exogenous supply system in their periodic-review inventory model. In this model, the replenishment lead time evolves over time. The optimal policy is shown to be of the (s, S) type but the parameters of the optimal policy change dynamically. Muharremoglu and Yang (2010) extend the stochastic lead time inventory models by considering a broader class of lead time processes including all existing lead time models.

In a competitive retail environment, excess demand may be partially lost rather than completely backlogged. Gruen et al. (2002) examine customers response to stockouts across eight retail categories and show that only 15% of customers will wait for replenishment when they observe stockouts and the remaining 85% will choose to buy a substitute or do not buy any product. Thus lost-sales inventory models might be more practical than backlogged inventory models. The studies of periodic-review lost-sales inventory models can be traced back to Bellman et al. (1955). In this paper, the authors consider a special case when the lead time equals to an review period and only consider an ordering cost and a penalty cost. Karlin and Scarf (1958) extend Bellman's inventory model by incorporating a holding cost and positive lead time. Morton (1969) further extends Karlin's results and derives the bounds of the optimal policy. Under a similar inventory model, Zipkin (2008) derives new bounds of the optimal policy by transforming state variables. In addition, Zipkin (2008) extends the analysis to several significant variations of the basic inventory model, including inventory models with capacity constraints, stochastic lead time and multiple demand classes separately. The periodic-review inventory models in the above-mentioned papers all assume a zero setup cost. A comprehensive review of lost-sales inventory models is done by Bijvank and Vis (2011).

For inventory systems in the real world, there might be some constraints imposed by some endogenous or exogenous factors. Beesack (1967) consider a finite-horizon periodic-review dynamic inventory model with a stockout constraint. This stockout constraint states that the ratio of the expected amount of stockouts to the expected demand over the finite-horizon cannot exceed a predetermined fraction. The optimal control policy is obtained by dynamic programming combined with a Lagrange multiplier. Shaoxiang and Lambrecht (1996), Gallego and Scheller-Wolf (2000) and Shaoxiang (2004) examine periodic-review inventory models with a supply capacity contract and a fixed setup cost. Partial characterizations of the optimal policies are provided in these three papers.

2.2 Periodic-review Inventory Models with a Piecewise Constant Setup Cost

Instead of assuming a constant setup cost for any order quantity, many researchers consider quantity-dependent setup costs in periodic-review dynamic inventory models.

Setup cost structure in (1.1) is a reasonable setup cost function in many cases. For example, the transportation cost can be modeled as (1.1) when products are delivered to customers by vehicles. Lippman (1969) considers a single-product periodic-review dynamic inventory model with a nondereasing and subadditive setup cost shown in (1.1). Lippman obtains a two-parameter optimal policy for the one-period model, but he only provides a partial characterization of the optimal policy for the multi-period model. Under the same setup cost structure, Iwaniec (1979) specifies the conditions under which the optimal policy is a full-batch ordering policy. The full-batch ordering policy is first studied by Veinott (1965). Under these conditions, Iwaniec proves that the sequence of critical numbers associated with the optimal full-batch ordering policy is bounded and nondecreaasing. Furthermore, the author specifies the limiting value that characterizes the optimal full-batch ordering policy for the infinite-horizon problem. Alp et al. (2013) consider a similar inventory model but allow partial-batch ordering flexibility in their inventory model. By introducing an alternative cost-accounting scheme, the authors provide a complete characterization of the optimal policy for the one-period problem, but they partially characterize the optimal policy for the infinite-horizon problem and propose two computationally efficient heuristic policies. Alp et al. (2003) provide a generalization of this problem but with deterministic demand and stochastic lead time.

In some inventory systems, the setup cost is defined in a supply contract signed after negotiations between suppliers and customers. Henig et al. (1997) examine the inventory control policies for a periodic-review inventory model with an ordering cost that is zero if the order quantity below a predetermined contract volume and linear in the excess quantity portion. Chao and Zipkin (2008) consider a same model but with a setup cost shown in (1.2). This setup cost can be interpreted as the cost of disruption for the suppliers as well as additional administrative costs with extra order quantities. The optimal policy for this inventory model is partially characterized and an effective heuristic policy is proposed. Zhou et al. (2009) consider a setup cost structure, under which customers are offered free shipping if the order quantity exceeds a certain quantity specified by suppliers. This setup cost structure may appear in e-commerce in which there is intense competition among the companies. Zhou et al. (2009) characterize the structural properties of the optimal policy for the single-period problem and propose a heuristic policy for the multi-period problem. The heuristic policy is demonstrated to be close to the optimal policy.

In some practical periodic-review inventory systems, the incurred setup cost may be a function of the order quantity in the previous period. Toy and Berk (2006) show that a process can be kept warm till next production period if the previous order size exceeds a threshold and otherwise is cold. The warm process allows manufacturers to avoid the setup cost for the current period. A similar warm/cold setup cost structure in food and other industries is discussed in Robinson and Sahin (2001). Caliskan-Demirag et al. (2013) consider a periodic-review inventory model with stochastic demand and a setup cost

$$K(\xi) = K_0 u(\xi) u(R - q),$$

where ξ is the order quantity of current period, q is the order quantity of the previous period, R is a threshold value and $u(x) = 1$ if $x > 0$ and 0 otherwise. Assuming a proportional cost and a linear holding and shortage cost, the authors partially characterize the optimal control policy and propose two heuristic policies.

2.3 Continuous-review Inventory Models with a Constant Setup Cost

There are several reasons for examining continuous-review inventory models. Browne and Zipkin (1991) list several reasons from the perspective of inventory management. Zipkin (1986) points out that continuous-review models lead to easy calculation of key performance measures. He et al. (2015) point out that it is more tractable to solve a Brownian control problem than to solve a dynamic program for the periodic-review inventory problem under a quantity-dependent setup cost.

Brownian inventory models have received considerable attention in the literature. Brownian demand processes in continuous-time inventory models are analogous to i.i.d. normally distributed demands in periodic-review inventory models. Bather (1966) is the first to introduce the Brownian inventory models. Assuming a convex holding and shortage cost along with an ordering cost that consists of a proportional cost and a fixed setup cost, the author demonstrates that the optimal policy for the average-cost inventory model with zero lead time is of the (s, S) type. Based on this pioneering paper, a lot of studies have extended this fundamental Brownian inventory model to more general scenarios but all under a constant setup cost assumption except for He et al. (2015). Sulem (1986) discusses an inventory model with a piecewise linear holding and shortage cost and zero lead time under the discounted cost criterion. Benkherouf (2007) considers a similar inventory model with a generalized holding and shortage cost $h(\cdot)$ with four conditions. The optimal policies are proved to be of the (s, S) type for these two inventory models. Feng and Muthuraman (2010) provide a computational method for solving impulse control problems with zero lead time. Bensoussan et al. (2005) and Benkherouf and Bensoussan (2009) consider a model with a demand process as a mixture of a Brownian motion and a compound Poisson process and demonstrate the optimal policy is of the (s, S) type. Except Feng and Muthuraman (2010) utilizing a computational method, the optimal policies of the inventory models considered in above papers are obtained by solving a set of quasi-variational inequalities (QVIs) derived from the Bellman equation as well as the boundary conditions imposed on the reorder level and the order-up-to level. These boundary conditions are known as smooth pasting conditions. A comprehensive account of smooth pasting and its applications is provided by Dixit (1993).

Harrison et al. (1983) prove the optimality of a control band policy (q, Q, S) for an inventory system whose demand is a drifted Brownian motion. The discounted-cost inventory model discussed in this paper allows both upward adjustments and downward adjustments and does not allow backlogs. In this paper, the authors adopt a two-step lower bound approach to prove the optimality of the control band policy and to obtain the optimal parameters. Compared to the QVI method, the lower bound approach provides a self-contained proof and thus becomes a widely used method for solving Brownian control problems. The technique of smooth pasting is also used in the lower bound approach to obtain the optimal policy. By the lower bound approach, Ormeci et al. (2008) tackle the average cost case directly, without vanishing the discount rate as in the previous literature. The authors further extend the optimality of the control band policy for the bounded inventory level control problem and the constrained order quantity control problem with Lagrangian relaxation techniques. The costs considered in both papers include an ordering cost, consisting of a fixed setup cost and a proportional cost, and a linear holding and shortage cost. Dai et al. (2013a,b) extend the optimality of the control band policy to a similar Brownian inventory model but with a convex holding and shortage cost under the average cost criterion and the discounted cost criterion respectively. Assuming that only upward adjustments are allowed, Yao et al. (2015) prove the optimality of (s, S) policies for a Brownian inventory model with a concave ordering cost by the lower bound approach. Wu and Chao (2013) apply the lower bound approach to examine a Brownian inventory model with production capacity constraints and prove the optimality of (s, S) policies.

Studies of stochastic lead time in continuous-review inventory models can be traced back to Zipkin (1986). In this paper, Zipkin examines both stationary and limiting distributions of the inventory level and the inventory position under a stochastic lead time assumption with no order crossing. However, the the author does not specify the optimal policy. Johansen and Thorstenson (1993) consider an inventory model with lead times following a gamma distribution. Kulkarni and Yan (2012) examine an inventory system in which the demand rate is modeled as a finite state continuous time Markov chain and with stochastic lead time. The authors provide algorithms for computing the best (r, Q) policy. Song et al. (2010) consider a single-item (r, Q) inventory system with a compound-Poisson demand process. The authors demonstrate how the optimal policy parameters as well as the long-run average cost changes in response to the changes of the distribution of lead time. Muthuraman et al. (2014) prove the optimality of (s, S) policy and obtain the limiting distribution of the inventory level for a Brownian inventory model with stochastic lead time by the QVI method.

Chapter 3

Optimal Policy Under the Long-Run Average Cost Criterion Without Backlogs

In this chapter, we follow the two-step lower bound approach to obtain an optimal policy for a continuous-review inventory model without backlogs under the long-run average cost criterion. The optimal policy is an (s, S) policy with $s = 0$ and we use $(0, S)$ policy to represent it in this thesis. In particular, Section 3.1 presents the continuous-review inventory model and assumptions of cost functions. In Section 3.2, we present the main results of this chapter. In Section 3.3, we establish a lower bound for the long-run average cost incurred by any admissible policy. We compute the long-run average cost under a $(0, S)$ policy in Section 3.4. In Section 3.5, we demonstrate how to select the optimal $(0, S)$ policy and prove the optimality of it. In Section 3.6, we provide a numerical analysis of the optimal $(0, S)$ policy.

3.1 Model and Assumptions

Consider a continuous-review inventory system whose inventory level at time $t \geq 0$ is denoted by $Z(t)$. We assume that the inventory system does not allow backlogs, which implies that the inventory controller is obliged to maintain $Z(t) \geq 0$ for all $t \geq 0$. We further assume that all demands must be satisfied and the lead time for each order is zero. Let $D(t)$ and $Y(t)$ be the cumulative demand quantity and the cumulative order quantity during time $[0, t]$, respectively. In our Brownian control model, only upward adjustments are allowed. Then, the inventory level at time $t \geq 0$ is given by

$$Z(t) = x - D(t) + Y(t),$$

where x is a non-negative real number. We set $Z(0-) = x$ and interpret x as the initial inventory level. We assume that the cumulative demand quantity process $D = \{D(t) : t \geq 0\}$ is a Brownian motion starting form $D(0) = 0$ with positive drift $\mu > 0$ and variance σ^2. Namely, the process D can be represented as

$$D(t) = \mu t - \sigma B(t),$$

where $B = \{B(t) : t \geq 0\}$ is a standard Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = \{\mathcal{F}(t) : t \geq 0\}$. Then, by the definition of $D(t)$, the inventory level at time t is given by

$$Z(t) = X(t) + Y(t), \tag{3.1}$$

where

$$X(t) = x - \mu t + \sigma B(t) \tag{3.2}$$

can be interpreted as the inventory level in the absence of control and it is a Brownian motion starting from $X(0) = x$ with drift $-\mu$ and variance σ^2.

An inventory order policy is specified by the cumulative order quantity process $Y = \{Y(t) : t \geq 0\}$. An ordering policy is said to be *admissible* if the ordering policy Y satisfies:

(i) for each sample path ω, $Y(\omega, \cdot)$ is a nondecreasing function that is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$;

(ii) $Y(t) \geq 0$ for all $t \geq 0$;

(iii) Process Y is adapted to the filtration $\mathbb{F}$, namely $Y(t)$ is $\mathcal{F}(t)$ measurable for all $t \geq 0$.

Let $\mathcal{U}$ denote the set of all admissible policies. The controller of the inventory system can control the inventory level by replenishing the inventory by any admissible policy $Y \in \mathcal{U}$. For any $t \geq 0$, $Y(t-)$ is the left limit at time t and let $Y(0-) = 0$ by convention. Then the controlled inventory level $Z(t)$ is right continuous on $[0, \infty)$ and has left limits on $(0, \infty)$. At any time $t \geq 0$, the controller can replenish the inventory by any desired amount, but the controller is obliged to keep $Z(t) \geq 0$ for all $t \geq 0$. This obligation implies that backlogs are not allowed in the inventory. For any admissible control policy Y, a time point t is said to be an *order time* if $\Delta Y(t) := Y(t) - Y(t-) > 0$. Let $N(t)$ be the cardinality of the set

$$\{u \in [0, t] : \Delta Y(u) = Y(u) - Y(u-) > 0\}.$$

We allow an order at time $t = 0$. With initial inventory level $Z(0-) = x$, the inventory level after the initial order can be represented by $Z(0) = x + Y(0)$, where $Y(0)$ can be positive or zero.

Two types of costs are incurred in this continuous-review inventory system: the inventory holding cost and the ordering cost. The inventory holding cost is incurred at a rate $h(z)$ when the inventory level is z. Since the controller is required to keep $Z(t) \geq 0$ for $t \geq 0$, h is defined on $[0, \infty)$. In the inventory model of this chapter, we assume the inventory holding cost rate function satisfies the following assumption.

Assumption 1. $h(\cdot)$ satisfies

(A1) $h(\cdot)$ is convex and $h(0) = 0$;

(A2) $h(\cdot)$ is continuously differentiable on $[0, \infty)$;

(A3) $h'(z) > 0$ for $z \geq 0$;

(A4) $h(\cdot)$ is polynomially bounded, i.e., there exist positive constants $a_i > 0$, $i = 0, 1$ and a positive integer n such that $h(z) \leq a_0 + a_1 z^n$ for all $z \geq 0$.

An ordering cost is incurred whenever an order is placed and this cost is a function of the order quantity $\xi > 0$. When an order with quantity $\xi > 0$ is placed, it incurs a *setup cost* of $K(\xi) > 0$ and a *proportional cost* of $k\xi$ with proportional cost rate $k > 0$. Let $\phi(\xi)$ denote the ordering cost with order quantity ξ. Then $\phi(\xi)$ is given by

$$\phi(\xi) = K(\xi) + k \cdot \xi. \tag{3.3}$$

Since $K(\xi) > 0$ for $\xi > 0$, we only need to consider policies with $N(t) < \infty$ for any $t > 0$. Otherwise, the total cost would be infinite during the time interval $[0, t]$. Namely, we only need to consider the policies that order finite times in any finite time interval, which implies that $Y(t)$ is piecewise constant on each sample path. Such a policy can be specified by a sequence of pairs $\{(T_i, \xi_i) : i = 0, 1, \ldots\}$ where T_i is the ith order time and $\xi_i = \Delta Y(T_i) = Y(T_i) - Y(T_i-)$ is the quantity of the ith order. By convention, we set $T_0 = 0$ and let ξ_0 be the quantity of the order placed at time zero. Put $\xi_0 = 0$ if no order is placed. Then, the ordering process $Y(t)$ can be represented as

$$Y(t) = \sum_{i=0}^{N(t)} \xi_i.$$

Therefore, investigating an optimal control policy given that $K(\xi) > 0$ for $\xi > 0$ is equivalent to exploring a sequence of order times together with corresponding ordering quantities $\{(T_i, \xi_i) : i = 0, 1, \ldots\}$, which turns out to be an *impulse control* problem for the Brownian model.

We consider the setup cost function defined in (1.5), i.e.,

$$K(\xi) = \sum_{i=1}^{M} K_i \cdot 1_{\{\xi \in (Q_i, Q_{i+1})\}} + \sum_{i=1}^{M-1} (K_i \wedge K_{i+1}) \cdot 1_{\{\xi = Q_{i+1}\}},$$

where

$$0 = Q_1 < Q_2 < \cdots < Q_{M-1} < Q_M < Q_{M+1} = \infty,$$

$$K_i > 0 \quad \text{for } i = 1, 2, \ldots, M,$$

$$K_i \neq K_{i+1} \quad \text{for } i = 1, 2, \ldots, M-1.$$

Our objective is to find an admissible inventory policy Y that minimizes the long-run average cost

$$AC(x, Y) = \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}_x\Big[\int_0^t h(Z(u))du + \sum_{i=0}^{N(t)}(K(\xi_i) + k\cdot\xi_i)\Big], \tag{3.4}$$

where $\mathbb{E}_x$ is the expectation operator conditioning on the initial inventory level $Z(0-) = x$.

3.2 Main Results

In this section, we present the main results of this chapter. Under the quantity-dependent setup cost defined in (1.5), an optimal policy for the Brownian inventory model in this chapter is a $(0, S)$ policy with $S > 0$. We propose an algorithm for computing the optimal order-up-to level.

We use $U(0, S)$ to denote such a $(0, S)$ policy. A $(0, S)$ policy can be specified by the sequence of pairs $\{(T_i, \xi_i) : i = 0, 1, \ldots\}$ as follows. With $T_0 = 0$, the ith order is placed at time $T_i = \inf\{t > T_{i-1} : Z(t-) = 0\}$ for $i = 1, 2, \ldots$ and the quantity of the ith order is given by $\xi_i = S$ for $i = 1, 2, \ldots$ and

$$\xi_0 = \begin{cases} S & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases}$$

The optimal order-up-to level S^* can be obtained by the following algorithm.

Step 1. For $z \geq 0$ and $A \in \mathbb{R}$, let $\lambda = \frac{2\mu}{\sigma^2}$ and

$$g_A(z) = A + \frac{\lambda}{\mu} \int_z^\infty h(y) e^{\lambda(z-y)} dy. \tag{3.5}$$

For $n = 1, \ldots, M$, obtain $\hat{A}_n$ and $\hat{S}_n > 0$ by solving

$$\int_0^{\hat{S}_n} [g_{\hat{A}_n}(y) + k] dy = -K_n, \tag{3.6}$$

and

$$g_{\hat{A}_n}(\hat{S}_n) = -k. \tag{3.7}$$

Step 2. Define

$$\mathcal{N}_< = \{n \in \{1, 2, \ldots, M\} : \hat{S}_n \leq Q_n\},$$
$$\mathcal{N}_= = \{n \in \{1, 2, \ldots, M\} : \hat{S}_n \in (Q_n, Q_{n+1})\},$$
$$\mathcal{N}_> = \{n \in \{1, 2, \ldots, M\} : \hat{S}_n \geq Q_{n+1}\},$$

and

$$S_n^* = \begin{cases} Q_n & \text{for } n \in \mathcal{N}_<, \\ \hat{S}_n & \text{for } n \in \mathcal{N}_=, \\ Q_{n+1} & \text{for } n \in \mathcal{N}_>. \end{cases} \tag{3.8}$$

Step 3. For $n \in \mathcal{N}_=$, let

$$A_n^* = \hat{A}_n.$$

For $n \in \mathcal{N}_> \cup \mathcal{N}_<$, obtain A_n^* by solving

$$\int_0^{S_n^*} [g_{A_n^*}(y) + k] dy = -K_n.$$

Step 4. Let $\mathcal{M} = \{1, \ldots, M\}$. Define

$$n^* = \min\{n \in \mathcal{M} : A_i^* \leq A_n^* \text{ for all } i \in \mathcal{M}\}. \tag{3.9}$$

Let

$$S^* = S_{n^*}^*, \quad A^* = A_{n^*}^* \quad \text{and} \quad K^* = K_{n^*}. \tag{3.10}$$

In Step 1, we obtain the parameter $\hat{S}_n$ associated with $\hat{A}_n$ by smoothness conditions similar with those in Bather (1966). This $(0, \hat{S}_n)$ policy is expected to be optimal when the setup cost is constant at K_n. However under the quantity-dependent setup cost, a quantity constraint is imposed on each setup cost value. Thus in Step 2 we obtain S_n^*, which is confined within the corresponding interval. We obtain the auxiliary parameter A_n^* in Step 3. Finally in Step 4, we select the optimal S^* by picking the largest A_n^*. Please see Section 3.5 for explicit derivations.

Then $(0, S^*)$ policy from the above algorithm is an optimal policy for our Brownian inventory model. The optimality of this $(0, S^*)$ policy is proved by the following theorem.

Theorem 3.1. *Assume the cost function h satisfies Assumption 1 and that the setup cost is given by* (1.5). *Control policy $U(0, S^*)$ obtained by Steps 1–4 is an optimal admissible policy that minimizes the long-run average cost given by* (3.4). *Namely, we have*

$$AC(x, U(0, S^*)) \leq AC(x, Y) \quad \text{for } x \geq 0 \text{ and } Y \in \mathcal{U}.$$

Moreover, the minimum long-run average cost is $AC(x, U(0, S^*)) = -\mu A^*$.

We will prove Theorem 3.1 in Section 3.5.2.

3.3 Lower Bound Under the Long-Run Average Cost Criterion

In this section, we propose and prove a theorem that establishes a lower bound for the long-run average cost incurred by any admissible control policy.

Theorem 3.2. *Assume that h satisfies Assumption 1. Let $f(\cdot) : [0, \infty) \to \mathbb{R}$ be twice continuously differentiable. Assume that there exists a positive real number γ such that*

$$\Gamma f(z) + h(z) \geq \gamma \quad \textit{for all } z \geq 0, \tag{3.11}$$

where

$$\Gamma f(z) = \frac{1}{2}\sigma^2 f''(z) - \mu f'(z).$$

We further assume that

$$f(z_1) - f(z_2) \leq K(z_2 - z_1) + k \cdot (z_2 - z_1) \quad \textit{for all } 0 \leq z_1 < z_2, \tag{3.12}$$

and $f'(\cdot)$ is polynomially bounded, i.e., there exist positive constants a_0, a_1 and a positive integer n such that

$$|f'(z)| < a_0 + a_1 z^n \quad \textit{for all } z \geq 0. \tag{3.13}$$

Then

$$AC(x, Y) \geq \gamma, \tag{3.14}$$

where $AC(x, Y)$, given by (3.4)*, is the long-run average cost under any admissible control policy Y .*

By Theorem 3.2, if we can find an admissible ordering policy whose long-run average cost γ satisfies the conditions in Theorem 3.2, we can conclude that this admissible control policy must be optimal among all admissible policies. In the lower bound approach, Theorem 3.2 is referred to as a verification theorem. In order to prove Theorem 3.2, we first need to show some technical results.

The following comparison theorem is a critical result for proving Theorem 3.2. It implies that a policy that is optimal among the policies subject to order-up-to bounds must be optimal among all admissible policies. Compared with the general admissible policies, policies subject to order-up-to bounds are analytically tractable.

For $m = 1, 2, \ldots$, let

$$\mathcal{U}_m = \{Y \in \mathcal{U} : Z(t) \leq m \text{ for all order time } t\}$$

represent the set of admissible policies with an order-up-to bound at m.

Theorem 3.3 (**Comparison Theorem**). *Assume that the holding cost rate function h is nondecreasing on $[0, \infty)$ and the setup cost function $K(\cdot)$ is bounded. Then for any admissible policy Y, there exists a sequence of admissible policies $\{Y_m \in \mathcal{U}_m : m = 1, 2, \ldots\}$ such that*

$$\lim_{m \to \infty} AC(x, Y_m) \leq AC(x, Y). \tag{3.15}$$

The Theorem 3.3 can be derived from Theorem 2 in He et al. (2015). Let $\overline{\mathcal{U}}$ be the set of all admissible policies subject to order-up-to bounds, namely,

$$\bar{\mathcal{U}} = \bigcup_{m=1}^{\infty} \mathcal{U}_m.$$

This comparison theorem implies that a policy that is optimal in $\overline{\mathcal{U}}$ must be optimal in $\mathcal{U}$. Therefore, we only need to search all admissible policies subject to order-up-to bounds for the optimal policy. Moreover, it is more convenient to prove the optimality among the admissible policies subject to order-up-to bounds.

The following lemma provides three important results that are important for proving Theorem 3.2.

Lemma 3.1. *Let $f(\cdot) : \mathbb{R} \to \mathbb{R}$ be a differentiable function and Z be the inventory process given in* (3.1) *with $Y \in \overline{\mathcal{U}}$. Assume there exist positive constants a_0, a_1 and a positive integer n such that*

$$|f'(z)| < a_0 + a_1 z^n \quad \textit{for all } z \geq 0.$$

Then,

$$\mathbb{E}_x[|f(Z(t))|] < \infty \quad \textit{for } t \geq 0, \tag{3.16}$$

and

$$\mathbb{E}_x\Big[\int_0^t (f'(Z(u)))^2 du\Big] < \infty \quad \textit{for } t \geq 0. \tag{3.17}$$

Moreover,

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}_x[f(Z(t))] = 0. \tag{3.18}$$

This lemma can be derived from Lemma 3 in He et al. (2015) directly.

Proof of Theorem 3.2. By Theorem 3.3, it suffices to consider an arbitrary policy $Y \in \overline{\mathcal{U}}$, namely it suffices to consider $Y \in \mathcal{U}_m$ for a fixed positive integer m. For any $Y \in \mathcal{U}_m$, recall that

$$Z(t) = x - \mu t + \sigma B(t) + Y(t).$$

By Itô's formula , we have

$$\begin{aligned} f(Z(t)) &= f(Z(0)) + \int_0^t \Gamma f(Z(u))du + \sigma \int_0^t f'(Z(u))dB(u) + \sum_{0<u\leq t} \Delta f(Z(u)) \\ &= f(Z(0-)) + \int_0^t \Gamma f(Z(u))du + \sigma \int_0^t f'(Z(u))dB(u) + \sum_{0\leq u\leq t} \Delta f(Z(u)) \end{aligned}$$

$$\begin{aligned}
&\geq f(Z(0-)) + \gamma t - \int_0^t h(Z(u))du + \sigma \int_0^t f'(Z(u))dB(u) + \sum_{0\leq u\leq t} \Delta f(Z(u)) \\
&\geq f(Z(0-)) + \gamma t - \int_0^t h(Z(u))du \\
&\quad +\sigma \int_0^t f'(Z(u))dB(u) - \sum_{i=0}^{N(t)} \phi(\xi_i), \qquad (3.19)
\end{aligned}$$

where the first inequality follows from (3.11), the second inequality results from (3.12) and ϕ is defined in (3.3). By (3.17) and Theorem 3.2.1 in Øksendal (2003), we have

$$\mathbb{E}_x\big[\int_0^t f'(Z(u))dB(u)\big] = 0.$$

Because (3.16) holds, we can take expectation on both sides of (3.19) and obtain

$$\mathbb{E}_x[f(Z(t))] \geq f(x) + \gamma t - \mathbb{E}_x[\int_0^t h(Z(u))du] - \mathbb{E}_x[\sum_{i=0}^{N(t)} \phi(\xi_i)]. \qquad (3.20)$$

Dividing both side by t and taking limit as $t \to \infty$ of both sides of (3.20), we have

$$AC(x, Y) + \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_x[f(Z(t))] \geq \gamma.$$

Then by (3.18), the conclusion $AC(x, Y) \geq \gamma$ follows. □

3.4 Long-Run Average Cost Under $(0, S)$ Policies

In this section, we will compute the long-run average cost under any $(0, S)$ policy, the policy with a reorder level zero and an order-up-to level S. By the definition of admissible policies, all $(0, S)$ policies are in $\bar{\mathcal{U}}$, the set of admissible policies subject to order-up-to bounds.

Theorem 3.4. *For any $S > 0$, if there exists a twice continuously differentiable function $V(\cdot) : [0, \infty) \to \mathbb{R}$ and a positive number γ such that*

$$\Gamma V(z) + h(z) = \gamma \text{ for } z \geq 0, \qquad (3.21)$$

$$V(S) - V(0) = -K(S) - k \cdot S, \tag{3.22}$$

$$V' \textit{ is polynomially bounded.} \tag{3.23}$$

Then the long-run average cost under $U(0, S)$ is given by

$$AC(x, U(0, S)) = \gamma. \tag{3.24}$$

Proof. By the Itô's formula together with (3.21), we have

$$\begin{aligned} V(Z(t)) &= V(Z(0-)) + \gamma t - \int_0^t h(Z(u))du + \sigma \int_0^t V'(Z(u))dB(u) \\ &\quad - \sum_{0 \le u \le t} \Delta V(Z(u)). \end{aligned} \tag{3.25}$$

Under the $(0, S)$ policy with $S > 0$, it follows from (3.22) that $\Delta V(Z(u)) = -K(S) - k \cdot S$ whenever $\Delta Z(u) > 0$ and $u \ge 0$. Then (3.25) turns out to be

$$V(Z(t)) = V(Z(0-)) + \gamma t - \int_0^t h(Z(u))du + \sigma \int_0^t V'(Z(u))dB(u) - \sum_{i=0}^{N(t)} \phi(\xi_i),$$

where ϕ is given by (3.3) and $\xi_i = S$ for $i = 1, 2, \ldots$ and

$$\xi_0 = \begin{cases} S & \text{if } x = 0, \\ 0 & \text{if } x > 0. \end{cases}$$

By (3.17) and Theorem 3.2.1 in Øksendal (2003), we have $\mathbb{E}_x[\int_0^t V'(Z(u))dB(u)] = 0$. Since (3.16) holds, we can take expectation of the above equation and by dividing t of both sides and taking limit as $t \to \infty$, we can obtain

$$\lim_{t\to\infty} \frac{1}{t} E_x[V(Z(t))] + AC(x, U(0, S)) = \gamma.$$

By (3.18) and (3.23), we can conclude that

$$AC(x, U(0, S)) = \gamma.$$

□

In the following proposition, we provide the solution to equations (3.21)–(3.23).

Proposition 3.1. *The solution to* (3.21)–(3.23) *is given by*

$$\gamma = \frac{1}{S}\Big[\mu K(S) + \mu k \cdot S + \lambda \int_0^S \int_z^\infty h(y)e^{\lambda(z-y)}dydz\Big], \tag{3.26}$$

and

$$V(z) = \int_0^z g(y)dy \quad \textit{for } z \geq 0, \tag{3.27}$$

with

$$g(z) = -\frac{\gamma}{\mu} + \frac{\lambda}{\mu}\int_z^\infty h(y)e^{\lambda(z-y)}dy, \tag{3.28}$$

where $\lambda = \frac{2\mu}{\sigma^2}$*. The solution* $V(z)$ *given by* (3.27) *together with* (3.28) *is unique up to addition by a constant.*

Proof. The ordinary differential equation (3.21) can be rewritten as

$$(e^{-\lambda z}g(z))' = \frac{2}{\sigma^2}(\gamma - h(z))e^{-\lambda z}.$$

By integrating both sides over the interval $[0, z]$ and dividing both sides by $e^{-\lambda z}$, we have

$$\begin{aligned} g(z) &= g(0)e^{\lambda z} + \frac{2\gamma}{\sigma^2}\int_0^z e^{\lambda(z-y)}dy - \frac{2}{\sigma^2}\int_0^z h(y)e^{\lambda(z-y)}dy \\ &= g(0)e^{\lambda z} + \frac{\gamma\lambda}{\mu}\int_0^z e^{\lambda(z-y)}dy - \frac{\lambda}{\mu}\int_0^z h(y)e^{\lambda(z-y)}dy. \end{aligned}$$

By (3.23), we have

$$\lim_{z\to\infty}\frac{g(z)}{e^{\lambda z}} = 0,$$

from which we can infer

$$g(0) = \frac{\lambda}{\mu}\int_0^\infty h(y)e^{-\lambda y}dy - \frac{\gamma\lambda}{\mu}\int_0^\infty e^{-\lambda y}dy.$$

Then, we can rewrite $g(z)$ as follows

$$g(z) = -\frac{\gamma}{\mu} + \frac{\lambda}{\mu}\int_z^\infty h(y)e^{\lambda(z-y)}dy.$$

Then the expression of γ in (3.26) can be derived from the equation (3.22) directly. $\square$

Remark 3.1. The relative value funtion $V(z)$ can be interpreted as the cost disadvantage of inventory level z relative to the reorder level 0. By the expression of $g(z)$ in (3.28) and the relationship between $g(z)$ and $V(z)$ in (3.27), we can obtain the expression for $V(z)$ as follows

$$V(z) = -\frac{\gamma z}{\mu} + \frac{\lambda}{\mu}\int_0^z \int_0^\infty h(u+y)e^{-\lambda u}dudy. \tag{3.29}$$

For $x \geq y$, define

$$\tau(x,y) = \inf\{t \geq 0 : X(0) = x, X(t) = y\},$$
$$W(x,y) = \mathbb{E}_x\big[\int_0^{\tau(x,y)} h(X(u))du\big].$$

Under a $(0, S)$ policy, $\tau(z, 0)$ defined above can be interpreted as the first order time, and $W(z, 0)$ can be interpreted as the expected holding and shortage cost during $[0, \tau(z, 0)]$. By Section 15.3 in Karlin and Taylor (1981), we have

$$W(z,0) = \frac{\lambda}{\mu}\int_0^z \int_0^\infty h(u+y)e^{-\lambda u}dudy.$$

By Theorem 5.32 in Serfozo (2009),

$$\tau(z,0) = \frac{z}{\mu}.$$

Then $V(z)$ can be rewritten as

$$V(z) = W(z, 0) - \gamma \mathbb{E}[\tau(z, 0)].$$

$W(z, 0)$ is the cost disadvantage of a system starting from time zero with initial inventory level $Z(0-) = z$ compared to a system starting from time $\tau(z, 0)$ with initial inventory level $Z(\tau(z, 0)-) = 0$. $\gamma \mathbb{E}[\tau(z, 0)]$ can be interpreted as the cost disadvantage of a system starting from time zero compared to a system starting from time $\tau(z, 0)$ with the same initial inventory level. Therefore, $V(z)$, as the difference of these two costs can be interpreted as the relative cost disadvantage of inventory level z compared to the reorder level zero.

3.5 Optimal Policy

For $z \geq 0$, let

$$g_0(z) = \frac{\lambda}{\mu} \int_z^{\infty} h(y) e^{\lambda(z-y)} dy. \tag{3.30}$$

By (3.5), we have $g_A(z) = g_0(z) + A$. Let $V_A(z) = \int_0^z g_A(y) dy$ for $z \geq 0$, where $g_A(z)$ is given by (3.5). By Proposition 3.1, $V_A'(z)$ is polynomially bounded on $[0, \infty)$. Furthermore, $V_A(z)$ satisfies the differential equation

$$\Gamma V_A(z) + h(z) = -\mu A.$$

For any $(0, S)$ policy, if $V_A(z)$ further satisfies

$$V_A(S) - V_A(0) = -K(S) - k \cdot S, \tag{3.31}$$

we can conclude by Theorem 3.4 that the long-run average cost under a $(0, S)$ policy is given by

$$AC(x, U(0, S)) = -\mu A. \tag{3.32}$$

This conclusion implies that in order to minimize the long-run average cost under a $(0, S)$ policy, we should maximize the value of A without violating (3.31).

In the rest of this chapter, we need the properties of $g_A(z)$ with respect to the auxiliary variable A for fixed $z \geq 0$. To make the notation clear, for fixed $z \geq 0$, let

$$\varphi_z(A) = g_A(z).$$

Then $\varphi_z'(A) = 1$.

In Section 3.5.1, we demonstrate how the four-step algorithm in Section 3.2 attains the maximum value of A by selecting the $(0, S^*)$ policy, the best policy among $(0, S)$ policies. In Section 3.5.2, we show that under this $(0, S^*)$ policy, the corresponding relative value function and the corresponding long-run average cost jointly satisfy the conditions specified in the lower bound theorem. Thus, the $(0, S^*)$ policy is an optimal policy for our Brownian inventory model in this chapter.

3.5.1 The Optimal $(0, S)$ Policy

In this subsection, we first show the monotonicity of $g_A(z)$ in Lemma 3.2, which is an important lemma for proving subsequent lemmas. Then we identify a set of $(0, S)$ policies $\{U(0, \hat{S}_n) : n = 1, \ldots, M\}$ by Lemma 3.3. However under the quantity-dependent setup cost, the obtained $\hat{S}_n$ may not fall into an interval from Q_n to Q_{n+1}. Therefore we obtain a set of modified $(0, S)$ policies $\{U(0, S_n^*) : n = 1, \ldots, M\}$ by the process shown in (3.8). Lemma 3.4 proves the existence of A_n^* together with S_n^* such that they jointly satisfy (3.42), which is equivalent to (3.31). Finally, we select the optimal $(0, S^*)$ policy out of the set $\{U(0, S_n^*) : n = 1, \ldots, M\}$ by (3.9) and (3.10).

First of all, we prove the monotonicity of $g_A(z)$ which in defined in (3.5).

Lemma 3.2. *Assume that h satisfies Assumption 1. Then for any $A \in \mathbb{R}$, $g_A(z)$ is strictly increasing in $z \in [0, \infty)$. Furthermore,*

$$\lim_{z \to \infty} g_A(z) = \infty. \tag{3.33}$$

Proof. It suffices to show $g'_A(z)$ is always positive for $A \in \mathbb{R}$ and $z \geq 0$. By taking derivative of $g_A(z)$ defined in (3.5), we obtain

$$\begin{aligned} g'_A(z) = g'_0(z) &= \frac{\lambda}{\mu}\Big[\lambda e^{\lambda z} \int_z^\infty h(y) e^{-\lambda y} dy - h(z)\Big] \\ &= -\frac{\lambda}{\mu}\Big[e^{\lambda z} \int_z^\infty h(y) d(e^{-\lambda y}) + h(z)\Big] \\ &= \frac{\lambda}{\mu} \int_z^\infty h'(y) e^{-\lambda(z-y)} dy \\ &> 0, \end{aligned}$$

where the inequality follows from (A3) in Assumption 1.

Next we prove (3.33). It suffices to show $\lim_{z \to \infty} g_0(z) = \infty$. By (3.30),

$$\begin{aligned} \lim_{z \to \infty} g_0(z) &= \lim_{z \to \infty} \frac{\lambda \int_z^\infty h(y) e^{-\lambda y} dy}{\mu e^{-\lambda z}} \\ &= \lim_{z \to \infty} \frac{-\lambda h(z) e^{-\lambda z}}{-\mu \lambda e^{-\lambda z}} \\ &= \lim_{z \to \infty} \frac{h(z)}{\mu} \\ &= \infty, \end{aligned}$$

where the second equality follows from L'Hôpital's Rule and the last equality follows from (A1)–(A3) in Assumption 1. □

By the following lemma, we identify a set of $(0, S)$ policies $\{U(0, \hat{S}_n) : n = 1, \ldots, M\}$.

Lemma 3.3. *Assume that h satisfies Assumption 1. For an arbitrary $\kappa > 0$, there exist a unique pair $(\hat{S}(\kappa), \hat{A}(\kappa))$ with $\hat{S}(\kappa) > 0$ such that*

$$\int_0^{\hat{S}(\kappa)} [g_{\hat{A}(\kappa)}(y) + k]dy = -\kappa, \tag{3.34}$$

$$g_{\hat{A}(\kappa)}(\hat{S}(\kappa)) = -k. \tag{3.35}$$

For any $0 < \kappa_i < \kappa_j$, the corresponding $(\hat{S}(\kappa_i), \hat{A}(\kappa_i))$ and $(\hat{S}(\kappa_j), \hat{A}(\kappa_j))$ satisfy

$$\hat{S}(\kappa_i) < \hat{S}(\kappa_j) \quad \textit{and} \quad \hat{A}(\kappa_i) > \hat{A}(\kappa_j). \tag{3.36}$$

Proof. Firstly, we show that there exists an $\overline{A} \in \mathbb{R}$ such that $g_A(0) < -k$ for $A \in (-\infty, \overline{A})$. Define $\overline{A}$ as

$$\overline{A} = -k - g_0(0) = -k - \frac{\lambda}{\mu}\int_0^\infty h(y)e^{-\lambda y}dy.$$

By (A4) in Assumption 1, $g_0(0)$ is bounded, which further implies that $\overline{A}$ is bounded. Therefore, $g_{\overline{A}}(0) = \overline{A} + g_0(0) = -k$. Then for any $A \in (-\infty, \overline{A})$, we have $g_A(0) < -k$. By Lemma 3.2, if $A \in [\overline{A}, \infty)$, $g_A(z) > -k$ for $z \in (0, \infty)$. Thus we only need to consider when $A \in (-\infty, \overline{A})$. Since (3.33) holds and $g_A(z)$ is strictly increasing in $z \in [0, \infty)$, there exists a unique $S(A) > 0$ such that $g_A(S(A)) = -k$ for any $A \in (-\infty, \overline{A})$. Furthermore, by the Implicit Function Theorem, $S(A)$ is a continuous function of A and the derivative $S'(A)$ exists, Define

$$\Lambda(A) = \int_0^{S(A)} [g_A(y) + k]dy. \tag{3.37}$$

Next, we prove that for any $\kappa > 0$, there exists a unique $\hat{A}(\kappa)$ such that $\Lambda(\hat{A}(\kappa)) = -\kappa$. To show the existence and uniqueness of $\hat{A}(\kappa)$, it suffices to show that

$$\Lambda'(A) > 0 \quad \text{for } A \in (-\infty, \overline{A}), \tag{3.38}$$

$$\lim_{A \to \overline{A}} \Lambda(A) = 0, \tag{3.39}$$

$$\lim_{A\to-\infty}\Lambda(A) = -\infty. \tag{3.40}$$

Taking derivative of (3.37) with respect to A,

$$\begin{aligned}
\Lambda'(A) &= \int_0^{S(A)} \varphi_z'(A)dy + [g_A(S(A)) + k]S'(A) \\
&= \int_0^{S(A)} 1dy \\
&= S(A) > 0 \quad \text{for } A \in (-\infty, \overline{A}).
\end{aligned} \tag{3.41}$$

Next we show (3.39). Since $g_A(z)$ is strictly increasing in z and $g_{\overline{A}}(0) = -k$, we can deduce that $\lim_{A\to\overline{A}} S(A) = 0$, which implies $\lim_{A\to\overline{A}} \Lambda(A) = 0$.

It remains to prove (3.40). Taking derivative of the equation $g_A(S(A)) = -k$, i.e.,

$$A + \frac{\lambda}{\mu}\int_{S(A)}^{\infty} h(y)e^{\lambda(S(A)-y)}dy = -k,$$

with respect to A for $A \in (-\infty, \overline{A})$, we have

$$S'(A) = -\frac{1}{g_A'(S(A))} < 0,$$

where the inequality follows from Lemma 3.2. Together with that $S(A) > 0$ for $A \in (-\infty, \overline{A})$, we have $\lim_{A\to-\infty} S(A) > 0$. Then (3.40) follows from (3.41) and $\lim_{A\to-\infty}\Lambda'(A) = \lim_{A\to-\infty} S(A) > 0$.

It remains to prove (3.36). By (3.34), (3.35) and the Implicit Function Theorem, the derivatives $\hat{A}'(\kappa)$ and $\hat{S}'(\kappa)$ exist. Thus to prove (3.36), it suffices to show

$$\hat{A}'(\kappa) < 0 \quad and \quad \hat{S}'(\kappa) > 0.$$

Taking derivative of (3.34) and (3.35) with respect to κ, we can obtain

$$[g_{\hat{A}(\kappa)}(\hat{S}(\kappa)) + k]\hat{S}'(\kappa) + \int_0^{\hat{S}(\kappa)} [\varphi_y'(\hat{A}(\kappa)) \cdot \hat{A}'(\kappa)]dy = -1,$$

and

$$\varphi'_{\hat{S}(\kappa)}(\hat{A}(\kappa)) \cdot \hat{A}'(\kappa) + g'_{\hat{A}(\kappa)}(\hat{S}(\kappa)) \cdot \hat{S}'(\kappa) = 0.$$

By equation (3.35) and $\varphi'_z(A) = 1$ for $z \in [0, \infty)$, we can conclude that

$$\hat{A}'(\kappa) = -\frac{1}{\hat{S}(\kappa)} < 0 \quad \text{and} \quad \hat{S}'(\kappa) = -\frac{\hat{A}'(\kappa)}{g'_{\hat{A}(\kappa)}(\hat{S}(\kappa))} > 0,$$

where the first inequality follows from $\hat{S}(\kappa) > 0$ and the second inequality follows from Lemma 3.2.

□

Then if $\kappa = K_n$ where $n = 1, \ldots, M$ and let $(\hat{S}_n, \hat{A}_n)$ denote the pair $(\hat{S}(K_n), \hat{A}(K_n))$, the conditions (3.34) and (3.35) are equivalent to conditions (3.6) and (3.7) in Step 1. Then for K_n where $n = 1, \ldots, M$, we obtain a set of $(0, S)$ policies $\{U(0, \hat{S}_n) : n = 1, \ldots, M\}$. When the setup cost is K_n, the quantity of an order is constrained to an interval from Q_n to Q_{n+1} (which by (1.5) might be (Q_n, Q_{n+1}), $(Q_n, Q_{n+1}]$, $[Q_n, Q_{n+1})$ or $[Q_n, Q_{n+1}]$). However, the obtained $\hat{S}_n$ might not fall into an interval from Q_n to Q_{n+1}. In Step 2, we define S_n^* in (3.8) based on the relative position of $\hat{S}_n$ to the interval (Q_n, Q_{n+1}). By the definition of S_n^*, we have $Q_n \leq S_n^* \leq Q_{n+1}$ and $0 < S_n^* < \infty$ where $n = 1, \ldots, M$. In the following lemma, we show that given such an S_n^*, there exists a unique A_n^* such that equation (3.31) holds.

Lemma 3.4. *For any $K_n > 0$ where $n = 1, \ldots, M$, together with S_n^* defined in* (3.8)*, there exists a unique $A_n^* \in \mathbb{R}$ such that*

$$\int_0^{S_n^*} [g_{A_n^*}(y) + k] dy = -K_n. \tag{3.42}$$

Furthermore,

$$A_n^* \leq \hat{A}_n. \tag{3.43}$$

Proof. Define

$$\Lambda_1(A) = \int_0^{S_n^*} [g_A(y) + k] dy. \tag{3.44}$$

To prove (3.42), it suffices to show that

$$\Lambda_1'(A) > 0, \tag{3.45}$$

$$\lim_{A \to \overline{A}} \Lambda_1(A) > 0, \tag{3.46}$$

$$\lim_{A \to -\infty} \Lambda_1(A) = -\infty. \tag{3.47}$$

By taking derivative of (3.44) with respect to A, we have

$$\Lambda_1'(A) = \int_0^{S_n^*} \varphi_y'(A) dy = \int_0^{S_n^*} 1 dy = S_n^* > 0,$$

where the inequality follows the fact that $S_n^* > 0$.

Now, we are going to show (3.46).

$$\begin{aligned} \lim_{A \to \overline{A}} \Lambda_1(A) &= \lim_{A \to \overline{A}} \int_0^{S_n^*} [g_A(y) + k] dy \\ &> \lim_{A \to \overline{A}} S_n^* [g_A(0) + k] \\ &= 0, \end{aligned}$$

where the inequality follows from Lemma 3.2 and the last equality follows from $g_{\overline{A}}(0) = -k$ and $0 < S_n^* < \infty$.

Next, we are going to show (3.47).

$$\begin{aligned} \lim_{A \to -\infty} \Lambda_1(A) &= \lim_{A \to -\infty} \int_0^{S_n^*} [g_A(y) + k] dy \\ &< \lim_{A \to -\infty} S_n^* \cdot [g_A(S_n^*) + k] \\ &= -\infty, \end{aligned}$$

where the inequality follows from Lemma 3.2 and the last equality follows from the fact $0 < S_n^* < \infty$ and the definition of $g_A(z)$ in (3.5).

Finally, we prove (3.43) by contradiction. Suppose $A_n^* > \hat{A}_n$, then $g_{A_n^*}(z) > g_{\hat{A}_n}(z)$ must hold for any $z \geq 0$ by the definition of $g_A(z)$ in (3.5). Since $g_{\hat{A}_n}(z) \leq -k$ for $z \in [0, \hat{S}_n]$ and $g_{\hat{A}_n}(z) > -k$ for $z \in (\hat{S}_n, \infty)$,

$$\begin{aligned}\int_0^{S_n^*} [g_{A_n^*}(y) + k]dy &> \int_0^{S_n^*} [g_{\hat{A}_n}(y) + k]dy \\ &\geq \int_0^{\hat{S}_n} [g_{\hat{A}_n}(y) + k]dy \\ &= -K_n,\end{aligned}$$

which contradicts with (3.42). Hence, we must have $A_n^* \leq \hat{A}_n$. □

In order to prove subsequent lemmas, we need the following lemma that shows the properties of $g_A(z)$ at the order-up-to level S_n^* of the $(0, S_n^*)$ policy.

Lemma 3.5. *For* $n \in \mathcal{N}_<$*,*

$$g_{A_n^*}(S_n^*) \geq -k. \tag{3.48}$$

For $n \in \mathcal{N}_=$*,*

$$g_{A_n^*}(S_n^*) = -k. \tag{3.49}$$

For $n \in \mathcal{N}_>$*,*

$$g_{A_n^*}(S_n^*) \leq -k. \tag{3.50}$$

Proof. For $n \in \mathcal{N}_=$, we have $S_n^* = \hat{S}_n$. Comparing Lemma 3.3 and Lemma 3.4, we have $A_n^* = \hat{A}_n$. Therefore, according to Lemma 3.3, we have $g_{A_n^*}(S_n^*) = -k$.

For $n \in \mathcal{N}_<$, we have $\hat{S}_n \leq Q_n = S_n^*$. We prove (3.48) for two cases: $\hat{S}_n = Q_n = S_n^*$ and $\hat{S}_n < Q_n = S_n^*$. If $\hat{S}_n = Q_n = S_n^*$, similar to $n \in \mathcal{N}_=$, we have $A_n^* = \hat{A}_n$.

Thus $g_{A_n^*}(S_n^*) = -k$. If $\hat{S}_n < Q_n = S_n^*$, we show (3.48) by contradiction. Suppose $g_{A_n^*}(S_n^*) < -k$, then

$$-K_n = \int_0^{S_n^*} [g_{A_n^*}(y) + k]dy < \int_0^{\hat{S}_n} [g_{A_n^*}(y) + k]dy \leq \int_0^{\hat{S}_n} [g_{\hat{A}_n}(y) + k]dy = -K_n,$$

where the first inequality follows from $\hat{S}_n < S_n^*$, $g_{A_n^*}(S_n^*) < -k$ and Lemma 3.2 and the second inequality follows from (3.43). Therefore, $g_{A_n^*}(S_n^*) \geq -k$ for $n \in \mathcal{N}_<$.

For $n \in \mathcal{N}_>$, we have $\hat{S}_n \geq Q_{n+1} = S_n^*$. We prove (3.50) for two cases: $\hat{S}_n = Q_{n+1} = S_n^*$ and $\hat{S}_n > Q_{n+1} = S_n^*$. If $\hat{S}_n = Q_{n+1} = S_n^*$, similar to $n \in \mathcal{N}_=$, we have $A_n^* = \hat{A}_n$. Thus $g_{A_n^*}(S_n^*) = -k$. If $\hat{S}_n > Q_{n+1} = S_n^*$, we show (3.50) by contradiction. Suppose $g_{A_n^*}(S_n^*) > -k$, then

$$-K_n = \int_0^{S_n^*} [g_{A_n^*}(y) + k]dy < \int_0^{\hat{S}_n} [g_{A_n^*}(y) + k]dy \leq \int_0^{\hat{S}_n} [g_{\hat{A}_n}(y) + k]dy = -K_n,$$

where the first inequality follows from $\hat{S}_n > S_n^*$, $g_{A_n^*}(S_n^*) > -k$ and Lemma 3.2 and the second inequality follows from (3.43). $\square$

According to the definition in of $\mathcal{N}_<$, $\mathcal{N}_=$ and $\mathcal{N}_>$ in Section 3.2, we can conclude that sets $\mathcal{N}_=$, $\mathcal{N}_>$ and $\mathcal{N}_<$ are disjoint and $\mathcal{N}_= \cup \mathcal{N}_< \cup \mathcal{N}_> = \{1, 2, \dots, M\}$.

Notice that for $n \in \mathcal{N}_> \cup \mathcal{N}_<$, we may not have $K(S_n^*) = K_n$ by the setup cost given in (1.5). We define

$$\begin{aligned}
\mathcal{N} &= \{n \in \mathcal{N}_> \cup \mathcal{N}_< : K(S_n^*) = K_n\}, \\
\overline{\mathcal{N}} &= \{n \in \mathcal{N}_> \cup \mathcal{N}_< : K(S_n^*) \neq K_n\}.
\end{aligned}$$

Then $\mathcal{N}_=$, $\mathcal{N}$ and $\overline{\mathcal{N}}$ are disjoint and $\mathcal{N}_= \cup \mathcal{N} \cup \overline{\mathcal{N}} = \{1, 2, \dots, M\}$.

For each S_n^*, we have $\int_0^{S_n^*} [g_{A_n^*}(y) + k]dy = -K_n$ by (3.42). However, $\int_0^{S_n^*} [g_{A_n^*}(y) + k]dy = -K(S_n^*)$ (equivalent to (3.31)) may not hold since the equation $K(S_n^*) = K_n$ may not hold. By the following lemma, we show that we should always have $K(S_{n^*}^*) = K_{n^*}$ if we select the n^* by (3.9).

To state the next lemma, we first define

$$\underline{\chi}(n) = \max\{i = 1, \ldots, n-1 : i \in \mathcal{N} \cup \mathcal{N}_=\} \text{ for } n \in \mathcal{N}_< \backslash \mathcal{N}, \quad (3.51)$$

$$\overline{\chi}(n) = \min\{i = n+1, \ldots, M : i \in \mathcal{N} \cup \mathcal{N}_=\} \text{ for } n \in \mathcal{N}_> \backslash \mathcal{N}. \quad (3.52)$$

Lemma 3.6. *For each* $n \in \mathcal{N}_< \backslash \mathcal{N}$*,* $\underline{\chi}(n)$ *defined in* (3.51) *exists and satisfies* $A_n^* < A_{\underline{\chi}(n)}^*$*; for each* $n \in \mathcal{N}_> \backslash \mathcal{N}$*,* $\overline{\chi}(n)$ *defined in* (3.52) *exists and satisfies* $A_n^* < A_{\overline{\chi}(n)}^*$.

Proof. For each $n \in \mathcal{N}_< \backslash \mathcal{N}$, we prove the existence of $\underline{\chi}(n)$ by contradiction. Suppose for some $n \in \mathcal{N}_< \backslash \mathcal{N}$, $\underline{\chi}(n)$ does not exist, namely, $i \notin \mathcal{N} \cup \mathcal{N}_=$ and $K(S_i^*) \neq K_i$ for $i = 1, \ldots, n-1$. Since $K_1 > 0$, Lemma 3.3 implies $\hat{S}_1 > 0 = Q_1$, from which we can deduce $1 \notin \mathcal{N}_<$ and $n \geq 2$. $n \in \mathcal{N}_< \backslash \mathcal{N}$ implies that $\hat{S}_n \leq Q_n = S_n^*$ and $K(S_n^*) = K_{n-1} < K_n$. By (3.36), we have $\hat{S}_{n-1} < \hat{S}_n \leq Q_n$, which together with $n-1 \notin \mathcal{N} \cup \mathcal{N}_=$ implies $n-1 \in \mathcal{N}_< \backslash \mathcal{N}$. By induction, we can obtian $\{1, \ldots, n-1\} \subset \mathcal{N}_< \backslash \mathcal{N}$, which contradicts the fact that $1 \notin \mathcal{N}_<$. Therefore, $\underline{\chi}(n)$ must exist.

For each $n \in \mathcal{N}_< \backslash \mathcal{N}$, we can conclude $\{\underline{\chi}(n)+1, \ldots, n\} \subset \mathcal{N}_< \backslash \mathcal{N}$ and $K_{\underline{\chi}(n)} < \cdots < K_n$ from above arguments. Then by (3.36), we have $\hat{S}_{\underline{\chi}(n)} < \hat{S}_{\underline{\chi}(n)+1} \leq Q_{\underline{\chi}(n)+1}$, which implies $\underline{\chi}(n) \in \mathcal{N}_< \cup \mathcal{N}_=$. By Lemma 3.2 and Lemma 3.5, we have

$$g_{A_{\underline{\chi}(n)}^*}(z) > -k \quad \text{for } z > S_{\underline{\chi}(n)}^*.$$

By (3.8), we have $S_{\underline{\chi}(n)}^* < S_n^*$. Then,

$$\int_0^{S_n^*} [g_{A_{\underline{\chi}(n)}^*}(y) + k] dy > \int_0^{S_{\underline{\chi}(n)}^*} [g_{A_{\underline{\chi}(n)}^*}(y) + k] dy = -K_{\underline{\chi}(n)}.$$

Furthermore, we have

$$\int_0^{S_n^*} [g_{A_{\underline{\chi}(n)}^*}(y) + k] dy = S_n^* A_{\underline{\chi}(n)}^* + \int_0^{S_n^*} [g_0(y) + k] dy,$$

and

$$\int_0^{S_n^*}[g_{A_n^*}(y)+k]dy = S_n^*A_n^* + \int_0^{S_n^*}[g_0(y)+k]dy = -K(S_n^*) = -K_{n-1}.$$

We can conclude that $A_n^* < A_{\underline{\chi}(n)}^*$ from the facts that $K_{\underline{\chi}(n)} \le K_{n-1}$ and that $S_n^* > 0$.

Using the fact that $Q_{M+1} = \infty$, we prove that $\overline{\chi}(n)$ exists and $A_n < A_{\overline{\chi}(n)}$ by similar rationales. Thus the details are omitted. □

We obtain the n^* by (3.9) and S^*, K^* by (3.10) . Notice that by Lemma 3.6, we always have $n^* \in \mathcal{N}_= \cup \mathcal{N}$, namely we always have $\int_0^{S^*}[g_{A^*}(y)+k]dy = -K(S^*)$. We will prove the optimality of this $(0, S^*)$ policy in Section 3.5.2.

3.5.2 Verification

In this subsection, we prove Theorem 3.1. Namely, we will prove that the $(0, S^*)$ policy obtained by the four-step algorithm in Section 3.2 is an optimal policy for our Brownian inventory model in this chapter. To prove Theorem 3.1, we need the following technical result.

Lemma 3.7. *If $K(z_2 - z_1) = K_n$, for $n = 1, 2, \ldots, M$ and $z_2 > z_1 \ge 0$, then*

$$\int_{z_1}^{z_2}[g_{A^*}(y)+k]dy \ge \int_0^{S_n^*}[g_{A_n^*}(y)+k]dy = -K_n. \tag{3.53}$$

Proof. Consider three cases: $n \in \mathcal{N}_=$, $n \in \mathcal{N}_<$ and $n \in \mathcal{N}_>$. For $n \in \mathcal{N}_=$, we have

$$\begin{aligned}\int_{z_1}^{z_2}[g_{A^*}(y)+k]dy &\ge \int_{z_1}^{z_2}[g_{A_n^*}(y)+k]dy \\ &\ge \int_0^{z_2-z_1}[g_{A_n^*}(y)+k]dy \\ &\ge \int_0^{S_n^*}[g_{A_n^*}(y)+k]dy \\ &= -K_n,\end{aligned}$$

where the first inequality follows from $A^* \ge A_n^*$, the second inequality follows from Lemma 3.2 and the third inequality follows from Lemma 3.2 and (3.49).

For $n \in \mathcal{N}_<$, we have $z_2 - z_1 \geq Q_n = S_n^*$ by (3.8). Then

$$\begin{aligned}
\int_{z_1}^{z_2} [g_{A^*}(y) + k]dy &\geq \int_{z_1}^{z_2} [g_{A_n^*}(y) + k]dy \\
&\geq \int_0^{z_2 - z_1} [g_{A_n^*}(y) + k]dy \\
&\geq \int_0^{S_n^*} [g_{A_n^*}(y) + k]dy \\
&= -K_n,
\end{aligned}$$

where the third inequality follows from Lemma 3.2, (3.48) and $z_2 - z_1 \geq S_n^*$.

For $n \in \mathcal{N}_>$, we have $x - y \leq Q_{n+1} = S_n^*$ by (3.8). Then

$$\begin{aligned}
\int_{z_1}^{z_2} [g_{A^*}(y) + k]dy &\geq \int_{z_1}^{z_2} [g_{A_n^*}(y) + k]dy \\
&\geq \int_0^{z_2 - z_1} [g_{A_n^*}(y) + k]dy \\
&\geq \int_0^{S_n^*} [g_{A_n^*}(y) + k]dy \\
&= -K_n,
\end{aligned}$$

where the third inequality follows from Lemma 3.2, (3.50) and $z_2 - z_1 \leq S_n^*$. □

Finally, we provide the proof of Theorem 3.1.

Proof of Theorem 3.1. Firstly, we show that $AC(x, U(0, S^*)) = -\mu A^*$ for $x \geq 0$. Namely, we need to show $V_{A^*}(\cdot)$ satisfies the conditions in Theorem 3.4. By the definition of $g_A(z)$ in (3.5) together with $V_A(z) = \int_0^z g_A(y)dy$ and Proposition 3.1, $V_{A^*}(\cdot)$ is twice continuously differentiable, $V'_{A^*}(\cdot)$ is polynomially bounded and

$$\Gamma V_{A^*}(z) + h(z) = -\mu A^* \quad \text{for } z \geq 0.$$

By Lemma 3.4 and Lemma 3.6, we have

$$\int_0^{S^*} [g_{A^*}(y) + k]dy = -K^* = -K(S^*),$$

which can be rewritten as

$$V_{A^*}(S^*) - V_{A^*}(0) = -K^* - k \cdot S^* = -K(S^*) - k \cdot S^*.$$

Therefore, we have $AC(x, U(0, S^*) = -\mu A^*$ for $x \geq 0$.

To show the optimality of the $(0, S^*)$ policy, it suffices to show that $V_{A^*}(\cdot)$ together with $\gamma = -\mu A^*$ satisfies all the conditions of the lower bound in Theorem 3.2. We have already shown that $V_{A^*}(\cdot)$ together with $\gamma = -\mu A^*$ satisfies (3.11) with equality, that $V_{A^*}(\cdot)$ is twice continuously differentiable and that $V'_{A^*}(\cdot)$ is polynomially bounded. It remains to show (3.12). By Lemma 3.7, for $z_2 > z_1 \geq 0$,

$$\int_{z_1}^{z_2} [g_{A^*}(y) + k] dy \geq -K(z_2 - z_1),$$

which can be rewritten as

$$V_{A^*}(z_1) - V_{A^*}(z_2) \leq K(z_2 - z_1) + k \cdot (z_2 - z_1).$$

Therefore,

$$AC(x, Y) \geq AC(x, U(0, S^*)) \text{ for all } x \geq 0 \text{ and } Y \in \mathcal{U}.$$

Furthermore,

$$AC(x, U(0, S^*)) = -\mu A^* \quad \text{for } x \geq 0.$$

□

3.6 Numerical Analysis

In the section, we conduct a numerical analysis of the optimal $(0, S)$ policy in this chapter. Furthermore, we compare this result with the result of EOQ model.

Assume that $h(z) = z^2$. Then (3.5) implies that

$$g_A(z) = A + \frac{z^2}{\mu} + \frac{2z}{\mu\lambda} + \frac{2}{\mu\lambda^2}, \quad \text{for } z \geq 0.$$

Set $\mu = 10$, $\sigma = 1$, $k = 1$, $M = 3$ and

$$K(\xi) = \begin{cases} 40 & \text{for } \xi \in (0, 10], \\ 50 & \text{for } \xi \in (10, 20), \\ 30 & \text{for } \xi \in [20, \infty). \end{cases}$$

Step 1. For $n = 1, 2, 3$, we obtain $\hat{S}_n$ and $\hat{A}_n$; see Table 3.1.

Table 3.1: Results of Step 1

n	(Q_n, Q_{n+1})	K_n	$\hat{S}_n$	$\hat{A}_n$
1	$(0, 10]$	40	8.4094	-8.1564
2	$(10, 20)$	50	9.0607	-9.3007
3	$[20, \infty)$	30	7.6382	-6.9111

Step 2. From Table 3.1, we have $\mathcal{N}_= = \{1\}$ and $\mathcal{N}_< = \{2, 3\}$. Thus we have $S_1^* = 8.4094$, $S_2^* = 10$ and $S_3^* = 20$.

Step 3. We obtain the parameter A_n^* for $n = 1, 2, 3$; see Table 3.2.

Table 3.2: Results of Step 3

n	(Q_n, Q_{n+1})	K_n	S_n^*	A_n^*
1	$(0, 10]$	40	8.4094	-8.1564
2	$(10, 20)$	50	10	-9.3838
3	$[20, \infty)$	30	20	-15.9338

Step 4. From Table 3.2, we obtain that $n^* = 1$. Therefore, $(0, S_1^*)$ with $S_1^* = 8.4094$ is an optimal policy and its long-run average cost is $\gamma^* = -\mu A_1^* = 81.564$.

Under the same assumption, we can obtain the optimal economic order quantity, $Q^* = 8.4343$ under the EOQ model by Perera et al. (2016). By (3.26), we can obtain the long-run average of this EOQ policy under the Brownian inventory policy is 81.565.

Table 3.3: Comparison of EOQ Model and Brownian Model

Model	Order Quantity	Cost
Brownian Model	8.4094	81.564
EOQ Model	8.4343	81.565

From Table 3.3, we conclude that we can save the long-run average cost by using our algorithm. Although the cost difference is small under this example, the saved cost can be huge if the units of the parameters are in millions or in billions.

Chapter 4

Optimal Policy Under the Discounted Cost Criterion Without Backlogs

We have specified the optimal control policy for a Brownian inventory system without backlogs under the long-run average cost criterion in the previous chapter. However, when we make decisions in the practical world, the time value of money should be considered in many cases. The rationale for this consideration is that the risk-free rate is positive most of the time. Therefore, it is more meaningful in the economic sense to search the optimal policy under the discounted cost criterion. In this chapter, we follow the two-step lower bound approach to obtain an optimal control policy for the continuous-review inventory under the discounted cost criterion. Section 4.1 presents the continuous-review model and assumptions. In Section 4.2, we present the main results of this chapter. In Section 4.3, we establish a lower bound for the expected infinite-horizon discounted cost incurred by any admissible policy. We compute the expected infinite-horizon discounted cost under a $(0, S)$ policy in Section 4.4. In Section 4.5, we demonstrate how to select the optimal $(0, S)$ policy and prove the optimality of it. In Section 4.6, we provide the proof of the Comparison Theorem, Theorem 4.3.

4.1 Model and Assumptions

In this chapter, we consider a continuous-review inventory model that is similar to the model in Chapter 3 but under the discounted cost criterion and a different assumption of the holding cost rate function. To obtain the optimal policy under the discounted cost criterion, we need different technical skills in both steps of the lower bound approach. In the inventory model of this chapter, we also assume that backlogs are not allowed and that the lead time for each order is zero. As in Chapter 3, let $D(t)$ and $Y(t)$ be the cumulative demand quantity and the cumulative order quantity during time $[0, t]$. We assume that only upward adjustments are allowed in our Brownian control model. Then, the inventory level at time $t \geq 0$ is given by

$$Z(t) = x - D(t) + Y(t).$$

The process D is given by

$$D(t) = \mu t - \sigma B(t).$$

For convenience, we repeat the expression of the inventory level at time t

$$Z(t) = X(t) + Y(t), \tag{4.1}$$

where

$$X(t) = x - \mu t + \sigma B(t) \tag{4.2}$$

can be interpreted as the inventory level in the absence of control.

Two types of costs are incurred in this continuous-review inventory system: the inventory holding cost and the ordering cost. The inventory holding cost is incurred at a rate $h(z)$ when the inventory level is z. Since the controller is required to keep $Z(t) \geq 0$

for $t \geq 0$, h is defined on $[0, \infty)$. In the inventory model of this chapter, we assume that the inventory holding cost rate function satisfies the following assumption.

Assumption 2. $h(\cdot)$ satisfies

(A1) $h(\cdot)$ is convex and $h(0) = 0$;

(A2) $h(\cdot)$ is continuously differentiable in $[0, \infty)$;

(A3) $h'(z) \geq 0$ for $z \geq 0$ and $\lim_{z \to \infty} h'(z) > \beta k$ where $\beta > 0$ is the discount rate;

(A4) $h''(\cdot)$ is polynomially bounded, i.e., there exist positive constants $a_i > 0$, $i = 0, 1$ and a positive integer n such that $h''(z) \leq a_0 + a_1 z^n$ for all $z \geq 0$.

An ordering cost is incurred whenever an order is placed and this cost is a function of the ordering quantity $\xi > 0$. When an order with quantity $\xi > 0$ is placed, it incurs a *setup cost* of $K(\xi) > 0$ given by (1.5) and a *proportional cost* of $k\xi$ with proportional cost rate $k > 0$. Let $\phi(\xi)$ denote the ordering cost with ordering quantity ξ. Then $\phi(\xi)$ is given by (3.3).

Since $K(\xi) > 0$ for $\xi > 0$, we only need to consider the policies with $N(t) < \infty$ for any $t > 0$. Otherwise, the total cost would be infinite in the time interval $[0, t]$. Then a policy can be specified by a sequence of pairs $\{(T_i, \xi_i) : i = 0, 1, \ldots\}$ where T_i is the ith order time and $\xi_i = \Delta Y(T_i) = Y(T_i) - Y(T_i-)$ is the quantity of the ith order. Therefore, investigating an optimal control policy given that $K(\xi) > 0$ for $\xi > 0$ is equivalent to exploring a sequence of ordering time together with corresponding ordering quantity, $\{(T_i, \xi_i) : i = 0, 1, \ldots\}$, which turns out be an *impulse control* problem for the Brownian model.

We aim to find an admissible inventory policy Y that minimizes the expected discounted cost over the infinite horizon

$$DC(x, Y) = \mathbb{E}_x \Big[\int_0^\infty e^{-\beta t} h(Z(t)) dt + \sum_{i=0}^{\infty} e^{-\beta T_i} (K(\xi_i) + k\xi_i) \Big], \tag{4.3}$$

where the constant $\beta > 0$ is the discounted rate and $\mathbb{E}_x$ is the expectation operator conditioning on the initial inventory level $Z(0-) = x$.

4.2 Main Results

In this section, we present the main results of this chapter. Under the quantity-dependent setup cost defined in (1.5), an optimal policy for the Brownian inventory model in this chapter is a $(0, S)$ policy with $S > 0$. We propose an algorithm for computing the optimal order-up-to level. We use $U(0, S)$ to denote a $(0, S)$ policy. The optimal order-up-to level S^* can be obtained by the following algorithm.

Step 1. For $z \geq 0$ and $B \in \mathbb{R}$, let $\lambda_1 = \frac{\mu+\sqrt{\mu^2+2\beta\sigma^2}}{\sigma^2}$, $\lambda_2 = \frac{-\mu+\sqrt{\mu^2+2\beta\sigma^2}}{\sigma^2}$,

$$\begin{aligned} V_B(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[-\frac{1}{\lambda_2^2}Be^{-\lambda_2 z} \\ &\quad + \int_0^z e^{-\lambda_2(z-y)}h(y)dy + \int_z^\infty e^{\lambda_1(z-y)}h(y)dy\Big]. \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} g_B(z) &= V_B'(z) \\ &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h'(y)dy \cdot e^{\lambda_1 z} \\ &\quad + \frac{e^{-\lambda_2 z}}{\lambda_2}\Big(B + \lambda_2\int_0^z e^{\lambda_2 y}h'(y)dy\Big)\Big]. \end{aligned} \tag{4.5}$$

For $n = 1, \ldots, M$, obtain $\hat{B}_n$ and $\hat{S}_n > 0$ by solving

$$\int_0^{\hat{S}_n}[g_{\hat{B}_n}(y) + k]dy = -K_n, \tag{4.6}$$

and

$$g_{\hat{B}_n}(\hat{S}_n) = -k. \tag{4.7}$$

Step 2. Define

$$\mathcal{N}_< = \{n \in \{1, 2, \ldots, M\} : \hat{S}_n \leq Q_n\},$$

$$\mathcal{N}_= = \{n \in \{1, 2, \ldots, M\} : \hat{S}_n \in (Q_n, Q_{n+1})\},$$

$$\mathcal{N}_> = \{n \in \{1, 2, \ldots, M\} : \hat{S}_n \geq Q_{n+1}\},$$

and

$$S_n^* = \begin{cases} Q_n & \text{for } n \in \mathcal{N}_<, \\ \hat{S}_n & \text{for } n \in \mathcal{N}_=, \\ Q_{n+1} & \text{for } n \in \mathcal{N}_>. \end{cases} \tag{4.8}$$

Step 3. For $n \in \mathcal{N}_=$, let

$$B_n^* = \hat{B}_n.$$

For $n \in \mathcal{N}_> \cup \mathcal{N}_<$, obtain B_n^* by solving

$$\int_0^{S_n^*} [g_{B_n^*}(y) + k] dy = -K_n.$$

Step 4. Let $\mathcal{M} = \{1, \ldots, M\}$. Define

$$n^* = \min\{n \in \mathcal{M} : B_i^* \leq B_n^* \text{ for all } i \in \mathcal{M}\}. \tag{4.9}$$

Let

$$S^* = S_{n^*}^*, \quad B^* = B_{n^*}^* \quad \text{and} \quad K^* = K_{n^*}. \tag{4.10}$$

In Step 1, we obtain the parameter $\hat{S}_n$ associated with $\hat{B}_n$ by smoothness conditions similar with those in Sulem (1986). This $(0, \hat{S}_n)$ policy is expected to be optimal when the setup cost is constant at K_n. However under the quantity-dependent setup cost, a quantity constraint is imposed on each setup cost value. Thus in Step 2 we obtain S_n^*, which is restricted to the corresponding interval. We obtain the auxiliary parameter B_n^* in Step 3. Finally in Step 4, we select the optimal S^* by picking the largest B_n^*. Please see Section 4.5 for explicit derivations. The main difference of the algorithm in this section and the algorithm in Section 3.2 is that we have different function $g(\cdot)$. This difference results from the discounted cost criterion adopted in this chapter.

Then the $(0, S^*)$ policy obtained from the above algorithm is an optimal policy for our Brownian inventory model. The optimality of this $(0, S^*)$ policy is proved by the following theorem.

Theorem 4.1. *Assume the cost function h satisfies Assumption 2 and that the setup cost is given by* (1.5)*. Control policy $U(0, S^*)$ obtained by Steps 1–4 is an optimal admissible policy that minimizes the discounted cost given by* (4.3)*. Namely, we have*

$$DC(x, U(0, S^*)) \leq DC(x, Y) \quad \textit{for } x \geq 0 \textit{ and } Y \in \mathcal{U}.$$

Moreover, the minimum discounted cost is $DC(x, U(0, S^*)) = V_{B^*}(x)$ for $x \geq 0$.

4.3 Lower Bound Under the Discounted Cost Criterion

In this section, we propose and prove a theorem that establishes a lower bound for the discounted cost incurred by any admissible control policy.

Theorem 4.2. *Assume that h satisfies Assumption 2. Let $f(\cdot) : [0, \infty) \to \mathbb{R}$ be twice continuously differentiable. Assume that*

$$\Gamma f(z) - \beta f(z) + h(z) \geq 0 \quad \textit{for all } z \geq 0, \tag{4.11}$$

where

$$\Gamma f(z) = \frac{1}{2}\sigma^2 f''(z) - \mu f'(z).$$

We further assume that

$$f(z_1) - f(z_2) \leq K(z_2 - z_1) + k \cdot (z_2 - z_1) \quad \textit{for all } 0 \leq z_1 < z_2, \tag{4.12}$$

and $f'(\cdot)$ is polynomially bounded, i.e., there exist positive constants a_0, a_1 and a positive integer n such that

$$|f'(z)| < a_0 + a_1 z^n \quad \textit{for all } z \geq 0. \tag{4.13}$$

Then

$$DC(x, Y) \geq f(x), \tag{4.14}$$

where $DC(x, Y)$, given by (4.3)*, is the total discounted cost over the infinite horizon under any admissible control policy Y.*

By Theorem 4.2, if we can find an admissible ordering policy Y whose cost function $DC(x, Y)$ satisfies the conditions of $f(\cdot)$ in Theorem 4.2, we can conclude that this admissible control policy must be optimal among all admissible policies. In the lower bound approach, Theorem 4.2 is referred to as a verification theorem. In order to prove Theorem 4.2, we first need to show some technical results.

The following comparison theorem is a critical result for proving Theorem 4.2. It implies that a policy that is optimal among the policies subject to order-up-to bounds must be optimal among all admissible policies. Compared with the general admissible policies, policies subject to order-up-to bounds are analytically tractable.

For $m = 1, 2, \ldots$, let $\mathcal{U}_m$ represents the set of admissible policies with an order-up-to bound at m.

Theorem 4.3 (Comparison Theorem)**.** *Assume that the holding cost rate function h is nondecreasing on $[0, \infty)$ and the setup cost function $K(\cdot)$ is bounded. Then for any admissible policy Y, there exists a sequence of admissible policies $\{Y_m \in \mathcal{U}_m : m = 1, 2, \ldots\}$ such that*

$$\lim_{m \to \infty} DC(x, Y_m) \leq DC(x, Y). \tag{4.15}$$

The proof of Theorem 4.3 is shown in Section 4.6. Let $\overline{\mathcal{U}}$ be the set of all admissible policies subject to order-up-to bounds. Theorem 4.3 implies that a policy that is optimal in $\overline{\mathcal{U}}$ must be optimal in $\mathcal{U}$. Therefore, we only need to search all admissible policies subject to order-up-to bounds for the optimal policy as in Chapter 3.

Proof of Theorem 4.2. By Theorem 4.3, it suffices to consider an arbitrary policy $Y \in \overline{\mathcal{U}}$, namely it suffices to consider $Y \in \mathcal{U}_m$ for a fixed positive integer m. For any $Y \in \mathcal{U}_m$, recall that

$$Z(t) = x - \mu t + \sigma B(t) + Y(t).$$

By Itô's formula , we have

$$\begin{aligned} e^{-\beta t} f(Z(t)) &= f(Z(0)) + \int_0^t e^{-\beta u}(\Gamma f(Z(u)) - \beta f(Z(u))du \\ &\quad + \sigma \int_0^t e^{-\beta u} f'(Z(u))dB(u) + \sum_{0<u\leq t} e^{-\beta u}\Delta f(Z(u)) \\ &= f(Z(0-)) + \int_0^t e^{-\beta u}(\Gamma f(Z(u)) - \beta f(Z(u))du \\ &\quad + \sigma \int_0^t e^{-\beta u} f'(Z(u))dB(u) + \sum_{0\leq u\leq t} e^{-\beta u}\Delta f(Z(u)) \end{aligned} \tag{4.16}$$

By (4.11) and (4.12), we have

$$\begin{aligned} e^{-\beta t} f(Z(t)) &\geq f(x) - \int_0^t e^{-\beta u} h(Z(u))du + \sigma \int_0^t e^{-\beta u} f'(Z(u))dB(u) \\ &\quad - \sum_{i=0}^{N(t)} e^{-\beta T_i}\phi(\xi_i), \end{aligned} \tag{4.17}$$

where ϕ is define in (3.3). By (3.17) and Theorem 3.2.1 in Øksendal (2003), we have

$$\mathbb{E}_x\Big[\int_0^t f'(Z(u))dB(u)\Big] = 0.$$

Since (3.16) holds, we can take expectation on both sides of (4.17) and obtain

$$\mathbb{E}_x[e^{-\beta t} f(Z(t))] \geq f(x) - \mathbb{E}_x\Big[\int_0^t e^{-\beta u} h(Z(u))du\Big] - \mathbb{E}_x\Big[\sum_{i=0}^{N(t)} e^{-\beta T_i}\phi(\xi_i)\Big].$$

By (4.13), (3.18) and taking limit as $t \to \infty$ of both sides of the above inequality, we can conclude that $DC(x, Y) \geq f(x)$. □

4.4 Expected Discounted Cost Under $(0, S)$ Policies

In this section, we will compute the discounted cost under any $(0, S)$ policy, the policy with a reorder level zero and an order-up-to level S. By the definition of admissible policies, all $(0, S)$ policies are in $\bar{\mathcal{U}}$, the set of admissible policies subject to order-up-to bounds.

Theorem 4.4. *For any $S > 0$, if there exists a twice continuously differentiable function $V(\cdot) : [0, \infty) \to \mathbb{R}$ such that*

$$\Gamma V(z) - \beta V(z) + h(z) = 0 \textit{ for } z \geq 0, \tag{4.18}$$

$$V(S) - V(0) = -K(S) - k \cdot S, \tag{4.19}$$

$$V' \textit{ is polynomially bounded.} \tag{4.20}$$

Then,

$$DC(x, U(0, S)) = V(x). \tag{4.21}$$

Proof. By the Itô's formula (see (4.16)), together with (4.18) and (4.19), we have

$$\begin{aligned} e^{-\beta t} V(Z(t)) &= V(Z(0)) - \int_0^t e^{-\beta u} h(Z(u))du \\ &\quad + \sigma \int_0^t e^{-\beta u} V'(Z(u))dB(u) - \sum_{i=0}^{N(t)} e^{-\beta T_i}\phi(\xi_i). \end{aligned} \tag{4.22}$$

We can take expectation on both sides of (4.22) because (3.16) holds. By (3.17) and Theorem 3.2.1 in Øksendal (2003), we can obtain

$$\begin{aligned}\mathbb{E}_x[e^{-\beta t}V(Z(t))] &= V(Z(0)) - \mathbb{E}_x\Big[\int_0^t e^{-\beta u}h(Z(u))du\Big] - \mathbb{E}_x\Big[\sum_{i=1}^{N(t)} e^{-\beta T_i}\phi(\xi_i)\Big] \\ &= V(Z(0)) - \mathbb{E}_x\Big[\int_0^t e^{-\beta u}h(Z(u))du\Big] - \mathbb{E}_x\Big[\sum_{i=0}^{N(t)} e^{-\beta T_i}\phi(\xi_i)\Big] \\ &\quad +\mathbb{E}_x[\phi(\xi_0)].\end{aligned}$$

By (3.18), (4.20) and letting $t \to \infty$, we can obtain

$$DC(x, U(0,S)) = \mathbb{E}_x[V(Z(0))] + \mathbb{E}_x[\phi(\xi_0)]. \tag{4.23}$$

The initial inventory level x in the inventory model of this chapter is require to be non-negative. If $x > 0$, we have $Z(0) = Z(0-) = x$ and $\xi_0 = 0$ under the $(0, S)$ policy. Then by (4.23), we have $DC(x, U(0,S)) = V(x)$. If $x = 0$, we have $Z(0) = S$ and $\xi_0 = S$. By (4.23) and (4.19), we have $DC(0, U(0,S)) = V(S) + K(S) + k \cdot S = V(0)$. Therefore, we can conclude that $DC(x, U(0,S)) = V(x)$. □

In the following proposition, we provide the solution to equations (4.18)–(4.20).

Proposition 4.1. *The solution to* (4.18)–(4.20) *is given by*

$$V(z) = B_0 e^{-\lambda_2 z} + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_0^z e^{-\lambda_2(z-y)}h(y)dy + \int_z^\infty e^{\lambda_1(z-y)}h(y)dy\Big] \tag{4.24}$$

for $z \geq 0$*, where*

$$\begin{aligned} B_0 &= \frac{1}{e^{-\lambda_2 S}-1}\Big\{ - K(S) - k\cdot S - \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_0^S e^{-\lambda_2(S-y)}h(y)dy \\ &\quad + \int_S^\infty e^{\lambda_1(S-y)}h(y)dy - \int_0^\infty e^{-\lambda_1 y}h(y)dy\Big]\Big\}, \end{aligned} \tag{4.25}$$

and

$$\lambda_1 = \frac{\mu + \sqrt{\mu^2 + 2\beta\sigma^2}}{\sigma^2},$$

$$\lambda_2 = \frac{-\mu + \sqrt{\mu^2 + 2\beta\sigma^2}}{\sigma^2}.$$

The solution $V(z)$ given by (4.24) *together with* (4.25) *is unique.*

Proof. Since $z = \lambda_1$ and $z = -\lambda_2$ are two solutions for the quadratic equation

$$\frac{1}{2}\sigma^2 z^2 - \mu z - \beta = 0,$$

then $g_1(z) = e^{\lambda_1 z}$ and $g_2(z) = e^{-\lambda_2 z}$ are two solutions for the homogeneous ordinary differential equation (ODE)

$$\Gamma g - \beta g = 0.$$

Let

$$w(z) = \det \begin{pmatrix} g_1(z) & g_2(z) \\ g_1'(z) & g_2'(z) \end{pmatrix} = -(\lambda_1 + \lambda_2) e^{(\lambda_1 - \lambda_2) z}$$

and

$$\begin{aligned} d_1(z) &= \int_0^z \frac{1}{w(y)} g_2(y) \frac{2}{\sigma^2} h(y) dy = -\frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \int_0^z e^{-\lambda_1 y} h(y) dy, \\ d_2(z) &= -\int_0^z \frac{1}{w(y)} g_1(y) \frac{2}{\sigma^2} h(y) dy = \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \int_0^z e^{\lambda_2 y} h(y) dy. \end{aligned}$$

Then the non-homogeneous ODE (4.18) has a particular solution

$$\begin{aligned} V_0(z) &= d_1(z) g_1(z) + d_2(z) g_2(z) \\ &= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\int_0^z e^{-\lambda_2 (z-y)} h(y) dy - \int_0^z e^{\lambda_1 (z-y)} h(y) dy \Big]. \end{aligned}$$

Then a general solution for (4.18) is

$$V(z) = A_0 e^{\lambda_1 z} + B_0 e^{-\lambda_2 z} + V_0(z).$$

By (4.20), we have $\lim_{z\to\infty} \frac{V(z)}{e^{\lambda_1 z}} = 0$, from which we can obtain

$$A_0 = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\int_0^\infty e^{-\lambda_1 y}h(y)dy.$$

Then $V(z)$ is given by

$$\begin{aligned} V(z) &= A_0 e^{\lambda_1 z} + B_0 e^{-\lambda_2 z} + V_0(z) \\ &= B_0 e^{-\lambda_2 z} + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_0^z e^{-\lambda_2(z-y)}h(y)dy + \int_z^\infty e^{\lambda_1(z-y)}h(y)dy\Big] \end{aligned}$$

Finally by (4.19), we can derive that B_0 is given by (4.25).

□

4.5 Optimal Policy

By (4.4),

$$\begin{aligned} V_B(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[-\frac{1}{\lambda_2^2}Be^{-\lambda_2 z} \\ &\quad + \int_0^z e^{-\lambda_2(z-y)}h(y)dy + \int_z^\infty e^{\lambda_1(z-y)}h(y)dy\Big], \\ &= B_1 e^{-\lambda_2 z} + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_0^z e^{-\lambda_2(z-y)}h(y)dy + \int_z^\infty e^{\lambda_1(z-y)}h(y)dy\Big] \end{aligned}$$

where

$$B_1 = -\frac{2}{\lambda_2^2\sigma^2(\lambda_1+\lambda_2)}B.$$

By Proposition 4.1, V_B' is polynomially bounded on $[0,\infty)$. Furthermore, $V_B(z)$ satisfies the differential equation

$$\Gamma V_B(z) - \beta V_B(z) + h(z) = 0 \text{ for } z \geq 0.$$

For any $(0, S)$ policy, if $V_B(z)$ further satisfies

$$V_B(S) - V_B(0) = -K(S) - k \cdot S, \tag{4.26}$$

we can conclude by Theorem 4.4 that the discounted cost under a $(0, S)$ policy is given by

$$DC(x, U(0, S)) = V_B(x). \tag{4.27}$$

For any fixed $z \geq 0$, $V_B(z)$ given by (4.4) is strictly decreasing in B. Thus in order to minimize the discounted cost under a $(0, S)$ policy, we should maximize the value of B without violating (4.26).

By applying integration by parts to (4.5), for $z \geq 0$

$$\begin{aligned} g_B(z) &= \frac{1}{\beta} h'(z) + \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\frac{1}{\lambda_1} \int_z^\infty e^{-\lambda_1 y} h''(y) dy \cdot e^{\lambda_1 z} \\ &\quad + \frac{e^{-\lambda_2 z}}{\lambda_2} \big(B - h'(0) - \int_0^z e^{\lambda_2 y} h''(y) dy \big) \Big], \end{aligned} \tag{4.28}$$

For $z \geq 0$, we have

$$\begin{aligned} g'_B(z) &= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\lambda_1 \int_z^\infty e^{-\lambda_1 y} h'(y) dy \cdot e^{\lambda_1 z} - \Big(B + \lambda_2 \int_0^z e^{\lambda_2 y} h'(y) dy \Big) e^{-\lambda_2 z} \Big] \\ &= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\int_z^\infty e^{-\lambda_1 y} h''(y) dy \cdot e^{\lambda_1 z} \\ &\quad - \Big(B - h'(0) - \int_0^z e^{\lambda_2 y} h''(y) dy \Big) e^{-\lambda_2 z} \Big]. \end{aligned} \tag{4.29}$$

In the rest of this chapter, we need to discuss properties of $g_B(z)$ with respect to the auxiliary variable B for fixed $z \geq 0$. To make the notation clear, for fixed $z \geq 0$, let

$$\varphi_z(B) = g_B(z).$$

Then by (4.5), we have

$$\varphi_z'(B) = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{e^{-\lambda_2 z}}{\lambda_2}. \tag{4.30}$$

In Section 4.5.1, we demonstrate how the four-step algorithm in Section 4.2 attains the maximum value of B by selecting the $(0, S^*)$ policy, the best policy among $(0, S)$ policies. In Section 4.5.2, we prove that under this $(0, S^*)$ policy, the corresponding discounted cost function satisfies the conditions specified by the lower bound theorem. Thus, the $(0, S^*)$ policy is an optimal policy for the Brownian inventory model in this chapter.

4.5.1 The Optimal $(0, S)$ Policy

In this subsection, we first show the monotonicity of $g_B(z)$ in Lemma 4.1, which is an important lemma for proving subsequent lemmas. Then we identify a set of $(0, S)$ policies $\{U(0, \hat{S}_n) : n = 1, \ldots, M\}$ by Lemma 4.2. However under the quantity-dependent setup cost (1.5), the $\hat{S}_n$ may not fall into an interval from Q_n to Q_{n+1}. We obtain a set of modified $(0, S)$ policies $\{U(0, S_n^*) : n = 1, \ldots, M\}$ by (4.8). Given the S_n^*, Lemma 4.3 proves the existence of B_n^* such that B_n^* and S_n^* jointly satisfy (4.26). Finally, we select the best $(0, S^*)$ policy out of the set $\{U(0, S_n^*) : n = 1, \ldots, M\}$ by (4.9) and (4.10).

Before stating the following lemma, we first define

$$\overline{B} = h'(0) + \int_0^\infty e^{-\lambda_1 y} h''(y) dy. \tag{4.31}$$

By $(A4)$ in Assumption 2, $\overline{B}$ is bounded.

Lemma 4.1. *Assume that h satisfies Assumption 2. Then for $B \in (-\infty, \overline{B})$, $g_B(z)$ is strictly increasing in $z \in [0, \infty)$. Furthermore,*

$$\lim_{z\to\infty} g_B(z) > k. \tag{4.32}$$

Proof. To show the monotonicity of $g_B(z)$ in $z \in [0,\infty)$, it suffices to show that $g_B'(z) > 0$ in $z \in [0,\infty)$ for $B \in (-\infty, \overline{B})$. By (4.29), for any $B \in (-\infty, \overline{B})$, we have

$$\begin{aligned}
g_B'(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy\cdot e^{\lambda_1 z} \\
&\quad -\big(B-h'(0)-\int_0^z e^{\lambda_2 y}h''(y)dy\big)e^{-\lambda_2 z}\Big] \\
&> \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy\cdot e^{\lambda_1 z} \\
&\quad -\big(\overline{B}-h'(0)-\int_0^z e^{\lambda_2 y}h''(y)dy\big)e^{-\lambda_2 z}\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy\cdot e^{\lambda_1 z} \\
&\quad -\big(\int_0^\infty e^{-\lambda_1 y}h''(y)dy-\int_0^z e^{\lambda_2 y}h''(y)dy\big)e^{-\lambda_2 z}\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy\cdot\big(e^{\lambda_1 z}-e^{-\lambda_2 z}\big) \\
&\quad +\int_0^z \big(e^{\lambda_2 y}-e^{-\lambda_1 y}\big)h''(y)dy\cdot e^{-\lambda_2 z}\Big] \\
&\geq 0,
\end{aligned}$$

where the second equality follows from the definition of $\overline{B}$ in (4.31) and the last inequality follows from Assumption 2, $\lambda_1, \lambda_2 > 0$ and $z \geq 0$.

Next, we prove (4.32). By (4.5)

$$\begin{aligned}
&\lim_{z\to\infty} g_B(z) \\
=\ &\lim_{z\to\infty}\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h'(y)dy\cdot e^{\lambda_1 z} \\
&\quad +\frac{1}{\lambda_2}\Big(B+\lambda_2\int_0^z e^{\lambda_2 y}h'(y)dy\Big)e^{-\lambda_2 z}\Big] \\
=\ &\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\lim_{z\to\infty}\Big[\frac{\int_z^\infty e^{-\lambda_1 y}h'(y)dy}{e^{-\lambda_1 z}}+\frac{\int_0^z e^{\lambda_2 y}h'(y)dy}{e^{\lambda_2 z}}\Big].
\end{aligned}$$

Since we have

$$\lim_{z\to\infty}\frac{\int_z^\infty e^{-\lambda_1 y}h'(y)dy}{e^{-\lambda_1 z}} = \lim_{z\to\infty}\frac{h'(z)}{\lambda_1}$$

and

$$\lim_{z\to\infty}\frac{\int_0^z e^{\lambda_2 y}h'(y)dy}{e^{\lambda_2 z}} = \lim_{z\to\infty}\frac{h'(z)}{\lambda_2}$$

by L'Hôpital's Rule, then

$$\begin{aligned}\lim_{z\to\infty} g_B(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\lim_{z\to\infty}\Big(\frac{h'(z)}{\lambda_1}+\frac{h'(z)}{\lambda_2}\Big)\\ &= \frac{1}{\beta}\lim_{z\to\infty} h'(z)\\ &> k,\end{aligned}$$

where the last equality comes from $\lambda_1 = \frac{\mu+\sqrt{\mu^2+2\beta\sigma^2}}{\sigma^2}$ and $\lambda_2 = \frac{-\mu+\sqrt{\mu^2+2\beta\sigma^2}}{\sigma^2}$, and the inequality follows from (A3) in Assumption 2. $\square$

By the following lemma, we identify a set of $(0, S)$ policies $\{U(0, \hat{S}_n) : n = 1, \ldots, M\}$.

Lemma 4.2. *Assume that h satisfies Assumption 2. For an arbitrary $\kappa > 0$, there exist a unique pair $(\hat{S}(\kappa), \hat{B}(\kappa))$ with $\hat{S}(\kappa) > 0$ such that $\hat{B}(\kappa) \in (-\infty, \overline{B})$,*

$$\int_0^{\hat{S}(\kappa)} [g_{\hat{B}(\kappa)}(y) + k]dy = -\kappa, \tag{4.33}$$

$$g_{\hat{B}(\kappa)}(\hat{S}(\kappa)) = -k. \tag{4.34}$$

For any $0 < \kappa_i < \kappa_j$, the corresponding $(\hat{S}(\kappa_i), \hat{B}(\kappa_i))$ and $(\hat{S}(\kappa_j), \hat{B}(\kappa_j))$ satisfy

$$\hat{S}(\kappa_i) < \hat{S}(\kappa_j) \quad \textit{and} \quad \hat{B}(\kappa_i) > \hat{B}(\kappa_j). \tag{4.35}$$

Proof. First of all, we show that if $B \in [\overline{B}, \infty)$, $g_B(z) > -k$ for $z \geq 0$. Since

$$\begin{aligned} g'_{\overline{B}}(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy \cdot e^{\lambda_1 z}\\ &\quad -\big(\overline{B} - h'(0) - \int_0^z e^{\lambda_2 y}h''(y)dy\big)e^{-\lambda_2 z}\Big]\\ &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy \cdot \big(e^{\lambda_1 z} - e^{-\lambda_2 z}\big)\end{aligned}$$

$$
\begin{aligned}
&+\int_0^z \left(e^{\lambda_2 y}-e^{-\lambda_1 y}\right)h''(y)dy\cdot e^{-\lambda_2 z}\Big] \\
\geq\ & 0,
\end{aligned}
$$

and

$$
\begin{aligned}
g_{\overline{B}}(0) &= \frac{1}{\beta}h'(0)+\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\frac{1}{\lambda_1}\int_0^\infty e^{-\lambda_1 y}h''(y)dy+\frac{1}{\lambda_2}(\overline{B}-h'(0+))\Big] \\
&= \frac{1}{\beta}h'(0)+\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\frac{1}{\lambda_1}\int_0^\infty e^{-\lambda_1 y}h''(y)dy+\frac{1}{\lambda_2}\int_0^\infty e^{-\lambda_1 y}h''(y)dy\Big] \\
&\geq 0,
\end{aligned}
$$

we can conclude that $g_{\overline{B}}(z)>0$ for $z\geq 0$. By (4.5), $g_B(z)$ is strictly increasing in B for any fixed $z\geq 0$, which implies that $g_B(z)\geq g_{\overline{B}}(z)>0>-k$ for $z\geq 0$ if $B\in[\overline{B},\infty)$. Thus, we only need to consider $B\in(-\infty,\overline{B})$.

By (4.30), we have $\varphi_0'(B)>0$. By (A4) in Assumption 2, there exist c_0 and c_1 such that $h'(z)\leq c_0+c_1z^{n+1}$ for all $z\geq 0$, which further implies that $\int_0^\infty e^{-\lambda_1 y}h'(y)dy$ is bounded. Therefore, we have

$$
\lim_{B\to-\infty} g_B(0) = \lim_{B\to-\infty}\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_0^\infty e^{-\lambda_1 y}h'(y)dy+\frac{B}{\lambda_2}\Big]=-\infty.
$$

Together with that $g_{\overline{B}}(0)\geq 0$, we can conclude that there exist a unique $\widetilde{B}\in(-\infty,\overline{B})$ such that

$$
g_{\widetilde{B}}(0)=-k. \tag{4.36}
$$

By Lemma 4.1 ,(4.36) and $\varphi_0'(B)>0$, we can conclude that for any $B\in[\widetilde{B},\infty)$, $g_B(z)>-k$ for $z>0$ and that for any $B\in(-\infty,\widetilde{B})$, there exists a unique $S(B)>0$ such that

$$
g_B(S(B))=-k.
$$

By the Implicit Function Theorem, $S(B)$ is a continuous function of B and the derivative $S'(B)$ exists. For $B \in (-\infty, \widetilde{B})$,define

$$\Lambda(B) = \int_0^{S(B)} [g_B(y) + k] dy. \tag{4.37}$$

Finally, we prove that for any $\kappa > 0$, there exists a unique $\hat{B}(\kappa)$ such that $\Lambda(\hat{B}(\kappa)) = -\kappa$. To show the existence and uniqueness of $\hat{B}(\kappa)$, it suffices to show that

$$\Lambda'(B) > 0 \quad \text{for } B \in (-\infty, \widetilde{B}), \tag{4.38}$$

$$\lim_{B \to \widetilde{B}} \Lambda(B) = 0, \tag{4.39}$$

$$\lim_{B \to -\infty} \Lambda(B) = -\infty. \tag{4.40}$$

Firstly, we will show (4.38). From the definition of $\Lambda(B)$ in (4.37), we have

$$\begin{aligned} \Lambda'(B) &= \frac{\mathrm{d}}{\mathrm{d}B}\Big(\int_0^{S(B)} [g_B(y) + k] dy\Big) \\ &= \int_0^{S(B)} \varphi_y'(B) dy + [g_B(S(B)) + k] S'(B) \\ &= \int_0^{S(B)} \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_2} e^{-\lambda_2 y} dy \\ &> 0. \end{aligned}$$

By (4.36), we can conclude that $\lim_{B \to \widetilde{B}} S(B) = 0$. Therefore, we have

$$\lim_{B \to \widetilde{B}} \Lambda(B) = 0.$$

It remains to prove (4.40). Taking the derivatives of the both sides of equation $g_B(S(B)) = -k$, i.e.,

$$\begin{aligned} &\frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\int_{S(B)}^{\infty} e^{-\lambda_1 y} h'(y) dy \cdot e^{\lambda_1 S(B)} \\ &+ \frac{e^{-\lambda_2 S(B)}}{\lambda_2} \Big(B + \lambda_2 \int_0^{S(B)} e^{\lambda_2 y} h'(y) dy \Big) \Big] = -k, \end{aligned}$$

with respect to B for $B \in (-\infty, \widetilde{B})$, we have

$$S'(B) = -\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{e^{-\lambda_2 S(B)}}{\lambda_2}\frac{1}{g'(S(B))} < 0,$$

where the inequality follows from Lemma 4.1. Together with that $S(B) > 0$ for $B \in (-\infty, \widetilde{B})$, we can conclude that $\lim_{B\to-\infty} S(B) > 0$. Then (4.40) follows from (4.38) and

$$\begin{aligned}
\lim_{B\to-\infty} \Lambda'(B) &= \lim_{B\to-\infty} \int_0^{S(B)} \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{-\lambda_2 y}dy \\
&= \lim_{B\to-\infty} \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2^2}[1-e^{-\lambda_2 S(B)}] \\
&> 0.
\end{aligned}$$

It remains to prove (4.35). By (4.33), (4.34) and the Implicit Function Theorem, the derivatives $\hat{B}'(\kappa)$ and $\hat{S}'(\kappa)$ exist. Then to prove (4.35), it suffices to show

$$\hat{B}'(\kappa) < 0 \quad \text{and} \quad \hat{S}'(\kappa) > 0.$$

Taking derivative of (4.33) and (4.34) with respect to κ, we can obtain

$$[g_{\hat{B}(\kappa)}(\hat{S}(\kappa)) + k]\hat{S}'(\kappa) + \int_0^{\hat{S}(\kappa)} [\varphi_y'(\hat{B}(\kappa)) \cdot \hat{B}'(\kappa)]dy = -1,$$

and

$$\varphi_{\hat{S}(\kappa)}'(\hat{B}(\kappa)) \cdot \hat{B}'(\kappa) + g_{\hat{B}(\kappa)}'(\hat{S}(\kappa)) \cdot \hat{S}'(\kappa) = 0.$$

Then by (4.30) and (4.34), we can conclude that

$$\hat{B}'(\kappa) = -\frac{\sigma^2(\lambda_1+\lambda_2)\lambda_2^2}{2(1-e^{-\lambda_2\hat{S}(\kappa)})} < 0 \quad \text{and} \quad \hat{S}'(\kappa) = -\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}\frac{e^{-\lambda_2\hat{S}(\kappa)}\hat{B}'(\kappa)}{g_{\hat{B}(\kappa)}'(\hat{S}(\kappa))} > 0,$$

where the first inequality follows from $\hat{S}(\kappa) > 0$ and $\lambda_1, \lambda_2 > 0$ and the second inequality follows from Lemma 4.1 and $\hat{B}(\kappa) \in (-\infty, \overline{B})$. □

If $\kappa = K_n$ where $n = 1, \ldots, M$ and let $(\hat{S}_n, \hat{B}_n)$ denote the pair $(\hat{S}(K_n), \hat{B}(K_n))$, the conditions (4.33) and (4.34) are equivalent to conditions (4.6) and (4.7) in Step 1. Then for K_n where $n = 1, \ldots, M$, we obtain a set of $(0, S)$ policies $\{U(0, \hat{S}_n) : n = 1, \ldots, M\}$. When the setup cost is K_n, the quantity of an order is constrained to an interval from Q_n to Q_{n+1} (which by (1.5) might be (Q_n, Q_{n+1}), $(Q_n, Q_{n+1}]$, $[Q_n, Q_{n+1})$ or $[Q_n, Q_{n+1}]$). However, the obtained $\hat{S}_n$ might not fall into an interval from Q_n to Q_{n+1}. In Step 2, we define S_n^* in (4.8) based on the relative position of $\hat{S}_n$ to the interval (Q_n, Q_{n+1}). By the definition of S_n^*, we have $Q_n \leq S_n^* \leq Q_{n+1}$ and $0 < S_n^* < \infty$ where $n = 1, \ldots, M$. In the following lemma, we show that given such an S_n^*, there exists a unique B_n^* such that equation (4.26) holds.

Lemma 4.3. *For any K_n where $n = 1, \ldots, M$, together with S_n^* defined in* (4.8)*, there exists a unique $B_n^* \in (-\infty, \overline{B})$ such that*

$$\int_0^{S_n^*} [g_{B_n^*}(y) + k]dy = -K_n. \tag{4.41}$$

Furthermore, we have

$$B_n^* \leq \hat{B}_n. \tag{4.42}$$

Proof. Define

$$\Lambda_1(B) = \int_0^{S_n^*} [g_B(y) + k]dy. \tag{4.43}$$

To prove (4.41), it suffices to show that

$$\begin{aligned} \Lambda_1'(B) &> 0, && (4.44) \\ \lim_{B \to \overline{B}} \Lambda_1(B) &> 0, && (4.45) \\ \lim_{B \to -\infty} \Lambda_1(B) &= -\infty. && (4.46) \end{aligned}$$

Firstly, we show the inequality (4.44). By taking derivative of (4.43) with regard to B, we have

$$\begin{aligned} \Lambda_1'(B) &= \int_0^{S_n^*} \varphi_y'(B)dy \\ &= \int_0^{S_n^*} \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{-\lambda_2 y}dy \\ &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2^2}\left(1-e^{-\lambda_2 S_n^*}\right) \\ &> 0, \end{aligned}$$

where the inequality follows the fact that $S_n^* > 0$. Then we show (4.45). By (4.36),

$$\lim_{B\to\overline{B}} g_B(0) > g_{\widetilde{B}}(0) = -k,$$

which together with Lemma 4.1 and $S_n^* > 0$ implies

$$\lim_{B\to\overline{B}} \Lambda_1(B) > \lim_{B\to\overline{B}} S_n^* \cdot [g_B(0)+k] > 0.$$

Next, we are going to show (4.46).

$$\begin{aligned} \lim_{B\to-\infty} \Lambda_1(B) &= \lim_{B\to-\infty} \int_0^{S_n^*} [g_B(y)+k]dy \\ &< \lim_{B\to-\infty} S_n^* \cdot [g_B(S_n^*)+k] \\ &= -\infty, \end{aligned}$$

where the inequality follows the monotonicity of $g_B(z)$ shown in Lemma 4.1 and the last equality follows from the fact $0 < S_n^* < \infty$ and (4.5).

Finally, we show (4.42) by contradiction. Suppose $B_n^* > \hat{B}_n$, then $g_{B_n^*}(z) > g_{\hat{B}_n}(z)$ must hold for any $z \geq 0$ by (4.5). Since $g_{\hat{B}_n}(z) \leq -k$ for $z \in [0, \hat{S}_n]$ and $g_{\hat{B}_n}(z) > -k$ for $z \in (\hat{S}_n, \infty)$,

$$\int_0^{S_n^*} [g_{B_n^*}(y)+k]dy \quad > \quad \int_0^{S_n^*} [g_{\hat{B}_n}(y)+k]dy$$

$$
\begin{aligned}
&\geq \int_0^{\hat{S}_n} [g_{\hat{B}_n}(y) + k]dy \\
&= -K_n,
\end{aligned}
$$

which contradicts with (4.41). Hence, we must have $B_n^* \leq \hat{B}_n$. $\square$

In order to prove subsequent lemmas, we need the following lemma that shows the properties of $g_B(z)$ at the order-up-to level S_n^* of the $(0, S_n^*)$ policy.

Lemma 4.4. *For $n \in \mathcal{N}_<$,*

$$
g_{B_n^*}(S_n^*) \geq -k. \tag{4.47}
$$

For $n \in \mathcal{N}_=$,

$$
g_{B_n^*}(S_n^*) = -k. \tag{4.48}
$$

For $n \in \mathcal{N}_>$,

$$
g_{B_n^*}(S_n^*) \leq -k. \tag{4.49}
$$

Proof. For $n \in \mathcal{N}_=$, we have $S_n^* = \hat{S}_n$. Comparing Lemma 4.2 and Lemma 4.3, we have $B_n^* = \hat{B}_n$. Therefore, according to Lemma 4.2, we have $g_{B_n^*}(S_n^*) = -k$.

For $n \in \mathcal{N}_<$, we have $\hat{S}_n \leq Q_n = S_n^*$. We prove (4.47) for two cases: $\hat{S}_n = Q_n = S_n^*$ and $\hat{S}_n < Q_n = S_n^*$. If $\hat{S}_n = Q_n = S_n^*$, similar to $n \in \mathcal{N}_=$, $B_n^* = \hat{B}_n$, which further implies $g_{B_n^*}(S_n^*) = -k$. If $\hat{S}_n < Q_n = S_n^*$, we show (4.47) by contradiction. Suppose $g_{B_n^*}(S_n^*) < -k$, then

$$
-K_n = \int_0^{S_n^*} [g_{B_n^*}(y) + k]dy < \int_0^{\hat{S}_n} [g_{B_n^*}(y) + k]dy \leq \int_0^{\hat{S}_n} [g_{\hat{B}_n}(y) + k]dy = -K_n,
$$

where the first inequality follows from $\hat{S}_n < S_n^*$, $g_{B_n^*}(S_n^*) < -k$ and Lemma 4.1 and the second inequality follows from (4.42). Therefore, $g_{B_n^*}(S_n^*) \geq -k$ for $n \in \mathcal{N}_<$.

For $n \in \mathcal{N}_>$, we have $\hat{S}_n \geq Q_{n+1} = S_n^*$. We prove (4.49) for two cases: $\hat{S}_n = Q_{n+1} = S_n^*$ and $\hat{S}_n > Q_{n+1} = S_n^*$. If $\hat{S}_n = Q_{n+1} = S_n^*$, similar to $n \in \mathcal{N}_=$, $B_n^* = \hat{B}_n$, which further implies $g_{B_n^*}(S_n^*) = -k$. If $\hat{S}_n > Q_{n+1} = S_n^*$, we show (4.49) by contradiction. Suppose $g_{B_n^*}(S_n^*) > -k$, then

$$-K_n = \int_0^{S_n^*} [g_{B_n^*}(y) + k]dy < \int_0^{\hat{S}_n} [g_{B_n^*}(y) + k]dy \leq \int_0^{\hat{S}_n} [g_{\hat{B}_n}(y) + k]dy = -K_n,$$

where the first inequality follows from $\hat{S}_n > S_n^*$, $g_{B_n^*}(S_n^*) > -k$ and Lemma 4.1 and the second inequality follows from (4.42). Therefore, $g_{B_n^*}(S_n^*) \leq -k$ for $n \in \mathcal{N}_>$. $\square$

According to the definition in of $\mathcal{N}_<$, $\mathcal{N}_=$ and $\mathcal{N}_>$ in Section 4.2, we can conclude that sets $\mathcal{N}_=$, $\mathcal{N}_>$ and $\mathcal{N}_<$ are disjoint and $\mathcal{N}_= \cup \mathcal{N}_< \cup \mathcal{N}_> = \{1, 2, \ldots, M\}$.

Notice that for $n \in \mathcal{N}_> \cup \mathcal{N}_<$, we may not have $K(S_n^*) = K_n$ by the setup cost given in (1.5). We define

$$\begin{aligned} \mathcal{N} &= \{n \in \mathcal{N}_> \cup \mathcal{N}_< : K(S_n^*) = K_n\}, \\ \overline{\mathcal{N}} &= \{n \in \mathcal{N}_> \cup \mathcal{N}_< : K(S_n^*) \neq K_n\}. \end{aligned}$$

Then $\mathcal{N}_=$, $\mathcal{N}$ and $\overline{\mathcal{N}}$ are disjoint and $\mathcal{N}_= \cup \mathcal{N} \cup \overline{\mathcal{N}} = \{1, 2, \ldots, M\}$.

For each S_n^*, we have $\int_0^{S_n^*} [g_{B_n^*}(y) + k]dy = -K_n$ by (4.41). However, $\int_0^{S_n^*} [g_{A_n^*}(y) + k]dy = -K(S_n^*)$ (equivalent to (4.26)) may not hold since the equation $K(S_n^*) = K_n$ may not hold. By the following lemma, we show that we should always have $K(S_{n^*}^*) = K_{n^*}$ if we select the n^* by (4.9).

To state the next lemma, we first define

$$\underline{\chi}(n) = \max\{i = 1, \ldots, n-1 : i \in \mathcal{N} \cup \mathcal{N}_=\} \text{ for } n \in \mathcal{N}_< \backslash \mathcal{N}, \tag{4.50}$$

$$\overline{\chi}(n) = \min\{i = n+1, \ldots, M : i \in \mathcal{N} \cup \mathcal{N}_=\} \text{ for } n \in \mathcal{N}_> \backslash \mathcal{N}. \tag{4.51}$$

Lemma 4.5. *For each $n \in \mathcal{N}_< \backslash \mathcal{N}$, $\underline{\chi}(n)$ defined in* (4.50) *exists and satisfies $B_n^* < B_{\underline{\chi}(n)}^*$; for each $n \in \mathcal{N}_> \backslash \mathcal{N}$, $\overline{\chi}(n)$ defined in* (4.51) *exists and satisfies $B_n^* < B_{\overline{\chi}(n)}^*$.*

Proof. For each $n \in \mathcal{N}_< \backslash \mathcal{N}$, we prove the existence of $\underline{\chi}(n)$ by contradiction. Suppose for some $n \in \mathcal{N}_< \backslash \mathcal{N}$, $\underline{\chi}(n)$ does not exist, namely, $i \notin \mathcal{N} \cup \mathcal{N}_=$ and $K(S_i^*) \neq K_i$ for $i = 1, \ldots, n-1$. Since $K_1 > 0$, Lemma 4.2 implies $\hat{S}_1 > 0 = Q_1$, from which we can deduce $1 \notin \mathcal{N}_<$ and $n \geq 2$. $n \in \mathcal{N}_< \backslash \mathcal{N}$ implies that $\hat{S}_n \leq Q_n = S_n^*$ and $K(S_n^*) = K_{n-1} < K_n$. By (4.35), we have $\hat{S}_{n-1} < \hat{S}_n \leq Q_n$, which together with $n-1 \notin \mathcal{N} \cup \mathcal{N}_=$ implies $n-1 \in \mathcal{N}_< \backslash \mathcal{N}$. By induction, we can obtian $\{1, \ldots, n-1\} \subset \mathcal{N}_< \backslash \mathcal{N}$, which contradicts the fact that $1 \notin \mathcal{N}_<$. Therefore, $\underline{\chi}(n)$ must exist.

For each $n \in \mathcal{N}_< \backslash \mathcal{N}$, we can conclude $\{\underline{\chi}(n)+1, \ldots, n\} \subset \mathcal{N}_< \backslash \mathcal{N}$ and $K_{\underline{\chi}(n)} < \cdots < K_n$ from above arguments. Then by (4.35), we have $\hat{S}_{\underline{\chi}(n)} < \hat{S}_{\underline{\chi}(n)+1} \leq Q_{\chi(n)+1}$, which implies that $\underline{\chi}(n) \in \mathcal{N}_< \cup \mathcal{N}_=$. By Lemma 4.1 and Lemma 4.4, we have

$$g_{B^*_{\underline{\chi}(n)}}(z) > -k \quad \text{for } z > S^*_{\underline{\chi}(n)}.$$

By (4.8), we have $S^*_{\underline{\chi}(n)} < S_n^*$. Then

$$\int_0^{S_n^*} [g_{B^*_{\underline{\chi}(n)}}(y) + k]dy > \int_0^{S^*_{\underline{\chi}(n)}} [g_{B^*_{\underline{\chi}(n)}}(y) + k]dy = -K_{\underline{\chi}(n)}.$$

Furthermore, we have

$$\int_0^{S_n^*} [g_{B_n^*}(y) + k]dy = -K(S_n^*) = -K_{n-1}.$$

We can conclude that $B_n^* < B^*_{\underline{\chi}(n)}$ from $K_{\underline{\chi}(n)} \leq K_{n-1}$ and (4.5).

Using the fact that $Q_{M+1} = \infty$, we prove that $\overline{\chi}(n)$ exists and $B_n < B_{\overline{\chi}(n)}$ by similar rationales. The details are omitted. □

We obtain the n^* by (4.9) and S^*, K^* by (4.10) . Notice that by Lemma 4.5, we always have $n^* \in \mathcal{N}_= \cup \mathcal{N}$, namely we always have $\int_0^{S^*} [g_{B^*}(y) + k]dy = -K(S^*)$. We will prove the optimality of this $(0, S^*)$ policy in Section 4.5.2.

4.5.2 Verification

In this subsection, we prove Theorem 4.1. Namely, we will prove that the $(0, S^*)$ policy obtained by the four-step algorithm in Section 4.2 is an optimal policy for our Brownian inventory model in this chapter. To prove Theorem 4.1, we need the following technical result.

Lemma 4.6. *If $K(z_2 - z_1) = K_n$, for $n = 1, 2, \ldots, M$ and $z_2 > z_1 \geq 0$, then*

$$\int_{z_1}^{z_2} [g_{B^*}(y) + k]dy \geq \int_0^{S_n^*} [g_{B_n^*}(y) + k]dy = -K_n. \tag{4.52}$$

Proof. Consider three cases: $n \in \mathcal{N}_=$, $n \in \mathcal{N}_<$ and $n \in \mathcal{N}_>$. For $n \in \mathcal{N}_=$, we have

$$\begin{aligned}
\int_{z_1}^{z_2} [g_{B^*}(y) + k]dy &\geq \int_{z_1}^{z_2} [g_{B_n^*}(y) + k]dy \\
&\geq \int_0^{z_2 - z_1} [g_{B_n^*}(y) + k]dy \\
&\geq \int_0^{S_n^*} [g_{B_n^*}(y) + k]dy \\
&= -K_n,
\end{aligned}$$

where the first inequality follows from $B^* \geq B_n^*$, the second inequality follows from Lemma 4.1 and the third inequality follows from Lemma 4.1 and (4.48).

For $n \in \mathcal{N}_<$, we have $z_2 - z_1 \geq S_n^*$. Then

$$\begin{aligned}
\int_{z_1}^{z_2} [g_{B^*}(y) + k]dy &\geq \int_{z_1}^{z_2} [g_{B_n^*}(y) + k]dy \\
&\geq \int_0^{z_2 - z_1} [g_{B_n^*}(y) + k]dy \\
&\geq \int_0^{S_n^*} [g_{B_n^*}(y) + k]dy \\
&= -K_n,
\end{aligned}$$

where the third inequality follows from Lemma 4.1, (4.47) and $z_2 - z_1 \geq S_n^*$.

For $n \in \mathcal{N}_>$, we have $z_2 - z_1 \le S_n^*$. Then

$$\begin{aligned}\int_{z_1}^{z_2} [g_{B^*}(y) + k]dy &\ge \int_{z_1}^{z_2} [g_{B_n^*}(y) + k]dy \\ &\ge \int_0^{z_2 - z_1} [g_{B_n^*}(y) + k]dy \\ &\ge \int_0^{S_n^*} [g_{B_n^*}(y) + k]dy \\ &= -K_n,\end{aligned}$$

where the third inequality follows from Lemma 4.1, (4.49) and $z_2 - z_1 \le S_n^*$. □

Finally, we provide the proof of Theorem 3.1.

Proof of Theorem 4.1. First, we show that $DC(x, U(0, S^*)) = V_{B^*}(x)$ for $x \ge 0$. Namely, we need to show $V_{B^*}(\cdot)$ satisfies the conditions in Theorem 4.4. By the definition of $V_B(z)$ in (4.4) and Proposition 4.1, $V_{B^*}(\cdot)$ is twice continuously differentiable, $V_B'(\cdot)$ is polynomially bounded and

$$\Gamma V_{B^*}(z) - \beta V_{B^*}(z) + h(z) = 0 \quad \text{for } z \ge 0.$$

By Lemma 4.3 and Lemma 4.5, we further have

$$\int_0^{S^*} [g_{B^*}(y) + k]dy = -K^* = -K(S^*),$$

which can be rewritten as

$$V_{B^*}(S^*) - V_{B^*}(0) = -K^* - k \cdot S^* = -K(S^*) - k \cdot S^*.$$

Therefore, $DC(x, U(0, S^*)) = V_{B^*}(x)$ for $x \ge 0$.

To show the optimality of the $(0, S^*)$ policy, it suffices to show that $V_{B^*}(\cdot)$ satisfies all the conditions of the lower bound in Theorem 4.2. We have already show that V_{B^*} satisfies (4.11) with equality, that V_{B^*} is twice continuously differentiable and that V_{B^*}'

is polynomially bounded. It remains to show (4.12). By Lemma 4.6, for $z_2 > z_1 \geq 0$

$$\int_{z_1}^{z_2} [g_{B^*}(y) + k]dy \geq -K(z_2 - z_1),$$

which can be rewritten as

$$V_{B^*}(z_1) - V_{B^*}(z_2) \leq K(z_2 - z_1) + k \cdot (z_2 - z_1).$$

Therefore,

$$DC(x, Y) \geq DC(x, U(0, S^*)) \text{ for all } x \geq 0 \text{ and } Y \in \mathcal{U}.$$

Furthermore,

$$DC(x, U(0, S^*)) = V_{B^*}(x) \quad \text{for } x \geq 0.$$

□

4.6 Policies Subject to Order-up-to Bounds

We prove Theorem 4.3 in this section. Let Y be an arbitrary admissible policy that $DC(x, Y)$ is finite, otherwise Theorem 4.3 holds directly. Based on this Y, we first construct a policy subject to an order-up-to bound m, i.e., $Y_m \in \mathcal{U}_m$, where m is a constant positive integer. Then, we prove that $\{Y_m \in \mathcal{U}_m : m = 1, 2, \ldots\}$ has a subsequence that satisfies (4.15).

For each admissible policy Y, we would construct a policy $Y_m \in \mathcal{U}_m$ that incurs less holding cost and less proportional cost. As m goes large, the discounted setup cost under Y_m should be asymptotically dominated by that under Y. Since the controller is obliged to keep $Z(t) \geq 0$, we also must make sure $Z_m(t) \geq 0$ under Y_m. Based on Y, such a Y_m is constructed as follows. The construction of modified policy Y_m follows from the procedure in He et al. (2015).

Let Y_m^c be the continuous part of Y_m. Under Y_m, the inventory level at time t is

$$Z_m(t) = X(t) + Y_m(t), \tag{4.53}$$

where $X(t)$ is given by (4.2) and

$$Y_m(t) = Y_m^c(t) + \sum_{0 \le u \le t} \Delta Y_m(u).$$

The continuous part of Y_m is constructed by

$$Y_m^c(t) = \int_0^t 1_{\{Z(u) \le m\}} dY^c(u), \tag{4.54}$$

where Y^c is the continuous part of Y. On each sample path, Y_m can have a jump either at a jump time of Y or at a hitting time of zero by Z_m. More specifically, let $J_m = \{t \ge 0 : \Delta Y_m(t) > 0\}$ be the set of jump times of Y_m, $J = \{t \ge 0 : \Delta Y(t) > 0\}$ be the jump times of Y, and $H_m = \{t \ge 0 : Z_m(t-) = 0\}$ be the set of hitting times of zero by Z_m. Then, $J_m \subset J \cup H_m$. The size of each jump of Y_m is specifies as follows:

(J1) $\Delta Y_m(t) = 0$ for $t \in J$, if $Z_m(t-) > m/2$;

(J2) $\Delta Y_m(t) = \Delta Y(t)$ for $t \in J$, if $Z_m(t-) \le m/2$ and $Z_m(t-) + \Delta Y(t) \le m$;

(J3) $\Delta Y_m(t) = m - Z_m(t-)$ for $t \in J$, if $Z_m(t-) \le m/2$ and $Z_m(t-) + \Delta Y(t) > m$;

(J4) $\Delta Y_m(t) = Z(t) \wedge m$ for $t \in H_m \backslash J$, where Z is the inventory process under policy Y given in (4.1).

In other words, Y_m does not make jumps when the inventory level is above $m/2$. If the inventory level is below $m/2$, Y_m has simultaneous jumps with Y. Each simultaneous jump takes the corresponding jump size of Y, as long as the inventory level will not exceed m after the jump. Otherwise, the simultaneous jump will replenish the inventory to level m. In addition, Y_m will jump when the inventory level reaches zero. When the inventory drop to zero, it will replenish the inventory level to $Z(t) \wedge m$, i.e., the minimum value of $Z(t)$ and m.

The following lemma state that compared with policy Y, the modified policy Y_m maintains a lower inventory level and the keeps $Z_m(t) \geq 0$.

Lemma 4.7. *Let Y be an admissible policy. For a fixed positive integer m, let Y_m be the policy constructed by* (4.54) *and (J1)–(J4). Then, $Z_m(t) \leq Z(t)$ for all $t \geq 0$ on each sample path, where Z is the inventory process under policy Y.*

Proof. It is trivial that $Y_m \in \mathcal{U}_m$. Since $Z_m(0-) = Z(0-) = x$, we can conclude that $Z_m(0) \leq Z(0)$ according to the construction principles $(J1) - (J4)$. For $t \geq 0$, define

$$\tau = \sup\{u \in [0,t] : u \in H_m \backslash J, \Delta Y_m(u) > 0\},$$

with the convention $\sup \emptyset = 0$. Then we have the relationship $Z_m(\tau) < Z(\tau)$ according to $(J4)$. If there exists some $u \in (\tau, t]$ such that $\Delta Y_m(u) > 0$, the jump at time u must belong to type (J2) or type (J3), which implies that $\Delta Y_m(u) \leq \Delta Y(u)$. Because Y^c is nondecreasing, it follows from (4.54) that $Y_m^c(t) - Y_m^c(\tau) \leq Y^c(t) - Y^c(\tau)$. Therefore, $Z_m(t) \leq Z(t)$ for all $t \geq 0$. □

Lemma 4.7 implies that the policy Y_m incurs less holding cost and less proportional cost. Next, we prove the comparison theorem by establishing asymptotic dominance between the discounted cost incurred by these two policies.

Proof of Theorem 4.3. Firstly, we show that the incurred holding cost under Y_m is less than the incurred holding cost under Y. Since h is nondecreasing in $[0, \infty)$ by Assumption 2, from Lemma 4.7, we have

$$\int_0^\infty e^{-\beta t} h(Z_m(t))dt \leq \int_0^\infty e^{-\beta t} h(Z(t))dt \quad \text{for } t \geq 0. \tag{4.55}$$

Namely, the holding cost under Y_m is less than the holding cost under Y.

Next, we show Y_m incurs less proportional cost than Y, i.e.,

$$PC_m[0,t] \leq PC[0,t] \quad \text{for all } t \geq 0,$$

where $PC_m[0,t]$ is the proportional cost under modified policy Y_m during time interval $[0,t]$ and $PC[0,t]$ is the proportional cost under policy Y. Before we prove this inequality, we first define some notations. Let $\zeta_2(t)$, $\zeta_3(t)$ and $\zeta_4(t)$ represent the cumulative quantity of (J2)-type jumps, (J3)-type jumps and (J4)-type jumps during time interval $[0,t]$ respectively. Then we have the relationship

$$Y_m(t) = \zeta_2(t) + \zeta_3(t) + \zeta_4(t).$$

Furthermore, we have $Y_m(t) \le Y(t)$ for all $t \ge 0$ by Lemma 4.7. Let $\tilde{\nu}_{i,m}$ for $i = 1, 2, \ldots$ represent the jumping times of positive (J4)-type jumps for constant m.For $t \ge \tilde{\nu}_{1,m}$, define $\eta(t)$ as

$$\eta(t) = \max\{i : \tilde{\nu}_{i,m} \le t\}.$$

Therefore, during time interval $(\tilde{\nu}_{\eta(t),m}, t]$, there is no positive (J4)-type jumps. According to (J1)–(J3), (J1)-type jumps, (J2)-type jumps and (J3)-type jumps under Y_m jump simultaneously with Y and jump quantities of these three types of jumps under Y_m are less or equal to the quantities of the simultaneous jumps under Y_m, which implies that $PC_m(\tilde{\nu}_{\eta(t),m}, t]$, the proportional cost incurred by Y_m during time interval $(\tilde{\nu}_{\eta(t),m}, t]$, is less or equal to $PC(\tilde{\nu}_{\eta(t),m}, t]$, the proportional cost incurred by Y during time interval $(\tilde{\nu}_{\eta(t),m}, t]$. Therefore,

$$\begin{aligned} & PC[0,t] - PC_m[0,t] \\ = \; & PC[0,\tilde{\nu}_{\eta(t),m}] + PC(\tilde{\nu}_{\eta(t),m}, t] - PC_m[0,\tilde{\nu}_{\eta(t),m}] - PC_m(\tilde{\nu}_{\eta(t),m}, t] \\ \ge \; & PC[0,\tilde{\nu}_{\eta(t),m}] - PC_m[0,\tilde{\nu}_{\eta(t),m}]. \end{aligned}$$

For $j = 1, 2, \ldots$, we show the following inequality by induction,

$$\begin{aligned} & PC[0,\tilde{\nu}_{j,m}] - PC_m[0,\tilde{\nu}_{j,m}] \\ \ge \; & k[Y(\tilde{\nu}_{j,m}) - \zeta_2(\tilde{\nu}_{j,m}) - \zeta_3(\tilde{\nu}_{j,m}) - \zeta_4(\tilde{\nu}_{j,m})]e^{-\beta\tilde{\nu}_{j,m}}. \end{aligned} \quad (4.56)$$

For $j = 1$, we have $\zeta_4(\tilde{\nu}_{1,m}-) = 0$. Since for jumps under Y_m of (J1) type, (J2) type and (J3) type jumps simultaneously with Y, we have $Y(\tilde{\nu}_{1,m}-) - Y_m(\tilde{\nu}_{1,m}-) = Y(\tilde{\nu}_{1,m}-) - \zeta_2(\tilde{\nu}_{1,m}-) - \zeta_3(\tilde{\nu}_{1,m}-)$, which is the amount of the order quantity under policy Y during $[0, \tilde{\nu}_{1,m})$ exceeds the order quantity under policy Y_m. Thus,

$$PC[0, \tilde{\nu}_{1,m}) - PC_m[0, \tilde{\nu}_{1,m}) \geq k[Y(\tilde{\nu}_{1,m}) - \zeta_2(\tilde{\nu}_{1,m}) - \zeta_3(\tilde{\nu}_{1,m})]e^{-\beta\tilde{\nu}_{1,m}}.$$

By the fact that Y does not jump at time point $\tilde{\nu}_{1,m}$ and that $\Delta\zeta_4(\tilde{\nu}_{1,m}) = \zeta_4(\tilde{\nu}_{1,m})$, we can obtain

$$\begin{aligned}
& PC[0, \tilde{\nu}_{1,m}] - PC_m[0, \tilde{\nu}_{1,m}] \\
\geq\ & k[Y(\tilde{\nu}_{1,m}) - \zeta_2(\tilde{\nu}_{1,m}) - \zeta_3(\tilde{\nu}_{1,m})]e^{-\beta\tilde{\nu}_{1,m}} - k\Delta\zeta_4(\tilde{\nu}_{1,m})e^{-\beta\tilde{\nu}_{1,m}} \\
=\ & k[Y(\tilde{\nu}_{1,m}) - \zeta_2(\tilde{\nu}_{1,m}) - \zeta_3(\tilde{\nu}_{1,m}) - \zeta_4(\tilde{\nu}_{1,m})]e^{-\beta\tilde{\nu}_{1,m}}.
\end{aligned}$$

Suppose (4.56) holds for $i \geq 1$, we will show (4.56) holds for $i + 1$.

$$\begin{aligned}
& PC[0, \tilde{\nu}_{i+1,m}] - PC_m[0, \tilde{\nu}_{i+1,m}] \\
=\ & PC(\tilde{\nu}_{i,m}, \tilde{\nu}_{i+1,m}] - PC_m(\tilde{\nu}_{i,m}, \tilde{\nu}_{i+1,m}] + PC[0, \tilde{\nu}_{i,m}] - PC_m[0, \tilde{\nu}_{i,m}] \\
\geq\ & PC(\tilde{\nu}_{i,m}, \tilde{\nu}_{i+1,m}] - PC_m(\tilde{\nu}_{i,m}, \tilde{\nu}_{i+1,m}] \\
& \quad + k[Y(\tilde{\nu}_{i,m}) - \zeta_2(\tilde{\nu}_{i,m}) - \zeta_3(\tilde{\nu}_{i,m}) - \zeta_4(\tilde{\nu}_{i,m})]e^{-\beta\tilde{\nu}_{i,m}} \\
\geq\ & PC(\tilde{\nu}_{i,m}, \tilde{\nu}_{i+1,m}] - PC_m(\tilde{\nu}_{i,m}, \tilde{\nu}_{i+1,m}] \\
& \quad + k[Y(\tilde{\nu}_{i,m}) - \zeta_2(\tilde{\nu}_{i,m}) - \zeta_3(\tilde{\nu}_{i,m}) - \zeta_4(\tilde{\nu}_{i,m})]e^{-\beta\tilde{\nu}_{i+1,m}} \\
\geq\ & k\Big[[Y(\tilde{\nu}_{i+1,m}) - Y(\tilde{\nu}_{i,m})] - [\zeta_2(\tilde{\nu}_{i+1,m}) - \zeta_2(\tilde{\nu}_{i,m})] \\
& \quad - [\zeta_3(\tilde{\nu}_{i+1,m}) - \zeta_3(\tilde{\nu}_{i,m})]\Big]e^{-\beta\tilde{\nu}_{i+1,m}} - k\Delta\zeta_4(\tilde{\nu}_{i+1,m})e^{-\beta\tilde{\nu}_{i+1,m}} \\
& \quad + k[Y(\tilde{\nu}_{i,m}) - \zeta_2(\tilde{\nu}_{i,m}) - \zeta_3(\tilde{\nu}_{i,m}) - \zeta_4(\tilde{\nu}_{i,m})]e^{-\beta\tilde{\nu}_{i+1,m}} \\
=\ & k[Y(\tilde{\nu}_{i+1,m}) - \zeta_2(\tilde{\nu}_{i+1,m}) - \zeta_3(\tilde{\nu}_{i+1,m}) - \zeta_4(\tilde{\nu}_{i+1,m})]e^{-\beta\tilde{\nu}_{i+1,m}},
\end{aligned}$$

where the second inequality follows from the fact that $\tilde{\nu}_{i+1,m} > \tilde{\nu}_{i,m}$ and $Y(t) \geq Y_m(t) = \zeta_2(t) + \zeta_3(t) + \zeta_4(t)$ for all $t \geq 0$ and the third inequality follows from the same rationale of proving case $i = 1$. Therefore, (4.56) holds for all $j \in \mathbb{Z}^+$, from which we can deduce

$$
\begin{aligned}
PC[0,t] - PC_m[0,t] &\geq PC[0,\tilde{\nu}_{\eta(t),m}] - PC_m[0,\tilde{\nu}_{\eta(t),m}] \\
&\geq [Y(\tilde{\nu}_{\eta(t),m}) - \zeta_2(\tilde{\nu}_{\eta(t),m}) - \zeta_3(\tilde{\nu}_{\eta(t),m}) - \zeta_4(\tilde{\nu}_{\eta(t),m})]e^{-\beta\tilde{\nu}_{\eta(t),m}} \\
&\geq 0,
\end{aligned}
$$

where the last inequality follows from the fact that $Y(t) \geq Y_m(t) = \zeta_2(t) + \zeta_3(t) + \zeta_4(t)$ for all $t \geq 0$ and $k \geq 0$. Namely the modified policy Y_m incurs less proportional cost.

Finally, we consider the setup cost. When a jump of type (J2) is made by Y_m, the setup cost is equal to that incurred by the simultaneous jump of Y. Consider two consecutive jumps of type (J3). Let τ_1 and τ_2 be the respective jump times with $0 \leq \tau_1 < \tau_2$. Because X has continuous sample paths and Y_m is nondecreasing, it follows from (4.53) that $X(\tau_1) - X(\tau_2) \geq Z_m(\tau_1) - Z_m(\tau_2-) \geq m/2$. Let

$$
\tau_3 = \inf\Big\{u \in (0, \tau_2 - \tau_1] : X(\tau_1 + u) = X(\tau_1) - \frac{m}{2}\Big\}.
$$

By the strong Markov property of Brownian motion, τ_3 has the same distribution as

$$
\bar{\tau} = \inf\Big\{u > 0 : -\mu u + \sigma B(u) = -\frac{m}{2}\Big\},
$$

where B is the standard Brownian motion. Then $\bar{\tau}$ is the first hitting time of $-m/2$ by a Brownian motion with drift $-\mu$. By Proposition 3.3 in Harrison (2013), the Laplace Transform $\mathbb{E}[e^{-\beta\bar{\tau}}]$ is given by

$$
\mathbb{E}[e^{-\beta\tau_3}] = \mathbb{E}[e^{-\beta\bar{\tau}}] = \exp\Big\{\frac{m(\mu - \sqrt{\mu^2 + 2\beta\sigma^2})}{2\sigma^2}\Big\}.
$$

Since $\tau_2 - \tau_1 \geq \tau_3$, it follows that $\mathbb{E}[e^{-\beta(\tau_2-\tau_1)}] \leq \exp\Big\{\frac{m(\mu-\sqrt{\mu^2+2\beta\sigma^2})}{2\sigma^2}\Big\}$, implying the expected discounted time between two consecutive (J3) jumps is less than $\exp\Big\{\frac{m(\mu-\sqrt{\mu^2+2\beta\sigma^2})}{2\sigma^2}\Big\}$.

Now consider two consecutive positive jumps of type (J4). Let $\tilde{\tau}_1$ and $\tilde{\tau}_2$ be the respective jump times with $0 \leq \tilde{\tau}_1 < \tilde{\tau}_2$. We would like to show that there exists some $\tilde{\tau}_0 \in [\tilde{\tau}_1, \tilde{\tau}_2)$ such that $Z_m(\tilde{\tau}_0) > m/2$. Since $\Delta Y_m(\tilde{\tau}_2) > 0$, we must have the relationship $Z_m(\tilde{\tau}_2-) \neq Z(\tilde{\tau}_2-)$. If $Z_m(\tilde{\tau}_1) = Z(\tilde{\tau}_1)$ holds, $\tilde{\tau}_0$ must exist. Otherwise, Y_m can only have jumps of type (J2) during $(\tilde{\tau}_1, \tilde{\tau}_2)$ and this yields $Z_m(\tilde{\tau}_2-) = Z(\tilde{\tau}_2-)$, a contradiction. If $Z_m(\tilde{\tau}_1) \neq Z(\tilde{\tau}_1)$, we have $Z_m(\tilde{\tau}_1) = m$ and thus set $\tilde{\tau}_0 = \tilde{\tau}_1$. Therefore, $Z_m(\tilde{\tau}_0) > m/2$ holds for some $\tilde{\tau}_0 \in [\tilde{\tau}_1, \tilde{\tau}_2)$. Define

$$\tilde{\tau}_3 = \inf\Big\{u \in (0, \tilde{\tau}_2 - \tilde{\tau}_0] : X(\tilde{\tau}_0 + u) = X(\tilde{\tau}_0) - \frac{m}{2}\Big\},$$

from which we can obtain

$$\mathbb{E}[e^{-\beta\tilde{\tau}_3}] = \mathbb{E}[e^{-\beta\bar{\tau}}] = \exp\Big\{\frac{m(\mu - \sqrt{\mu^2 + 2\beta\sigma^2})}{2\sigma^2}\Big\}.$$

Since $\tilde{\tau}_2 - \tilde{\tau}_1 \geq \tilde{\tau}_3$ it follows that $\mathbb{E}[e^{-\beta(\tilde{\tau}_2 - \tilde{\tau}_1)}] \leq \exp\Big\{\frac{m(\mu-\sqrt{\mu^2+2\beta\sigma^2})}{2\sigma^2}\Big\}$, implying the expected discounted time between two consecutive (J4) jumps is less than $\exp\Big\{\frac{m(\mu-\sqrt{\mu^2+2\beta\sigma^2})}{2\sigma^2}\Big\}$.

Let N be a positive number such that $K(\xi) < N$ for any $\xi > 0$. Let $\nu_{i,m}$ for $i = 1, 2, \ldots$ represent the jumping times of type (J3) jumps. Similarly, we use $\nu_{i,m}$ short for $\nu_{i,m}(m)$ in our thesis. By the strong Markov property of Brownian motion and the discussion above,

$$\begin{aligned} & DC(x, Y_m) - DC(x, Y) \\ \leq\ & N\mathbb{E}_x\Big[\sum_{i=1}^{\infty} e^{-\beta\nu_{i,m}} + \sum_{i=1}^{\infty} e^{-\beta\tilde{\nu}_{i,m}}\Big] \\ \leq\ & N\frac{1}{1 - \exp\Big\{\frac{m(\mu-\sqrt{\mu^2+2\beta\sigma^2})}{2\sigma^2}\Big\}}\Big(\mathbb{E}_x[e^{-\beta\nu_{1,m}}] + \mathbb{E}_x[e^{-\beta\tilde{\nu}_{1,m}}]\Big). \end{aligned} \tag{4.57}$$

Next, we show that $\lim_{m\to\infty} e^{-\beta\nu_{1,m}} = 0$ in probability by contradiction. Suppose not, i.e., there exist $\varepsilon_1, \delta_1 > 0$ such that for any $N_1 > 0$, there exists $m > N_1$,

$$\mathbb{P}(\Omega_{1,m}) \geq \delta_1,$$

where $\Omega_{1,m} = \{\omega : e^{-\beta\nu_{1,m}} > \varepsilon_1\}$. By the definition of (J3)-type jumps, $\Delta Y(\nu_{1,m}) \geq m/2$. Then for the sample path $\omega \in \Omega_{1,m}$, the present value of the proportional cost incurred by the jump at time $\nu_{1,m}$ under policy Y exceeds $e^{-\beta\nu_{1,m}} \cdot \frac{km}{2}$. Hence,

$$DC(x, Y) \geq \mathbb{E}_x[e^{-\beta\nu_{1,m}} \cdot \frac{km}{2}] \geq \frac{km}{2}\mathbb{E}_x[e^{-\beta\nu_{1,m}}] \geq \frac{km}{2}\varepsilon_1 \cdot \delta_1.$$

Since m can be arbitrarily large, we must have $DC(x, Y) = \infty$, a contradiction. Therefore, we have $\lim_{m\to\infty} e^{-\beta\nu_{1,m}} = 0$ in probability. Since $\nu_{1,m} > 0$, $e^{-\beta\nu_{1,m}}$ is bounded by 1. Then by the Bounded Convergence Theorem, we have $\lim_{m\to\infty} \mathbb{E}_x[e^{-\beta\nu_{1,m}}] = 0$.

Finally, we show $\lim_{m\to\infty} \mathbb{E}_x[e^{-\beta\tilde{\nu}_{1,m}}] = 0$ by two steps. Let $\hat{\nu}_{1,m}$ represent the first jumping time of the (J1)-type jump. In the first step, we show that $\tilde{\nu}_{1,m} \geq \min\{\nu_{1,m}, \hat{\nu}_{1,m}\}$ for each sample path under any $m > 0$ by contradiction. Suppose that $\tilde{\nu}_{1,m} < \min\{\nu_{1,m}, \hat{\nu}_{1,m}\}$ for some sample path. Then there are only jumps of type (J2) before the first positive jump of (J4) type. By the definition of (J2)-type jumps, we have the equation $Z_m(\tilde{\nu}_{1,m}-) = Z(\tilde{\nu}_{1,m}-)$. Since $\tilde{\nu}_{1,m} \in H_m \backslash J$, we must have $Z(\tilde{\nu}_{1,m}-) = Z(\tilde{\nu}_{1,m}) = Z_m(\tilde{\nu}_{1,m}-) = 0$, implying that $\Delta Y_m(\tilde{\nu}_{1,m}) = 0$, a contradiction. In the second step, we prove that $\lim_{m\to\infty} e^{-\beta\hat{\nu}_{1,m}} = 0$ in probability by contradiction. Suppose not, i.e., there exist $\varepsilon_2, \delta_2 > 0$ such that for any $N_2 > 0$, there exists $m > N_2$,

$$\mathbb{P}(\Omega_{2,m}) \geq \delta_2,$$

where $\Omega_{2,m} = \{\omega : e^{-\beta\hat{\nu}_{1,m}} > \varepsilon_2\}$. By the definition of (J1)-type jumps and Lemma 4.7, we have $Z(\hat{\nu}_{1,m}-) \geq Z_m(\hat{\nu}_{1,m}-) > m/2$. Then for the sample path $\omega \in \Omega_{2,m}$. the present value of the cost incurred under control policy Y during time interval $[0, \hat{\nu}_{1,m})$

must exceed $e^{-\beta\tilde{\nu}_{1,m}} \cdot \frac{k(m-2x)}{2}$. Hence,

$$DC(x,Y) \geq \mathbb{E}_x[e^{-\beta\hat{\nu}_{1,m}} \cdot \frac{k(m-2x)}{2}] \geq \frac{k(m-2x)}{2}\mathbb{E}_x[e^{-\beta\hat{\nu}_{1,m}}] \geq \frac{k(m-2x)}{2}\varepsilon_2 \cdot \delta_2.$$

Since m can be arbitrarily large and the initial inventory level x is finite, we must have $DC(x,Y) = \infty$, a contradiction. By the discussion above, we must have $\lim_{m\to\infty} e^{-\beta\tilde{\nu}_{1,m}} = 0$ in probability. Since $\tilde{\nu}_{1,m} > 0$, $e^{-\beta\tilde{\nu}_{1,m}}$ is bounded by 1. Then by the Bounded Convergence Theorem, we have $\lim_{m\to\infty} \mathbb{E}_x[e^{-\beta\tilde{\nu}_{1,m}}] = 0$.

Then by taking limsup as $m \to \infty$ of both sides of (4.57), we have

$$\limsup_{m\to\infty} DC(x, Y_m) \leq DC(x, Y).$$

By the Bolzano-Weierstrass theorem, $\{DC(x, Y_m) : m = 1, 2, \ldots\}$ has a convergent subsequence, so the inequality (4.15) holds. $\square$

Chapter 5

Optimal Policy Under the Discounted Cost Criterion with Backlogs and Non-negative Initial Inventory Level

In the previous two chapters, we have shown the optimal control policy for Brownian inventory models without backlogs under the long-run average cost criterion and the discounted cost criterion. However, in some inventory systems, backlogs are allowed and the shortage cost is incurred. In this chapter, we follow the two-step lower bound approach to obtain an optimal control policy for the continuous-review inventory model with backlogs under the discounted cost criterion. The optimal policy for a similar inventory model but under the long-run average cost criterion is discussed in He et al. (2015). In Section 5.1, we present the continuous-review model and assumptions. Section 5.2 presents the main results of this chapter. In Section 5.3, we establish a lower bound for the expected infinite-horizon discounted cost incurred by an arbitrary admissible policy. We compute the expected infinite-horizon discounted cost under an (s, S) policy in Section 5.4. In Section 5.5, we demonstrate how to select the optimal (s, S) policy and prove the optimality of it.

5.1 Model and Assumptions

In this chapter, we consider a continuous-review inventory model that is similar to the model in Chapter 4 but we assume that all unsatisfied demands are backlogged. Because we allow backlogs in the inventory system, we need to consider negative inventory level in the both steps of the lower bound approach. Let $D(t)$ and $Y(t)$ be the cumulative demand quantity and the cumulative order quantity during time $[0, t]$. We assume only upward adjustments are allowed in our Brownian control model. Then the inventory level at time $t \geq 0$ is given by

$$Z(t) = x - D(t) + Y(t),$$

where x is the initial inventory level. We assume the initial inventory level is non-negative in our model, namely $x \geq 0$. We leave the $x < 0$ case for the future. The process D can be represented as

$$D(t) = \mu t - \sigma B(t).$$

For convenience, we repeat the expression of the inventory level at time t

$$Z(t) = X(t) + Y(t), \tag{5.1}$$

where

$$X(t) = x - \mu t + \sigma B(t) \tag{5.2}$$

can be interpreted as the inventory level in the absence of control. Since we assume that all unsatisfied demands are backlogged, the inventory level at time t can be negative.

Two types of costs are incurred in this continuous-review inventory system: the inventory holding and shortage cost and the ordering cost. The inventory holding and shortage cost is incurred at a rate $h(z)$ when the inventory level is z. Since we assume that all unsatisfied demands are backlogged, h is defined on the real line $\mathbb{R}$. In the inventory model of this chapter, we assume that the inventory holding cost rate function h satisfies the following assumption.

Assumption 3. $h(\cdot)$ satisfies

(A1) $h(\cdot)$ is convex and $h(0) = 0$;

(A2) $h(\cdot)$ is twice continuously differentiable except at 0;

(A3) $h'(z) \leq 0$ for $z < 0$ and $\lim_{x \to -\infty} h'(x) < -\beta k - \beta \max_{1 \leq i \leq M} \{\frac{K_i}{Q_{i+1}}\}$ for $x < 0$; $h'(z) \geq 0$ for $z > 0$ and $\lim_{x \to +\infty} h'(x) > k\beta$ for $x > 0$, where $\beta > 0$ is the discount rate;

(A4) $h'(\cdot)$ and $h''(\cdot)$ are polynomially bounded. Namely, there exist positive constants $a_i > 0$, $i = 0, 1$ and a positive integer n such that $h''(z) \leq a_0 + a_1|z|^n$ for all $z \in \mathcal{R}$.

An ordering cost is incurred whenever an order is placed and this cost is a function of the ordering quantity $\xi > 0$. When an order with quantity $\xi > 0$ is placed, it incurs a *setup cost* of $K(\xi) > 0$ given by (1.5) and a *proportional cost* of $k\xi$ with proportional cost rate $k > 0$. Let $\phi(\xi)$ denote the ordering cost with ordering quantity ξ. Then $\phi(\xi)$ is given by (3.3).

Since $K(\xi) > 0$ for $\xi > 0$, we only need to consider the policies with $N(t) < \infty$ for any $t \in \mathbb{R}^+$, where $N(t)$ is cardinality of the set $\{u \in [0, t] : \Delta Y(u) = Y(u) - Y(u-) > 0\}$. Otherwise, the total cost would be infinite in the time interval $[0, t]$. Then a policy can be specified by a sequence of pairs $\{(T_i, \xi_i) : i = 0, 1, \ldots\}$ where T_i is the ith order time and $\xi_i = \Delta Y(T_i) = Y(T_i) - Y(T_i-)$ is the quantity of the ith order. Therefore, investigating an optimal control policy given that $K(\xi) > 0$ for $\xi > 0$ is equivalent to exploring a sequence of ordering time together with corresponding ordering quantity, $\{(T_i, \xi_i) : i = 0, 1, \ldots\}$, which turns out be an *impulse control* problem for the Brownian model.

We aim to find an admissible inventory policy Y that minimizes the expected discounted cost over the infinite horizon

$$DC(x, Y) = \mathbb{E}_x\Big[\int_0^\infty e^{-\beta t} h(Z(t))dt + \sum_{i=0}^\infty e^{-\beta T_i}(K(\xi_i) + k\xi_i)\Big], \tag{5.3}$$

where the constant $\beta > 0$ is the discounted rate and $\mathbb{E}_x$ is the expectation operator conditioning on the initial inventory level $Z(0-) = x > 0$.

5.2 Main Results

In this section, we present the main results of this chapter. Under the quantity-dependent setup cost defined in (1.5), an optimal policy for the Brownian inventory model with non-negative initial inventory level in this chapter is an (s, S) policy. We propose an algorithm for computing the optimal reorder level s and the order-up-to level S. We use $U(s, S)$ to denote an (s, S) policy. The optimal reorder level s^* and order-up-to level S^* can be obtained by the following algorithm.

Step 1. For $z \in \mathbb{R}$ and $B \in \mathbb{R}$, put $\lambda_1 = \frac{\mu+\sqrt{\mu^2+2\beta\sigma^2}}{\sigma^2}$, $\lambda_2 = \frac{-\mu+\sqrt{\mu^2+2\beta\sigma^2}}{\sigma^2}$. Let

$$\begin{aligned} V_B(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[-\frac{1}{\lambda_2^2}Be^{-\lambda_2 z} \\ &\quad + \int_0^z e^{-\lambda_2(z-y)}h(y)dy + \int_z^\infty e^{\lambda_1(z-y)}h(y)dy\Big]. \end{aligned} \tag{5.4}$$

$$\begin{aligned} g_B(z) &= V_B'(z) \\ &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h'(y)dy \cdot e^{\lambda_1 z} \\ &\quad + \frac{e^{-\lambda_2 z}}{\lambda_2}\Big(B + \lambda_2\int_0^z e^{\lambda_2 y}h'(y)dy\Big)\Big], \end{aligned} \tag{5.5}$$

Then

$$\begin{aligned} g_B'(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\lambda_1\int_z^\infty e^{-\lambda_1 y}h'(y)dy \cdot e^{\lambda_1 z} \\ &\quad - \Big(B + \lambda_2\int_0^z e^{\lambda_2 y}h'(y)dy\Big)e^{-\lambda_2 z}\Big]. \end{aligned} \tag{5.6}$$

For $n = 1, \ldots, M$, obtain $\hat{s}_n$, $\hat{S}_n$ and $\hat{B}_n$ by solving

$$\textstyle\int_{\hat{s}_n}^{\hat{S}_n}[g_{\hat{B}_n}(y) + k]dy = -K_n, \tag{5.7}$$

$$g_{\hat{B}_n}(\hat{s}_n) = g_{\hat{B}_n}(\hat{S}_n) = -k, \tag{5.8}$$

$$\hat{s}_n < \hat{S}_n. \tag{5.9}$$

Put $\hat{Q}_n = \hat{S}_n - \hat{s}_n$.

Step 2. Define

$$\mathcal{N}_< = \{n \in \{1, 2, \ldots, M\} : \hat{S}_n - \hat{s}_n \leq Q_n\},$$

$$\mathcal{N}_= = \{n \in \{1, 2, \ldots, M\} : \hat{S}_n - \hat{s}_n \in (Q_n, Q_{n+1})\},$$

$$\mathcal{N}_> = \{n \in \{1, 2, \ldots, M\} : \hat{S}_n - \hat{s}_n \geq Q_{n+1}\},$$

and

$$Q_n^* = \begin{cases} Q_n & \text{for } n \in \mathcal{N}_<, \\ \hat{S}_n - \hat{s}_n & \text{for } n \in \mathcal{N}_=, \\ Q_{n+1} & \text{for } n \in \mathcal{N}_>. \end{cases} \tag{5.10}$$

Step 3. For $n \in \mathcal{N}_=$, let

$$s_n^* = \hat{s}_n, \quad S_n^* = \hat{S}_n, \quad B_n^* = \hat{B}_n.$$

For $n \in \mathcal{N}_> \cup \mathcal{N}_<$, obtain (s_n^*, S_n^*, B_n^*) by solving

$$\begin{cases} S_n^* - s_n^* = Q_n^*, \\ \int_{s_n^*}^{S_n^*} [g_{B_n^*}(z) + k] dz = -K_n, \\ g_{B_n^*}(s_n^*) = g_{B_n^*}(S_n^*). \end{cases}$$

Step 4. Let $\mathcal{M} = \{1, \ldots, M\}$. Define

$$n^* = \min\{n \in \mathcal{M} : B_i^* \leq B_n^* \text{ for all } i \in \mathcal{M}\}. \tag{5.11}$$

Let

$$s^* = s^*_{n^*}, \quad S^* = S^*_{n^*}, \quad B^* = B^*_{n^*} \quad \text{and} \quad K^* = K_{n^*}. \tag{5.12}$$

In Step 1, we obtain the parameters $\hat{s}_n$, $\hat{S}_n$ associated with $\hat{B}_n$ by smoothness conditions similar with those in Sulem (1986). This $(\hat{s}_n, \hat{S}_n)$ policy is expected to be optimal when the setup cost is constant at K_n. However under the quantity-dependent setup cost, a quantity constraint is imposed on each setup cost value. Thus in Step 2 we obtain s^*_n and S^*_n, whose difference is confined with the corresponding interval. We obtain the auxiliary parameter B^*_n in Step 3. Finally in Step 4, we select the optimal s^* and S^* by picking the largest B^*_n. Please see Section 5.5 for explicit derivations. The difference between the algorithm is this section and the algorithm in Section 4.2 is that we have one more parameter s because the expected optimal policy in this chapter is of the (s, S) type.

Then the (s^*, S^*) policy obtained from the above algorithm is an optimal policy for our Brownian inventory model in this chapter. The optimality of this (s^*, S^*) policy is proved by the following theorem.

Theorem 5.1. *Assume the cost function h satisfies Assumption 3 and that the setup cost is given by* (1.5). *Control policy $U(s^*, S^*)$ obtained by Steps 1–4 is an optimal admissible policy that minimizes the discounted cost given by* (5.3) *if the initial inventory is non-negative. Namely, we have*

$$DC(x, U(s^*, S^*)) \leq DC(x, Y) \quad \textit{for } x \geq 0 \textit{ and } Y \in \mathcal{U}.$$

Moreover, the minimum discounted cost is $DC(x, U(s^*, S^*)) = V_{B^*}(x)$ for $x \geq 0$.

5.3 Lower Bound

In this section, we propose and prove a theorem that establishes a lower bound for the discounted cost incurred by any admissible control policy.

Theorem 5.2. *Assume that h satisfies Assumption 3. Let $f(\cdot) : \mathbb{R} \to \mathbb{R}$ be continuously differentiable with f' absolutely continuous. Assume that*

$$\Gamma f(z) - \beta f(z) + h(z) \geq 0 \quad \textit{for all } z \in \mathbb{R}, \tag{5.13}$$

where

$$\Gamma f(z) = \frac{1}{2}\sigma^2 f''(z) - \mu f'(z).$$

We further assume that

$$f(z_1) - f(z_2) \leq K(z_2 - z_1) + k \cdot (z_2 - z_1) \quad \textit{for all } z_1 < z_2, \tag{5.14}$$

and $f'(\cdot)$ is polynomially bounded, i.e., there exist positive constants a_0, a_1 and a positive integer n such that

$$|f'(z)| < a_0 + a_1 z^n \quad \textit{for } z \geq 0, \tag{5.15}$$

and

$$|f'(z)| < a_0 \quad \textit{for } z < 0. \tag{5.16}$$

Then

$$DC(x, Y) \geq f(x), \tag{5.17}$$

where $DC(x, Y)$, given by (5.3)*, is the total discounted cost over the infinite horizon under any admissible control policy Y.*

By Theorem 5.2, if we can find an admissible ordering policy Y whose cost function $DC(x, Y)$ satisfies the conditions of $f(\cdot)$ in Theorem 5.2, we can conclude that this control policy must be optimal among all the admissible policies. In the lower bound approach, Theorem 5.2 is referred as a verification theorem. In order to prove Theorem 5.2, we first need to show some technical results. Let $\mathcal{U}_m$ represent the set of admissible policies with order-up-to bound at m.

Theorem 5.3 (Comparison Theorem)**.** *Assume that the hold cost rate function h is nondecreasing on $[0,\infty)$ and the setup cost function $K(\cdot)$ is bounded. Then for any admissible policy Y, there exists a sequence of admissible policies $\{Y_m \in \mathcal{U}_m : m = 1, 2, \ldots\}$ such that*

$$\lim_{m\to\infty} DC(x, Y_m) \le DC(x, Y) \quad \textit{for all } x \in \mathbb{R}. \tag{5.18}$$

The proof of Theorem 5.3 follows the same rationale of Section 4.6 with $Z_m(t) = Z(t)$ when $Z_m(t) < 0$ shown in Lemma 8 of He et al. (2015). Let $\overline{\mathcal{U}}$ be the set of all admissible policies subject to order-up-to bound. Theorem 5.3 implies that a policy that is optimal in $\overline{\mathcal{U}}$ must be optimal in $\mathcal{U}$. Therefore, we only need to search all admissible policies subject to order-up-to bounds for the optimal policy as in Chapter 3 and in Chapter 4.

The following lemma provides three important results that are important for proving Theorem 5.2.

Lemma 5.1. *Let $f(\cdot) : \mathbb{R} \to \mathbb{R}$ be a differentiable function and Z be the inventory process given in* (5.1) *with $Y \in \overline{\mathcal{U}}$. Assume there exist positive constants a_0, a_1 and a positive integer n such that*

$$|f'(z)| < a_0 + a_1|z|^n \quad \textit{for all } z \in \mathbb{R}.$$

Then,

$$\mathbb{E}_x[|f(Z(t))|] < \infty \quad \textit{for } t \ge 0, \tag{5.19}$$

and

$$\mathbb{E}_x\Big[\int_0^t (e^{-\beta s}f'(Z(s)))^2 ds\Big] < \infty \quad \textit{for } t \geq 0. \tag{5.20}$$

Moreover,

$$\lim_{t\to\infty} \mathbb{E}_x[e^{-\beta t}|f(Z(t))\cdot 1_{\{Z(t)\geq 0\}}|] = 0. \tag{5.21}$$

This lemma can be derived form Lemma 3 in He et al. (2015) directly.

Proof of Theorem 5.2. By Theorem 5.3, it suffices to consider an arbitrary policy $Y \in \overline{\mathcal{U}}$. For any $Y \in \mathcal{U}_m$, recall that

$$Z(t) = x - \mu t + \sigma B(t) + Y(t).$$

By Itô's formula , we have

$$\begin{aligned} e^{-\beta t} f(Z(t)) &= f(Z(0)) + \int_0^t e^{-\beta s}(\Gamma f(Z(s)) - \beta f(Z(s))ds \\ &\quad + \sigma \int_0^t e^{-\beta s} f'(Z(s))dB(s) + \sum_{0<s\leq t} e^{-\beta s}\Delta f(Z(s)) \\ &= f(Z(0-)) + \int_0^t e^{-\beta s}(\Gamma f(Z(s)) - \beta f(Z(s))ds \\ &\quad + \sigma \int_0^t e^{-\beta s} f'(Z(s))dB(s) + \sum_{0\leq s\leq t} e^{-\beta s}\Delta f(Z(s)). \end{aligned}$$

By (5.13) and (5.14), we have

$$\begin{aligned} e^{-\beta t} f(Z(t)) &\geq f(x) - \int_0^t e^{-\beta s} h(Z(s))ds + \sigma \int_0^t e^{-\beta s} f'(Z(s))dB(s) \\ &\quad - \sum_{i=0}^{N(t)} e^{-\beta T_i}\phi(\xi_i). \end{aligned} \tag{5.22}$$

By (5.20) and Theorem 3.2.1 in Øksendal (2003),

$$\mathbb{E}_x\Big[\int_0^t e^{-\beta s} f'(Z(s))dB(s)\Big] = 0.$$

Since (5.19) holds, we can take expectation on both sides of (5.22) and obtain

$$\mathbb{E}_x[e^{-\beta t}f(Z(t))] \;\geq\; f(x) - \mathbb{E}_x[\int_0^t e^{-\beta s}h(Z(s))ds] - \mathbb{E}_x[\sum_{i=0}^{N(t)} e^{-\beta T_i}\phi(\xi_i)].$$

Taking limit as $t \to \infty$ of both the sides of the above inequality, we have

$$\liminf_{t\to\infty} \mathbb{E}_x[e^{-\beta t}f(Z(t))] + \liminf_{t\to\infty} \mathbb{E}_x[\int_0^t e^{-\beta s}h(Z(s))ds + \sum_{i=0}^{N(t)} e^{-\beta T_i}\phi(\xi_i)] \geq f(x).$$

It follows from (5.3) that

$$\liminf_{t\to\infty} \mathbb{E}_x[e^{-\beta t}f(Z(t))] + DC(x,Y) \geq f(x). \tag{5.23}$$

If $\liminf_{t\to\infty} \mathbb{E}_x[e^{-\beta t}f(Z(t))] \leq 0$, then $DC(x,Y) \geq f(x)$ follows from (5.23). If $\liminf_{t\to\infty} \mathbb{E}_x[e^{-\beta t}f(Z(t))] > c$ for a positive constant c, we show that $DC(x,Y) = \infty$, from which the conclusion $DC(x,Y) \geq f(x)$ follows.

When $\liminf_{t\to\infty} \mathbb{E}_x[e^{-\beta t}f(Z(t))] > c$, it follows from (5.21) that

$$\liminf_{t\to\infty} \mathbb{E}_x[e^{-\beta t}f(Z(t)) \cdot 1_{\{Z(t)<0\}}] > c,$$

and thus there sufficiently large t_c such that for $t > t_c$ we have

$$\mathbb{E}_x[e^{-\beta t}f(Z(t)) \cdot 1_{\{Z(t)<0\}}] > \frac{1}{2}c.$$

By (5.16), there exists a real number d_0 such that $|f(z)| < a_0|z|+d_0$ for $z < 0$. Therefore, for $t > t_c$ we must have

$$\mathbb{E}_x[|Z(t)|] > \frac{ce^{\beta t} - 2d_0}{2a_0}. \tag{5.24}$$

By the assumption that h is convex and $h'(z) > 0$ for $z > 0$ and (A3) in Assumption 3, we can find positive constants $d_1 > 0$ and $d_2 > 0$ such that $h(z) \geq d_1|z| - d_2$ for all

$z \in \mathbb{R}$. Therefore,

$$\mathbb{E}_x\Big[\int_0^\infty e^{-\beta t} h(Z(t))dt\Big] \geq \mathbb{E}_x\Big[\int_0^\infty e^{-\beta t} d_1|Z(t)|dt\Big] - d_2/\beta,$$

where by (5.24) and Tonelli's theorem, the right side must be positive infinite. Hence, $DC(x,Y) \geq f(x)$ follows from that $DC(x,Y) = \infty$.

5.4 Expected Discounted Cost Under (s,S) Policies

In this section, we will compute the discounted cost under any (s,S) policy. Under an (s,S) policy, the controller replenishes the inventory level to the order-up-to level S immediately when it drops below or equal to the reorder level s. By the definition of admissible policies, all (s,S) policies are in $\bar{\mathcal{U}}$, the set of admissible policies subject to order-up-to bounds.

Theorem 5.4. *For any $(s,S) \in \mathbb{R}^2$, if there exists a twice continuous differentiable function $V(\cdot) : \mathbb{R} \to \mathbb{R}$ and positive constants a_0, a_1 and a positive integer n such that*

$$\Gamma V(z) - \beta V(z) + h(z) = 0, \quad \textit{for } z \in \mathbb{R} \tag{5.25}$$

$$V(S) - V(s) = -K(S-s) - k\cdot(S-s), \tag{5.26}$$

$$|V'(z)| < a_0 + a_1 z^n, \quad \textit{for } z \geq 0. \tag{5.27}$$

Then, the total discounted cost under an (s,S) policy with initial inventory level x is given by

$$DC(x, U(s,S)) = \begin{cases} V(x) & \textit{for } x \in (s,\infty), \\ V(S) + K(S-x) + k\cdot(S-x) & \textit{for } x \in (-\infty, s]. \end{cases} \tag{5.28}$$

Proof. By Itô's formula (see (5.22)) together with (5.25) and (5.26), we have

$$e^{-\beta t}V(Z(t)) \quad = \quad V(Z(0)) + \sigma\int_0^t e^{-\beta t}V'(Z(s))dB(s)$$

$$- \int_0^t e^{-\beta t} h(Z(s))ds - \sum_{i=1}^{N(t)} e^{-\beta T_i} \phi(\xi_i). \tag{5.29}$$

By (5.20) and Theorem 3.2.1 in Øksendal (2003), we have

$$\mathbb{E}_x \Big[\int_0^t e^{-\beta s} f'(Z(s))dB(s) \Big] = 0.$$

We can take expectation of both sides of (5.29) because (5.19) holds. By taking expectation of (5.29) and taking limit as $t \to \infty$, we have

$$\lim_{t\to\infty} \mathbb{E}_x[e^{-\beta t} V(Z(t))] + DC(x, U(s,S)) = V(Z(0)) + \mathbb{E}_x[\phi(\xi_0)]. \tag{5.30}$$

Under an (s, S) policy, we have

$$|V(Z(t))| \leq |V(Z(t)) \cdot 1_{\{Z(t)\geq 0\}}| + \max\{|V(z)| : (s \wedge 0) \leq z \leq 0\}.$$

The first term in (5.30), $\lim_{t\to\infty} \mathbb{E}_x[e^{-\beta t} V(Z(t))] = 0$ follows from (5.21). Together with the fact that $\max\{|V(z)| : (s \wedge 0) \leq z \leq 0\}$ is bounded, (5.30) can be reduced to

$$DC(x, U(s,S)) = V(Z(0)) + \mathbb{E}_x[\phi(\xi_0)]. \tag{5.31}$$

By the definition of (s, S) policies, if $x \in (s, \infty)$, $Z(0) = Z(0-) = x$ and $\xi_0 = 0$. Then by (5.31), we have

$$DC(x, U(s,S)) = V(x).$$

If $x \in (-\infty, s]$, $Z(0) = S$ and $\xi_0 = S - x$ by the definition of (s, S) policies. Then by (5.31), we have

$$DC(x, U(s,S)) = V(S) + K(S - x) + k \cdot (S - x).$$

□

Proposition 5.1. *The solution to* (5.25)–(5.27) *is given by*

$$V(z) = B_0 e^{-\lambda_2 z} + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_0^z e^{-\lambda_2(z-y)}h(y)dy + \int_z^\infty e^{\lambda_1(z-y)}h(y)dy\Big], \quad (5.32)$$

where

$$\begin{aligned} B_0 &= \frac{1}{e^{-\lambda_2 S} - e^{-\lambda_2 s}}\Big\{ -K(S-s) - k\cdot(S-s) - \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_0^S e^{-\lambda_2(S-y)}h(y)dy \\ &\quad - \int_0^s e^{-\lambda_2(s-y)}h(y)dy + \int_S^\infty e^{\lambda_1(S-y)}h(y)dy - \int_s^\infty e^{\lambda_1(s-y)}h(y)dy\Big]\Big\}, \quad (5.33) \end{aligned}$$

and

$$\begin{aligned} \lambda_1 &= \frac{\mu + \sqrt{\mu^2 + 2\beta\sigma^2}}{\sigma^2}, \\ \lambda_2 &= \frac{-\mu + \sqrt{\mu^2 + 2\beta\sigma^2}}{\sigma^2}. \end{aligned}$$

The solution $V(z)$ given by (5.32) *together with* (5.33) *is unique.*

Proof. By the proof of Proposition 4.1, (5.32) is the solution to (5.25) and (5.27). By (5.26), we can derive that B_0 is given by (5.33). □

5.5 Optimal Policy

By (5.4),

$$\begin{aligned} V_B(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[-\frac{1}{\lambda_2^2}Be^{-\lambda_2 z} \\ &\quad + \int_0^z e^{-\lambda_2(z-y)}h(y)dy + \int_z^\infty e^{\lambda_1(z-y)}h(y)dy\Big] \\ &= B_1 e^{-\lambda_2 z} + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_0^z e^{-\lambda_2(z-y)}h(y)dy + \int_z^\infty e^{\lambda_1(z-y)}h(y)dy\Big] \end{aligned}$$

where

$$B_1 = -\frac{2}{\lambda_2^2\sigma^2(\lambda_1+\lambda_2)}B.$$

By Proposition 5.1, $V_B'(z)$ is polynomially bounded on $[0,\infty)$. Furthermore, $V_B(z)$ satisfies the differential equation

$$\Gamma V_B(z) - \beta V_B(z) + h(z) = 0 \text{ for } z \in \mathbb{R}.$$

For any (s,S) policy, if $V_B(z)$ further satisfies

$$V_B(S) - V_B(s) = -K(S-s) - k \cdot (S-s), \tag{5.34}$$

we can conclude by Theorem 5.4 that the discounted cost under an (s,S) policy is given by

$$DC(x, U(s,S)) = \begin{cases} V_B(x) & \text{for } x \in (s,\infty), \\ V_B(S) + K(S-x) + k \cdot (S-x) & \text{for } x \in (-\infty, s]. \end{cases} \tag{5.35}$$

For any fixed $z \in \mathbb{R}$, $V_B(z)$ given by (5.4) is strictly decreasing in B. Thus in order to minimize the discounted cost under an (s,S) policy, we should maximize the value of B without violating (5.34).

By applying integration by parts to (5.5),

$$g_B(z) = \begin{cases} \frac{1}{\beta}h'(z) + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \cdot \Big[\frac{1}{\lambda_1}\big(h'(0+) - h'(0-) \\ \quad + \int_z^0 e^{-\lambda_1 y}h''(y)dy + \int_0^\infty e^{-\lambda_1 y}h''(y)dy\big)e^{\lambda_1 z} \\ \quad + \frac{1}{\lambda_2}\big(B - h'(0-) + \int_z^0 e^{\lambda_2 y}h''(y)dy\big)e^{-\lambda_2 z}\Big], & \text{for } z < 0, \\ \frac{1}{\beta}h'(0+) + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \cdot \Big[\frac{1}{\lambda_1}\int_0^\infty e^{-\lambda_1 y}h''(y)dy \\ \quad + \frac{1}{\lambda_2}\big(B - h'(0+)\big)\Big], & \text{for } z = 0, \\ \frac{1}{\beta}h'(z) + \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \cdot \Big[\frac{1}{\lambda_1}\int_z^\infty e^{-\lambda_1 y}h''(y)dy \cdot e^{\lambda_1 z} \\ \quad + \frac{1}{\lambda_2}\big(B - h'(0+) - \int_0^z e^{\lambda_2 y}h''(y)dy\big)e^{-\lambda_2 z}\Big], & \text{for } z > 0. \end{cases} \tag{5.36}$$

Furthermore, by applying integration by parts to (5.6),

$$
g'_B(z) = \begin{cases} \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\big[\big(h'(0+) - h'(0-) + \int_z^0 e^{-\lambda_1 y} h''(y)dy + \int_0^\infty e^{-\lambda_1 y} h''(y)dy\big)e^{\lambda_1 z} \\ \quad -\big(B - h'(0-) + \int_z^0 e^{\lambda_2 y} h''(y)dy\big)e^{-\lambda_2 z}\big], \quad \text{for } z < 0, \\ \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\big[\int_z^\infty e^{-\lambda_1 y} h''(y)dy \cdot e^{\lambda_1 z} \\ \quad -\big(B - h'(0+) - \int_0^z e^{\lambda_2 y} h''(y)dy\big)e^{-\lambda_2 z}\big], \quad \text{for } z \geq 0. \end{cases} \tag{5.37}
$$

Then, we can obtain

$$
g''_B(z) = \begin{cases} \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\big[\lambda_1\big(h'(0+) - h'(0-) + \int_z^0 e^{-\lambda_1 y} h''(y)dy + \int_0^\infty e^{-\lambda_1 y} h''(y)dy\big)e^{\lambda_1 z} \\ \quad +\lambda_2\big(B - h'(0-) + \int_z^0 e^{\lambda_2 y} h''(y)dy\big)e^{-\lambda_2 z}\big], \quad \text{for } z < 0, \\ \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\big[\lambda_1\int_z^\infty e^{-\lambda_1 y} h''(y)dy \cdot e^{\lambda_1 z} \\ \quad +\lambda_2\big(B - h'(0+) - \int_0^z e^{\lambda_2 y} h''(y)dy\big)e^{-\lambda_2 z}\big], \quad \text{for } z \geq 0. \end{cases} \tag{5.38}
$$

In the rest of this chapter, we need to discuss properties of $g_B(z)$ with respect to the auxiliary variable B for fixed $z \in \mathbb{R}$. To make the notation clear, for fixed $z \in \mathbb{R}$, let

$$
\varphi_z(B) = g_B(z).
$$

Then by (5.5), we have

$$
\varphi'_z(B) = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{e^{-\lambda_2 z}}{\lambda_2}. \tag{5.39}
$$

In Section 5.5.1, we demonstrate how the four-step algorithm in Section 5.2 attains the maximum value of B by selecting the (s^*, S^*) policy, the best policy among (s, S) policies. In Section 5.5.2, we prove that under this (s^*, S^*) policy, the corresponding discounted cost function satisfies the conditions specified by the lower bound theorem if the initial inventory level is non-negative. Thus, this (s^*, S^*) should be an optimal policy for the Brownian inventory model in this chapter.

5.5.1 The Optimal (s, S) Policy

In this subsection, we first specify an interval for the auxiliary variable B and discuss the properties of the boundaries of the interval in Lemma 5.2. Within this interval, we show the property of $g_B(z)$ in Lemma 5.3, which is an important lemma for proving subsequent lemmas. After showing some technical results in Lemma 5.4 and Lemma 5.5, we identify a set of (s, S) policies $\{U(\hat{s}_n, \hat{S}_n) : n = 1, \dots, M\}$ by Lemma 5.6. However under the quantity-dependent setup cost (1.5), the order quantity under $U(\hat{s}_n, \hat{S}_n)$, $\hat{Q}_n = \hat{S}_n - \hat{s}_n$, may not fall into an interval from Q_n to Q_{n+1}. Thus, we obtain a set of modified (s, S) policies $\{U(s_n^*, S_n^*) : n = 1, \dots, M\}$ by (5.10). Given the Q_n^* obtained by (5.10), Lemma 5.7 proves the existence and uniqueness of (s_n^*, S_n^*, B_n^*) such that they jointly satisfy (5.34). Finally, we select the optimal (s^*, S^*) policy out of the set $\{U(s_n^*, S_n^*) : n = 1, \dots, M\}$ by (5.11) and (5.12).

Let

$$\underline{B} = h'(0-) - \int_{-\infty}^{0} e^{\lambda_2 y} h''(y) dy \quad \text{and} \quad \overline{B} = h'(0+) + \int_{0}^{\infty} e^{-\lambda_1 y} h''(y) dy. \tag{5.40}$$

Lemma 5.2. *For $\underline{B}$ and $\overline{B}$ defined in* (5.40)*, we have*

$$\underline{B} < 0 < \overline{B}.$$

Proof. By Assumption 3, we have $h'(0-) \leq 0$ and $\int_{-\infty}^{0} e^{\lambda_2 y} h''(y) dy \geq 0$. Then by (5.40), we have $\underline{B} \leq 0$. If $\int_{-\infty}^{0} e^{\lambda_2 y} h''(y) dy < 0$, we have the strict inequality $\underline{B} < 0$. Otherwise if $\int_{-\infty}^{0} e^{\lambda_2 y} h''(y) dy = 0$, it suffices to show that $h'(0-) < 0$, which implies that $\underline{B} < 0$. Since $\int_{-\infty}^{0} e^{\lambda_2 y} h''(y) dy = 0$, we must have $h''(z) = 0$ for $z \in (-\infty, 0)$. Together with (A3) in Assumption 3, we have $h'(0-) < -k\beta < 0$. Thus $\underline{B} < 0$. The conclusion $0 < \overline{B}$ follows from the same rationale and the proof is omitted. $\square$

In the next lemma, we show the monotone intervals of $g_B(z)$.

Lemma 5.3. *For $B \in (\underline{B}, \overline{B})$, there exists a unique $z^*(B)$ such that*

$$\begin{cases} g_B'(z) < 0 & \textit{for } z < z^*(B), \\ g_B'(z) = 0 & \textit{for } z = z^*(B), \\ g_B'(z) > 0 & \textit{for } z > z^*(B). \end{cases} \tag{5.41}$$

In particular, $z^*(B) \in (-\infty, 0)$. Furthermore,

$$\lim_{z\to-\infty} g_B(z) = \infty \quad \textit{and} \quad \lim_{z\to\infty} g_B(z) > k. \tag{5.42}$$

Proof. We first prove (5.41) for two cases: $B \in (\underline{B}, h'(0-))$ and $B \in [h'(0-), \overline{B})$. Note that $\underline{B} \leq h'(0-)$ by the definition of $\underline{B}$ in (5.40). Without loss of generality, we assume that these two intervals are nonempty.

(i) For $B \in (\underline{B}, h'(0-))$. Since $\int_z^0 e^{\lambda_2 y} h''(y)dy$ is continuous and decreasing for $z \in (-\infty, 0)$, there exists an $x_1 \in (-\infty, 0)$ such that

$$B = h'(0-) - \int_{x_1}^0 e^{\lambda_2 y} h''(y)dy. \tag{5.43}$$

Note that x_1 may be not unique. With any x_1 that satisfies (5.43), we next prove that

$$g_B''(z) > 0 \quad \text{for } z \in (-\infty, x_1], \tag{5.44}$$

$$\lim_{z\to-\infty} g_B'(z) = -\infty, \tag{5.45}$$

$$\lim_{z\uparrow x_1} g_B'(z) > 0 \quad \textit{and} \tag{5.46}$$

$$g_B'(z) > 0 \quad \text{for } z \in (x_1, \infty), \tag{5.47}$$

which together imply that there exists a unique $z^*(B)$ such that (5.41) holds and $z^*(B) < x_1 < 0$.

We first prove (5.44). For $z \in (-\infty, x_1]$, we have

$$B - h'(0-) + \int_z^0 e^{\lambda_2 y} h''(y)dy = \int_z^{x_1} e^{\lambda_2 y} h''(y)dy \geq 0, \tag{5.48}$$

and

$$
\begin{aligned}
&h'(0+) - h'(0-) + \int_z^0 e^{-\lambda_1 y} h''(y)dy + \int_0^\infty e^{-\lambda_1 y} h''(y)dy \\
&= \overline{B} - h'(0-) + \int_z^0 e^{-\lambda_1 y} h''(y)dy \\
&> 0, \qquad (5.49)
\end{aligned}
$$

where the inequality follows from $\overline{B} > 0$ and $h'(0-) \leq 0$. Therefore, the definition of $g''_B(z)$ in (5.38) and inequalities (5.48)–(5.49) together imply that (5.44) holds.

Secondly we prove (5.45). We have

$$
\begin{aligned}
&\lim_{z\to-\infty} \frac{g'_B(z)}{e^{-\lambda_2 z}} \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} \lim_{z\to-\infty} \Big[\frac{h'(0+) - h'(0-) + \int_z^0 e^{-\lambda_1 y} h''(y)dy + \int_0^\infty e^{-\lambda_1 y} h''(y)dy}{e^{-(\lambda_1+\lambda_2)z}} \\
&\quad -(B - h'(0-) + \int_z^0 e^{\lambda_2 y} h''(y)dy)\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\lim_{z\to-\infty} \frac{\int_z^0 e^{-\lambda_1 y} h''(y)dy}{e^{-(\lambda_1+\lambda_2)z}} - (B - \underline{B})\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\lim_{z\to-\infty} \frac{h''(z)}{(\lambda_1+\lambda_2)e^{-\lambda_2 z}} - (B - \underline{B})\Big] \\
&= -\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}(B - \underline{B}) \\
&< 0,
\end{aligned}
$$

where the second equality follows from the fact that $\int_0^\infty e^{-\lambda_1 y} h''(y)dy < \infty$ by Assumption 3, the third equality follows from L'Hôpital's Rule and the last equality follows from Assumption 3. Hence, $g'_B(z)$ is in the same order with $-e^{-\lambda_2 z}$ when z goes to $-\infty$, which implies (5.45).

Next, we check (5.46). We have

$$
\begin{aligned}
\lim_{z\uparrow x_1} g'_B(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} e^{\lambda_1 z}\big(h'(0+) - h'(0-) + \int_{x_1}^0 e^{-\lambda_1 y} h''(y)dy \\
&\qquad + \int_0^\infty e^{-\lambda_1 y} h''(y)dy\big) \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2} e^{\lambda_1 z}\big(\overline{B} - h'(0-) + \int_{x_1}^0 e^{-\lambda_1 y} h''(y)dy\big)
\end{aligned}
$$

$$> \quad 0,$$

where the inequality follows from $\overline{B} > 0$ and $h'(0-) \leq 0$.

It remains to prove (5.47). We consider two cases: $z \in (x_1, 0)$ and $z \in [0, \infty)$. For $z \in (x_1, 0)$, note that (5.49) also holds. Then the inequality (5.49) together with

$$B - h'(0-) + \int_z^0 e^{\lambda_2 y} h''(y)dy = \int_z^{x_1} e^{\lambda_2 y} h''(y)dy \leq 0$$

implies that (5.47) holds for $z \in (x_1, 0)$. For $z \in [0, \infty)$, we have $B < h'(0-) \leq h'(0+)$. Then

$$B - h'(0+) - \int_0^z e^{\lambda_2 y} h''(y)dy < 0 \quad \text{for } z \geq 0,$$

which together with the definition of $g'_B(z)$ in (5.37) implies that (5.47) holds for $z \geq 0$.

(ii) For $B \in [h'(0-), \overline{B})$, we will show that

$$g''_B(z) > 0 \quad \text{for } z \in (-\infty, 0), \tag{5.50}$$

$$\lim_{z \to -\infty} g'_B(z) = -\infty, \tag{5.51}$$

$$\lim_{z \uparrow 0} g'_B(z) > 0 \quad \textit{and} \tag{5.52}$$

$$g'_B(z) > 0 \quad \text{for } z \in [0, \infty), \tag{5.53}$$

which together imply that there exists a unique $z^*(B)$ such that (5.41) holds and $z^*(B) < 0$.

Firstly, we prove (5.50). For $z \in (-\infty, 0)$, we first note that (5.49) also holds. Then

$$B - h'(0-) + \int_z^0 e^{\lambda_2 y} h''(y)dy \geq 0,$$

which together with (5.49) implies that $g''_B(z) > 0$ for $z \in (-\infty, 0)$.

The proof of (5.51) is same with (5.45) in Case (i).

Furthermore, we prove (5.52). We have

$$\begin{aligned}
\lim_{z\uparrow 0} g_B'(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[h'(0+)-h'(0-)+\int_0^\infty e^{-\lambda_1 y}h''(y)dy - B + h'(0-)\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}(\overline{B}-B) \\
&> 0.
\end{aligned}$$

We are going to prove (5.53). For $z\in[0,\infty)$, we have

$$\begin{aligned}
g_B'(z) &= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy\cdot e^{\lambda_1 z} \\
&\quad -\big(B-h'(0+)-\int_0^z e^{\lambda_2 y}h''(y)dy\big)e^{-\lambda_2 z}\Big] \\
&> \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy\cdot e^{\lambda_1 z} \\
&\quad -\big(\overline{B}-h'(0+)-\int_0^z e^{\lambda_2 y}h''(y)dy\big)e^{-\lambda_2 z}\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy\cdot e^{\lambda_1 z} \\
&\quad -\big(\int_0^\infty e^{-\lambda_1 y}h''(y)dy-\int_0^z e^{\lambda_2 y}h''(y)dy\big)e^{-\lambda_2 z}\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h''(y)dy\cdot\big(e^{\lambda_1 z}-e^{-\lambda_2 z}\big) \\
&\quad +\int_0^z\big(e^{\lambda_2 y}-e^{-\lambda_1 y}\big)h''(y)dy\cdot e^{-\lambda_2 z}\Big] \\
&\geq 0,
\end{aligned}$$

where the first inequality follows from $B<\overline{B}$ and the second equality follows from the definition of $\overline{B}$ in (5.40).

We have finished the proof of (5.41) and $z^*(B)<0$. It remains to prove (5.42). Firstly, (5.41), (5.45) and (5.51) immediately imply that $\lim_{z\to-\infty} g_B(z)=\infty$, which is the first part of (5.42). Furthermore, it follows from (5.5) that

$$\begin{aligned}
&\lim_{z\to\infty} g_B(z) \\
&= \lim_{z\to\infty}\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\int_z^\infty e^{-\lambda_1 y}h'(y)dy\cdot e^{\lambda_1 z}+\frac{1}{\lambda_2}\Big(B+\lambda_2\int_0^z e^{\lambda_2 y}h'(y)dy\Big)e^{-\lambda_2 z}\Big] \\
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\lim_{z\to\infty}\frac{\int_z^\infty e^{-\lambda_1 y}h'(y)dy}{e^{-\lambda_1 z}}+\lim_{z\to\infty}\frac{\int_0^z e^{\lambda_2 y}h'(y)dy}{e^{\lambda_2 z}}\Big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\Big[\lim_{z\to\infty}\frac{h'(z)}{\lambda_1}+\lim_{z\to\infty}\frac{h'(z)}{\lambda_2}\Big]\\
&= \frac{1}{\beta}\lim_{z\to\infty}h'(z)\\
&> k,
\end{aligned}
$$

where the third equality follows from L'Hôpital's Rule, the last equality follows from $\lambda_1 = \frac{\mu+\sqrt{\mu^2+2\beta\sigma^2}}{\sigma^2}$ and $\lambda_2 = \frac{-\mu+\sqrt{\mu^2+2\beta\sigma^2}}{\sigma^2}$, and the inequality follows from Assumption 3. □

For $B \in (\underline{B}, \overline{B})$, $g_B(z)$ attains it global minimum at point $z^*(B)$. With respect to this minimum point $z^*(B)$, we have the following property.

Lemma 5.4. *For $B \in (\underline{B}, \overline{B})$, $z^*(B)$ is continuous and strictly increasing in B. Furthermore,*

$$\lim_{B\uparrow\overline{B}} z^*(B) = 0 \quad \textit{and} \quad \lim_{B\downarrow\underline{B}} z^*(B) = -\infty.$$

Proof. By Lemma 5.3, $z^*(B)$ is the solution of $g_B'(z) = 0$. Then by the Implicit Function Theorem, $z^*(B)$ is continuous in B and the derivative $z^{*\prime}(B)$ exists. By the fact that $z^*(B) < 0$ shown in Lemma 5.3 and the expressions of $g_B'(z)$ in (5.6) and (5.37), $z^*(B)$ is the solution of

$$\lambda_1 \int_z^\infty e^{-\lambda_1 y}h'(y)dy \cdot e^{\lambda_1 z} = \Big(B + \lambda_2\int_0^z e^{\lambda_2 y}h'(y)dy\Big)e^{-\lambda_2 z}, \tag{5.54}$$

or equivalently the solution of

$$
\begin{aligned}
&\Big(h'(0+) - h'(0-) + \int_z^0 e^{-\lambda_1 y}h''(y)dy + \int_0^\infty e^{-\lambda_1 y}h''(y)dy\Big)e^{\lambda_1 z}\\
= \;&\Big(B - h'(0-) + \int_z^0 e^{\lambda_2 y}h''(y)dy\Big)e^{-\lambda_2 z}.
\end{aligned} \tag{5.55}
$$

Applying the Implicit Function Theorem to (5.55), we have

$$z^{*\prime}(B) = \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{e^{-\lambda_2 z^*(B)}}{g_B''(z^*(B))}. \tag{5.56}$$

By (5.44) and (5.50), we have

$$g''_B(z^*(B)) > 0,$$

which implies

$$z^{*\prime}(B) > 0. \tag{5.57}$$

Define

$$z^*(\overline{B}) = \lim_{B \uparrow \overline{B}} z^*(B).$$

Since $z^*(B) < 0$ for $B \in (\underline{B}, \overline{B})$, we have

$$z^*(\overline{B}) = \lim_{B \uparrow \overline{B}} z^*(B) \leq 0.$$

Suppose that $z^*(\overline{B}) < 0$. By (5.55), we have

$$\begin{aligned} & (\overline{B} - h'(0-))(e^{\lambda_1 z^*(\overline{B})} - e^{-\lambda_2 z^*(\overline{B})}) \\ = & \int_{z^*(\overline{B})}^{0} (e^{\lambda_2 (y - z^*(\overline{B}))} - e^{-\lambda_1 (y - z^*(\overline{B}))}) h''(y) dy. \end{aligned} \tag{5.58}$$

Since $z^*(\overline{B}) < 0$, we have

$$(\overline{B} - h'(0-))(e^{\lambda_1 z^*(\overline{B})} - e^{-\lambda_2 z^*(\overline{B})}) < 0,$$
$$\int_{z^*(\overline{B})}^{0} (e^{\lambda_2 (y - z^*(\overline{B}))} - e^{-\lambda_1 (y - z^*(\overline{B}))}) h''(y) dy \geq 0,$$

which contradicts with (5.58). Therefore, we have $z^*(\overline{B}) = \lim_{B \uparrow \overline{B}} z^*(B) = 0$.

Next we show that

$$z^*(\underline{B}) = \lim_{B \downarrow \underline{B}} z^*(B) = -\infty.$$

Since $z^*(B)$ is monotone in B, the limit exists. Now suppose that $z^*(\underline{B}) > -\infty$. By (5.55), we have

$$
\begin{aligned}
& \big(h'(0+) - h'(0-) + \int_{z^*(\underline{B})}^{0} e^{-\lambda_1 y} h''(y)dy + \int_0^\infty e^{-\lambda_1 y} h''(y)dy\big) e^{\lambda_1 z^*(\underline{B})} \\
= \;& \big(\underline{B} - h'(0-) + \int_{z^*(\underline{B})}^{0} e^{\lambda_2 y} h''(y)dy\big) e^{-\lambda_2 z^*(\underline{B})},
\end{aligned}
$$

where

$$
\begin{aligned}
& \big(\underline{B} - h'(0-) + \int_{z^*(\underline{B})}^{0} e^{\lambda_2 y} h''(y)dy\big) e^{-\lambda_2 z^*(\underline{B})} \\
= \;& -\int_{-\infty}^{z^*(\underline{B})} e^{\lambda_2 y} h''(y)dy \cdot e^{-\lambda_2 z^*(\underline{B})} \leq 0,
\end{aligned}
$$

contradicts with that

$$
\begin{aligned}
& \big(h'(0+) - h'(0-) + \int_{z^*(\underline{B})}^{0} e^{-\lambda_1 y} h''(y)dy + \int_0^\infty e^{-\lambda_1 y} h''(y)dy\big) e^{\lambda_1 z^*(\underline{B})} \\
= \;& \overline{B} - h'(0-) + \int_{z^*(\underline{B})}^{0} e^{-\lambda_1 y} h''(y)dy > 0,
\end{aligned}
$$

where the inequality follows from $\underline{B} < 0 < \overline{B}$. Hence, we have

$$
\lim_{B \downarrow \underline{B}} z^*(B) = -\infty.
$$

□

The technical result shown in the following lemma is important to prove its subsequent lemma.

Lemma 5.5. *There exists a unique* $\tilde{B} \in (\underline{B}, \overline{B})$ *such that*

$$
g_{\tilde{B}}(z^*(\tilde{B})) = -k. \tag{5.59}
$$

Proof. Since

$$
\begin{aligned}
\frac{\mathrm{d} g_B(z^*(B))}{\mathrm{d} B} &= g_B'(z^*(B)) z^{*\prime}(B) + \varphi'_{z^*(B)}(B) \\
&= \varphi'_{z^*(B)}(B) \\
&= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_2} e^{-\lambda_2 z^*(B)} > 0,
\end{aligned}
$$

it suffices to show that

$$
\begin{aligned}
\lim_{B \downarrow \underline{B}} g_B(z^*(B)) &< -k, && (5.60) \\
\lim_{B \uparrow \overline{B}} g_B(z^*(B)) &> -k. && (5.61)
\end{aligned}
$$

Firstly, we will show (5.60). By Lemma 5.4 and L'Hôpital's Rule, we have

$$
\begin{aligned}
\lim_{B \downarrow \underline{B}} \int_{z^*(B)}^{\infty} e^{-\lambda_1 y} h'(y) dy e^{\lambda_1 z^*(B)} &= \lim_{z \to -\infty} \int_{z}^{\infty} e^{-\lambda_1 y} h'(y) dy e^{\lambda_1 z} \\
&= \lim_{z \to -\infty} \frac{h'(z)}{\lambda_1},
\end{aligned}
$$

which together with (5.54) implies that

$$
\lim_{B \downarrow \underline{B}} \Big(B + \lambda_2 \int_0^{z^*(B)} e^{\lambda_2 y} h'(y) dy \Big) e^{-\lambda_2 z^*(B)} = \lim_{z \to -\infty} h'(z).
$$

Then by the expression of $g_B(z)$ in (5.5), we can obtain

$$
\begin{aligned}
\lim_{B \downarrow \underline{B}} g_B(z^*(B)) &= \lim_{B \downarrow \underline{B}} \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\int_{z^*(B)}^{\infty} e^{-\lambda_1 y} h'(y) dy \cdot e^{\lambda_1 z^*(B)} \\
&\quad + \frac{1}{\lambda_2} \Big(B + \lambda_2 \int_0^{z^*(B)} e^{\lambda_2 y} h'(y) dy \Big) e^{-\lambda_2 z^*(B)} \Big] \\
&= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\lim_{z \to -\infty} \frac{h'(z)}{\lambda_1} + \lim_{z \to -\infty} \frac{h'(z)}{\lambda_2} \Big] \\
&= \lim_{z \to -\infty} \frac{h'(z)}{\beta} \\
&< -k,
\end{aligned}
$$

where the inequality follows from (A3) in Assumption 3.

Finally, we prove (5.61). By Lemma 5.4 and (5.36), we have

$$
\begin{aligned}
\lim_{B \uparrow \overline{B}} g_B(z^*(B)) &= \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\frac{1}{\lambda_1} (\overline{B} - h'(0-)) + \frac{1}{\lambda_2} (\overline{B} - h'(0-)) \Big] + \frac{h'(0-)}{\beta} \\
&= \frac{\overline{B}}{\beta} > 0,
\end{aligned}
$$

where the inequality follows from $\overline{B} > 0$. □

Under the assumption that the setup cost for any order quantity is constant, Sulem (1986) specified the necessary and sufficient conditions for the optimal (s, S) policy that minimizes the total discounted cost. By the following lemma, we identify a set of (s, S) policies $\{U(\hat{s}_n, \hat{S}_n) : n = 1, \ldots, M\}$ with similar conditions.

Lemma 5.6. *For an arbitrary $\kappa > 0$, there exists a unique triplet $(\hat{s}(\kappa), \hat{S}(\kappa), \hat{B}(\kappa))$, such that*

$$
\int_{\hat{s}(\kappa)}^{\hat{S}(\kappa)} [g_{\hat{B}(\kappa)}(y) + k] dy = -\kappa, \tag{5.62}
$$

$$
g_{\hat{B}(\kappa)}(\hat{s}(\kappa)) = g_{\hat{B}(\kappa)}(\hat{S}(\kappa)) = -k, \tag{5.63}
$$

$$
\hat{s}(\kappa) < \hat{S}(\kappa). \tag{5.64}
$$

In particular, $\underline{B} < \hat{B}(\kappa) < \tilde{B}$. Furthermore for any $0 < \kappa_i < \kappa_j$, the corresponding $(\hat{s}(\kappa_i), \hat{S}(\kappa_i), \hat{B}(\kappa_i))$ and $(\hat{s}(\kappa_j), \hat{S}(\kappa_j), \hat{B}(\kappa_j))$ satisfy

$$
\hat{s}(\kappa_j) < \hat{s}(\kappa_i), \quad \hat{S}(\kappa_i) < \hat{S}(\kappa_j) \quad \text{and} \quad \hat{B}(\kappa_i) > \hat{B}(\kappa_j). \tag{5.65}
$$

Proof. Firstly, we show that we only need to consider $B \in (\underline{B}, \tilde{B})$. By Lemma 5.3 and Lemma 5.5, we have $g_B(z) > -k$ for $z \in \mathbb{R}$ if $B \in (\tilde{B}, \overline{B})$. By (5.5), $g_B(z)$ is strictly increasing in B for any fixed $z \in \mathbb{R}$. Therefore, $g_B(z) > -k$ for $z \in \mathbb{R}$ if $B > \tilde{B}$. Furthermore by Lemma 5.3 and Lemma 5.5, we have $g_{\tilde{B}}(z) > -k$ except at the point $z^*(\tilde{B})$. Thus if $B \geq \tilde{B}$, there do not exist s and S with $s < S$ such that

$g_B(s) = g_B(S) = -k$. If $B \in (-\infty, \underline{B}]$, we have

$$\begin{aligned} B - h'(0-) + \int_z^0 e^{\lambda_2 y} h''(y) dy &\leq \underline{B} - h'(0-) + \int_z^0 e^{\lambda_2 y} h''(y) dy \\ &= -\int_{-\infty}^z e^{\lambda_2 y} h''(y) dy \\ &< 0 \quad \text{for } z < 0. \end{aligned}$$

Together with (5.37) and (5.49), we can conclude that $g'_B(z) > 0$ for $z < 0$. Furthermore, we have

$$B - h'(0+) - \int_0^z e^{\lambda_2 y} h''(y) dy \leq \underline{B} - h'(0+) - \int_0^z e^{\lambda_2 y} h''(y) dy < 0 \quad \text{for } z \geq 0.$$

Together with (5.37), we can conclude that $g'_B(z) > 0$ for $z \geq 0$. Thus if $B \in (-\infty, \underline{B}]$, $g_B(z)$ is strictly increasing for $z \in \mathbb{R}$, which implies that there do not exist s and S with $s < S$ such that $g_B(s) = g_B(S)$. Therefore, we only need to consider $B \in (\underline{B}, \tilde{B})$.

By Lemma 5.3 and Lemma 5.5, for $B \in (\underline{B}, \tilde{B})$, we have $g_B(z^*(B)) < -k$. Furthermore, there exist unique $s(B)$ and $S(B)$ such that

$$s(B) < S(B) \quad \text{and} \quad g_B(s(B)) = g_B(S(B)) = -k.$$

By Lemma 5.3, we can conclude that $s(B) < z^*(B) < S(B)$, $g'_B(s(B)) < 0$ and $g'_B(S(B)) > 0$. By the Implicit Function Theorem, the derivatives $s'(B)$ and $S'(B)$ exist. For $B \in (\underline{B}, \tilde{B})$, define

$$\Lambda(B) = \int_{s(B)}^{S(B)} [g_B(y) + k] dy. \tag{5.66}$$

Next, we prove that for any $\kappa > 0$, there exists a unique $\hat{B}(\kappa)$ such that $\Lambda(\hat{B}(\kappa)) = -\kappa$. To show the existence and uniqueness of $\hat{B}(\kappa)$, it suffices to show that

$$\Lambda'(B) > 0 \quad \text{for } B \in (\underline{B}, \tilde{B}), \tag{5.67}$$

$$\lim_{B \uparrow \tilde{B}} \Lambda(B) = 0, \tag{5.68}$$

$$\lim_{B \downarrow \underline{B}} \Lambda(B) = -\infty. \tag{5.69}$$

Firstly, we show (5.67). By the definition of $\Lambda(B)$ in (5.66), we have

$$\begin{aligned}
& \Lambda'(B) \\
= & \int_{s(B)}^{S(B)} \varphi'_y(B)dy + [g_B(S(B)) + k]S'(B) - [g_B(s(B)) + k]s'(B) \\
= & \int_{s(B)}^{S(B)} \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_2} e^{-\lambda_2 y} dy \\
> & \ 0.
\end{aligned}$$

Since $g_B(z^*(\tilde{B})) = -k$, we can conclude that $\lim_{B \uparrow \tilde{B}} s(B) = \lim_{B \uparrow \tilde{B}} S(B) = z^*(\tilde{B})$. Therefore, we have

$$\lim_{B \uparrow \tilde{B}} \Lambda(B) = 0.$$

It remains to prove (5.69). Since we have $\lim_{B \downarrow \underline{B}} z^*(B) = -\infty$ by Lemma 5.4 and $s(B) < z^*(B)$, then $\lim_{B \downarrow \underline{B}} s(B) = -\infty$. Taking derivatives of the both sides of equation $g_B(S(B)) = -k$, i.e.,

$$\begin{aligned}
& \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\int_{S(B)}^{\infty} e^{-\lambda_1 y} h'(y) dy \cdot e^{\lambda_1 S(B)} \\
& \quad + \frac{e^{-\lambda_2 S(B)}}{\lambda_2} \Big(B + \lambda_2 \int_0^{S(B)} e^{\lambda_2 y} h'(y) dy \Big) \Big] = -k,
\end{aligned}$$

with respect to B for $B \in (\underline{B}, \tilde{B})$, we have

$$S'(B) = -\frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{e^{-\lambda_2 S(B)}}{\lambda_2} \frac{1}{g'(S(B))} < 0, \tag{5.70}$$

which implies that $\lim_{B \downarrow \underline{B}} S(B) > -\infty$. Then (5.69) follow from (5.67) and

$$\begin{aligned}
\lim_{B \downarrow \underline{B}} \Lambda'(B) &= \lim_{B \downarrow \underline{B}} \int_{s(B)}^{S(B)} \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_2} e^{-\lambda_2 y} dy \\
&= \lim_{B \downarrow \underline{B}} \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{1}{\lambda_2^2} [e^{-\lambda_2 s(B)} - e^{-\lambda_2 S(B)}]
\end{aligned}$$

$$= \infty.$$

It remains to prove (5.65). By (5.62), (5.63) and the Implicit Function Theorem, the derivatives $s'(\kappa)$, $S'(\kappa)$ and $B'(\kappa)$ exist. Then to prove (5.65), it suffices to show

$$s'(\kappa) < 0, \quad S'(\kappa) > 0, \quad \text{and} \quad B'(\kappa) < 0.$$

Taking derivatives of (5.62) and (5.63) with respect to κ, we can obtain

$$[g_{\hat{B}(\kappa)}(\hat{S}(\kappa)) + k]\hat{S}'(\kappa) - [g_{\hat{B}(\kappa)}(\hat{s}(\kappa)) + k]\hat{s}'(\kappa) + \int_{\hat{s}(\kappa)}^{\hat{S}(\kappa)} [\varphi'_y(\hat{B}(\kappa)) \cdot \hat{B}'(\kappa)]dy = -1,$$

and

$$\begin{aligned} \varphi'_{\hat{S}(\kappa)}(\hat{B}(\kappa)) \cdot \hat{B}'(\kappa) + g'_{\hat{B}(\kappa)}(\hat{S}(\kappa)) \cdot \hat{S}'(\kappa) &= 0, \\ \varphi'_{\hat{s}(\kappa)}(\hat{B}(\kappa)) \cdot \hat{B}'(\kappa) + g'_{\hat{B}(\kappa)}(\hat{s}(\kappa)) \cdot \hat{s}'(\kappa) &= 0, \end{aligned}$$

Then by (5.39) and (5.63), we can conclude that

$$\begin{aligned} \hat{B}'(\kappa) &= -\frac{\sigma^2(\lambda_1+\lambda_2)\lambda_2^2}{2(e^{-\lambda_2\hat{s}(\kappa)} - e^{-\lambda_2\hat{S}(\kappa)})} < 0, \\ \hat{s}'(\kappa) &= -\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}\frac{e^{-\lambda_2\hat{s}(\kappa)}\hat{B}'(\kappa)}{g'_{\hat{B}(\kappa)}(\hat{s}(\kappa))} < 0, \\ \hat{S}'(\kappa) &= -\frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}\frac{e^{-\lambda_2\hat{S}(\kappa)}\hat{B}'(\kappa)}{g'_{\hat{B}(\kappa)}(\hat{S}(\kappa))} > 0, \end{aligned}$$

where the first inequality follows from $\hat{s}(\kappa) < \hat{S}(\kappa)$ and $\lambda_1, \lambda_2 > 0$, the second inequality follows from $g'_{\hat{B}(\kappa)}(\hat{s}(\kappa)) < 0$ and the third inequality follows form $g'_{\hat{B}(\kappa)}(\hat{S}(\kappa)) > 0$. □

If $\kappa = K_n$ where $n = 1, \ldots, M$ and let $(\hat{s}_n, \hat{S}_n, \hat{B}_n)$ denote the triplet $(\hat{s}(K_n), \hat{S}(K_n), \hat{B}(K_n))$ and z_n^* denote $z^*(\hat{B}(K_n))$, the conditions (5.62)–(5.64) are equivalent to conditions (5.7)–(5.9) in Step 1. Then for $\kappa = K_n$ where $n = 1, \ldots, M$, we can obtain a set of (s, S) policies $\{U(\hat{s}_n, \hat{S}_n) : n = 1, \ldots, M\}$ by Lemma 5.6.

When the setup cost is K_n, the quantity of an order is constrained to an interval from Q_n to Q_{n+1} (which by (1.5) might be (Q_n, Q_{n+1}), $(Q_n, Q_{n+1}]$, $[Q_n, Q_{n+1})$ or $[Q_n, Q_{n+1}]$). However, the order quantity for each order under the obtained $(\hat{s}_n, \hat{S}_n)$ policy, $\hat{S}_n - \hat{s}_n$, might not fall into an interval from Q_n to Q_{n+1}. In Step 2, we define Q_n^* in (5.10) based on the relative position of $\hat{S}_n - \hat{s}_n$ to the interval (Q_n, Q_{n+1}). By the definition of Q_n^*, we have $Q_n \leq Q_n^* \leq Q_{n+1}$ and $0 < Q_n^* < \infty$ where $n = 1, \ldots, M$. In the following lemma, we show that given such an Q_n^*, there exists a unique triplet (s_n^*, S_n^*, B_n^*) such that (5.34) holds.

Lemma 5.7. *For any given $K_n > 0$ where $n = 1, \ldots, M$, together with Q_n^* defined in* (5.10)*, there exists a unique triplet (s_n^*, S_n^*, B_n^*) with $s_n^* < S_n^*$ such that*

$$S_n^* - s_n^* \quad = \quad Q_n^*, \tag{5.71}$$

$$\int_{s_n^*}^{S_n^*} [g_{B_n^*}(z) + k] dz \quad = \quad -K_n, \tag{5.72}$$

$$g_{B_n^*}(s_n^*) \quad = \quad g_{B_n^*}(S_n^*). \tag{5.73}$$

Furthermore, we have $s_n^* < 0$ and

$$\underline{B} < B_n^* \leq \hat{B}_n. \tag{5.74}$$

Proof. Since we have proved that $g_B(z)$ is strictly increasing in z when $B \in (-\infty, \underline{B}]$ and that $g_B(z) \geq -k$ for $z \in \mathbb{R}$ when $B \in [\tilde{B}, \infty)$ in the proof of Lemma 5.6, we only need consider when $B \in (\underline{B}, \tilde{B})$. For $n \in \mathcal{N}_=$, $Q_n^* = \hat{S}_n - \hat{s}_n$. Put $s_n^* = \hat{s}_n$, $S_n^* = \hat{S}_n$ and $B_n^* = \hat{B}_n$. Then by Lemma 5.6, the triplet (s_n^*, S_n^*, B_n^*) satisfies (5.71)–(5.73). Next, we consider two cases: $n \in \mathcal{N}_>$ and $n \in \mathcal{N}_>$.

(i) For $n \in \mathcal{N}_>$, we have $Q_n^* = Q_{n+1}$ by (5.10). By Lemma 5.3, for any $B \in (\underline{B}, \tilde{B})$ and given $Q_{n+1} > 0$, there exist unique $s_1(B)$ and $S_1(B)$ with $s_1(B) < S_1(B)$ and $S_1(B) - s_1(B) = Q_{n+1}$ such that $g_B(s_1(B)) = g_B(S_1(B))$. Then we can conclude that $s_1(B) < z^*(B) < S_1(B)$ by Lemma 5.3 and that the derivatives $s_1'(B)$ and $S_1'(B)$ exist by the Implicit Function Theorem. Define

$$\Lambda_1(B) = \int_{s_1(B)}^{S_1(B)} [g_B(y) + k]dy. \tag{5.75}$$

It suffices to show

$$\begin{aligned} \Lambda_1(\hat{B}_n) &\geq -K_n, & (5.76) \\ \lim_{B\downarrow \underline{B}} \Lambda_1(B) &< -K_n, & (5.77) \\ \Lambda_1'(B) &> 0. & (5.78) \end{aligned}$$

Firstly, we will show (5.76). By Lemma 5.6,we have $g_{\hat{B}_n}(\hat{s}_n) = g_{\hat{B}_n}(\hat{S}_n) = -k$. Together with Lemma 5.3, the fact that $g_{\hat{B}_n}(s_1(\hat{B}_n)) = g_{\hat{B}_n}(S_1(\hat{B}_n))$ and that $\hat{S}_n - \hat{s}_n \geq Q_{n+1}$, we can conclude that $\hat{s}_n \leq s_1(\hat{B}_n) < S_1(\hat{B}_n) \leq \hat{S}_n$. Then by Lemma 5.3 and Lemma 5.6,

$$\begin{aligned} \Lambda_1(\hat{B}_n) &= \int_{s_1(\hat{B}_n)}^{S_1(\hat{B}_n)} [g_{\hat{B}_n}(y) + k]dy \\ &\geq \int_{\hat{s}_n}^{\hat{S}_n} [g_{\hat{B}_n}(y) + k]dy \\ &= -K_n. \end{aligned}$$

Next, we will show (5.77). By Lemma 5.3, we have

$$\lim_{B\downarrow \underline{B}} \Lambda_1(B) < Q_{n+1} \cdot [\lim_{B\downarrow \underline{B}} g_B(S_1(B)) + k] \leq Q_{n+1} \cdot [\lim_{B\downarrow \underline{B}} g_B(z^*(B) + Q_{n+1}) + k].$$

It suffices to show that

$$\lim_{B\downarrow \underline{B}} g_B(z^*(B) + Q_{n+1}) < -k - \frac{K_n}{Q_{n+1}}.$$

By definition of $g_B(z)$ in (5.5), we have

$$\lim_{B\downarrow \underline{B}} g_B(z^*(B) + Q_{n+1})$$

$$= \lim_{B \downarrow \underline{B}} \frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \Big[\int_{z^*(B)+Q_{i+1}}^{\infty} e^{-\lambda_1 y} h'(y) dy \cdot e^{\lambda_1 (z^*(B)+Q_{n+1})}$$
$$+ \frac{1}{\lambda_2} \Big(B + \lambda_2 \int_0^{z^*(B)+Q_{n+1}} e^{\lambda_2 y} h'(y) dy \Big) e^{-\lambda_2 (z^*(B)+Q_{n+1})} \Big]. \qquad (5.79)$$

For the first term in (5.79), we have

$$\begin{aligned} & \lim_{B \downarrow \underline{B}} \int_{z^*(B)+Q_{n+1}}^{\infty} e^{-\lambda_1 y} h'(y) dy \cdot e^{\lambda_1 (z^*(B)+Q_{n+1})} \\ = & \lim_{z \to -\infty} \frac{\int_{z+Q_{n+1}}^{\infty} e^{-\lambda_1 y} h'(y) dy}{e^{-\lambda_1 (z+Q_{n+1})}} \\ = & \lim_{z \to -\infty} \frac{h'(z+Q_{n+1})}{\lambda_1}, \end{aligned}$$

where the first equality follows from Lemma 5.4 and the second equality is due to L'Hôpital's Rule. For the second term in (5.79), by (5.37) the fact that $g'(z^*(B) + Q_{n+1}) > 0$ and the expression of g'_B in (5.6), we have

$$\begin{aligned} & \lim_{B \downarrow \underline{B}} \frac{1}{\lambda_2} \Big(B + \lambda_2 \int_0^{z^*(B)+Q_{n+1}} e^{\lambda_2 y} h'(y) dy \Big) e^{-\lambda_2 (z^*(B)+Q_{n+1})} \\ < & \lim_{B \downarrow \underline{B}} \frac{\lambda_1}{\lambda_2} \int_{z^*(B)+Q_{n+1}}^{\infty} e^{-\lambda_1 y} h'(y) dy \cdot e^{\lambda_1 (z^*(B)+Q_{n+1})} \\ = & \lim_{z \to -\infty} \frac{h'(z+Q_{n+1})}{\lambda_2}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \lim_{B \downarrow \underline{B}} g_B(z^*(B) + Q_{n+1}) & < \lim_{z \to -\infty} \frac{h'(z+Q_{n+1})}{\beta} \\ & < -k - \frac{K_n}{Q_{n+1}}, \end{aligned}$$

where the last inequality follows from (A3) in Assumption 3.

Finally, we prove (5.78). Taking derivatives of the both sides of (5.75) with respect to B, we have

$$\Lambda'_1(B) = \int_{s_1(B)}^{S_1(B)} \varphi'_y(B) dy + [g_B(s_1(B)) + k] s'_1(B) - [g_B(S_1(B)) + k] S'_1(B)$$

$$
\begin{aligned}
&= \int_{s_1(B)}^{S_1(B)} \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{-\lambda_2 y}dy \\
&> 0,
\end{aligned}
$$

where the second equality follows from $g_B(S_1(B)) = g_B(s_1(B))$ and $S_1(B) = s_1(B) + Q_{n+1}$.

(ii) For $n \in \mathcal{N}_<$, we have $Q_n^* = Q_n$ by (5.10). By Lemma 5.3, for any $B \in (\underline{B}, \tilde{B})$ and given $Q_n > 0$, there exist unique $s_2(B)$ and $S_2(B)$ with $s_2(B) < S_2(B)$ and $S_2(B) - s_2(B) = Q_n$ such that $g_B(s_2(B)) = g_B(S_2(B))$. Then we can conclude that$s_2(B) < z^*(B) < S_2(B)$ by Lemma 5.3 and that the derivatives $s_2'(B)$ and $S_2'(B)$ exist by the Implicit Function Theorem. Define

$$
\Lambda_2(B) = \int_{s_2(B)}^{S_2(B)} [g_B(y) + k]dy. \tag{5.80}
$$

It suffices to show

$$
\begin{aligned}
\Lambda_2'(B) &> 0, && (5.81)\\
\Lambda_2(\hat{B}_n) &\geq -K_n, && (5.82)\\
\lim_{B\downarrow\underline{B}} \Lambda_2(B) &< -K_n. && (5.83)
\end{aligned}
$$

Firstly, we prove (5.81). Taking derivatives of the both sides of (5.80) with respect to B, we have

$$
\begin{aligned}
\Lambda_2'(B) &= \int_{s_2(B)}^{S_2(B)} \varphi_y'(B)dy + [g_B(s_2(B)) + k]s_2'(B) - [g_B(S_2(B)) + k]S_2'(B) \\
&= \int_{s_2(B)}^{S_2(B)} \frac{2}{\sigma^2}\frac{1}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}e^{-\lambda_2 y}dy \\
&> 0,
\end{aligned}
$$

where the second equality follows from $g_B(S_2(B)) = g_B(s_2(B))$ and $S_2(B) = s_2(B) + Q_n$.

Nest, we will show (5.82). By Lemma 5.6,we have $g_{\hat{B}_n}(\hat{s}_n) = g_{\hat{B}_n}(\hat{S}_n) = -k$. Together with Lemma 5.3, the fact that $g_{\hat{B}}(s_2(\hat{B}_n)) = g_{\hat{B}}(S_2(\hat{B}_n))$ and that $\hat{S}_n - \hat{s}_n \leq Q_n$, we have $s_2(\hat{B}_n) \leq \hat{s}_n < \hat{S}_n \leq S_2(\hat{B}_n)$. Then by Lemma 5.3 and Lemma 5.6,

$$\begin{aligned}\Lambda_2(\hat{B}_n) &= \int_{s_2(\hat{B}_n)}^{S_2(\hat{B}_n)} [g_{\hat{B}_n}(y) + k]dy \\ &\geq \int_{\hat{s}_n}^{\hat{S}_n} [g_{\hat{B}_n}(y) + k]dy \\ &= -K_n.\end{aligned}$$

Finally, we will show (5.83). As in the proof of Lemma 5.6, for $B \in (\underline{B}, \tilde{B})$ there exist unique $s(B)$ and $S(B)$ such that $s(B) < S(B)$ and $g_B(s(B)) = g_B(S(B)) = -k$. Next, we show that there exists a unique $\acute{B} \in (\underline{B}, \hat{B}_n]$ such that $S(\acute{B}) - s(\acute{B}) = Q_n$. By the definition of $s(B)$ and $S(B)$, we have $s(\hat{B}_n) = \hat{s}_n$ and $S(\hat{B}_n) = \hat{S}_n$. Then $S(\hat{B}_n) - s(\hat{B}_n) = \hat{S}_n - \hat{s}_n \leq Q_n$. By (5.70) and $\lim_{B \downarrow \underline{B}} s(B) < \lim_{B \downarrow \underline{B}} z^*(B) = -\infty$, we have $\lim_{B \downarrow \underline{B}} (S(B) - s(B)) = \infty$. Taking derivatives of the both sides of equation $g_B(s(B)) = -k$ with respect to B for $B \in (\underline{B}, \tilde{B})$, we have

$$s'(B) = -\frac{2}{\sigma^2} \frac{1}{\lambda_1 + \lambda_2} \frac{e^{-\lambda_2 S(B)}}{\lambda_2} \frac{1}{g'(s(B))} > 0, \tag{5.84}$$

where the inequality follows from $g'(s(B)) < 0$. Therefore, we have $S'(B) - s'(B) > 0$ for $B \in (\underline{B}, \tilde{B})$. Then there there exists a unique $\acute{B} \in (\underline{B}, \hat{B}_n]$ such that $S(\acute{B}) - s(\acute{B}) = Q_n$. Namely, $g_{\acute{B}}(s_3(\acute{B})) = g_{\acute{B}}(s_3(\acute{B}) + Q_n) = -k$. Therefore, $s_2(\acute{B}) = s(\acute{B})$ and $S_2(\acute{B}) = S(\acute{B})$. By the definition of g_B in (5.5) and $\acute{B} \leq \hat{B}_n$, we have $g_{\acute{B}}(z) \leq g_{\hat{B}_n}(z)$ for $z \in \mathbb{R}$. Then by $g_{\hat{B}_n}(\hat{s}_n) = g_{\hat{B}_n}(\hat{S}_n) = -k$ and $g_{\acute{B}}(s_2(\acute{B})) = g_{\acute{B}}(S_2(\acute{B})) = -k$, we have

$$s_2(\acute{B}) \leq \hat{s}_n < \hat{S}_n \leq S_2(\acute{B}).$$

Therefore,

$$\begin{aligned}
\Lambda_2(\acute{B}) &= \int_{s_2(\acute{B})}^{S_2(\acute{B})} [g_{\acute{B}}(y) + k]dy \\
&\leq \int_{\hat{s}_n}^{\hat{S}_n} [g_{\acute{B}}(y) + k]dy \\
&\leq \int_{\hat{s}_n}^{\hat{S}_n} [g_{\hat{B}_n}(y) + k]dy \\
&= -K_n,
\end{aligned}$$

which together with (5.81) implies (5.83). □

In order to prove subsequent lemmas, we need the following lemma that shows the properties of $g_B(z)$ at the reorder level s_n^* and the order-up-to level S_n^* of the (s_n^*, S_n^*) policy.

Lemma 5.8. *For $n \in \mathcal{N}_<$,*

$$g_{B_n^*}(s_n^*) = g_{B_n^*}(S_n^*) \geq -k. \tag{5.85}$$

For $n \in \mathcal{N}_=$,

$$g_{B_n^*}(s_n^*) = g_{B_n^*}(S_n^*) = -k. \tag{5.86}$$

For $n \in \mathcal{N}_>$,

$$g_{B_n^*}(s_n^*) = g_{B_n^*}(S_n^*) \leq -k. \tag{5.87}$$

Proof. For $n \in \mathcal{N}_=$, we have $s_n^* = \hat{s}_n$, $S_n^* = \hat{S}_n$. Comparing Lemma 5.6 and Lemma 5.7, we have $\hat{B}_n = B_n^*$. Then by Lemma 5.6, we have $g_{B_n^*}(s_n^*) = g_{B_n^*}(S_n^*) = -k$.

For $n \in \mathcal{N}_<$, we have $\hat{S}_n - \hat{s}_n \leq Q_n$. We prove (5.85) for two cases: $\hat{S}_n - \hat{s}_n = Q_n$ and $\hat{S}_n - \hat{s}_n < Q_n$. If $\hat{S}_n - \hat{s}_n = Q_n$, similar to $n \in \mathcal{N}_=$, we have $\hat{B}_n = B_n^*$, which further implies $g_{B_n^*}(s_n^*) = g_{B^*}(S_n^*) = -k$. If $\hat{S}_n - \hat{s}_n < Q_n = S_n^* - s_n^*$, we have $B_n^* \leq \hat{B}_n$ by Lemma 5.7. Then by Lemma 5.3, Lemma 5.4, $\hat{s}_n < z^*(\hat{B}_n) < \hat{S}_n$ and

$s_n^* < z^*(B_n^*) < S_n^*$, we have $s_n^* < \hat{s}_n < \hat{S}_n < S_n^*$. We will prove (5.85) by contradiction. Suppose $g_{B_n^*}(s_n^*) = g_{B_n^*}(S_n^*) < -k$, we have

$$-K_n = \int_{s_n^*}^{S_n^*} [g_{B_n^*}(y) + k]dy < \int_{\hat{s}_n}^{\hat{S}_n} [g_{B_n^*}(y) + k]dy \leq \int_{\hat{s}_n}^{\hat{S}_n} [g_{\hat{B}_n}(y) + k]dy = -K_n,$$

where the first equality follows from (5.72), the first inequality follows from Lemma 5.3, the second inequality follows from the fact $B_n^* \leq \hat{B}_n$ and the last equality follows from (5.62). Therefore, $g_{B_n^*}(s_n^*) = g_{B_n^*}(S_n^*) \geq -k$ for $n \in \mathcal{N}_<$.

For $n \in \mathcal{N}_>$, we have $\hat{S}_n - \hat{s}_n \geq Q_{n+1}$. We prove (5.87) for two cases: $\hat{S}_n - \hat{s}_n = Q_{n+1}$ and $\hat{S}_n - \hat{s}_n > Q_{n+1}$. If $\hat{S}_n - \hat{s}_n = Q_{n+1}$, similar to $n \in \mathcal{N}_=$, we have $\hat{B}_n = B_n^*$, which further implies $g_{B_n^*}(s_n^*) = g_{B^*}(S_n^*) = -k$. If $\hat{S}_n - \hat{s}_n > Q_{n+1}$, we have $S_n^* - s_n^* = Q_{n+1} < \hat{S}_n - \hat{s}_n$. By Lemma 5.3, Lemma 5.4, $\hat{s}_n < z^*(\hat{B}_n) < \hat{S}_n$ and $s_n^* < z^*(B_n^*) < S_n^*$, we have $\hat{s}_n < s_n^* < S_n^* < \hat{S}_n$. Then by Lemma 5.3 and $B_n^* \leq \hat{B}_n$, we have $g_{B_n^*}(S_n^*) = g_{B_n^*}(s_n^*) < g_{B_n^*}(\hat{s}_n) \leq g_{\hat{B}_n}(\hat{s}_n) = -k$. □

According to the definition of $\mathcal{N}_<$, $\mathcal{N}_=$ and $\mathcal{N}_>$ in Section 5.2, we can conclude that sets $\mathcal{N}_=$, $\mathcal{N}_>$ and $\mathcal{N}_<$ are mutually disjoint and $\mathcal{N}_= \cup \mathcal{N}_< \cup \mathcal{N}_> = \{1, 2, \ldots, M\}$.

Notice that for $n \in \mathcal{N}_> \cup \mathcal{N}_<$, we may not have $K(Q_n^*) = K_n$ by the setup cost given in (1.5). We define

$$\begin{aligned} \mathcal{N} &= \{n \in \mathcal{N}_> \cup \mathcal{N}_< : K(Q_n^*) = K_n\}, \\ \overline{\mathcal{N}} &= \{n \in \mathcal{N}_> \cup \mathcal{N}_< : K(Q_n^*) \neq K_n\}. \end{aligned}$$

Then $\mathcal{N}_=$, $\mathcal{N}$ and $\overline{\mathcal{N}}$ are mutually disjoint and $\mathcal{N}_= \cup \mathcal{N} \cup \overline{\mathcal{N}} = \{1, 2, \ldots, M\}$.

For each S_n^*, we have $\int_{s_n^*}^{S_n^*} [g_{B_n^*}(y) + k]dy = -K_n$ by (5.72). However, $\int_{s_n^*}^{S_n^*} [g_{B_n^*}(y) + k]dy = -K(Q_n^*)$ (equivalent to (5.34)) may not hold since the equation $K(Q_n^*) = K_n$ may not hold. By the following lemma, we show that we should always have $K(S_{n^*}^*) = K_{n^*}$ if we select the n^* by (5.11).

To state the next lemma, we first define

$$\underline{\chi}(n) = \max\{i = 1, \ldots, n-1 : i \in \mathcal{N} \cup \mathcal{N}_=\} \text{ for } n \in \mathcal{N}_< \backslash \mathcal{N}, \tag{5.88}$$

$$\overline{\chi}(n) = \min\{i = n+1, \ldots, M : i \in \mathcal{N} \cup \mathcal{N}_=\} \text{ for } n \in \mathcal{N}_> \backslash \mathcal{N}. \tag{5.89}$$

Lemma 5.9. *For each $n \in \mathcal{N}_< \backslash \mathcal{N}$, $\underline{\chi}(n)$ defined in* (5.88) *exists and satisfies $B_n^* < B_{\underline{\chi}(n)}^*$; for each $n \in \mathcal{N}_> \backslash \mathcal{N}$, $\overline{\chi}(n)$ defined in* (5.89) *exists and satisfies $B_n^* < B_{\overline{\chi}(n)}^*$.*

Proof. For each $n \in \mathcal{N}_< \backslash \mathcal{N}$, we prove the existence of $\underline{\chi}(n)$ by contradiction. Suppose for some $n \in \mathcal{N}_< \backslash \mathcal{N}$, $\underline{\chi}(n)$ does not exist, namely, $i \notin \mathcal{N} \cup \mathcal{N}_=$ and $K(Q_i^*) \neq K_i$ for $i = 1, \ldots, n-1$. Since $K_1 > 0$, Lemma 5.6 implies $\hat{Q}_1 > 0 = Q_1$, from which we can deduce $1 \notin \mathcal{N}_<$ and $n \geq 2$. For $n \in \mathcal{N}_< \backslash \mathcal{N}$, we have $\hat{Q}_n \leq Q_n = Q_n^*$ and $K(Q_n^*) = K_{n-1} < K_n$. By (5.65), we have $\hat{Q}_{n-1} < \hat{Q}_n \leq Q_n$, which together with $n-1 \notin \mathcal{N} \cup \mathcal{N}_=$ implies $n-1 \in \mathcal{N}_< \backslash \mathcal{N}$. By induction, we can obtain $\{1, \ldots, n-1\} \subset \mathcal{N}_< \backslash \mathcal{N}$, which contradicts the fact that $1 \notin \mathcal{N}_<$. Therefore, $\underline{\chi}(n)$ must exist.

For each $n \in \mathcal{N}_< \backslash \mathcal{N}$, we can conclude $\{\underline{\chi}(n)+1, \ldots, n\} \subset \mathcal{N}_< \backslash \mathcal{N}$ and $K_{\underline{\chi}(n)} < \cdots < K_n$ from above arguments. Then by (5.65), we have $\hat{Q}_{\underline{\chi}(n)} < \hat{Q}_{\underline{\chi}(n)+1} \leq Q_{\underline{\chi}(n)}$, which implies that $\underline{\chi}(n) \in \mathcal{N}_< \cup \mathcal{N}_=$. By Lemma 5.3 and Lemma 5.8, we have

$$g_{B_{\underline{\chi}(n)}^*}(z) > -k \quad \text{for } z < s_{\underline{\chi}(n)}^* \text{ and } z > S_{\underline{\chi}(n)}^*.$$

By (5.10), we have $Q_{\underline{\chi}(n)}^* < Q_n^*$. Then we have $s_n^* < s_{\underline{\chi}(n)}^* < S_{\underline{\chi}(n)}^* < S_n^*$. Therefore,

$$\int_{s_n^*}^{S_n^*} [g_{B_{\underline{\chi}(n)}^*}(y) + k] dy > \int_{s_{\underline{\chi}(n)}^*}^{S_{\underline{\chi}(n)}^*} [g_{B_{\underline{\chi}(n)}^*}(y) + k] dy = -K_{\underline{\chi}(n)}.$$

Furthermore, we have

$$\int_{s_n^*}^{S_n^*} [g_{B_n^*}(y) + k] dy = -K(S_n^*) = -K_{n-1}.$$

We can conclude that $B_n^* < B_{\underline{\chi}(n)}^*$ from $K_{\underline{\chi}(n)} \leq K_{n-1}$ and (5.5).

Using the fact that $Q_{M+1} = +\infty$, we prove that $\overline{\chi}(n)$ exists and $B_n < B_{\overline{\chi}(n)}$ by similar rationales. The details are omitted. $\square$

We obtain the n^* by (5.11) and s^*, S^*, B^*, K^* by (5.12) . Notice that by Lemma 5.9, we always have $n^* \in \mathcal{N}_= \cup \mathcal{N}$, namely we always have $\int_{s^*}^{S^*}[g_{B^*}(y)+k]dy = -K(S^*-s^*)$. We will prove the optimality of this (s^*,S^*) policy in Section 5.5.2.

5.5.2 Verification

In this subsection, we prove Theorem 5.1. Namely, we will prove that the (s^*,S^*) policy obtained by the four-step algorithm in Section 5.2 is an optimal policy for our Brownian inventory model in this chapter. To prove Theorem 5.1, we need the following technical result.

Lemma 5.10. *Assume that cost function h satisfies Assumption 3. If $K(z_2-z_1)=K_n$, for $n \in 1,2,\ldots,M$ and $z_2>z_1$, then*

$$\int_{z_1}^{z_2}[g_{B^*}(y)+k]dy \geq \int_{s_n^*}^{S_n^*}[g_{B_n^*}(y)+k]dy = -K_n.$$

Proof. By Lemma 5.3, for any $B \in (\underline{B},\overline{B})$ and any constant $Q \in (0,+\infty)$, there exist unique $\alpha_1(Q),\alpha_2(Q)$ with $\alpha_2(Q)-\alpha_1(Q)=Q$ such that $g_B(\alpha_1(Q))=g_B(\alpha_2(Q))$ and

$$\begin{cases} g_B(z) < g_B(\alpha_1(Q)) & \text{for } z \in (\alpha_1(Q),\alpha_2(Q)), \\ g_B(z) > g_B(\alpha_1(Q)) & \text{for } z \notin [\alpha_1(Q),\alpha_2(Q)], \end{cases} \tag{5.90}$$

For any $z_2>z_1$, letting $Q=z_2-z_1$, we can get $\alpha_1(z_2-z_1)$ and $\alpha_2(z_2-z_1)$. Therefore, for any $B \in (\underline{B},\overline{B})$ and $z_2>z_1$, we have

$$\begin{aligned} \int_{z_1}^{z_2}[g_B(y)+k]dy &= \int_{[z_1,z_2]\cap[\alpha_1(z_2-z_1),\alpha_2(z_2-z_1)]}[g_B(y)+k]dy \\ &\quad + \int_{[z_1,z_2]\cap[\alpha_1(z_2-z_1),\alpha_2(z_2-z_1)]^c}[g_B(y)+k]dy \\ &\geq \int_{[z_1,z_2]\cap[\alpha_1(z_2-z_1),\alpha_2(z_2-z_1)]}[g_B(y)+k]dy \\ &\quad +[g_B(\alpha_1(z_2-z_1))+k]v([z_1,z_2]\cap[\alpha_1(z_2-z_1),\alpha_2(z_2-z_1)]^c) \\ &= \int_{[z_1,z_2]\cap[\alpha_1(z_2-z_1),\alpha_2(z_2-z_1)]}[g_B(y)+k]dy \\ &\quad +[g_B(\alpha_1(z_2-z_1))+k]v([z_1,z_2]^c\cap[\alpha_1(z_2-z_1),\alpha_2(z_2-z_1)]) \end{aligned}$$

$$
\begin{aligned}
&\geq \int_{[z_1,z_2]\cap[\alpha_1(z_2-z_1),\alpha_2(z_2-z_1)]} [g_B(y)+k]dy \\
&\qquad + \int_{[z_1,z_2]^c\cap[\alpha_1(z_2-z_1),\alpha_2(z_2-z_1)]} [g_B(y)+k]dy \\
&= \int_{\alpha_1(z_2-z_1)}^{\alpha_2(z_2-z_1)} [g_B(y)+k]dy, \qquad (5.91)
\end{aligned}
$$

where v is the Lebesgue measure in $\mathbb{R}$, the two inequalities follow from (5.90) and the second equality follows from

$$
\begin{aligned}
& v([z_1, z_2] \cap [\alpha_1(z_2 - z_1), \alpha_2(z_2 - z_1)]^c) \\
= \; & v([z_1, z_2]) - v([z_1, z_2] \cap [\alpha_1(z_2 - z_1), \alpha_2(z_2 - z_1)]) \\
= \; & v([\alpha_1(z_2 - z_1), \alpha_2(z_2 - z_1)]) - v([z_1, z_2] \cap [\alpha_1(z_2 - z_1), \alpha_2(z_2 - z_1)]^c) \\
= \; & v([z_1, z_2]^c \cap [\alpha_1(z_2 - z_1), \alpha_2(z_2 - z_1)]).
\end{aligned}
$$

Next, we show

$$
\int_{\alpha_1(z_2-z_1)}^{\alpha_2(z_2-z_1)} [g_{B_n^*}(y)+k]dy \geq \int_{s_n^*}^{S_n^*} [g_{B_n^*}(y)+k]dy. \qquad (5.92)
$$

The proof of the this inequality will be divided into three cases: $n \in \mathcal{N}_=$, $n \in \mathcal{N}_<$ and $n \in \mathcal{N}_>$.

For $n \in \mathcal{N}_=$, we have $s_n^* = \hat{s}_n < \hat{S}_n = S_n^*$, which together with Lemma 5.3 and (5.86) implies that

$$
\int_{\alpha_1(z_2-z_1)}^{\alpha_2(z_2-z_1)} [g_{B_n^*}(z)+k]dz \geq \int_{s_n^*}^{S_n^*} [g_{B_n^*}(z)+k]dz.
$$

For $n \in \mathcal{N}_<$, we have

$$
S_n^* - s_n^* = Q_n^* = Q_n \leq z_2 - z_1 = \alpha_2(z_2 - z_1) - \alpha_1(z_2 - z_1),
$$

which further implies

$$\alpha_1(z_2 - z_1) \leq s_n^* < S_n^* \leq \alpha_2(z_2 - z_1).$$

Together with (5.85), we have

$$\int_{\alpha_1(z_2-z_1)}^{\alpha_2(z_2-z_1)} [g_{B_n^*}(z) + k]dz \geq \int_{s_n^*}^{S_n^*} [g_{B_n^*}(z) + k]dz.$$

For $n \in \mathcal{N}_>$, we have

$$S_n^* - s_n^* = Q_n^* = Q_{n+1} \geq z_2 - z_1 = \alpha_2(z_2 - z_1) - \alpha_1(z_2 - z_1),$$

which further implies

$$s_n^* \leq \alpha_1(z_2 - z_1) < \alpha_2(z_2 - z_1) \leq S_n^*.$$

Together with (5.87), we have

$$\int_{\alpha_1(z_2-z_1)}^{\alpha_2(z_2-z_1)} [g_{B_n^*}(z) + k]dz \geq \int_{s_n^*}^{S_n^*} [g_{B_n^*}(z) + k]dz.$$

Therefore, for each n and z_1, z_2 that satisfy $K(z_2 - z_1) = K_n$, we have

$$\begin{aligned}
\int_{z_1}^{z_2} [g_{B^*}(y) + k]dy &\geq \int_{z_1}^{z_2} [g_{B_n^*}(y) + k]dy \\
&\geq \int_{\alpha_1(z_2-z_1)}^{\alpha_2(z_2-z_1)} [g_{B_n^*}(y) + k]dy \\
&\geq \int_{s_n^*}^{S_n^*} [g_{B_n^*}(y) + k]dy \\
&= -K_n,
\end{aligned}$$

where the first inequality follows from the definition of n^* in (5.11), the second inequality follows from (5.91) and the last inequality follows from (5.92). □

By Theorem 5.4, Lemma 5.7 and (5.11), we can conclude that $DC(x, U(s^*, S^*)) = V_{B^*}(x)$ for $x \geq 0$. However, V_{B^*} does not satisfy (5.16). Next, we will construct a function $\bar{V}(z)$ for $z \in \mathbb{R}$ such that $\bar{V}(z) = V_{B^*}(z)$ for $z \geq 0$ and it satisfies (5.16). We will also show that $\bar{V}(z)$ satisfies all the conditions in Theorem 5.2 in the proof of Theorem 5.1. The function $\bar{V}(z)$ for $z \in \mathbb{R}$ is constructed by the following procedure. By Lemma 5.3, Lemma 5.7 and (5.11), there exists $\bar{s} < z^*(B^*) < \bar{S}$ such that $g_{B^*}(\bar{s}) = g_{B^*}(\bar{S}) = -k$. By Lemma 5.3, there exist $z_0 < \bar{s}$ such that

$$\int_{z_0}^{\bar{S}} [g_{B^*}(y) + k] dy = 0. \tag{5.93}$$

Furthermore, we have $g_{B^*}(z_0) > -k$ and $z_0 < z^*(B^*) < 0$. Then, function $\bar{V}(z)$ is defined as

$$\bar{V}(z) = \begin{cases} V_{B^*}(z) & \text{for } z \geq z_0, \\ V_{B^*}(z_0) + g_{B^*}(z_0)(z - z_0) & \text{for } z < z_0, \end{cases} \tag{5.94}$$

where the $V_{B^*}(z)$ is defined in (5.4) and $g_{B^*}(z)$ is defined in (5.5).

To prove that $\bar{V}(Z)$ satisfies all the conditions in Theorem 5.2, we need the following lemma.

Lemma 5.11. *For $B \in (\underline{B}, \tilde{B})$ and $z^*(B)$ defined in Lemma 5.3,*

$$h'(z^*(B)) < -\beta k.$$

Proof. By (5.55), we have

$$\begin{aligned} & \big(B - h'(0-) + \int_{z^*(B)}^{0} e^{\lambda_2 y} h''(y) dy\big) e^{-\lambda_2 z^*(B)} \\ = & \big(h'(0+) - h'(0-) + \int_{z^*(B)}^{0} e^{-\lambda_1 y} h''(y) dy + \int_{0}^{\infty} e^{-\lambda_1 y} h''(y) dy\big) e^{\lambda_1 z^*(B)} \\ = & \big(\overline{B} - h'(0-) + \int_{z^*(B)}^{0} e^{\lambda_2 y} h''(y) dy\big) e^{\lambda_1 z^*(B)}) > 0, \end{aligned}$$

where the second equality follows from the definition of $\overline{B}$ and the inequality follows from $\overline{B} > 0$ and $z^*(B) < 0$. Together with (5.36), we have

$$\frac{1}{\beta}h'(z^*(B)) < g_B(z^*(B)) < -k,$$

where the inequality follows form Lemma 5.5 □

Proof of Theorem 5.1. Firstly, we show that $DC(x, U(s^*, S^*)) = V_{B^*}(x) = \bar{V}(x)$ for $x \geq 0$. Namely, we need to show V_{B^*} satisfies the conditions in Theorem 5.4. By definition of $V_B(z)$ in (5.4) and Proposition 5.1, $V_{B^*}(z)$ is twice continuously differentiable, $V_B'(z)$ is polynomially bounded in $[0, \infty)$ and

$$\Gamma V_{B^*}(z) - \beta V_{B^*}(z) + h(z) = 0 \quad \text{for } z \in \mathbb{R}.$$

By Lemma 5.7 and Lemma 5.9, we further have

$$\int_{s^*}^{S^*} [g_{B^*}(y) + k]dy = -K^* = -K(S^* - s^*),$$

which can be rewritten as

$$V_{B^*}(S^*) - V_{B^*}(s^*) = -K^* - k \cdot (S^* - s^*) = -K(S^* - s^*) - k \cdot (S^* - s^*).$$

Since $s^* < 0$, we have $DC(x, U(0, S^*)) = V_{B^*}(x) = \bar{V}(x)$ for $x \geq 0$ by Theorem 5.4.

To show the optimality of the (s^*, S^*) policy if the initial inventory level $x \geq 0$, it suffices to show that $\bar{V}(z)$ satisfies all the conditions of the lower bound in Theorem 5.2. By the definition of $\bar{V}(z)$ in (5.94), $\bar{V}(z)$ is continuously differentiable and $\bar{V}'(z)$ is absolutely continuous. Since $\bar{V}(z) = V_{B^*}(z)$ for $z \geq 0 > z_0$, $\bar{V}'(z)$ is polynomial bounded in $[0, \infty)$. In addition, for $z_0 \leq z \leq 0$, $\bar{V}'(z) = g_{B^*}(z)$ is continuous in $z \in [z_0, 0]$, which implies that $\bar{V}'(z)$ is bounded in $[z_0, 0]$ and that for $z < z_0$, $\bar{V}'(z) = g_{B^*}(z_0)$ is a constant. Thus $\bar{V}(z)$ satisfies (5.15) and (5.16).

It remains to show $\bar{V}(z)$ satisfies (5.13) and (5.14). Firstly, we show that (5.13) is satisfied. For $z \in \mathbb{R}$, we have

$$\frac{1}{2}\sigma^2 V''_{B^*}(z) - \mu V'_{B^*}(z) - \beta V_{B^*}(z) + h(z) = 0,$$

which implies that $\Gamma\bar{V}(z) - \beta\bar{V}(z) + h(z) = 0$ for $z \geq z_0$. Then we have

$$\frac{1}{2}\sigma^2 V''_{B^*}(z_0) - \mu V'_{B^*}(z_0) - \beta V_{B^*}(z_0) + h(z_0) = 0,$$

which together with $V''_{B^*}(z_0) = g'_{B^*}(z_0) < 0$ implies

$$-\mu g_{B^*}(z_0) - \beta V_{B^*}(z_0) + h(z_0) > 0. \tag{5.95}$$

For $z < z_0$,

$$\begin{aligned}
& \frac{1}{2}\sigma^2 \bar{V}''(z) - \mu \bar{V}'(z) - \beta \bar{V}(z) + h(z) \\
= \;& -\mu g_{B^*}(z_0) - \beta[V_{B^*}(z_0) + g_{B^*}(z_0)(z - z_0)] + h(z) \\
= \;& -\mu g_{B^*}(z_0) - \beta V_{B^*}(z_0) + h(z_0) - \beta g_{B^*}(z_0)(z - z_0) + h(z) - h(z_0) \\
> \;& h(z) - h(z_0) - \beta g_{B^*}(z_0)(z - z_0) \\
> \;& 0,
\end{aligned}$$

where the first inequality follows from (5.95) and the last inequality follows from the convexity of h, Lemma 5.11, $z_0 < z^*(B^*)$ and $g_{B^*}(z_0) > -k$.

Finally, we show that (5.14) is satisfied. We will show this inequality for three cases: $z_2 > z_1 \geq z_0$, $z_0 > z_2 > z_1$ and $z_2 \geq z_0 > z_1$. If $z_2 > z_1 \geq z_0$,

$$\bar{V}(z_1) - \bar{V}(z_2) \;=\; V_{B^*}(z_1) - V_{B^*}(z_2) \leq K(z_2 - z_1) + k \cdot (z_2 - z_1),$$

where the inequality follows from Lemma 5.10. If $z_0 > z_2 > z_1$,

$$\bar{V}(z_1) - \bar{V}(z_2) \;=\; g_{B^*}(z_0)(z_1 - z_2)$$

$$
\begin{aligned}
&< -k \cdot (z_1 - z_2) \\
&< K(z_2 - z_1) + k \cdot (z_2 - z_1),
\end{aligned}
$$

where the first inequality follows from that $g_{B^*}(z_0) > -k$ and the second inequality follows from that $K(\xi) > 0$ for any $\xi > 0$. If $z_2 \geq z_0 > z_1$,

$$
\begin{aligned}
& \bar{V}(z_1) - \bar{V}(z_2) \\
= \ & V_{B^*}(z_0) + g_{B^*}(z_0)(z_1 - z_0) - V_{B^*}(z_2) \\
= \ & -\int_{z_0}^{z_2} [g_{B^*}(y) + k]dy + k \cdot (z_2 - z_1) + k \cdot (z_1 - z_0) + g_{B^*}(z_0)(z_1 - z_0) \\
\leq \ & -\int_{z_0}^{\bar{S}} [g_{B^*}(y) + k]dy + k \cdot (z_2 - z_1) + k \cdot (z_1 - z_0) + g_{B^*}(z_0)(z_1 - z_0) \\
= \ & k \cdot (z_2 - z_1) + [g_{B^*}(z_0) + k](z_1 - z_0) \\
< \ & k \cdot (z_2 - z_1) \\
< \ & K(z_2 - z_1) + k \cdot (z_2 - z_1),
\end{aligned}
$$

where the first inequality follows from Lemma 5.3, $g_{B^*}(\bar{s}) = g_{B^*}(\bar{S}) = -k$ and $z_0 < \bar{s}$ and the second inequality follows from $g_{B^*}(z_0) > -k$ and $z_0 > z_1$.

Therefore, we have $DC(x, Y) \geq \bar{V}(x) = DC(x, U(s^*, S^*))$ for $x \geq 0$. Namely, (s^*, S^*) policy is an optimal policy that minimizes the total discounted cost for the Brownian inventory system with non-negative initial inventory level among all the admissible policies. □

Chapter 6

Conclusions and Future Work

We investigated optimal control policies for three continuous-review inventory models with general quantity-dependent setup costs.

In Chapter 3, we obtained an optimal control policy that minimizes the long-run average cost for a continuous-review inventory model without backlogs. We proved that a $(0, S)$ policy is an optimal policy among all admissible policies by the lower bound approach. This result is consistent with the optimality of the (s, S) policy for the inventory model in He et al. (2015), but the lower barrier of the optimal policy for our model is specified at 0. This difference can be attributed to the assumption that the backlogs are not allowed in our model. Furthermore, we introduced a four-step algorithm for computing the optimal order-up-to level S. In Chapter 4, we obtained an optimal control policy that minimizes the discounted cost over the infinite horizon for a continuous-review inventory model. The assumptions of the inventory model are similar to the model discussed in Chapter 3 except that we assumed a slightly different holding cost rate function. The optimal policy is also of the $(0, S)$ type. We provided a comparison theorem under the discounted cost criterion. With this comparison theorem, we may examine a tractable subset of admissible policies to obtain a globally optimal policy. Together with the comparison theorem under the long-run average cost criterion in He et al. (2015), the lower bound method for Brownian control problems has been improved and may be applied to more general inventory models.

An optimal control policy that minimizes the total discounted cost for a continuous-review inventory model with backlogs and a non-negative initial inventory level was obtained in Chapter 5. When the initial inventory level is non-negative, it was proved that a control policy of the (s, S) type is an optimal policy among all admissible control policies. When the initial inventory level is negative, the optimal policy structure is still unknown to us. The obstacle that prevents us from considering this case is the discontinuity of the initial ordering cost resulted from the step setup cost function, which further renders the total discounted cost function discontinuous. In the absence of the smoothness condition, the lower bound theorem may not be applicable when the initial inventory level is negative. A more general lower bound theorem is required for solving this problem. We leave it to our future work.

All the inventory models in this thesis assumed only upward adjustments. However, there are applications that require downward adjustments as well. For example, the manager of the inventory system may sign contracts with the suppliers and these contracts allow the manager to return the goods at some cost. Another example is the cash management problem, in which the inventory of the cash can be adjusted upwards and downwards by buying and selling risky assets. Thus, a possible extension of our models is to consider both upward and downward adjustments in the Brownian continuous-review inventory models with quantity-dependent setup costs. See Harrison et al. (1983), Ormeci et al. (2008) and Dai et al. (2013a,b) for studies considering both upward and downward adjustments in Brownian continuous-review inventory models with constant setup costs.

In the inventory models of this thesis, we assumed demand processes to be Brownian motions. One possible extension of our work is to apply the improved lower bound approach to more general demand processes, such as mean-reverting diffusions (Cadenillas et al., 2010).

The lead time in our inventory models was assumed to be zero. However, delivery lags could be random in the practice. Thus, another challenging extension is to consider stochastic lead times in the continuous-review inventory models with quantity dependent setup costs. See Song et al. (2010) and Muthuraman et al. (2014) for studies considering stochastic lead times in Brownian continuous-review inventory models with constant setup costs.

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