THREE ESSAYS ON GAME THEORY

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DECLARATION

I hereby declare that the thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Qian Xuewen
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Contents

Summary

Game theory is a tool to analyze agents' interactive strategic behavior in various situations and it is widely applied in economic study. The study of game theory includes the analysis of rational strategic behavior, based on which various solution concepts are developed. This thesis consists of three essays on game theory and analyzes both equilibrium and non-equilibrium solution concepts.

In the first essay, we study the existence and uniqueness (or order independence) of iterated elimination procedure from a choice-theoretic viewpoint. We show a general existence result of iterated elimination procedure on an abstract reduction system. We identify a sufficient condition of "Monotonicity*" for the order independence and, in (in)finite games, we provide a full characterization of Monotonicity*. We also demonstrate that our approach is applicable to any form of iterated elimination processes in arbitrary strategic games, e.g., iterated strict dominance, iterated weak dominance, rationalizability, etc.

In the second essay, we study Pearce's (1984) extensive form rationalizability (EFR), as a special kind of iterated elimination procedure, under general preferences. The main result of this paper shows that in generic perfect information games, for any model of conditional preference that admits all subjective expected utilities consistent with Bayesian updating and satisfies a rather weak monotonicity condition, the EFR strategy profiles yield the backward induction outcome. Moreover, if the model admits all

preferences (satisfying the rather weak monotonicity condition), then the elimination procedure associated with EFR coincides with the backward iterated dominance procedure.

In the third essay, we follow Blume and Zame (Econometrica 62:783-794, 1994) to study the relationship between perfectly and sequentially rational strategic behavior from the point of view of semi-algebraic geometry. We present a unified framework for analyzing rational strategic behavior, with diversiform structures of beliefs, in extensive games. In this paper, we show a general "generic" equivalence theorem between perfect rationality and sequential rationality, which is applicable to various solution concepts such as equilibrium, rationalizability, iterated dominance and mutually acceptable course of action (MACA).

1 A unified approach to iterated elimination procedures

1.1 Introduction

It is important and useful to define solution concepts by using iterated elimination procedures in game theory and economics. Notably, iterated elimination of strictly dominated strategies (IESDS), iterated elimination of weakly dominated strategies (IEWDS), iterated elimination of never-best responses (IENBR), and the backward induction principle are extensively discussed in game theory; see, e.g., Osborne and Rubinstein (1994, Chapter 4).[1] Intuitively, the iterated elimination procedures are profoundly related to the assumption of "common knowledge of rationality".

Most of the research in the literature has been restricted to specific forms of iterated elimination procedures in finite or CC games (where CC means strategy sets are compact and payoff functions are continuous). While IESDS always exists and is order independent in CC games, it could be ill-behaved in arbitrary strategic games (see Dufwenberg and Stegeman (2002) and Chen et al. (2007)). To generally examine this issue, we here consider any form of iterated elimination procedures in arbitrary strate-

[1] In contrast to the fixed-point method used in the equilibrium approach, this alternative approach develops solution concepts by using iterative procedures, for example, Bernheim (1984) and Pearce's (1984) notion of rationalizability, Dekel and Fudenberg's (1990) iterative procedure, Borgers's (1994) iterated pure-strategy dominance, Gul's (1996) τ-theories, Asheim and Dufwenberg's (2003) concept of a fully permissible set, Ambrus's (2006) definition of coalitional rationalizability, Cubitt and Sugden's (2011) reasoning-based iterative procedure, Halpern and Pass's (2012) iterated regret-minimization procedure, and Hillas and Samet's (2014) iterative elimination of flaws of weakly dominated strategies.

gic games. In this paper, we study the existence and uniqueness of final outcomes (or order independence) of iterated elimination procedure from a choice-theoretic viewpoint.[2] More specifically, we adopt the classical theory of choice in which the set of outcomes is formalized by a choice rule that specifies the acceptable/desirable choices. The reduction relation specifies that any feasible reduction from a given set is a deletion of elements outside the choice set. We consider all finite and transfinite sequences of reduction in any arbitrary abstract reduction system. For example, iterated deletion of strictly dominated strategies can be viewed as an (in)finite sequence of reduction in an abstract reduction system associated with the strict domination relation. Infinite and transfinite sequences of reduction has significance in economic theory and game theory. For example, the Nash equilibrium in standard Cournot game (Moulin, 1984) can be solved by an infinite sequences of IESDS. Moreover, Lipman (1994) pointed out that the strategic implication of "common knowledge of rationality" can only be characterized by an uncountably infinite iterated elimination of never best replies.

We show the existence of iterated elimination procedure for any arbitrary abstract reduction system (Theorem 1(a)). Except for the Zemerlo Fraenkel (ZF) axioms of set theory, our proof of the existence requires neither the Axiom of Choice nor the Well-Ordering Principle. Our existence

[2]Duggan and Le Breton (2014) modeled a player's decision as a choice set and analyzed set-valued solution concepts in finite games. Trost (2014) formulated each player's decision as an individual choice problem under uncertainty and offered some epistemic motivation for order-independent elimination procedures in finite games.

theorem implies that there always exists an iterative elimination procedure for any arbitrary game by allowing a transfinite reduction sequence. In addition, under a (strong) condition of Monotonicity, the iterated elimination procedure is order independent and preserves all "fixed-points" (Theorem 1(b)).

In this paper, we follow Gilboa et al. (1990) to seek weaker sufficient conditions for order independence that can be used for various forms of iterated elimination procedure including finite and infinite elimination processes used in game theory. The major feature of this paper is that we impose no restrictions on the structure of games, possibly with infinite strategy spaces and discontinuous payoff functions. In the literature on game theory, most of the discussions on order independence focused on finite reduction sequences (in finite games); see, e.g., Gilboa et al. (1990), Osborne and Rubinstein (1994), Marx and Swinkels (1997), Asheim and Dufwenberg (2003), Apt (2004, 2011), Ambrus (2006, 2009), Tercieux (2006), Oyama and Tercieux (2009), Cubitt and Sugden (2011), Chen and Micali (2013) and Hillas and Samet (2014). Only a few of the research papers, e.g., Lipman (1994), Ritzberger (2002), Dufwenberg and Stegeman (2002), Green (2011), Chen et al. (2007, 2015), dealt with order independence for infinite reduction sequences in infinite games, with restrictions to the iterated strict dominance or rationalizability.[3] In this paper, we identify a fairly weak condition of "Monotonicity*" for the order independence

[3] See also Arieli (2010) and Halpern and Pass (2012) for related discussions on (infinitely) iterative elimination procedures.

on any arbitrary abstract reduction system (Theorem 2).

Roughly speaking, Monotonicity* requires that: along a reduction sequence, no undesirable alternative (which is outside a choice set) be changed to a desirable alternative after removing some of the undesirable alternatives, that is, choice sets are never expansive along an elimination path. The following example shows that, while iterated weak dominance is, in general, not an order-independent elimination procedure, it can be order independent and satisfy Monotonicity* in some particular game. Consider a two-person game (where player 1 chooses a row and player 2 chooses a column):

	y_1	y_2	y_3
x_1	$1,1$	$1,1$	$0,0$
x_2	$0,1$	$1,1$	$2,1$
x_3	$0,0$	$0,0$	$2,1$

For any subset X of strategy profiles, choice set $c(X)$ consists of all weakly undominated strategy profiles in the reduced game with strategy-profile space X. The iterated weak dominance yields a unique outcome path: $\{x_1, x_2, x_3\} \times \{y_1, y_2, y_3\} \to \{x_1, x_2\} \times \{y_1, y_2, y_3\} \to \{x_1, x_2\} \times \{y_1, y_2\} \to \{x_1\} \times \{y_1, y_2\}$, *along which the choice rule c satisfies a monotonicity property: $c(Y) \subseteq c(X)$ if $Y \subseteq X$.* (In this game, the choice rule c fails to satisfy the monotonicity property off this outcome path, e.g., $c(\{x_2\} \times \{y_2, y_3\}) \not\subseteq c(\{x_2, x_3\} \times \{y_2, y_3\})$.) That is, the game of this example satisfies the sufficient condition of Monotonicity* for order independence.[4]

[4]This example fails to satisfy the original "nice weak dominance" condition due to

In finite games, iterated strict dominance is indeed an order-independent elimination procedure, which actually satisfies Monotonicity* (because each strictly dominated strategy in any finite game remains to be strictly dominated in a reduced game after eliminating some of the strictly dominated strategies). However, in infinite games, iterated strict dominance might not be order-independent; the order dependence problem in infinite games is much more complicated and deeper (see Dufwenberg and Stegeman (2002) for extensive discussions). In particular, Monotonicity* may fail to be satisfied in this case: a strictly dominated strategy in an infinite game can be changed to a strictly undominated strategy after eliminating some of the strictly dominated strategies.[5] Our main result of Theorem 2 implies that, if Monotonicity* holds, iterated strict dominance must be order independent in both finite games and infinite games. Exploring sufficient conditions for order independence for any kind of finitely and transfinitely iterated elimination procedures is the main focus of this paper.

We also apply our analysis to game theory. In finite games, Apt (2011) offered a uniform proof of order independence for various strategy elimination procedures based on Newman's (1942) Lemma; see also Apt (2004). We obtain Apt's (2011) Theorem 1 as a corollary of Theorem 2 (Corollary 1). In addition, we demonstrate how to apply our analysis of order

Marx and Swinkels (1997); see Section 3.1 for more discussions.

[5]For example, consider a simple one-person game where the strategy space is $X = (0,1)$ and the payoff function is $u(x) = x$ for every strategy x. Obviously, every strategy is strictly dominated and choice set $c(X) = \varnothing$. We can eliminate in round one all strategies except a particular strategy x in $(0,1)$. Thus, $c(\{x\}) = \{x\} \nsubseteq c(X)$, which violates Monotonicity*.

independence to some of the iterated elimination processes discussed in the literature, including iterated strict dominance, iterated weak dominance and rationalizability.

In infinite games, we provide a full characterization of Monotonicity* by Hereditarity* (Theorem 3). In contrast to the Monotonicity* property on choice sets of desirable alternatives, Hereditarity* is a property for sets of undesirable alternatives – i.e. dominated elements under an abstract dominance relation – which can be often used in the context of games. Along the lines of Jackson's (1992) idea of "boundedness" which requires any eliminated strategy to be justified by an undominated dominator, we introduce a novel and useful definition of "closed under dominance* (CD*)" games, including all compact and own-uppersemicontinuous games, to establish an order independence result in infinite games. In CD* games, we show that Gilboa et al.'s (1990) procedure is an order-independent iterated elimination procedure (Corollary 4). In the special case of finite games, we also show that the result holds true under a simple form of 1-CD* games (Corollaries 2 and 3).

The rest of the paper is organized as follows. In Section 2, we define the iterated elimination procedure on an abstract reduction system and establish its existence. We investigate the uniqueness of iterated elimination procedure and show the order independence result under Monotonicity*. In Section 3, we apply our analysis to finite and infinite games. We provide a full characterization of Monotonicity* by Hereditarity*. We also show an order independence result in the class of CD* games. Section 4 concludes. To facilitate reading, all the proofs are relegated to the appendixes.

1.2 Iterated elimination procedures

Consider an arbitrary set S of alternatives.[6] A *choice rule on* S is a function $c : 2^S \to 2^S$ which designates a *choice set* $c(X) \subseteq X$ for each subset $X \subseteq S$. For the purpose of this paper, we do not require the nonemptiness of choice sets. (Note that, for arbitrary function $f : 2^S \to 2^S$, we can define a choice rule c_f on S by $c_f(X) = X \cap f(X)$ for all $X \subseteq S$.) We interpret that, when faced with the set X of alternatives, all elements in the choice set $c(X)$ are regarded as "choosable/acceptable" outcomes – the alternatives that can be chosen; cf. Sen (1993, p.499). Throughout this paper, we denote by X and Y subsets of S. A choice rule c is said to satisfy *Monotonicity* if

$$[Y \subseteq X] \Rightarrow [c(Y) \subseteq c(X)];$$

that is, there are no fewer acceptable outcomes available within a wider scope of feasible alternatives.

We define the *reduction relation* $\to$ for the choice problem (S, c) as follows:

$$X \to Y \text{ iff } c(X) \subseteq Y \subseteq X.$$

That is, X can be reduced to Y iff no element in $c(X)$ is eliminated from X to a subset Y of X. Apparently, we allow $X \to X$ for any $X \subseteq S$. We

[6] Throughout this paper, we assume that sets satisfy the ZF axioms (cf., e.g., Jech 2003, p.3).

denote by $(S, \rightarrow)$ the *abstract reduction system* for the choice problem (S, c). We define the iterated elimination procedure on the abstract reduction system $(S, \rightarrow)$, possibly by using a transfinite process of reduction.[7] Let 0 denote the first element in an ordinal Λ, and let $\lambda + 1$ denote the successor to λ in Λ.

[7]Since the set S may be infinite, it is natural and necessary for us to consider a transfinite sequence of reduction on $(S, \rightarrow)$. Lipman (1994) demonstrated that, in infinite games, we need the transfinite induction to deal with the strategic implication of "common knowledge of rationality"; see also Chen et al. (2007, Example 1) and Green (2011) for more discussions.

Definition 1. An *iterated elimination process (IEP) for the choice problem* (S, c) is a reduction sequence $\{X^\lambda\}_{\lambda \leq \Lambda}$ on $(S, \rightarrow)$ such that

(a) $X^0 = S$,

(b) $X^\lambda \rightarrow X^{\lambda+1}$ (and $X^\lambda = \cap_{\lambda' < \lambda} X^{\lambda'}$ for a limit ordinal λ), and

(c) $X^\Lambda \rightarrow X$ only if $X^\Lambda = X$.

In Definition 1(c), the "stopping" condition: $X^\Lambda \rightarrow X$ only if $X^\Lambda = X$ expresses the idea that no elements in X^Λ can be eliminated for further consideration; it is equivalent to $X^\Lambda = c\left(X^\Lambda\right)$. An IEP $\{X^\lambda\}_{\lambda \leq \Lambda}$ for (S, c) is "fast" if $X^{\lambda+1} = c\left(X^\lambda\right)$ and $X^\Lambda \neq X^\lambda$ for all $\lambda < \Lambda$. Definition 1 does not require the elimination of *all* elements outside the choice set $c\left(X^\lambda\right)$ in each round of reduction since $c\left(X^\lambda\right) \subseteq X^{\lambda+1} \subseteq X^\lambda$; in particular, it allows for the elimination of no elements in some round of reduction: $X^{\lambda+1} = X^\lambda$.

The following theorem asserts that, for any arbitrary set S and choice rule c on S, there is always an iterated elimination process in Definition 1. Under Monotonicity, the iterated elimination procedure is *order-independent*: that is, all IEPs yield a unique set of final outcomes. Furthermore, every IEP results in all "fixed-points" of c and, thus, it preserves all elements $x = c(x)$.

Theorem 1. *(a) For any problem (S, c), there exists a (fast) IEP on $(S, \rightarrow)$. (b) Suppose that c satisfies Monotonicity. Then, the iterated elimination procedure is order-independent; moreover, if $\left\{X^\lambda\right\}_{\lambda \leq \Lambda}$ is an IEP for (S, c), then $X^\Lambda = \cup_{Z = c(Z)} Z$.*

We would like to point out that, except for the ZF axioms, our proof of the existence of iterated elimination procedure does not require the Axiom of Choice. The proof improves, if applied to iterated strict dominance in games, the existence proofs in Chen et al. (2007, 2015) which rely on either the Axiom of Choice or the Well-Ordering Principle.

The iterated elimination procedure is in general order-dependent: iterated elimination processes in Definition 1 may generate different sets of outcomes. For instance, some of the most prominent iterated elimination procedures such as iterated elimination of weakly dominated strategies (IEWDS) fail to be order independent. Under Monotonicity, Theorem 1(b) asserts that the iterated elimination procedure must be order-independent. While this result is simple, it is useful to determine the order independence for many iterated elimination processes used in game theory, which preserve Nash equilibria by Theorem 1(b). See, for example, Apt's (2004) related discussions on the order independence of various forms of iterated dominance in finite games, Ritzberger's (2002) Theorem 5.1 for the order independence of iterated strict dominance in the class of CC games (where strategy sets are compact and payoff functions are continuous), and Chen et al.'s (2007) Theorem 1 for the order independence of iterated strict dominance* in arbitrary games.

Nevertheless, iterated elimination of strictly dominated strategies (IESDS) fails to satisfy the property of monotonicity (because no element in a singleton of a strictly dominated strategy can be strictly dominated by using this dominated strategy). Monotonicity is not a necessary condition for order independence, as illustrated by the example in Introduction. In particular, the monotonicity property seems to be an unnecessary requirement for a circumstance that never occurs in performing the iterated elimination. We offer a weaker version of monotonicity for order independence, which we call "monotonicity*": it requires the monotonicity property *only along the iterated reduction sequence starting at* S. Let $\rightarrow^*$ denote the *indirect reduction relation* induced by $\rightarrow$. That is, $X \rightarrow^* Y$ iff there is a reduction sequence $\{X^\lambda\}_{\lambda \leq \Lambda}$ such that $X^0 = X$, $X^\lambda \rightarrow X^{\lambda+1}$ (and $X^\lambda = \cap_{\lambda' < \lambda} X^{\lambda'}$ for a limit ordinal λ), and $X^\Lambda = Y$. In other words, $X \rightarrow^* Y$ via a reduction sequence $\left\{X^\lambda\right\}_{\lambda \leq \Lambda}$.

Monotonicity*. $[S \rightarrow^* X \rightarrow^* Y] \Rightarrow [c(Y) \subseteq c(X)]$.

That is, the Monotonicity* property requires that, along a reduction sequence through X to Y, choice sets should not be expansive – i.e., no undesirable alternative outside $c(X)$ can be changed to a desirable one in $c(Y)$ during the phase of reduction: $X \rightarrow^* Y$. If we restrict our attention only to the one-step-ahead reduction, we obtain a simpler and weaker version of *1-Monotonicity**: $[S \rightarrow^* X \rightarrow Y] \Rightarrow [c(Y) \subseteq c(X)]$. The central result of this paper is that, under the weak condition of Monotonicity*, the iterated elimination procedure is order-independent. Moreover, 1-Monotonicity* is sufficient for order independence in the finite case.

Theorem 2. *(a) Suppose that c satisfies Monotonicity*. Then, the iterated* *elimination procedure for the problem (S, c) is order-independent. (b) Let S be a finite set. If c satisfies 1-Monotonicity*, then c satisfies Monotonicity* and, thus, the iterated elimination procedure is order-independent.*

Remark. Monotonicity implies Monotonicity* which in turn implies 1-Monotonicity*. 1-Monotonicity* is closely related to the Aizerman property used in the choice-theoretic literature: $[c(X) \subseteq Y \subseteq X] \Rightarrow [c(Y) \subseteq c(X)]$; see, e.g., Moulin (1985). In 1-Monotonicity*, we relax the premise of the Aizerman property only for the set X resulting from a reduction sequence starting at S. In Monotonicity*, we strengthen the premise of the Aizerman property by considering the indirect reduction relation $\rightarrow^*$ because the iterated elimination procedure may use a transfinite sequence of reduction in infinite sets.

The following example shows that, in the infinite set case, 1-Monotonicity* is not sufficient for order independence.

Example 1. Consider an infinite set $S = \mathbb{N} \cup \{-1\}$. The choice rule c is defined as: for subsets $N \subseteq \mathbb{N}$,

$$c(N \cup \Psi) = \begin{cases} N \backslash \min N & \text{if } |N| > 1 \\ N & \text{if } |N| = 1 \\ \Psi & \text{if } N = \varnothing \end{cases},$$

where Ψ is $\{-1\}$ or $\varnothing$. In this example there are two IEPs which generate different outcomes:

1. $X^0 = S$, $X^n = \mathbb{N} \backslash \{0, 1, ..., n-1\}$ $\forall n \in \mathbb{N}$, and $X^{\mathbb{N}} = \cap_{n \in \mathbb{N}} X^n = \varnothing$.
2. $\widetilde{X}^0 = S$, $\widetilde{X}^n = S \backslash \{0, 1, ..., n-1\}$ $\forall n \in \mathbb{N}$, and $\widetilde{X}^{\mathbb{N}} = \cap_{n \in \mathbb{N}} \widetilde{X}^n = \{-1\}$.

The choice rule c satisfies 1-Monotonicity*. However, c fails to satisfy Monotonicity* since $S \to^* \{-1\}$ and $\{-1\} = c(\{-1\}) \nsubseteq c(S) = \mathbb{N} \backslash \{0\}$. (Note: $\cap_{n\in\mathbb{N}} c\left(\widetilde{X}^n\right) \neq c\left(\cap_{n\in\mathbb{N}} \widetilde{X}^n\right)$.)

1.3 Applications in game theory

Consider an arbitrary (strategic) game:

$$G \equiv (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N}),$$

where N is an (in)finite set of players, S_i is an (in)finite set of player i's strategies, and $u_i\colon \times_{i\in N} S_i \to \mathbb{R}$ is player i's arbitrary payoff function. Let $S \equiv \times_{i\in N} S_i = S_i \times S_{-i}$. For $X \subseteq S$ let $X_i = \{s_i \in S_i : (s_i, x_{-i}) \in X\}$ and $X_{-i} = \{s_{-i} \in S_{-i} : (x_i, s_{-i}) \in X\}$. For game G, let c be a choice rule on S and for $X \subseteq S$ define

$$DOM(X) \equiv X \backslash c(X).$$

We consider the abstract reduction system $(S, \to)$ for the choice problem (S, c).

1.3.1 Finite games

Hereditarity Consider a finite game $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$. In finite games, along the lines of Gilboa et al.'s (1990) approach, Apt (2011) presented an easy-to-apply condition of "hereditarity" for order independence; Apt (2011) demonstrated that many of order-independence results for iterated elimination procedures in finite games can be obtained by checking the hereditarity property. We state this condition in our choice-based setting as follows: Let $X, Y \subseteq S$.

HEREDITARITY. $[X \to Y] \Rightarrow [(Y \cap DOM(X)) \subseteq DOM(Y)]$.

That is, Hereditarity says if x is dominated but not eliminated (i.e., $x \in DOM(X) \cap Y$), then it is still dominated after eliminating some of the dominated elements (i.e., $x \in DOM(Y)$). For example, under the strict domination relation, a finite game satisfies Hereditarity because every strictly dominated strategy in a finite game has an undominated dominator, remaining in any reduced game after eliminating some of the strictly dominated strategies, which strictly dominates the former dominated strategy in this reduced game. Since Hereditarity actually implies 1-Monotonicity* (see the proof of Corollary 1), Apt's (2011, Theorem 1) order-independence result follows immediately from Theorem 2(b).[8] Since the requirement of 1-Monotonicity* is only imposed on the elimination path, 1-Monotonicity* does not imply hereditarity (see the example in the introduction).

Corollary 1. *Suppose that G is a finite game. Under Hereditarity, the iterated elimination procedure is order-independent.*

1-CD* games Motivated by Jackson's (1992) idea of "boundedness" that requires that strategies be eliminated only by undominated strategies, Dufwenberg and Stegeman (2002) introduced a definition of "games

[8] Apt (2011) considered the class of finite sequences of reduction under a variety of dominance relations in finite games and showed this result by using Newman's (1942) Lemma.

closed under dominance (CD games)" for the strict dominance and showed, through an example, that IESDS can be order-dependent in CD games. Roughly speaking, CD games satisfy the property that at any point in a finite-step sequence of deletions, any dominated strategy has an undominated dominator. At a conceptual level, this definition of CD games does not rule out the possibility of order dependence of an elimination procedure. The following example shows this point.[9]

Example 2. Consider the two-person game:

	y_1	y_2
x_1	$1, 1$	$0, 0$
x_2	$1, 1$	$2, 1$

The game is a CD game under the weak dominance because, in every reduced game, any weakly dominated strategy is weakly dominated by a weakly undominated strategy. However, the iterated weak dominance is not order independent. At a conceptual level, the order independence of iterated strict dominance should not be attributed to the CD property, but due to the fact that Hereditarity holds for iterated strict dominance in finite games.

Nevertheless, the strict domination relation in finite games satisfies a stronger 1-CD* property that every strictly dominated strategy in a finite game has an undominated dominator, remaining to be an undominated

[9]By the transitivity of weak dominance, any finite game is a CD game; see Jackson (1992, p.763). This definition of CD games cannot be expected to solve the problem of order dependence in nature.

dominator in any reduced game after eliminating some of the strictly dominated strategies, which strictly dominates the former dominated strategy in this reduced game. In fact, if an abstract domination relation satisfies Monotonicity like ones under rationalizability and the strict dominance* as defined below, then CD implies this stronger 1-CD* property, which is sufficient for order independence of the iterated elimination procedure defined by using reduced games because it implies Hereditarity.

We follow Jackson's (1992) idea of "boundedness" to introduce the notion of 1-CD* games to solve the problem of order dependence under an arbitrary domination relation. Consider a finite game $G \equiv (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$. For $X \subseteq S = \times_{i \in N} S_i$, let $\succ_X$ denote an *abstract domination relation* on S given X. For profiles $x, y \in S$, $y \succ_X x$ is interpreted to mean that y dominates x conditionally on X; see also Luo (2001) for more discussions. For instance, $y \succ_X x$ can represent the strict domination relation: "y strictly dominates x given X", that is, $y \succ_X x$ iff there exists $i \in N$ such that $u_i(y_i, z_{-i}) > u_i(x_i, z_{-i})$ for all $z_{-i} \in X_{-i}$. Define $c(X) = X \backslash DOM(X)$, where

$$DOM(X) = \{x \in X : y \succ_X x \text{ for some } y \in X\}.$$

For any $X, Y \subseteq S$, let

$$DOM^Y(X) = \{x \in X : \ y \succ_X x \text{ for some } y \in Y\}.$$

Now consider the abstract reduction system $(S, \rightarrow)$ for the problem (S, c). We say that game G (under an abstract domination relation $\succ_X$) is *one-*

step-ahead-deletion closed under dominance (1-CD*)* if $S \rightarrow^* X \rightarrow Y$ and $y \succ_X x$ for some $y \in X$ and $x \in Y$ imply that: there exists $z^* \in Y$ such that $z \not\succ_Y z^* \succ_Y x$ for all $z \in Y$, i.e.,

$$[S \rightarrow^* X \rightarrow Y] \Rightarrow \left[(Y \cap DOM(X)) \subseteq DOM^{c(Y)}(Y)\right].$$

That is, at any stage in a sequence of deletions, any dominated element surviving the one-step-ahead deletion has an undominated dominator which continues dominating it at the end of the deletion. In other words, under an abstract domination relation, any dominated strategy that is not eliminated has an undominated dominator during the phase transition of one-step-ahead deletion.[10]

Note that a 1-CD* game must be a CD game because the former also satisfies the property: $S \rightarrow^* X \rightarrow X$ and $y \succ_X x$ for some $x, y \in X$ imply that: there exists $z^* \in X$ such that $z \not\succ_X z^* \succ_X x$ for all $z \in X$, i.e., $[S \rightarrow^* X] \Rightarrow \left[DOM(X) = DOM^{c(X)}(X)\right]$. Thus, the notion of 1-CD* games can be viewed as a "dynamic" version of CD games by extending the CD concept to the one-step-ahead reduction transition. The notion of 1-CD* games can exclude problematic games with the problem of order dependence; for instance, the game of Example 2 is not a 1-CD* game under the weak domination relation since $S \rightarrow X = \{x_1,\ x_2\} \times \{y_1\}$ and

[10] That is, a 1-CD* game has a "boundedness" property that there exists the undominated dominator which is not eliminated in the one-step-ahead deletion. This property is related to Gilboa, Kalai, and Zemel's (1990) (GKZ) notion of reduction that requires that the dominator is one which is not eliminated. Corollary 3 shows that, in 1-CD* games, the GKZ elimination procedure is equivalent to the reduction procedure discussed in this paper. In a finite game, 1-CD* is equivalent to CD*, the proof is similar to that of Theorem 2(b).

$(x_2, y_1) \succ_S (x_1, y_1)$ but $(x_2, y_1) \not\succ_X (x_1, y_1)$. The following corollary asserts that the iterated elimination procedure is order independent for any finite 1-CD* game.

Corollary 2. *Suppose that G is a finite 1-CD* game. Then, the iterated elimination procedure is order-independent.*

Gilboa, Kalai and Zemel (1990) (GKZ) studied a variety of elimination procedures and provide sufficient conditions for order independence. The key requirement for the GKZ procedure is that for any element x that is eliminated there exists an element y that dominates x and is not eliminated. More precisely, the GKZ procedure is an elimination procedure on the abstract reduction system $\left(S, \rightarrow^{\text{GKZ}}\right)$ associated with an abstract domination relation $\succ_X$, where, for subsets $X, Y \subseteq S$,

$$X \rightarrow^{\text{GKZ}} Y \text{ iff } Y \subseteq X \text{ and } X \backslash Y \subseteq DOM^Y(X).$$

That is, $X \rightarrow^{\text{GKZ}} Y$ iff every eliminated element $x \in X \backslash Y$ has a dominator $y \in Y$ (i.e. $x \in DOM^Y(X)$). Apparently, $X \rightarrow^{\text{GKZ}} Y$ implies $X \rightarrow Y$, since $DOM^Y(X) \subseteq DOM(X)$; the GKZ procedure can be viewed as a special form of the iterated elimination procedure in Definition 1. Moreover, if $DOM(X) \subseteq DOM^{c(X)}(X)$, $X \rightarrow^{\text{GKZ}} Y$ iff $X \rightarrow Y$. The following Corollary states that, in finite 1-CD* games, the iterated elimination procedure in Definition 1 is precisely the GKZ procedure, which is an order-independent procedure as proved by GKZ.

Corollary 3. *Suppose that G is a finite 1-CD* game. Then, the GKZ* *procedure is equivalent to the iterated elimination procedure in Definition 1 and, thus, the GKZ procedure is order independent.*

Related literature We demonstrate how to apply our analysis of order independence to some of the iterated elimination processes for finite games discussed in the literature, including iterated strict dominance, iterated weak dominance and rationalizability. For any subset X of strategy profiles, we can define the choice set $c(X)$ in the following ways.

1. [**strict dominance**] $c(X) = X \backslash DOM(X)$ where[11]

$$DOM(X) = \{x \in X : \exists i \in N \ \exists \sigma_i \in \Delta(X_i) \text{ s.t.}$$
$$u_i(\sigma_i, x'_{-i}) > u_i(x_i, x'_{-i}) \ \forall x'_{-i} \in X_{-i}\}.$$

   That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is strictly dominated by no mixed strategy in $\Delta(X_i)$. Since every strictly dominated strategy x_i in a finite game has an undominated dominator, remaining in a reduced game after eliminating some of the strictly dominated strategies, which strictly dominates x_i in that reduced game, $(Y \cap DOM(X)) \subseteq DOM(Y)$ for $c(X) \subseteq Y \subseteq X$. Thus, Hereditarity holds. By Corollary 1, IESDS is an order-independent procedure. (Under the strict dominance relation, 1-CD* actually holds true, but Monotonicity fails to be satisfied, e.g., $c(x) = x \notin c(X)$ for $x \in X \backslash c(X)$.)

[11] We denote by $\Delta(X_i)$ the probability space on X_i and by $u_i(\sigma_i, x_{-i})$ the expected payoff of player i under a mixed strategy $\sigma_i \in \Delta(X_{-i})$.

2. **[weak dominance]** $c(X) = X \backslash DOM(X)$ where $DOM(X) =$

$$\{x \in X : \exists i \in N \; \exists \sigma_i \in \Delta(X_i) \text{ s.t. } u_i\left(\sigma_i, x'_{-i}\right) \geq u_i\left(x_i, x'_{-i}\right) \; \forall x'_{-i} \in X_{-i}$$

$$\text{and } u_i\left(\sigma_i, x'_{-i}\right) > u_i\left(x_i, x'_{-i}\right) \text{ for some } x'_{-i} \in X_{-i}\}.$$

   That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is weakly dominated by no mixed strategy in $\Delta(X_i)$. The IEWDS procedure may not be order independent in general; see, e.g., Example 2.

3. **[strict dominance*]** $c(X) = X \backslash DOM^S(X)$ where

$$DOM^S(X) = \{x \in X : \exists i \in N \; \exists s_i^* \in S_i \text{ s.t.}$$

$$u_i\left(s_i^*, x'_{-i}\right) > u_i\left(x_i, x'_{-i}\right) \; \forall x'_{-i} \in X_{-i}\}.$$

   That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is strictly dominated by no strategy in S_i; see, e.g., Milgrom and Roberts (1990), Ritzberger (2002) and Chen et al. (2007). Since every strictly dominated strategy in a finite game has an undominated dominator which strictly dominates that dominated strategy in each of subgames, $\left(Y \cap DOM^S(X)\right) \subseteq DOM^S(Y)$ for $Y \subseteq X$. Thus, $c(Y) \subseteq c(X)$ if $Y \subseteq X$. That is, Monotonicity holds. By Theorem 1(b), The IESDS* procedure is order independent and preserves Nash equilibria.

4. **[pure-strategy dominance]** $c(X) = X \backslash DOM(X)$ where $DOM(X) =$

$$\{x \in X : \exists i \in N \; \forall Z_{-i} \subseteq X_{-i} \; \exists s_i^* \in S_i \text{ s.t. } u_i(s_i^*, z_{-i}) \geq u_i(x_i, z_{-i})$$

$$\forall z_{-i} \in Z_{-i} \text{ and } u_i(s_i^*, z_{-i}) > u_i(x_i, z_{-i}) \text{ for some } z_{-i} \in Z_{-i}\}.$$

That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is undominated in the sense of Borgers (1994). Under the pure-strategy dominance relation, since every dominated strategy in a finite game is clearly dominated in each subgame, $(Y \cap DOM(X)) \subseteq DOM(Y)$ for $Y \subseteq X$. Thus, Hereditarity holds. By Corollary 1, the pure-strategy dominance is an order-independent reduced procedure.

5. **[rationalizability]** $c(X) = X \cap BR(X)$ where[12]

$$BR(X) = \{s \in S : \forall i \in N \; \exists \mu_i \in \Delta(X_{-i}) \text{ s.t.}$$
$$u_i(s_i, \mu_i) \geq u_i(s_i', \mu_i) \;\; \forall s_i' \in S_i\}.$$

That is, $c(X)$ consists of all elements in X where each player i's strategy is a best response to some probabilistic belief in $\Delta(X_{-i})$. Since $BR(Y) \subseteq BR(X)$ for $Y \subseteq X$, $c(Y) \subseteq c(X)$ if $Y \subseteq X$. That is, Monotonicity holds. By Theorem 1(b), rationalizability is an order-independent elimination of never best response strategies which preserves Nash equilibria.

6. **[c-rationalizability]** Ambrus (2006) proposed a solution concept of "coalitional rationalizability (c-rationalizability)" in finite games by an iterative procedure of restrictions of strategies. The procedure is analogous to iterative elimination of never best response strategies, but operates on implicit agreements by coalitions. More specifically, let X and Z be product-form subsets of strategy profiles. Z is a *supported restriction* by coalition $J \subseteq N$ given X if (i) $Z_j \subseteq X_j$ for $j \in J$ and $Z_i = X_i$ for $i \notin J$

[12] We denote by $\Delta(X_{-i})$ the probability space on X_{-i} and by $u_i(x_i, \mu_i)$ the expected payoff of player i under a probabilistic belief $\mu_i \in \Delta(X_{-i})$.

and (ii) for $j \in J$, $x_j \in X_j \backslash Z_j$ implies that

$$\max_{f_{-j} \in \Delta(X_{-j})} u_j(x_j, f_{-j}) < \max_{s_j \in S_j} u_j(s_j, g_{-j}) \;\; \forall g_{-j} \in \Delta(Z_{-j}) \;\text{ with } g_{-j}^{-J} = f_{-j}^{-J}$$

where f_{-j}^{-J} and g_{-j}^{-J} are the marginal distributions of f_{-j} and g_{-j} over S_{-J} respectively. Let $\mathcal{F}(X)$ be the set of all the supported restrictions given X. We define the choice rule c for c-rationalizability by

$$c(X) = \cap_{Z \in \mathcal{F}(X)} Z.$$

Ambrus (2006) defined c-rationalizability by a (fast) iterated elimination procedure associated with this choice rule c; that is, in each elimination round the intersection of all supported restrictions is retained (see also Ambrus (2009) and Luo and Yang (2012) for more discussions). Ambrus (2006, Proposition 5) showed an order independence result, *under the restriction that each elimination round must be an intersection of some supported restrictions.* Since the choice rule c satisfies 1-Monotonicity* (see Lemma 4 in Appendix), by Theorem 2(b), Ambrus's (2006) notion of c-rationalizability is an order-independent procedure, without the aforementioned restriction.

7. **[HS-weak dominance]** $c(X) = X \backslash DOM(X)$ where

$$DOM(X) = \{x \in X : \exists i \in N \; \exists \sigma_i \in \Delta(S_i) \text{ s.t. } u_i(\sigma_i, x_{-i}) > u_i(x)$$
$$\text{and } u_i(\sigma_i, x'_{-i}) \geq u_i(x_i, x'_{-i}) \;\; \forall x'_{-i} \in X_{-i}\}.$$

That is, $c(X)$ consists of all strategy profiles in X where each player i's strategy is not weakly undominated in the sense of Hillas and Samet (2014, Definition 4). Under the HS-weak dominance relation, since every dominated strategy in a finite game has an undominated dominator which dominates that dominated strategy in each of subgames, $(Y \cap DOM(X)) \subseteq DOM(Y)$ for $c(X) \subseteq Y \subseteq X$. That is, Hereditarity holds. By Corollary 1, the HS-weak dominance is an order-independent procedure; see Hillas and Samet's (2014) Proposition 2.

8. **[IECFA]** Asheim and Dufwenberg (2003) defined the concept of a "fully permissible set", which captures an idea of "common certain belief" that each player avoids using a weakly dominated strategy, by an "iterative elimination of choice sets under full admissible consistency (IECFA)" in finite games. In a two-player game, the iterated elimination procedure can be simply defined on $\Sigma = \Sigma_1 \times \Sigma_2$ (instead of S), where $\Sigma_i = 2^{S_i} \backslash \{\varnothing\}$. For any nonempty $\Xi = \Xi_1 \times \Xi_2 \subseteq \Sigma$, define $c(\Xi) = c_1(\Xi_2) \times c_2(\Xi_1)$, where

$$c_i(\Xi_j) \equiv \{Q_i \in \Sigma_i : \exists (\varnothing \neq) \mathcal{Q} \subseteq \Xi_j \text{ s.t. } Q_i = S_i \backslash D_i(\mathcal{Q})\} \text{ and}$$

$$D_i(\mathcal{Q}) \equiv \left\{s_i \in S_i : \exists x_i \in \Delta(S_i) \text{ weakly dominates } s_i \text{ on } S_j \text{ or } \cup_{Q_j \in \mathcal{Q}} Q_j\right\}.$$

As Asheim and Dufwenberg (2003) pointed out, Monotonicity holds for the choice rule c. By Theorem 1(b), IECFA is an order independent procedure.

9. **[RBEU]** Cubitt and Sugden (2011) offered an iterative procedure of "reasoning-based expected utility procedure (RBEU)" for solving finite games. RBEU uses a sequence of "accumulation" and "elimination" op-

erations to categorize strategies as permissible and impermissible; some strategies remain uncategorized when the procedure halts. Cubitt and Sugden (2011) demonstrated that RBEU can delete more strategies than IESDS, while avoiding the order dependence problem associated with IEWDS. Formally, a sequence of categorizations $\{S(k)\}_{k=0}^{\infty}$ is inductively defined as: (i) $S(0) \equiv (\varnothing, \varnothing)$ and (ii) for all $k \geq 1$, $S(k) \equiv (S^{+}(k), S^{-}(k))$ such that, for all $i \in N$,

$$S_i^{+}(k) \subseteq \left\{s_i \in S_i : \ \forall \mu \in \Delta_{k-1}^{*}, \, u_i(s_i, \mu) \geq u_i(s_i', \mu) \ \text{ for all } s_i' \in S_i\right\};$$

$$S_i^{-}(k) \subseteq \left\{s_i \in S_i : \ \forall \mu \in \Delta_{k-1}^{*}, \, u_i(s_i', \mu) > u_i(s_i, \mu) \ \text{ for some } s_i' \in S_i\right\},$$

where $\Delta_{k-1}^{*} = \{\mu \in \Delta(S_{-i}) : \mu\left(S_{-i}^{-}(k-1)\right) = 0$ and $\mu(s_{-i}) > 0 \ \forall s_{-i} \in S_{-i}^{+}(k-1)\}$. Given a sequence $\{S(k)\}_{k=0}^{\infty}$, we can alternatively define a class of IEPs under the choice rule $c(X) = X \setminus \cup_{k \in \left\{k' : \, X \subseteq S \setminus S_i^{-}(k')\right\}} S_i^{-}(k)$. Apparently, Monotonicity holds for this choice rule c and, by Theorem 1(b), it generates an order-independent reduced procedure, which leads to a unique set of final outcomes by Cubitt and Sugden's (2011) Proposition 2.

10. [**nice weak dominance**] Let X denote a product set of S (i.e., $X = \times_{i \in N} X_i \subseteq S$). We say that player i's strategy $s_i \in S_i$ is *nicely weakly dominated (NWD)* on X, if there exists $s_i^{*} \in S_i$ such that for all $x_{-i} \in X_{-i}$, either $u_i(s_i^{*}, x_{-i}) > u_i(s_i, x_{-i})$ or $u(s_i^{*}, x_{-i}) = u(s_i, x_{-i})$; and the former inequality holds for some $x_{-i} \in X_{-i}$.[13] Define:

[13]That is, s_i is weakly dominated by s_i^{*} and the game satisfies the "transference of decisionmaker indifference (TDI)" condition: whenever a player is indifferent between two profiles that differ only in the player's strategy, that indifference is transferred to

$$c(X) \equiv \times_{i \in N} c_i(X) \equiv \times_{i \in N} (X_i \backslash DOM_i(X)) \text{ where}$$

$$DOM_i(X) = \{x_i \in X_i : x_i \text{ is NWD on } X \text{ by some } x_i^* \in X_i\}.$$

That is, $c_i(X)$ consists of all player i's strategies which are not nicely weakly dominated in the sense of Marx and Swinkels (1997, Definition 2). Marx and Swinkels (1997) showed that iterated elimination of nicely weakly dominated strategies is "outcome" order independent. This choice rule c for the nice weak dominance satisfies a variant of 1-Monotonicity* which gives rise to the desirable order-independence result (see Appendix II: NWD).

1.3.2 Infinite games

Hereditarity*: a full characterization Consider an infinite game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$. It seems natural and necessary to account for both finite and transfinite sequences of deletions. We present a variant form of "hereditarity" for order independence in infinite games.

HEREDITARITY*. $[S \to^* X \to^* Y] \Rightarrow [(Y \cap DOM(X)) \subseteq DOM(Y)]$.

Hereditarity* is a property for the domination relation used in a game. The following result shows that the Hereditarity* property provides a full characterization for Monotonicity* in the context of a game. Therefore, Hereditarity* provides an alternative sufficient condition for order independence in infinite games.

Theorem 3. *Hereditarity* and Monotonicity* are equivalent. Under*

the opponents as well.

Hereditarity, the iterated elimination procedure is order independent.*

CD* games In infinite games, restricted to iterated elimination of strictly dominated strategies (IESDS), Dufwenberg and Stegeman (2002) wrote, "More surprising, ... requiring that strategies be eliminated only by undominated strategies (a variation on Jackson's (1992) idea of "boundedness") does not solve the problem of order dependence ... We concluded that the problems of IESDS in infinite games are deeper than the possible nonexistence of the "best" dominating strategy." The following example, taken from Dufwenberg and Stegeman (2002), shows that the GKZ procedure cannot solve the problems of order dependence in CD games.

Example 3. Consider a two-person game: $G \equiv (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$, where $N = \{1,2\}$, $S_1 = S_2 = [0,1]\backslash\{1/3\}$, and $u_i : S_i \times S_j \mapsto \mathbb{R}$ for $i, j \in N$ and $i \neq j$, defined by

$$u_i(s_i, s_j) = s_i(1 - s_i - s_j) \text{ if } s_j \in \mathbb{Q},$$

$$u_i(s_i, s_j) = s_i(1 - s_i - 1/3) \text{ if } s_j \notin \mathbb{Q},$$

where $\mathbb{Q}$ denotes the set of rational numbers in $[0,1]$. Dufwenberg and Stegeman (2002, p.2017) showed that this game is a CD game. However, the IESDS procedure fails to be order independent; for example, there are two IESDS processes that generate different outcomes: Let $\langle a,b\rangle^2 \equiv [a,b]\backslash\{1/3\} \times [a,b]\backslash\{1/3\}$.

1. $X^0 = S = \langle a^0, b^0\rangle^2$, $X^n = \langle a^n, b^n\rangle^2$ (where $a^n = (1 - b^{n-1})/2$ and $b^n = (1 - a^{n-1})/2$), and $X^{\mathbb{N}} = \cap_{n\in\mathbb{N}} X^n = \varnothing$.

2. $\widetilde{X}^0 = S$, $\widetilde{X}^n = X^n \cup \{(q,q)\}$ for $n \in \mathbb{N}$, and $\widetilde{X}^{\mathbb{N}} = \cap_{n\in\mathbb{N}} \widetilde{X}^n = \{(q,q)\}$ (where $q \notin \mathbb{Q}$).

Particularly, for all $n \in \mathbb{N}$, $c\left(\widetilde{X}^n\right) = X^{n+1}$. Thus, $\varnothing = \cap_{n\in\mathbb{N}} c\left(\widetilde{X}^n\right) \not\supseteq c\left(\widetilde{X}^{\mathbb{N}}\right) = \{(q,q)\}$. That is, this game lacks the continuity at the limit point of deletions; the choice rule c is explosive at the limit point. In Example 1 in Section 2, the choice rule c also displays "upward jumps" at the limit point: $\cap_{n\in\mathbb{N}} c\left(\widetilde{X}^n\right) \subset c\left(\widetilde{X}^{\mathbb{N}}\right)$. (Note: This game of Example 3 is a 1-CD* game.)

In order to get rid of the problem of order dependence in games with infinite strategy sets, we need to introduce a stronger notion of CD* games. We say that game G (under an abstract domination relation $\succ_X$) is *closed under dominance* (CD*)* if $S \to^* X \to^* Y$ and $y \succ_X x$ for some $y \in X$ and $x \in Y$ imply that: there exists $z^* \in Y$ such that $z \not\succ_Y z^* \succ_Y x$ for all $z \in Y$, i.e., $[S \to^* X \to^* Y] \Rightarrow \left[(Y \cap DOM(X)) \subseteq DOM^{c(Y)}(Y)\right]$. That is, at any point in any valid sequence of deletions, any dominated element surviving the deletion process has an undominated dominator at the end point of the deletion which dominates it.

The following result asserts that, in the class of CD* games, there is no problem of order dependence. In particular, the GKZ procedure (by allowing an transfinite sequence of elimination) is always order independent. Under the strict dominance relation, all compact and own-uppersemicontinuous (COUSC) games are CD* and, hence, the IESDS procedure in Definition 1 is well-defined and order independent.

Corollary 4. *(a) Suppose that G is a CD* game. Then, the GKZ procedure is equivalent to the iterated elimination procedure in Definition 1 which is order independent. (b) Under the strict dominance relation, any compact and own-uppersemicontinuous game is a CD* game and, thus, the IESDS procedure defined in Definition 1 exists and is order independent.*

We have pointed out, in finite games, that the orginal definition of CD games might be conceptually little relevant to the property of order independence. We have thereby introduced a useful notion of CD* games for solving the problem of order dependence in infinite games. The definition of CD* not only captures Jackson's (1992) idea of "boundedness" that strategies are eliminated only by undominated strategies, but also it is immune from the problem of "discontinuity" at limit points as demonstrated in Examples 1 and 3. In CD* games, we have shown that the GKZ procedure is an order-independent iterated elimination procedure. The main driving force for order independence comes from the Hereditarity*/Monotonicity* property exhibited by a CD* game. While the class of CD* games is exclusive of all problematic games with the problem of order dependence, it abounds under the strict dominance, including all compact and own-uppersemicontinuous games. Consequently, the IESDS procedure in Definition 1 is always well-defined and order independent in COUSC games. (We would like to mention that Chen et al. (2007) presented an alternative definition of IESDS* by setting $c(X) = X \backslash DOM^S(X)$ and showed that IESDS* is well-defined and order independent in infinite games.)

1.4 Concluding remarks

In infinite games, Dufwenberg and Stegeman (2002) pointed out that the notion of IESDS might be an ill-behaved order-dependent procedure, even in the class of CD games where any dominated strategy has an undominated dominator, and they concluded that the problems of IESDS in infinite games are deeper than the possible nonexistence of the "best" dominating strategy. One major focus of this paper is to study various (transfinitely) iterated elimination procedures in the infinite case. We have shown a general existence of iterated elimination procedure. Following Gilboa et al.'s (1990) pioneering work, we have identified a fairly weak condition of Monotonicity* for the order independence on an abstract reduction system, which is closely related to the Aizerman property used in the choice-theoretic literature.

We have demonstrated that our approach is applicable to any form of iterated elimination processes in arbitrary strategic games. In addition, we have provided a full characterization of Monotonicity* by Hereditarity* in (in)finite games. We have also introduced a useful and variant notion of CD* games, including all compact and own-uppersemicontinuous (COUSC) games, and shown that the GKZ procedure is an order-independent iterated elimination procedure in CD* games. In particular, the IESDS procedure in Definition 1 is always well-defined and order independent in COUSC games.

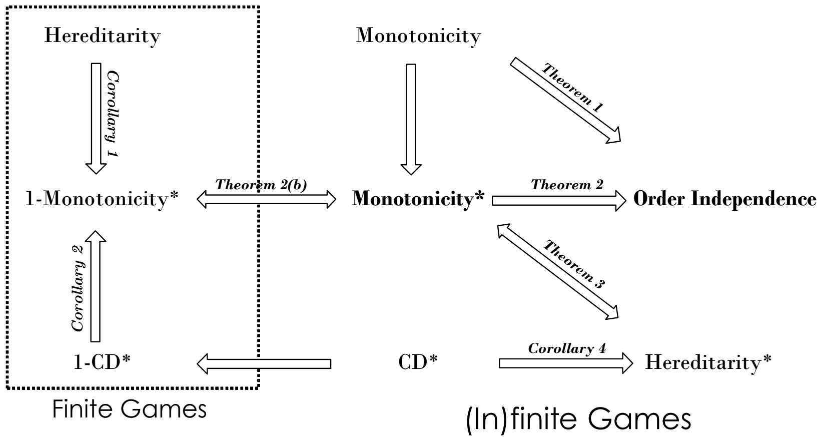

Figure. 1. Relationship between different conditions for order independence

We would like to point out that, except for the ZF axioms, the existence of iterated elimination procedure does not require the Axiom of Choice or the Well-Ordering Principle; this result improves the previous existence results of iterated elimination procedure in infinite games (e.g., Arieli (2012), Ritzberger (2002), Dufwenberg and Stegeman (2002) and Chen et al. (2007, 2015)). Our analysis of strategic games is completely topology-free and with no measure-theoretic assumption on the structure of the game, and it is applicable to any kind of iterated dominance in arbitrary games. Our framework in this paper can also be used to analyze the order independence of various forms of iterated elimination procedures in mixed extensions of finite games or general preference models used in game theory (cf. Chen et al. (2015)). Alternatively, we can define $c(X) = X \backslash DOM^S(X)$, which consists of all the elements in X that are undominated by any element in S. Our analysis in this paper is applicable to this alternative definition.

To close this paper, we would like to point out some possible extensions of this paper for future research. In this paper, we consider the order independence in terms of strategy profiles resulting from iterated elimination processes for games. Several papers discuss a slight variant of order independence in terms of "payoff outcomes" in finite strategic games (see, e.g., Marx and Swinkels (1997)) or in terms of "outcomes of play" in finite extensive games (see, e.g., Chen and Micali (2013) and Heifetz and Perea (2015)). The extension of this paper to such a variant of order independence in infinite games is an important subject for further research; Appendix II is an attempt in this direction. As we have emphasized, in this paper we focus on the existence and order independence of iterated elimination procedure. The exploration of iterated elimination procedures from an epistemic perspective is also an intriguing topic worth further investigation (see, e.g., Brandenburger et al. (2008)). Finally, Monotonicity* is not necessary for order independence (see Example 4 in Appendix I) and it is certainly interesting to explore the necessary and sufficient condition for order independence of iterated elimination procedure in general situations.

Appendix I: Proofs & Example 4

Proof of Theorem 1. (a) By transfinite recursion (see, e.g., Jech 2003, p.22), we define a sequence $\left\{X^{\lambda}\right\}_{\lambda\in Ord}$ (where Ord is the class of all ordinals) by

$$X^{0}=S,\; X^{\lambda+1}=c\left(X^{\lambda}\right),\text{ and } X^{\lambda}=\cap_{\lambda'<\lambda}X^{\lambda'} \text{ for a limit ordinal } \lambda. \quad (1)$$

By the Axiom Schema of Separation (see, e.g., Jech 2003, p.7), $\left\{X^{\lambda} : \lambda \in Ord\right\}$ is a set because it is a subclass of the power set of S. Suppose, to the contrary, that $X^{\lambda} \neq X^{\lambda'}$ for any $\lambda \neq \lambda'$, then there is a bijection from $\left\{X^{\lambda} : \lambda \in Ord\right\}$ to Ord. By the Axiom Schema of Replacement (see, e.g., Jech 2003, p.13), Ord is a set, contradicting the fact that Ord is not a set. By (1), it follows that $X^{\Lambda} = X^{\Lambda+1} = c\left(X^{\Lambda}\right)$ for some $\Lambda \in Ord$. Let $\Lambda^0 = \inf\left\{\Lambda \in Ord : X^{\Lambda} = X^{\Lambda+1} = c\left(X^{\Lambda}\right)\right\}$. Then the sequence $\left\{X^{\lambda}\right\}_{\lambda \leq \Lambda^0}$ is a fast IEP on $(S, \rightarrow)$.

(b) Let $Z = c(Z)$. Obviously, $Z \subseteq X^0$. Assume, by induction, that $Z \subseteq X^{\lambda'}$ for all $\lambda' < \lambda$. By monotonicity, $c(Z) \subseteq c\left(X^{\lambda'}\right)$ for all $\lambda' < \lambda$. Therefore, $Z = c(Z) \subseteq X^{\lambda}$. That is, $Z \subseteq X^{\lambda}$ for all $\lambda \leq \Lambda$. Therefore, $X^{\Lambda} \supseteq \cup_{Z=c(Z)} Z$. Since $X^{\Lambda} = c\left(X^{\Lambda}\right)$, $X^{\Lambda} = \cup_{Z=c(Z)} Z$. □

To prove Theorem 2, we need the following three lemmas.

Lemma 1. *If $S \rightarrow^* X$ and $S \rightarrow^* Y$ imply that there exists T such that $X \rightarrow^* T$ and $Y \rightarrow^* T$, then the iterated elimination procedure is order independent.*

Proof. Assume by absurdity that there are two IEPs: $S \rightarrow^* X = c(X)$ and $S \rightarrow^* Y = c(Y)$, but $X \neq Y$. Then there exists T such that $X \rightarrow^* T$ and $Y \rightarrow^* T$. Therefore, $X = T = Y$. A contradiction. □

Lemma 2. *If c satisfies Monotonicity*, $S \rightarrow^* X \rightarrow Y$ implies $Y \rightarrow c(X)$.*

Proof. Let $S \rightarrow^* X \rightarrow Y$. Since c satisfies Monotonicity*, $c(Y) \subseteq c(X)$. Since $X \rightarrow Y$, $c(Y) \subseteq c(X) \subseteq Y \subseteq X$. By the definition of $\rightarrow$, $Y \rightarrow c(X)$.

□

Lemma 3. *Suppose $S \to^* X$ via a reduction sequence $\left\{X^{\lambda}\right\}_{\lambda \leq \Lambda}$. Then $c(X) \subseteq \cap_{\lambda<\Lambda} c\left(X^{\lambda}\right)$ if c satisfies Monotonicity*.*

Proof. Since c satisfies Monotonicity*, $c(X) \subseteq c\left(X^{\lambda}\right)$ for all $\lambda < \Lambda$. Therefore, $c(X) \subseteq \cap_{\lambda<\Lambda} c\left(X^{\lambda}\right)$. $\square$

Proof of Theorem 2.[14] (a) Let $S \to^* X$ via a reduction sequence $\left\{X^{\lambda}\right\}_{\lambda \leq \Lambda}$ and $S \to^* Y$ via a reduction sequence $\left\{Y^{\lambda}\right\}_{\lambda \leq \Lambda}$. We say the "*diamond property holds (for $\left\{X^{\lambda}\right\}_{\lambda \leq \Lambda}$ and $\left\{Y^{\lambda}\right\}_{\lambda \leq \Lambda}$)*" if there exists an $\Lambda \times \Lambda$-diamond grid $\left\{S^{\alpha\beta}\right\}_{\alpha \leq \Lambda;\, \beta \leq \Lambda}$ such that

1. for all $\lambda \leq \Lambda$, $S^{\lambda 0} = X^{\lambda}$ and $S^{0\lambda} = Y^{\lambda}$;

2. for all $\alpha, \beta \leq \Lambda$, $\left\{S^{\alpha\lambda}\right\}_{\lambda \leq \Lambda}$ and $\left\{S^{\lambda\beta}\right\}_{\lambda \leq \Lambda}$are reduction sequences.

That is, the diamond structure spreads over a grid of $\Lambda \times \Lambda$ fractals (cf. Figure 2).

[14]Our proof is inspired by Gilboa et al.'s (1990) idea for order independence of finite reduction sequences. We show that, under Monotonicity*, any pair of transfinite reduction sequences has a nice "diamond" property which gives rise to the order independence result.

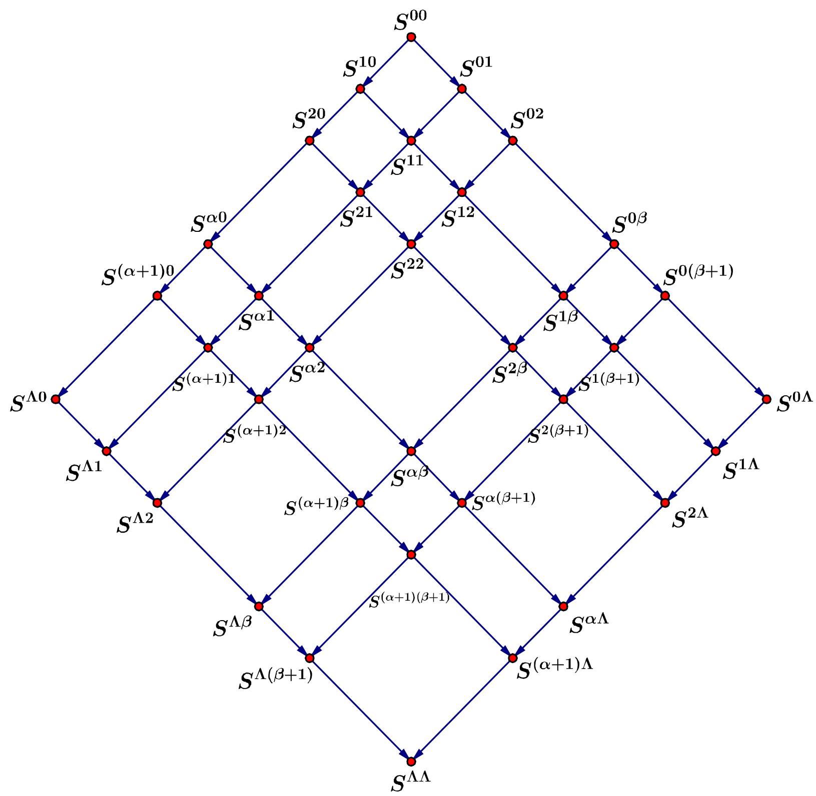

Figure. 2. A grid of $\Lambda \times \Lambda$ fractals

Observe that: $S \rightarrow^* X$ and $S \rightarrow^* Y$ iff there exists an ordinal Λ such that $S \rightarrow^* X$ via a reduction sequence $\left\{X^\lambda\right\}_{\lambda \leq \Lambda}$ and $S \rightarrow^* Y$ via a reduction sequence $\left\{Y^\lambda\right\}_{\lambda \leq \Lambda}$. By Lemma 1, it suffices to show that the diamond property holds true. We show it by (transfinite) induction on Λ. If $\Lambda = 1$, then $S \rightarrow X$ and $S \rightarrow Y$. By Lemma 2, $X \rightarrow c(S)$ and $Y \rightarrow c(S)$. Now assume that the diamond property holds for all $\lambda < \Lambda$. We distinguish two cases.

Case 1: Λ is a limit ordinal. Define $S^{\Lambda 0} \equiv X^\Lambda$ and $S^{\Lambda\beta} \equiv \cap_{\alpha<\Lambda} S^{\alpha\beta}$ for all $\beta < \Lambda$ and $\beta \neq 0$. Since $X^\Lambda = \cap_{\lambda<\Lambda} X^\lambda$, $S^{\Lambda 0} = \cap_{\alpha<\Lambda} S^{\alpha 0}$. By the induction hypothesis, for all $\beta < \Lambda$, we have

$$\left[S^{\alpha\beta} \to S^{\alpha(\beta+1)}\ \forall\alpha < \Lambda\right] \Leftrightarrow \left[c\left(S^{\alpha\beta}\right) \subseteq S^{\alpha(\beta+1)} \subseteq S^{\alpha\beta}\ \ \forall\alpha < \Lambda\right]$$
$$\Rightarrow \left[\cap_{\alpha<\Lambda} c\left(S^{\alpha\beta}\right) \subseteq \cap_{\alpha<\Lambda} S^{\alpha(\beta+1)} \subseteq \cap_{\alpha<\Lambda} S^{\alpha\beta}\right]$$
$$\Leftrightarrow \left[\cap_{\alpha<\Lambda} c\left(S^{\alpha\beta}\right) \subseteq S^{\Lambda(\beta+1)} \subseteq S^{\Lambda\beta}\right].$$

By Lemma 3, $c\left(S^{\Lambda\beta}\right) \subseteq \cap_{\alpha<\Lambda} c\left(S^{\alpha\beta}\right) \subseteq S^{\Lambda(\beta+1)} \subseteq S^{\Lambda\beta}$. Therefore, $S^{\Lambda\beta} \to S^{\Lambda(\beta+1)}$ for all $\beta < \Lambda$. (If β is a limit ordinal, $S^{\Lambda\beta} = \cap_{\alpha<\Lambda} S^{\alpha\beta} = \cap_{\alpha<\Lambda} \cap_{\beta'<\beta} S^{\alpha\beta'} = \cap_{\beta'<\beta} \cap_{\alpha<\Lambda} S^{\alpha\beta'} = \cap_{\beta'<\beta} S^{\Lambda\beta'}$.) Define $S^{\Lambda\Lambda} \equiv \cap_{\beta<\Lambda} S^{\Lambda\beta} = \cap_{\beta<\Lambda} \cap_{\alpha<\Lambda} S^{\alpha\beta}$. We find a reduction sequence $\left\{S^{\Lambda\beta}\right\}_{\beta\leq\Lambda}$. Similarly, we find a reduction sequence $\left\{S^{\alpha\Lambda}\right\}_{\alpha\leq\Lambda}$ with $S^{\Lambda\Lambda} = \cap_{\alpha<\Lambda} \cap_{\beta<\Lambda} S^{\alpha\beta} = \cap_{\alpha<\Lambda} S^{\alpha\Lambda}$.

Case 2: Λ is a successor ordinal. By the induction hypothesis, there exists $(\Lambda-1) \times (\Lambda-1)$-diamond grid $\left\{S^{\alpha\beta}\right\}_{\alpha\leq\Lambda-1;\ \beta\leq\Lambda-1}$ for $\left\{X^{\lambda}\right\}_{\lambda\leq\Lambda-1}$ and $\left\{Y^{\lambda}\right\}_{\lambda\leq\Lambda-1}$. Define $S^{\Lambda 0} \equiv X^{\Lambda}$ and $S^{\Lambda(\beta+1)} \equiv c\left(S^{(\Lambda-1)\beta}\right)$ (and $S^{\Lambda\beta} \equiv \cap_{\beta'<\beta} S^{\Lambda\beta'}$ if β is a limit ordinal) for all $\beta \leq \Lambda-1$. Since $X^{\Lambda-1} \to X^{\Lambda}$, by the induction hypothesis, $S \to^* S^{(\Lambda-1)0} \to S^{\Lambda 0}$ and $S \to^* S^{(\Lambda-1)0} \to S^{(\Lambda-1)1}$. By Lemma 2, $S^{\Lambda 0} \to S^{\Lambda 1}$ and $S^{(\Lambda-1)1} \to S^{\Lambda 1}$. Again by induction on $\beta \leq \Lambda - 1$, we have $S^{\Lambda\beta} \to S^{\Lambda(\beta+1)}$ for all $\beta \leq \Lambda - 1$ and $S^{(\Lambda-1)\beta} \to S^{\Lambda\beta}$ for any $\beta \leq \Lambda - 1$ (if β is a limit ordinal, the proof is totally similar to Case 1). Therefore, $\left\{S^{\Lambda\beta}\right\}_{\beta\leq\Lambda}$ and $\left\{S^{\alpha\beta}\right\}_{\alpha\leq\Lambda}$ for any $\beta \leq \Lambda - 1$ are reduction sequences. Similarly, we can find a reduction sequence $\left\{S^{\alpha\Lambda}\right\}_{\alpha\leq\Lambda}$ such that $\left\{S^{\alpha\beta}\right\}_{\beta\leq\Lambda}$ for any $\alpha \leq \Lambda - 1$ is a reduction sequence. That is, there exists an $\Lambda \times \Lambda$-diamond grid $\left\{S^{\alpha\beta}\right\}_{\alpha\leq\Lambda;\ \beta\leq\Lambda}$ for $\left\{X^{\lambda}\right\}_{\lambda\leq\Lambda}$ and $\left\{Y^{\lambda}\right\}_{\lambda\leq\Lambda}$. Therefore, the diamond property holds.

(b) Consider $S \to^* X \to^* Y$. By Theorem 2(a), it suffices to check $c(Y) \subseteq c(X)$. Since S is a finite set, there exists a natural number N such that $X \to^* Y$ via a finite reduction sequence $\{X^n\}_{n \leq N}$ (with $X^{n-1} \neq X^n$ for all $n \leq N$). By 1-Monotonicity*, $c(X^1) \subseteq c(X^0) = c(X)$. Assume inductively that $c(X^{n-1}) \subseteq c(X)$ for all $n \leq N$. Since $S \to^* X^{n-1} \to X^n$, $c(X^n) \subseteq c(X^{n-1}) \subseteq c(X)$ by 1-Monotonicity*. Thus, $c(Y) = c\left(X^N\right) \subseteq c(X)$. □

Proof of Corollary 1. Suppose that $X \to Y$. That is, $c(X) \subseteq Y \subseteq X$. By Hereditarity, we have

$$\begin{aligned}
[(Y \cap DOM(X)) \subseteq DOM(Y)] &\Leftrightarrow [Y \backslash (Y \cap DOM(X)) \supseteq Y \backslash DOM(Y)] \\
&\Leftrightarrow [Y \backslash DOM(X) \supseteq Y \backslash DOM(Y)] \\
&\Rightarrow [X \backslash DOM(X) \supseteq Y \backslash DOM(Y)].
\end{aligned}$$

That is, $c(Y) \subseteq c(X)$ if $X \to Y$. By Theorem 2(b), the iterated elimination procedure for G is order-independent. □

Proof of Corollary 2. Suppose that $S \to^* X \to Y$. Since $X \to Y$, $c(X) \subseteq Y \subseteq X$. Since G is a 1-CD* game, $Y \cap DOM(X) \subseteq DOM^{c(Y)}(Y) \subseteq DOM(Y)$. By the proof of Corollary 1, we obtain that $S \to^* X \to Y$ implies $c(Y) \subseteq c(X)$. By Theorem 2(b), the iterated elimination procedure for G is order independent. □

Proof of Corollary 3. Suppose that $S \to^* X \to Y$. Then $c(X) \subseteq Y \subseteq X$. Since G is 1-CD* and $S \to^* X \to X$, $DOM(X) = DOM^{c(X)}(X)$. Thus,

$DOM(X) = DOM^{Y}(X) = DOM^{c(X)}(X)$. Therefore, we obtain

$$\begin{aligned}[X \to Y] &\Leftrightarrow [Y \subseteq X \text{ and } X \backslash Y \subseteq DOM(X)] \\ &\Leftrightarrow \left[Y \subseteq X \text{ and } X \backslash Y \subseteq DOM^{Y}(X)\right] \\ &\Leftrightarrow \left[X \to^{\text{GKZ}} Y\right].\end{aligned}$$

That is, for any finite 1-CD* game, the GKZ procedure is equivalent to the iterated elimination procedure in Definition 1. By Corollary 2, the GKZ procedure is order independent. □

Lemma 4. *The choice rule c for c-rationalizability satisfies 1-Monotonicity*.*

Proof of Lemma 4. Let $X \searrow_J Z$ denote "supported restriction Z by coalition J given X". Consider $X \to Y$. Then $X \supseteq Y \supseteq c(X) = \cap_{Z \in \mathcal{F}(X)} Z \neq \varnothing$ by Ambrus's (2006) Proposition 1. Since $Y \cap Z \supseteq Y \cap c(X) \neq \varnothing$ for $Z \in \mathcal{F}(X)$, by Ambrus's (2006) Lemmas 1 and 2, $Y \searrow_J (Y \cap Z)$. Then $Y \cap Z \in \mathcal{F}(Y)$ for all Z in $\mathcal{F}(X)$. Thus, $c(Y) = \cap_{Z \in \mathcal{F}(Y)} Z \subseteq \cap_{Z \in \mathcal{F}(X)} (Y \cap Z) \subseteq \cap_{Z \in \mathcal{F}(X)} Z = c(X)$. □

Proof of Theorem 3. Suppose $S \to^{*} X \to^{*} Y$. Then $Y \subseteq X$. Thus, we have

$$\begin{aligned}[(Y \cap DOM(X)) \subseteq DOM(Y)] &\Leftrightarrow [Y \backslash (Y \cap DOM(X)) \supseteq Y \backslash DOM(Y)] \\ &\Leftrightarrow [Y \backslash DOM(X) \supseteq Y \backslash DOM(Y)] \\ &\Leftrightarrow [X \backslash DOM(X) \supseteq Y \backslash DOM(Y)].\end{aligned}$$

That is, $(Y \cap DOM(X)) \subseteq DOM(Y)$ iff $c(X) \supseteq c(Y)$. Therefore, Hereditarity* and Monotonicity* are equivalent. By Theorem 2(a), Hereditarity* implies that the iterated elimination procedure is order independent. □

Proof of Corollary 4. (a) Since any CD* game is 1-CD*, by Corollary 3, the GKZ procedure is equivalent to the iterated elimination procedure in Definition 1. Suppose $S \to^* X \to^* Y$. Since G is a CD* game, $(Y \cap DOM(X)) \subseteq DOM^{c(Y)}(Y) \subseteq DOM(Y)$. That is, Hereditarity* holds. By Theorem 3, the GKZ procedure is order independent.

(b) Suppose that $S \to^* X \to^* Y$ via a reduction sequence $\{X^\lambda\}_{\lambda \leq \Lambda}$. Let $y \succ_X x$ for some $y \in X$ and $x \in Y$. Then, $\exists i \in N$ such that $u_i(y_i, x_{-i}) > u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$. Since G is a COUSC game, by the proof of Dufwenberg and Stegeman's (2002) Lemma, $\exists z^* \in S$ such that, for all $y' \in Y$, (i) $u_i\left(z_i^*, y'_{-i}\right) > u_i\left(x_i, y'_{-i}\right)$ and (ii) $u_j(z_j^*, x_{-j}) \geq u_j(s_j, x_{-j})$ for all $j \in N$ and all $s_j \in S_j$. Since $x \in Y \subseteq X^\lambda$, $z^* \in X^\lambda$ for all $\lambda < \Lambda$. Thus, $z^* \in \cap_{\lambda<\Lambda} X^\lambda = Y$. By (i) and (ii), $z^* \succ_Y x$ and $z^* \in c(Y)$. Therefore, $(Y \cap DOM(X)) \subseteq DOM^{c(Y)}(Y)$, i.e., G is a CD* game. By Theorem 1(a) and Corollary 4(a), the IESDS procedure defined in Definition 1 exists and is order independent in COUSC games. □

Example 4 [Monotonicity* is not necessary for order independence]. Consider $S = \{x, y, z\}$ with the following choice function c (we abuse notation by writing, for example, xy instead of $\{x, y\}$):

$X \subseteq S$	xyz	xy	xz	yz	x	y	z
$c(X)$	x	x	z	z	$\varnothing$	$\varnothing$	$\varnothing$

Any IEP leads to $\varnothing$. But, $S \to xz$ fails to imply $c(xz) \subseteq c(S)$.

Appendix II: NWD

In this appendix, we offer an alternative proof of Marx and Swinkels's (1997) result on the "outcome" order independence for nice weak dominance (NWD).

Consider a finite game $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$. Let $X = \times_{i\in N} X_i \subseteq S$. A *snapshot of* X is defined as a game $(N, \{\Sigma_i\}_{i\in N}, \{v_i\}_{i\in N})$ such that, for all $i \in N$, there are surjections $e_i : X_i \mapsto \Sigma_i$ satisfying for all $\sigma \in \Sigma$, $v(\sigma) = u(x)\ \forall x \in e^{-1}(\sigma)$, where $\Sigma = \times_{i\in N}\Sigma_i$, $e(x) = (e_i(x_i))_{i\in N}$ and $e^{-1}(\sigma) = \{x \in X : e(x) = \sigma\}$. Denote $X \sim Y$ if they have a common snapshot.

Claim NWD. *For any finite game G, iterated elimination of nicely weakly dominated strategies are outcome order independent – i.e., if $\{X^\lambda\}_{\lambda\le\Lambda}$ and $\{Y^\lambda\}_{\lambda\le\Lambda'}$ are two finite IEPs of product sets of S, then $X^\Lambda \sim Y^{\Lambda'}$.*

Lemma NWD. *(a) $X \sim X' \Rightarrow c(X) \sim c(X')$. (b) $X \to Y \Rightarrow Y \to Z$ for some $Z \sim c(X)$ (and $Z \supseteq c(Y)$). (c) Let $[X] \to [Y]$ denote "$X' \to Y'$ for some $X' \sim X$ and $Y' \sim Y$". Then $[X] \to [Y] \Rightarrow [Y] \to [c(X)]$.*

Proof of Lemma NWD. (a) Suppose that $(N, \{\Sigma_i\}_{i\in N}, \{v_i\}_{i\in N})$ is a common snapshot of X and X' (via the corresponding surjections e and e'). Then, for all $\sigma \in \Sigma$, $e_i^{-1}(\sigma_i) \subseteq DOM_i(X) \Leftrightarrow \sigma_i \in DOM_i(\Sigma) \Leftrightarrow e_i'^{-1}(\sigma_i) \subseteq DOM_i(X')\ \forall i \in N$. Thus, for all $i \in N$, $c_i(X) = \cup_{\sigma_i\in c_i(\Sigma)} e_i^{-1}(\sigma_i)$ and $c_i(X') = \cup_{\sigma_i\in c_i(\Sigma)} e_i'^{-1}(\sigma_i)$. Therefore, $c(X)$ and $c(X')$ have a common snapshot $(N, \{c_i(\Sigma)\}_{i\in N}, \{v_i\}_{i\in N})$ by the (restricted) surjections $e|_{c(X)}$ and

$e'|_{c(X')}$, respectively. That is, $c(X) \sim c(X')$.

(b) Let $Z = \times_{i \in N} (c_i(X) \cup c_i(Y))$. Since $X \to Y$, $c(X) \subseteq Y \subseteq X$. So $c(Y) \subseteq Z \subseteq Y$, i.e., $Y \to Z$. It remains to show $Z \sim c(X)$. Let $z_i \in c_i(Y) \backslash c_i(X)$. Then, z_i is NWD on X by some $\widetilde{z}_i \in X_i$ and $u(z_i, y_{-i}) = u(\widetilde{z}_i, y_{-i})$ $\forall y_{-i} \in Y_{-i} \subseteq X_{-i}$. Since $Z \subseteq Y$, $u(z_i, z_{-i}) = u(\widetilde{z}_i, z_{-i})$ $\forall z_{-i} \in Z_{-i}$. By finiteness of X and transitivity of NWD, there exists $\widetilde{z}_i \in c_i(X)$ such that $u(z_i, z_{-i}) = u(\widetilde{z}_i, z_{-i})$ $\forall z_{-i} \in Z_{-i}$. Therefore, Z has a snapshot $(N, \{c_i(X)\}_{i \in N}, \{u_i\}_{i \in N})$ thorough the surjection $\widetilde{e}_i : Z_i \mapsto c_i(X)$ such that

$$\widetilde{e}_i(z_i) = \begin{cases} z_i, & \text{if } z_i \in c_i(X) \\ \widetilde{z}_i, & \text{if } z_i \in c_i(Y) \backslash c_i(X) \end{cases}.$$

Hence, $Z \sim c(X)$. By construction of Z, $c(Y) \subseteq Z \sim c(X)$.

(c) Suppose $[X] \to [Y]$. Then, there exist $X' \sim X$ and $Y' \sim Y$ such that $X' \to Y'$. By (b), $Y' \to Z'$ for some $Z' \sim c(X')$. By (a), $c(X) \sim c(X')$. Therefore, $[Y] \to [c(X)]$. □

Proof of Claim NWD. Without loss of generality, assume that $\{X^\lambda\}_{\lambda \leq \Lambda}$ and $\{Y^\lambda\}_{\lambda \leq \Lambda}$ be two finite IEPs of product sets of S. Then $[X^\lambda] \to [X^{\lambda+1}]$ and $[Y^\lambda] \to [Y^{\lambda+1}]$ for all $\lambda < \Lambda$. Similarly to the proof of Theorem 2(a), we can show, by Lemma NWD(c), that $\Lambda \times \Lambda$-grid $\{[S^{\alpha\beta}]\}_{\alpha \leq \Lambda; \beta \leq \Lambda}$ holds and $[X^\Lambda] = [S^{\Lambda\Lambda}] = [Y^\Lambda]$. □

2 An indistinguishability result on extensive form rationalizability under general preferences

2.1 Introduction

The game-theoretic solution concept of rationalizability, proposed by Bernheim (1984) and Pearce (1984), is logical implication of common knowledge of rationality in the standard expected utility. Pearce (1984) also put forward the notion of extensive form rationalizability (EFR) as an extension to extensive games. Roughly speaking, EFR strategies are those surviving a process of iterated elimination procedure. In each step, all strategies that are "never sequentially-best responses" are eliminated. The procedure stops when no such strategy exists. Battigalli (1997) pointed out that EFR can be characterized as the sequential rationality to a hierarchies of beliefs systems which conform to the best rationalizable principle: Players' beliefs conditional upon observing a history must be consistent with the highest degree of "strategic sophistication" of their opponents. The best-rationalization principle is also related to the epistemic condition of "strong belief in rationality" (Battigalli and Siniscalchi, 2002). In this sense, EFR fundamentally differs from the notion of backward induction (BI) whose epistemic foundation is rationality and common belief of future rationality (Perea, 2014). Indeed, as the example in the next section suggests, EFR and BI might predict different strategies even in "generic"[1] perfect information (PI) games. However, as pointed out by Reny (1992), Battigalli (1997) and Heifetz and Perea (2015), in generic PI games, BI and EFR yield the same terminal node. Since EFR formalizes a fashion of forward induction reasoning: "A player should use all information she acquired about her opponents' past behavior in order to improve her prediction of their future, simultaneous, and past (unobserved) behavior, relying on the assumption that they are rational. (Battigalli, 1997, p.41)" This result is of fundamental significance. As an outside observer, one only observes the realization of the actual outcome or terminal node after the play of the game. The equivalence result, in terms of outcomes, implies that it is impossible to tell whether players' underlying reasoning procedures conform to forward induction or backward induction, since both yield exactly the same observation.[2]

Pearce's notion of EFR presumes Bayesian rationality: each player chooses a strategy that maximizes the subjective expected utility (SEU) axiomatized by Savage (1954). However, the Ellsberg Paradox and related experimental evidences cast a doubt on the SEU model. Namely, decision makers usually have ambiguity aversion concerns which cannot be captured by the SEU model. Many generalizations of the SEU model are motivated then to relax Bayesian rationality. To name a few, for example, ordinal expected utility, probabilistically sophisticated preferences, Choquet ex-

[1] Roughly speaking, "generic" means for each player, payoffs are all different among different terminal nodes.

[2] See also Heifetz and Perea (2015) for more discussions on this issue.

pected utility theory and the multi-priors model. It is then meaningful to characterize EFR in a general preference model. Moreover, it is also interesting to examine whether the outcome equivalence result still holds in a broad sense. This paper thus aims to study these two questions.

Epstein and Wang (1996) offered a unified approach to study decision makers' rational strategic behavior under quite general preference setting (regular preference model). Their approach is applicable to study various notions of solution concepts in game theory. For example, Epstein (1997) extended rationalizability to general preference models. Epstein and Wang's (1996) approach is virtually static and might not be directly applied to study the notion of EFR in the dynamic setting of extensive games. This paper then extends Epstein and Wang's (1996) approach and formulates "a model of conditional preference" which is applicable to EFR. Roughly speaking, each player holds a conditional preference system which specifies a conditional preference at each of his/her own information set. "A model of conditional preference" thus is a collection of admissible conditional preference systems of all players. The behavioral assumption embedded in the notion of conditional preference systems is the consequentialism. That is, conditioned on an information set h, strategic choices at information sets incompatible with h are irrelevant for the conditional preference at h.[3] Throughout this paper, only a rather weak condition, *constant monotonicity (CM)*, is imposed on the preference model. Roughly speaking, it only

[3]The empirical study by Dominiak et al. (2012) show that more subjects act in line with consequentialism than with dynamic consistency and that this result is even stronger among ambiguity averse subjects.

requires that a constant act **a** with payoff a must be strictly preferred to another constant act **b** with payoff b whenever $a > b$. In games situation, it implies that a strategy is strictly preferred to another strategy if the former strategy assures a constant payoff strictly higher than the latter one does. On the other hand, CM is silent on the preference order of strategies with payoff uncertainties. Note that CM preferences might not have utility representations and Epstein's (1997) regular preferences satisfy CM. Therefore, preference models considered in this paper is rather general.

One main result in this paper shows that in generic PI games, for any model of conditional preferences which admits all SEU preferences (consistent with Bayesian updating) and satisfies CM, the EFR strategy profiles yield the backward induction outcome (Theorem 1). That is, the outcome equivalence result in the literature remains true for a broad class of preference models. This result has rich implications. On the one hand, one may interpret it as an indistinguishable result. From an outside observer's point of view, neither strategy choices nor the preferences of players can be observed. The dynamic version of three-color Ellsberg experiment, viewed as a single-person game, shows that different preferences may yield different observable implications. However, in generic PI games, EFR in a class of very general preference models yields the same set of outcomes of SEU preferences. The research line of indistinguishability/distinguishability appeals its significance in the literature: Bergemann and Morris (2009) applied "strategic distinguishability" to robust virtual implementation. On the other hand, the notion of EFR captures forward induction arguments which refine equilibrium concepts by restricting beliefs on off-equilibrium path (see Battigalli (1997); Battigalli and Siniscalchi (2002)). Theorem 1 implies that these refinements have no impact on behavior along equilibrium path for generic PI games, and the strategic implication of forward induction and backward induction coincides, i.e., backward induction reasoning and forward induction reasoning are also observationally indistinguishable. One may also see the backward induction outcome as a robust implication of EFR under general model of conditional preferences. Since CM condition is rather weak, the result covers all preference models discussed in the literature, e.g., the probabilistic sophistication model, the multi-priors model, the Choquet expected utility model, the ordinal expected utility model, the lexicographic preference model, and the "strongly monotonic" preference model (see Chen and Luo, 2012). Theorem 1 can be regarded as a generalization to Battigalli's (1997) Theorem 4. Battigalli's (1997) proof, based on properties of Kohlberg and Mertens' (1986) "fully stable sets", can not be directly adapted to general preferences. We generalize the result by investigating two crutial properties of EFR. The first one regards the "dominance solvability" of EFR. We show that in generic PI games, EFR yields a unique terminal node under any preference model satifying CM. (See Lemma 1 in appendix.) That is, CM is sufficient to dominance solvability. This sufficient condition might be the weakest one since CM is the weakest requirement for rational preferences as far as we know. The second one says that if the preference model admits SEU model, then EFR outcomes under such model include the one under SEU model. (Lemma 2 in appendix.) This result is implied from the "outcome order independence" of EFR under SEU model shown by Chen and Micali (2013) and Luo et al. (2016).

Another main result shows that if the model admits all CM preferences, then the elimination procedure associated with EFR coincides with the backward iterated dominance procedure (Theorem 2). That is, under the "largest" model of CM preferences, the procedure to characterize EFR is exactly the one inspired by backward induction reasoning. As mentioned earlier, EFR and BI are conceptually different and have different implications of strategic choice for players. Nevertheless, this result shows that the conflict between those two solution concepts could be mitigated if we enlarge the underlying preference model.

The rest of this paper is organized as follows. In section 2, an illustrative example demonstrates the main results in this paper. Section 3 sets up the analytical framework. Section 4 presents the main theorems. Section 5 dicusses the non-generic case. Section 6 offers concluding remarks. To facilitate reading, all the proofs are relegated to the Appendix.

2.2 An illustrative example

The following two-person centipede game, due to game Γ_3 in Battigalli (1997), demonstrates the main results in this paper for generic PI games (where generic means that no same payoff is assigned to two distinct terminal nodes for any player).

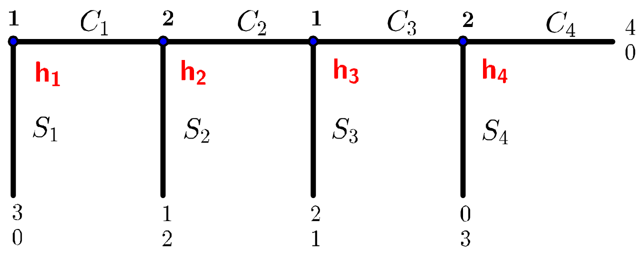

Fig. 1. A two-person game

Apparently, (S_1S_3, S_2S_4) is the unique BI strategy profile. Pearce's notion of EFR, or EFR with respect to SEU model, is the set of strategy profiles surviving iteratively eliminating never "sequential best replies" in the following way. In the first step, C_2C_4 is no better response than C_2S_4 conditioned on h_4; C_1S_3 is no better response than S_1S_3 conditioned on h_1 because by playing C_1S_3, player 1 either gets a payoff 1 or 2 while S_1S_3 secures the payoff 3. In the second step, player 2 considers that if h_2 is reached by a "rational" play, then it must be the case that player 1 chose C_1C_3. Therefore, C_2S_4 becomes player 2's only rational play. In the third step, C_1C_3 is no better response than S_1S_3 conditioned on h_1. Consequently, EFR set is $\{S_1S_3, S_1C_3\} \times \{C_2S_4\}$ which predicts the same terminal node (S_1) as the BI does. Note that the elimination procedure only involves strict dominance relation between pure strategies conditioned on reachable information sets. Thus EFR under the "strongly monotonic" preference model is exactly the same as that under SEU model.

Now consider a model which admits all CM preferences. Consequently, a strategy is strictly preferred to another strategy if both strategies "conditionally" assure constant payoffs and the former payoff is strictly higher than the latter one, and no specific preference ordering is primitively assumed once there is uncertainty involved (at the conditional). Particularly, in the first step, C_1S_3 could be an optimal reply to a CM preference which exhibits extremely ambiguity-seeking behavior. It is easy to see that under CM preference model, $C_2C_4; C_1C_3; C_1S_3$ are consecutively eliminated and left with the EFR set $\{S_1S_3, S_1C_3\} \times \{S_2S_4, S_2C_4\}$ whose outcome is the same as the BI outcome. Moreover, the elimination procedure exactly matches with the *backward iterated dominance procedure*: In each step, if backward induction deletes action a at node h, then delete all the strategies reaching h and choosing a (see, e.g., Osborne and Rubinstein, page 108).

Note that EFR set under SEU model is not a subset of that under CM model. It means that EFR is not monotone in preference model. The absence of monotonicity is related the "order dependent" issue of EFR. Suppose in each step along the EFR procedure, instead of eliminating all of the never "sequential best replies", we only eliminate some of them and stops when there is no more never "sequential best replies" left. This corresponds to a new elimination order. Different elimination orders may deliver different final sets. In particular, in the above example, the EFR procedure under CM model can be regarded as an alternative eliminition order of EFR under SEU model. Due to such ill behavior, the relationship between EFR under different preference model is unclear and thus it is difficult to find a straightforward proof for the indistinguishability result. However, EFR procedure under SEU model is "outcome order independent" in the sense that different eliminition order yields the same set of terminal nodes. This property provides us a short cut to relate EFR outcomes among different preference models. In fact, this property plays an important role in our proof.

2.3 Set-up

Consider a (finite) extensive form with perfect recall:[4]

$$\Gamma = (N, V, \{H_i\}_{i\in N}, \{A_h\}_{h\in\cup_{i\in N}H_i}),$$

where $N = \{1, 2, \cdots, n\}$ is the set of players, V is the set of nodes, A_h is the set of actions available at information set h and H_i is the set of information sets for player $i \in N$. Let $Z \subseteq V$ denote the set of terminal nodes. A payoff function for player i is a function $u_i : Z \to \mathbb{R}$. The game $\Gamma(u)$ is specified by the extensive form Γ and the payoffs $u \equiv (u_i)_{i\in N}$.[5]

A (pure) strategy of player i is a function that assigns an action in A_h to each information set $h \in H_i$. Let S_i denote the set of strategies of player i and $S \equiv \Pi_{i\in N} S_i \equiv S_i \times S_{-i}$ denote the set of strategy profiles.

[4]Since the formal description of an extensive form is by now standard (see, for instance, Kreps and Wilson (1982) and Osborne and Rubinstein (1994)), we here include the necessary notation only. We note that our approach in this paper can be easily extend to games with nature moves.

[5]To make the paper easy to read, we specify numerical payoffs to terminal nodes. However, it is not neccessary to do so. The analysis and main results in this paper apply as long as players are endowed with preference orderings over terminal nodes.

For a strategy profile $s \equiv (s_i)_{i \in N} \equiv (s_i, s_{-i}) \in S$, let $g(s)$ denote the corresponding terminal node realized.

Consider an information set h of player i. A pure strategy profile s reaches h if it reaches some node in h. The set of all profiles reaching h is denoted by $S(h)$, whose projections on S_i and S_{-i} are denoted by $S_i(h)$ and $S_{-i}(h)$ respectively. We say s_i (respectively s_{-i}) reaches h if $s_i \in S_i(h)$ (respectively $s_{-i} \in S_{-i}(h)$). By perfect recall, there is a unique sequence of actions of player i which leads to h, hence, $S(h) = S_i(h) \times S_{-i}(h)$.

From the decision theory point of view, each player is a decision maker who deals with opponents' strategic uncertainty and coordinates his/her sequential moves in the extensive game situation. The uncertainty partially resolves as the play of the game progresses. Specifically, once an information set $h \in H_i$ is realized and player i is about to move therein, he/she concludes that none of the strategy profiles which exclude h could be possibly played. That is, player i considers that h must be reachable by opponents' moves. Accordingly, player i restricts the opponents' strategic uncertainty to the set $S_{-i}(h)$. Each player is endowed with a *conditional preference system (cps)* to account for this process of uncertainty resolution. Formally, player i holds a cps $\succeq_i \equiv (\succeq_h)_{h \in H_i}$ such that for all $h \in H_i$, the complement of $S(h)$ is null in the sense of Savage (1954) for the conditional preference $\succeq_h$.[6] That is, payoffs on those terminal nodes incompatible with h are irrelevant for the conditional preference $\succeq_h$. One may refer to this

[6] $\forall s_i, s_i', t_i, t_i' \in S_i$, if $u_i(g(s_i, s_{-i})) = u_i(g(s_i', s_{-i}))$ and $u_i(g(t_i, s_{-i})) = u_i(g(t_i', s_{-i}))$ $\forall s_{-i} \in S_{-i}(h)$, then $s_i \succeq_h' t_i \Leftrightarrow s_i' \succeq_h t_i'$.

formulation as a form of consequentialism. It embodies the idea that a preference conditional on an event should not depend on the consequences outside of that event. For more discussions, see, for example, Epstein and Le Breton (1993), Ghirardato (2002) and Hanany and Klibanoff (2007). Conditional preference systems generalize the conditional probability systems introduced by Renyi (1992).

Suppose player i holds a "working hypothesis" which states that strategy profiles being played are conformed to a subset $S' \subseteq S$. If the realization of $h \in H_i$ does not falsify this hypothesis, i.e., $S' \cap S(h) \neq \emptyset$, then at h, player i will naturally maintain the hypothesis which provides additional information. Otherwise, player i cannot maintain both consequentialism and the working hypothesis. Since consequentialism is primitively assumed in this paper, player i must abandon the latter in counterfactual cases. We say player i's cps $\succeq_i$ knows a subset $S' \subseteq S$ if he/she is sure of S' whenever S' is not falsified. Formally, we adopts the following definition:

Definition 1. Player i's cps $\succeq_i$ *knows* $S' \subseteq S$ if $\forall h \in H_i$, the complement of $S' \cap S(h)$ is Savage-null for $\succeq_h$whenever $S' \cap S(h) \neq \emptyset$.

This knowledge notion in this paper is in the same spirit of the "strong belief" operator in Battigalli and Siniscalchi (2002) and generalizes it to general preferences.

Consider a "*model of conditional preferences*" $\mathcal{P}(\Gamma(u)) \equiv \{\mathcal{P}_i(\cdot)\}_{i \in N}$ on $\Gamma(u)$, where $\mathcal{P}_i(\cdot)$ is defined for any subset in product form $S' = \times_{i \in N} S'_i \subseteq$

S, and $\mathcal{P}_i(S')$ is interpreted as i's admissible cps which know S'.[7] Therefore, for every collection of cps in $\{\mathcal{P}_i(S')\}_{i\in N}$, the "reduced game" S' serves as a common "working hypothesis".

Definition 2. The *constantly monotone model* $\mathcal{P}^{CM}(\cdot)$ is defined as follows. For every $S' = \times_{i\in N} S'_i \subseteq S$, a cps $(\succeq_h)_{h\in H_i} \in \mathcal{P}_i^{CM}(S')$ if and only if it knows S' and for all $h \in H_i$, it satisfies *constant monotonicity (CM)*: $\forall s_i, s'_i \in S_i$,

$$[\forall s_{-i} \in S_{-i}(h), \, u_i(g(s_i, s_{-i})) = r > r' = u_i(g(s'_i, s_{-i}))] \Rightarrow [s_i \succ_h s'_i].$$

That is, CM requires that a strategy is strictly preferred to another strategy if the former strategy assures a constant payoff strictly higher than the latter one does. The restriction takes effect only under the case without any payoff uncertainty. Note that a CM preference is rather weak and might not have a utility representation. Any preference which violates CM would be considered as either trivial or irrational. In this sense, CM is self-evident and therefore no essential behavioral assumption, except consequentialism, is imposed on preferences throughout this paper. The following examples demonstrate that the analytical framework in this paper can be applied to games where the players have different kinds of preferences, including the standard SEU model (with Bayesian updating) and regular preference

[7] Since strategic implications are mainly concerns in this paper, for simplicity, we adopt a simple version of preference model here. We note that we can start from the state space and extend Epstein and Wang's (1996) way to construct a "model of conditional preference" from the primitive state space in dynamic setting. An advantage of the definition used in this paper is that it permits sharp results that can be interpreted as reflecting exclusively the more liberal meaning for the behavior in various game situations. (see Chen et al. (2016)) Note that, throughout this paper, no utility representation is assumed.

model as special cases. It is also applicable to model strategic behavior such as interactive ambiguity in extensive games.[8]

Example 1. The *SEU model*, denoted as $\mathcal{P}^{SEU}(\cdot)$, is defined through SEU representation based on conditional probability systems (see Renyi (1992)). For every $S' = \times_{i \in N} S'_i \subseteq S$, say $\mu \equiv (\mu_h)_{h \in H_i}$ is a conditional probability system over S'_{-i} if conditions (i)-(iii) hold for all $h, h' \in H_i$:

(i) μ_h is a probability distribution on $S_{-i}(h)$;

(ii) $\mu_h \left(S'_{-i} \cap S_{-i}(h) \right) = 1$ whenever $S'_{-i} \cap S_{-i}(h) \neq \emptyset$;

(iii) if $S''_{-i} \subseteq S_{-i}(h') \subseteq S_{-i}(h)$, then $\mu_h \left(S''_{-i} \right) = \mu_{h'} \left(S''_{-i} \right) \mu_h \left(S_{-i}(h') \right)$.[9]

A cps $(\succeq_h)_{h \in H_i} \in \mathcal{P}_i^{SEU}(S')$ if and only if for all $h \in H_i$, $\succeq_h$has an SEU representation as

$$u_i(s_i, \mu_h) = \sum_{s_{-i} \in S'_{-i}} u_i \left(g \left(s_i, s_{-i} \right) \right) \mu_h \left(s_{-i} \right), \forall s_i \in S_i,$$

for some conditional probability system μ over S'_{-i}.

Example 2. The *regular preference model*, denoted as $\mathcal{P}^{Reg}(\cdot)$, is defined as follows. Conditioned on every $h \in H_i$, each strategy s_i can be identified with an act $\mathbf{s}_i(\cdot)|_h$ on $S_{-i}(h)$. Formally, $\forall s_{-i} \in S_{-i}(h)$, $\mathbf{s}_i(\cdot)|_h$ maps s_{-i} to $u_i(g(s_i, s_{-i}))$. For every $S' = \times_{i \in N} S'_i \subseteq S$, a cps $(\succeq_h)_{h \in H_i} \in \mathcal{P}_i^{Reg}(S')$ if and only if it knows S' and for all $h \in H_i$, there exists a regular preference

[8] See Ahn (2007) and Kajii (2005) for more discussions.

[9] Condition (iii) says that if the information set h' follows h, then $\mu_{h'}$ is updated from μ_h by Bayes rule.

$\succsim_h$on $S_{-i}(h)$ (see Epstein 1997, pp.6-7) such that $s_i \succeq_h s_i' \Leftrightarrow \mathbf{s}_i(\cdot)|_h \succsim_h \mathbf{s}_i'(\cdot)|_h$, $\forall s_i, s_i' \in S_i$.

Both SEU model and regular preference model satisfy CM and thus are submodels of $\mathcal{P}^{CM}(\cdot)$.

2.4 EFR and indistinguishability

Bayesian rationality is the usual behavioral assumption made in the literature; that is, each player forms a prior probability distribution over opponents' play and chooses a strategy to maximize the corresponding expected utility. In the dynamic counterpart, a Bayesian-rational player is assumed to choose a strategy which is a sequential best reply with respect to some conditional probability systems. The model of conditional preference in this paper considerably relaxes the behavioral assumption and might accommodate dynamic Ellsberg paradox. The following definition is the "sequential rationality" condition adopted in this paper for an arbitrary model of conditional preference.

Definition 3. Given $\langle \Gamma(u), \mathcal{P}(\cdot)\rangle$ and $S' = \times_{i\in N} S_i' \subseteq S$, $s_i \in S_i$ is a $\mathcal{P}$-*best reply* on S' if there exists a cps $\succeq_i \in \mathcal{P}_i(S')$ such that the following condition holds: $\forall h \in H_i$ reached by s_i, $s_i \succeq_h s_i'$ for all $s_i' \in S_i(h) \cap S_i'$.

That is, given a hypothetical "reduced game" S', s_i is $\mathcal{P}$-best reply on S' if it can be supported by some $\succeq_i$, which knows S', in the following sense: s_i must be the most preferred strategy conditioned on every information set not precluded by itself, compared to all the other strategies in the reduced game which reach that information set. Denote $r_i(\mathcal{P}_i(S'))$ as the set of all $\mathcal{P}$-best replies on S' for player i and denote $r(\mathcal{P}(S')) \equiv \times_{i \in N} r_i(\mathcal{P}_i(S'))$. The following definition extends Pearce's notion of EFR to arbitrary model of conditional preferences.

Definition 4. Given $\langle \Gamma(u), \mathcal{P}(\cdot) \rangle$, let $S^0 = S$. Define $S^1, ..., S^{n+1}$ inductively as $S^{n+1} = S^n \cap r(\mathcal{P}(S^n))$, then $S^\infty = \cap_{n \geq 0} S^n$ is the set of $\mathcal{P}$-extensive form rationalizable ($\mathcal{P}$*-EFR*) strategy profiles.

That is, after iteratively eliminating never $\mathcal{P}$-best replies, the set left is $\mathcal{P}$*-EFR*. $\mathcal{P}^{SEU}$*-EFR* characterizes EFR in Pearce (1984).

Following Battigalli (1997), say $\Gamma(u)$ is without relevant ties if $\forall i \in N$, $\forall s_i, s_i' \in S_i$ and $\forall s_{-i} \in S_{-i}$,

$$g(s_i, s_{-i}) \neq g(s_i', s_{-i}) \Rightarrow u_i(g(s_i, s_{-i})) \neq u_i(g(s_i', s_{-i})).$$

In the above definition, the terminal nodes $g(s_i, s_{-i})$ and $g(s_i', s_{-i})$ (if different) are called relevant terminal nodes for player i. Without relevant ties means that there can not be payoff ties at relevant terminal nodes for player i.

Theorem 1. *Given $\langle \Gamma(u), \mathcal{P}(\cdot) \rangle$, if $\Gamma(u)$ is a perfect information game without relevant ties and $\mathcal{P}^{SEU}(\cdot) \subseteq \mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, then the outcome of $\mathcal{P}$-EFR strategies ($g(\mathcal{P}$-EFR$)$) is unique, which is the same as the backward induction outcome.*

Theorem 1 generalizes Battigalli's (1997) Theorem 4 whose proof uses some properties of Kohlberg and Mertens' (1986) "fully stable sets". The proof for Theorem 1 in this paper relies on two results. The first one states that for any model of conditional preference $\mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, the set of $\mathcal{P}$-*EFR* strategy profiles reaches a unique terminal node. (See Lemma 1 in appendix.) The second one is the "outcome order independence" of $\mathcal{P}^{SEU}$-*EFR*, i.e., all iterative elimination orders of never $\mathcal{P}^{SEU}$-best replies reaches the same set of terminal nodes, a result shown by Chen and Micali (2013). $\mathcal{P}$-*EFR* is a special elimination order in the sense that all never $\mathcal{P}$-best replies are eliminated in each step. An arbitrary elimination order would be eliminating some of the never $\mathcal{P}$-best replies in each step and it stops when there is no never $\mathcal{P}$-best reply left. (See Definition and Lemma 2 in appendix.)

Remark. Accroding to proofs in the appendix, Theorem 1 can be further strengthened to arbitrary elimination orders. Formally speaking: Given $\langle \Gamma(u), \mathcal{P}(\cdot) \rangle$, if $\Gamma(u)$ is a PI game without relevant ties and $\mathcal{P}^{SEU}(\cdot) \subseteq \mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, then for an aribrary elimination order of never $\mathcal{P}$-best replies, the survived set yields the unique backward induction outcome.

Theorem 1 can be interpreted as an indistinguishable result: EFR strategic behavior under a class of very general preference models is observationally indistinguishable from that under the SEU model. Furthermore, base on the observed play of the game, one can not distinguish players using EFR reasoning process from those using reasoning process of backward induction. Note that without relevant ties is a generic condition, the indistinguishability thus generically holds for PI games. Chen and Luo (2012) showed an indistinguishable result for compact Hausdoff strategic game under "concave like" condition. It is worth to note that the result in this paper is topological free. Lo (2000) provided an indistinguishability result to all the models of preference that satisfy Savage's axiom P3, which is a form of monotonicity. The CM condition in this paper is even weaker and covers almost all preference models discussed in the literature.

Theorem 2. *Given $\langle \Gamma(u), \mathcal{P}^{CM}(\cdot) \rangle$, if $\Gamma(u)$ is a perfect information game without ties*[10]*, then $\mathcal{P}^{CM}$-EFR is the same as the set survives backward iterated dominance procedure.*

As demonstrated in the illustrative example in section 2, S_1S_3 is "sequential rational" if and only if S_1C_3 is because both yield the outcome S_1 regardless of the strategic choice of player 2. Therefore, EFR is conceptually not possible to deliver the BI strategy profile. However, Theorem 2 draws a connection between $\mathcal{P}^{CM}$-*EFR* and the backward iterated dominance procedure. That is, under $\mathcal{P}^{CM}(\cdot)$, the procedure to compute EFR is exactly the one inspired by BI reasoning. In this sense, enlarging the preference model to $\mathcal{P}^{CM}(\cdot)$ could partially mitigate the conflict between EFR and BI.

[10] An extensive game is without ties if for any two different terminal nodes z and z', $u_i(z) \neq u_i(z')$ for all i. A game without relevant ties might have ties.

2.5 Non-genericity

For a non-generic PI game, the indistinguishability might fail. Consider the three-person PI game depicted in the following game tree.

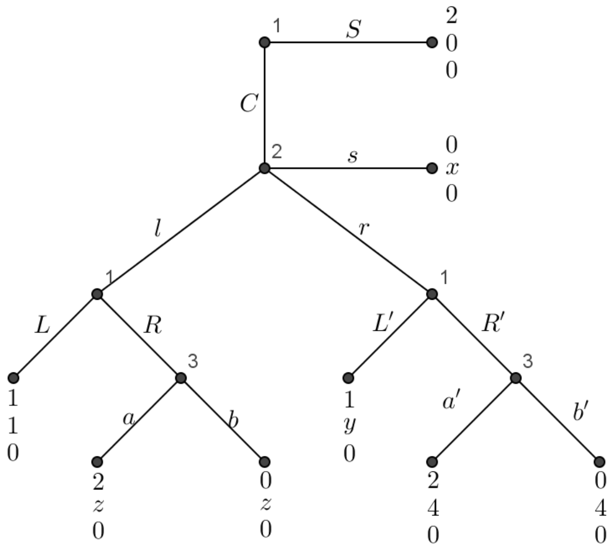

Fig. 2. A three-person game $\Gamma(x, y, z)$, where

$x, y, z \in \mathbb{R}$

Player 3 can be interpreted as nature moves which generate payoff uncertainties for player 1 but not player 2. Consider 4 different preference models: SEU, ordinal expeted utility model (OEU), multi-prior (MP) and strongly monotone (Mon) preference model. By playing CLL', player 1's payoff is either 0 or 1, both are less than 2, the payoff guaranteed by playing S in the first place. Therefore, any strongly monotone preference can not support CLL'. It can be checked that for all 4 models, in the fisrt step, EFR only eliminates CLL' regardless the values of x, y, z. In the second step, player 2's strategy choices and payoff uncertainties can be summarized in the following table.

	CLR'	CRL'	CRR'
s	x	x	x
l	1	z	z
r	4	y	4

Different combinations of (x, y, z) might deliver different strategic implications for player 2 in the second step. Finally, the following table describes non-rationalizable outcomes under different models and games.

	SEU	OEU	MP	Mon
$\Gamma(2,1,4)$	$\backslash Cs$			
$\Gamma(1,1,4)$	$\backslash Cs$	$\backslash Cs$		
$\Gamma(1,0,0)$	$\backslash ClL$	$\backslash ClL$	$\backslash ClL$	

The result shows that any pair of preference models could be distinguished in some game. Note that this game is non-generic and not dominance solvable. Consequently, the mutiplicity of rationalizable outcomes lead to distinguishability. Hence, this example shows dominance solvability is crutial to the indistinguishability.

2.6 Concluding Remarks

In this paper, we have formulated a model of conditional preferences and applied it to analyze the solution concept of EFR in extensive games. The main result show that behavioral implications of EFR are observationally indistinguishable among all preference models which admit SEU and satisfy constant monotonicity in generic PI games, and the EFR outcome is further indistinguishable from the BI outcome. Through out this paper, we impose a rather weak condition (probably the weakest), constant monotonicity, on preference models. All regular preferences satisfy this condition and our result is applicable for many preference models discussed in the literature, e.g., the probabilistic sophistication model, the multi-priors model, the Choquet expected utility model, the ordinal expected utility model and the lexicographic preference model.

Unlike other indistinguishable results in literatures mentioned above, our result does not rely on any topological or algebraic structure. Instead, our result is based on the idea of consequentialism embedded in the conditional preferences which are rich enough to regulate rational behavior. In this respect, our result is sharp and sheds light on important and fundamental issues on rational strategic behavior in dynamic context.

Appendix: Proofs

Definition. Given $\langle \Gamma(u), \mathcal{P}(\cdot) \rangle$, a decreasing sequence of product sets $\{D^k\}_{k\geq 0}$ is an *elimination order of never $\mathcal{P}$-best replies* (*EON-$\mathcal{P}$*) if the following conditions hold: (i) $D^0 = S$, (ii) $\forall k \geq 0, \forall s_i \in D_i^k \backslash D_i^{k+1}$, $s_i \notin r_i(\mathcal{P}_i(D^k))$, (iii) $D^\infty \subset r(\mathcal{P}(D^\infty))$ where $D^\infty \equiv \cap_{k\geq 0} D^k$.

Lemma 1. Given $\langle \Gamma(u), \mathcal{P}(\cdot) \rangle$, if $\Gamma(u)$ is a PI game without relevant ties and $\mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, then for an arbitrary *EON-$\mathcal{P}$*, $\{D^k\}_{k\geq 0}$, there is at most one outcome in D^∞ i.e., $|g(D^\infty)| \leq 1$. ($|\cdot|$ denotes the cardinality)

Proof of Lemma 1. Let $\mathcal{Y} = \{h \in \cup_i H_i : |g(D^\infty \cap S(h))| > 1\}$. Suppose on the contrary $|g(D^\infty)| > 1$, then $\mathcal{Y}$ includes the initial history and thus is nonempty. Pick $\underline{h} \in \mathcal{Y}$ such that it has no successor in $\mathcal{Y}$. $|g'(D^\infty \cap S(\underline{h}))| > 1 \Rightarrow \exists s, s' \in D^\infty \cap S(\underline{h})$ such that $g(s) \neq g(s')$. Without loss of generality, assume $\underline{h} \in H_i$. Then $\forall x_{-i} \in D^\infty_{-i} \cap S_{-i}(\underline{h})$, $g(s_i, x_{-i}) = g(s)$. Otherwise, since D^∞ is a product set, we can find a successor of $\underline{h}$ in $\mathcal{Y}$. Similarly, we have $\forall x_{-i} \in D^\infty_{-i} \cap S_{-i}(\underline{h})$, $g(s'_i, x_{-i}) = g(s')$. Since the game is without relevant ties, $u_i(g(s)) \neq u_i(g(s'))$. Without loss of generality, assume $u_i(g(s')) > u_i(g(s))$. $\forall \succeq_i \in \mathcal{P}_i(D^\infty) \subseteq \mathcal{P}_i^{CM}(D^\infty)$, since $D^\infty \cap S(\underline{h}) \neq \emptyset$, the complement of $D^\infty \cap S(\underline{h})$ is Savage-null for $\succeq_{\underline{h}}$. By constant monotonicity, $s'_i \succ_{\underline{h}} s_i$. Therefore $s_i \notin \mathcal{P}_i(D^\infty)$, which contradicts $D^\infty \subseteq r(\mathcal{P}(D^\infty))$. $\square$

Lemma 2. Given $\langle \Gamma(u), \mathcal{P}^{SEU}(\cdot) \rangle$, Let $\{D^k\}_{k\geq 0}$ and $\{\bar{D}^k\}_{k\geq 0}$ be elimination orders of never $\mathcal{P}^{SEU}$-best replies, then $g(D^\infty) = g(\bar{D}^\infty)$.

Proof of Lemma 2. According to Shimoji and Watson (1998), $\{D^k\}_{k\geq 0}$ and $\{\bar{D}^k\}_{k\geq 0}$ are elimination orders of conditional dominated strategies. According to Chen and Micali (2013), the elimination is order independent, i.e., $g(D^\infty) = g(\bar{D}^\infty)$. $\square$

Proof of Theorem 1. Let $\{BI^k\}_{k\geq 0}$ be the backward iterated dominance procedure. It is easy to see that $\{BI^k\}_{k\geq 0}$ is an elimination order of never $\mathcal{P}^{SEU}$-best replies and $g(BI^\infty)$ is the unique backward induction outcome. Consider an arbitrary elimination order of never $\mathcal{P}$-best replies, $\{D^0, ..., D^K\}$. $\forall k \geq K$, define $D^{k+1} = D^k \cap r\left(\mathcal{P}^{SEU}\left(D^k\right)\right)$. Since $\mathcal{P}^{SEU} \subseteq \mathcal{P}$, $\{D^k\}_{k\geq 0}$ is an elimination order of never $\mathcal{P}^{SEU}$-best replies. By

Lemma 2, $g(BI^{\infty}) = g(D^{\infty}) \subseteq g(D^K)$. Since $\mathcal{P}(\cdot) \subseteq \mathcal{P}^{CM}(\cdot)$, by Lemma 1, $\left|g(D^K)\right| \leq 1$. Thus, $g(D^K) = g(BI^{\infty})$. The result follows since $\mathcal{P}$-*EFR* is the result of a special elimination order of never $\mathcal{P}$-best replies. □

Proof of Theorem 2. Let $\{BI^k\}_{k\geq 0}$ be the backward iterated dominance procedure. Define $\left\{S^k\right\}_{k\geq 0}$ inductively as $S^0 = S$ and $S^{k+1} = S^k \cap r\left(\mathcal{P}^{CM}\left(S^k\right)\right)$ for all $k \geq 0$. It suffices to show $BI^k = S^k$ for all $k \geq 0$. The equivalence is trivial when $k = 0$, and suppose it holds for $0, 1, ..., k$. It suffices to show (i) $s \in BI^k \backslash BI^{k+1} \Rightarrow s \in S^k \backslash S^{k+1}$; (ii) $s \in B^{k+1} \Rightarrow s \in S^{k+1}$. Let H^1 denote the set of last decision nodes. For all $l \geq 1$, inductively define H^{l+1} as the set of last decision nodes in $(\cup_{i\in N} H_i) \backslash H^l$. Denote a_h^* as the action prescribed by the backward induction at h. By definition,

$$BI^k = \left\{s \in S : \text{if } s \text{ reaches some } h \in \cup_{l=1}^k H^l, \text{ then } s_h = a_h^*\right\}$$

(i) $\forall s \in BI^k \backslash BI^{k+1}$, s reaches some $h^* \in H^{k+1}$ and $s_{h^*} \neq a_{h^*}^*$. Without loss of generality, assume $h^* \in H_i$. $\forall x \in BI^k \cap S(h^*)$, $x_h = a_h^*$ for all h which follows h^*. Therefore, (s_i, x_{-i}) reaches the same terminal nodes for all $x \in BI^k \cap S(h^*)$. Let $s^* = (s_{-h^*}, a_{h^*}^*)$, then $s_i^* \in BI^k \cap S(h^*)$. Similarly, (s_i^*, x_{-i}) reaches the same terminal node for all $x \in BI^k \cap S(h^*)$. Since $\Gamma(u)$ has no ties, by backward induction, $s_i^* \succeq_{h^*} s_i$ for all $\succeq_i \in \mathcal{P}^{CM}\left(BI^k\right)$. Thus $s \in r\left(S^k\right)$. By induction hypothesis, $s \in S^k \backslash S^{k+1}$.

(ii) $\forall s \in BI^{k+1}$. $\forall h \in \cup_{l=1}^{k+1} H^l$, if s reaches h, then $s_{h'} = a_{h'}^*$ for all h' equal to h or follows h, therefore, s is the most preference action for any constantly monotone $\succeq_h$. $\forall h \notin \cup_{l=1}^{k+1} H^l$, either A_h is a singleton or there are strategy profiles in $BI^k \cap S(h)$ leads to different terminal nodes. In the first case, s is the most preference action for any constantly monotone $\succeq_h$. In the second case, there exist $\succeq_h$which is certain of $BI^k \cap S(h)$ and supports the non-constant act $\mathbf{s}(\cdot)|_h$ as the most preferred one. Overall, $s \in r\left(B^k\right)$. By induction hypothesis, $s \in S^{k+1}$. $\square$

3 Generic equivalence between perfectly and sequentially rational strategic behavior

3.1 Introduction

In dealing with imperfection in (finite) extensive games, Selten (1975) introduced the notion of (trembling-hand) perfect equilibrium. A perfect equilibrium is an equilibrium that takes the possibility of off-the-equilibrium play into account by assuming that the players, through the idea of "trembling hand", may choose all unintended strategies, albeit with small probabilities. In the spirit of Selten's (1975) perfectness, Kreps and Wilson (1982) proposed an alternative notion of sequential equilibrium, by imposing the so-called "sequential consistency" and "sequential rationality" on the behavior of every player. Sequential equilibrium is more inclusive and weaker than perfect equilibrium: every perfect equilibrium must be sequential. Kreps and Wilson (1982, Section 7) pointed out that the two concepts lead to similar prescriptions for equilibrium play: For each particular game form and for almost all assignments of payoffs to the terminal nodes, almost all sequential equilibria are perfect equilibria, and the sets of sequential and perfect equilibria fail to coincide only at payoffs where the perfect equilibrium correspondence fails to be upper hemi-continuous. Blume and Zame (1994) (hereafter BZ94) strengthened Kreps and Wilson's (1982) result and showed that: For almost all assignments of payoffs to the terminal nodes, the sets of sequential and perfect equilibria are identical. The research line of genericity in game theory sheds light on important and fundamental issues on rational strategic behavior; e.g., Harsanyi (1973) justified the stability of mixed strategy equilibria in generic games by Sard's Theorem. Kreps and Wilson (1982) and Kohlberg and Mertens (1986) also applied Sard's Theorem and the Regular Value Theorem in differential topology to study equilibrium distributions over terminal nodes and the generic finiteness of equilibria components (see also Govindan and Wilson (2001, 2006, 2012), Govindan and McLennan (2001), Hillas and Kohlberg (2002), Haller and Lagunoff (2002), McKelvey and McLennan (1996), and Pimienta and Shen (2014) for more discussions).

BZ94 obtained the "generic" equivalence result by exploiting a special semi-algebraic structure of the graphs of the perfect and sequential equilibrium correspondences, because graphs of the two correspondences can each be written as a subset of a Euclidean space defined by a finite number of polynomial equalities and inequalities. As they pointed out,[1]

> "We believe that, just as differential topology has proved to be the right tool for studying the fine structure of the Walrasian equilibrium correspondence, so will real algebraic geometry prove to be the right tool for studying the fine structure of game-theoretic equilibrium correspondences. (BZ94, p.784)"

In this paper, we follow BZ94 to study the relationship between perfectly

[1] van Damme (1992, Theorem 2.6.1) presented an "almost all" theorem: In "almost all" normal form games, Nash equilibria are "regular" equilibria (hence proper equilibria). Nevertheless, as van Damme (1992, p.45) pointed out, the analysis "is of limited value for the study of extensive form games as any nontrivial such game gives rise to a nongeneric normal form."

and sequentially rational strategic behavior in a broad sense, from the point of view of semi-algebraic geometry. We establish a general "generic" equivalence theorem between perfect rationality and sequential rationality in (finite) extensive games (Theorem 1). More specifically, we show that the difference between the perfectly and sequentially rational correspondences under very feasible behavioral assumptions occurs only for "nongeneric" payoffs (which are included in a lower-dimensional semi-algebraic payoffs set). We also apply our general "generic" equivalence theorem to various solution concepts such as equilibrium, rationalizability, iterated dominance and MACA (Greenberg et al. (2009)); in particular, we obtain a variety of generic equivalence results as corollaries of Theorem 1 (Corollaries 1-4).

In a special class of "generic" games with perfect information (i.e., it is not a "nongeneric" case where, for some player, a same payoff is assigned to two distinct terminal nodes), it is fairly easy to see that perfect/sequential equilibrium yields the unique backward induction outcome (in terms of strategy profiles). In other words, sequential and perfect equilibria are generically identical in games with perfect information. The similar result indeed holds true for the notion of perfect/sequential rationalizability. That is, in the class of "generic" games with perfect information, both perfect/sequential equilibrium and rationalizability lead to the unique backward induction outcome, excluding a lower-dimensional set of payoffs (see Example in Section 2).

In this paper, we provide a unified approach to the "generic" relationship between perfectly and sequentially rational strategic behavior. We present a general framework to accommodate diversiform structures of beliefs for different solution concepts and distinct players in games by restrictions on the scope of trembling sequences (specified by sets $\mathfrak{X}$). The graphs of perfectly and sequentially rational correspondences are related respectively to the closure and vertical closure of a set $\mathcal{R}^{\mathfrak{X}}$ of "perfectly-rational states" (Proposition 1). Based upon Generic Local Triviality in semi-algebraic geometry, we show that the closure and vertical closure of a semi-algebraic set almost coincide (Proposition 2). Consequently, perfectly and sequentially rational correspondences under the "structures of beliefs" $\mathfrak{X}$ are generically identical (Theorem 1). Our approach of this paper is rather feasible and applicable to various solution concepts, *as long as belief structures* $\mathfrak{X}$ *are semi-algebraic.* In this paper, we show that the belief structures behind many solution concepts in the literature are indeed semi-algebraic; for instance, if $\mathfrak{X}$ is restricted to a "common" trembling sequence for all players, Theorem 1 delivers BZ94's "generic" equivalence result for perfect and sequential equilibria.

One major feature of this paper is that, unlike BZ94, our approach does not rely directly on semi-algebraic properties of specific solution concepts. More specifically, BZ94's approach relies on the semi-algebraic property of sets of perfect/sequential equilibria, which are defined by polynomial equalities and inequalities, in finite dimensional Euclidean spaces. However, it is less clear that other kinds of perfect/sequential solution concepts – such as the sets of perfect/sequential rationalizable strategies – are semi-algebraic. Rather than working directly on the semi-algebraic property of solution concepts, we here take a different approach by exploiting the semi-algebraic property of the primitive set of "perfectly-rational states", which delivers a more general and fundamental generic equivalence between sequential and perfect rational behavior (Theorem 1). Moreover, BZ94 defined perfect and sequential equilibria by using "perturbed games" possibly with payoff perturbations (see Kreps and Wilson (1982)); our definitions in this paper are based on an alternative idea of "trembling strategies" possibly with payoff perturbations, so that our approach is feasible and applicable to unified solution concepts of "perfect-MACA" and "sequential-MACA" suggested by Greenberg et al. (2009) in complex situations. As a matter of fact, our paper provides an alternative approach to the study of the "generic" relationship between perfect and sequential equilibria in BZ94 (cf. also Section 5 for more discussions). From a technical perspective, BZ94 showed the generic equivalence result by using the generic continuity property of semi-algebraic correspondence; our proof is direct and dependent on the fact that the closure and vertical closure of a semi-algebraic set are generically identical (Proposition 2).

The rest of the paper is organized as follows. In Section 2, we provide an illustrative example to explain the general generic equivalence relationship between perfectly and sequentially rational strategic behavior. In Section 3, we present an analytical framework. In Section 4, we show a general "generic" equivalence theorem. We also obtain equivalence results for various solution concepts, as corollaries of the general generic equivalence theorem. Section 5 concludes. To facilitate reading, all the proofs are relegated to Appendix.

3.2 An Illustrative Example

The following two-person game demonstrates that there is a general relationship of "generic" equivalence between perfectly and sequentially rational strategic behavior (where a "generic" case means that no same payoff is assigned to distinct terminal nodes for each player).[2]

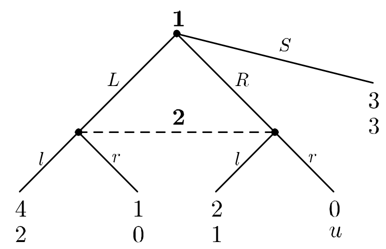

Figure. 1. A two-person game $\Gamma(u)$ *where* $u \in \mathbb{R}$.

Apparently, L dominates R (for player 1); l dominates r (for player 2) if payoffs $u \leq 1$. It is easy to see that sequential equilibrium differs from perfect equilibrium only at "nongeneric" payoff $u = 1$. Moreover, the difference between perfect and sequential equilibria occurs only for "nongeneric" payoff(s) that are resided in a lower-dimensional payoffs space. BZ94 showed that: For "almost all" or "generic" assignments of payoffs to the terminal nodes, the sets of sequential and perfect equilibria are identical. (This

[2]More precisely, a statement is "generically" true if it is false only for a lower dimensional subset of the payoff vector space.

example shows that there is no "generic" equivalence relationship between Myerson's (1978) proper equilibrium and perfect equilibrium: for "generic" payoffs $u > 1$, (S, r) is a perfect equilibrium but not a proper equilibrium.)

This sort of "generic" equivalence relationship indeed holds true for perfectly and sequentially rational strategic behavior in a broad sense: that is, "sequential rationality" differs from "perfect rationality" only at "nongeneric" payoff $u = 1$. For simplicity, we restrict attention to player 2's behavior in the game in Figure 1. Clearly, strategy r is not perfectly rational for player 2 since l (weakly) dominates r at "nongeneric" payoff $u = 1$. But, r is sequentially rational when $u = 1$ if player 2 holds a belief assessment $(p, 1 - p) = (0, 1)$ at his information set; this belief assessment can be generated by a "trembling sequence" $x^{\varepsilon} \equiv \varepsilon^2 L + \varepsilon R + (1 - \varepsilon - \varepsilon^2) S$ as $\varepsilon \to 0$. Note that, although r is not optimal along the "trembling sequence" x^{ε}, it can be optimal by a slight perturbation on payoff u. (For instance, r is optimal along the "trembling sequence" x^{ε} under perturbed payoff $u^{\varepsilon} = 1 + 2\varepsilon$.) In other words, r can be perfectly rational under payoff perturbations. Subsequently, sequentially rational strategy r can be obtained from a limit point of "perfectly-rational states" $(x^{\varepsilon}, u^{\varepsilon}, r)$, i.e., $\lim_{\varepsilon \to 0} (x^{\varepsilon}, u^{\varepsilon}, r) = (1 \circ S, 1, r)$.

In fact, every sequentially rational strategy can be characterized by a limit point of perfectly-rational states (see Lemma 1 in Appendix), while every perfectly rational strategy is naturally associated with a limit point of perfectly-rational states, *without payoff perturbations*. That is, the set of sequentially (resp. perfectly) rational strategies can be characterized by the closure (resp. vertical closure) of the set of perfectly-rational states (see Proposition 1 in Section 3). By Generic Local Triviality in semi-algebraic geometry, the closure and vertical closure of the set of perfectly rational states are almost the same (see Proposition 2 in Section 4). Consequently, we obtain our central result of this paper: sequential rationality differs from perfect rationality only at "nongeneric" payoffs (see Theorem 1 in Section 4). This result is applicable to various kinds of solution concepts discussed in the literature, such as equilibrium, rationalizability, iterated dominance and MACA (see Corollaries 1-4 in Section 4). For example, if the belief structure allows different players to have distinct trembling sequences, our Theorem 1 yields a "generic" equivalence result for perfect and sequential rationalizability (Corollary 1); if the "structure of beliefs" is restricted to a "common" trembling sequence for all players, our Theorem 1 delivers BZ94's "generic" equivalence result for perfect and sequential equilibria (Corollary 2).

3.3 An analytical framework

3.3.1 Set-up

We consider a (finite) extensive form with perfect recall:[3]

$$\Gamma = (N, V, H, \{A_h\}_{h \in H}),$$

[3]Since the formal description of an extensive form is by now standard (see, for instance, Kreps and Wilson (1982) and Osborne and Rubinstein (1994)), we here include the necessary notation only. We note that our approach in this paper can be easily extend to games with nature moves.

where $N = \{1, 2, \cdots, n\}$ is the set of players, V is the set of nodes, H is the set of information sets, A_h is the set of actions available at information set h. Let $Z \subseteq V$ denote the set of terminal nodes. A payoff function for player i is a function $u_i : Z \to \mathbb{R}$. Let $U = \Pi_{i \in N} U_i$ where $U_i = \mathbb{R}^Z$ is the space of player i's payoff functions. The game $\Gamma(u)$ is specified by the extensive form Γ and the payoffs $u \in U$.

A mixed action at information set h is a probability distribution over the actions in A_h. Let $\mathbb{Y}_h$ denote the set of mixed actions at h (i.e. $\mathbb{Y}_h = \Delta(A_h)$). The set of player i's (behavior) strategies is $\mathbb{Y}_i = \Pi_{h \in H_i} \mathbb{Y}_h$ (where H_i is the set of player i's information sets). Let $\mathbb{Y} = \Pi_{i \in N} \mathbb{Y}_i$ and $\mathbb{Y}_{-i} = \Pi_{j \neq i} \mathbb{Y}_j$. (For a profile $y \in \mathbb{Y}$, we also write $y = (y_i, y_{-i}) = (y_h, y_{-h})$.)

The sets $\mathbb{Y}$, $\mathbb{Y}_i$, $\mathbb{Y}_{-i}$ and $\mathbb{Y}_{-h}$ can be viewed as semi-algebraic sets, which are defined by linear equalities and inequalities, in finite dimensional Euclidean spaces.[4] Fix a terminal node z, the probability $\Pr(z|y)$ that z is reached (from the initial node) is a polynomial function of $y \in \mathbb{Y}$. In game $\Gamma(u)$, i's expected payoff from $y \in Y$ is defined as: $v_i(y, u_i) = \Sigma_{z \in Z} u_i(z) \Pr(z|y)$, which is semi-algebraic on $\mathbb{Y} \times U_i$.

3.3.2 Perfect rationality and sequential rationality

Consider a game $\Gamma(u)$. For a strategy-profile vector $\mathbf{x} \in \mathbb{Y}^n$, we write $\mathbf{x} \equiv ({}^i x)_{i \in N}$ such that ${}^i x \in \mathbb{Y}$ for each player i. Let $int(\mathbb{Y})$ denote the

[4] A set $X \subseteq \mathbb{R}^n$ is *semi-algebraic* if it is the finite union of sets of the form $\{x \in \mathbb{R}^n : f_1(x) = 0, \cdots, f_k(x) = 0 \text{ and } g_1(x) > 0, \cdots, g_m(x) > 0\}$, where the f_i and g_j are polynomials with real coefficients. A correspondence is *semi-algebraic* if and only if its graph is a semi-algebraic set.

set of completely-mixed-strategy profiles,[5] and let $\mathfrak{X} \subseteq [int(\mathbb{Y})]^n$. In this paper, we use $\mathfrak{X}$ to allude to a "structure of beliefs", to which the trembling way of "beliefs" or "conjectures" sequence for players confines. Let $\mathbf{x}^t \xrightarrow{\mathfrak{X}} \mathbf{x}$ denote a sequence $\{\mathbf{x}^t\}_{t=0}^{\infty}$ in $\mathfrak{X}$ which converges to $\mathbf{x}$ in $\mathbb{Y}^n$. (Note: We allow two players i and j to have distinct trembling sequences ${}^i x^t \rightsquigarrow {}^i x$ and ${}^j x^t \rightsquigarrow {}^j x$, respectively. We use $y^t \rightsquigarrow y$ to denote a trembling sequence $\{y^t\}_{t=0}^{\infty}$ in $int(\mathbb{Y})$ which converges to y in $\mathbb{Y}$ as $t \to \infty$.) An $\mathfrak{X}$-*assessment* is a profile-and-distributions vector $(\mathbf{x}, \mu) \equiv ({}^i x, \mu_i)_{i \in N}$ such that there exist a sequence $\mathbf{x}^t \xrightarrow{\mathfrak{X}} \mathbf{x}$ and, for each player i, $\mu_i^t \to \mu_i$ where μ_i^t is a collection of distributions over i's information sets, derived from ${}^i x^t$ in $int(\mathbb{Y})$ using Bayes' rule. Let $B_i({}^i x, u_i)$ denote the set of player i's "locally" best responses to ${}^i x \in \mathbb{Y}$, i.e., $B_i({}^i x, u_i) \equiv$

$$\left\{ y_i \in \mathbb{Y}_i : \forall h \in H_i,\, v_i\left(\left(y_h,\ {}^i x_{-h}\right), u_i\right) \geq v_i\left(\left(a_h,\ {}^i x_{-h}\right), u_i\right)\ \forall a_h \in A_h \right\}.$$

Definition 1. Let $Y \subseteq \mathbb{Y}$ and $\mathfrak{X} \subseteq [int(\mathbb{Y})]^n$.

(a) [**Perfect Rationality**] A strategy profile $y \in \mathbb{Y}$ is *perfectly rational with respect to* $(Y,\ \mathfrak{X})$ if there exists $\mathbf{x}^t \xrightarrow{\mathfrak{X}} \mathbf{x}$ such that, for each player i, ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i \in B_i\left({}^i x^t, u_i\right) \forall t$.

(b) [**Sequential Rationality**] A strategy profile $y \in \mathbb{Y}$ is *sequentially rational with respect to* $(Y,\ \mathfrak{X})$ if there exists an $\mathfrak{X}$-assessment $(\mathbf{x}, \mu)$ such that for all i and $h \in H_i$, ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i \in \arg\max_{y_i' \in \mathbb{Y}_i}$

[5] A completely-mixed-strategy profile $y \in \mathbb{Y}$ assigns strictly positive probability to every action at every information set.

$v_i\left(y_i', \left({}^i x, \mu_i\right), u_i | h\right)$.[6]

That is, a strategy profile y is perfectly rational with respect to $(Y,\ \mathfrak{X})$ if there exists a sequence $\{\mathbf{x}^t\}_{t=0}^{\infty}$ of trembling-beliefs profiles for all players in the belief structure $\mathfrak{X}$ (which converges to $\mathbf{x} = ({}^i x)_{i \in N}$ in $\mathbb{Y}^n$) such that, for each player i, the limit opponent-strategy profile ${}^i x_{-i}$ resides in the scope Y_{-i} of opponents' plausible choices and the limit strategy ${}^i x_i$ is consistent with y_i which is a "locally" best response along the trembling beliefs sequence $\{{}^i x^t\}_{t=0}^{\infty}$. Similarly, a strategy profile y is sequentially rational with respect to $(Y,\ \mathfrak{X})$ if there exists an $\mathfrak{X}$-assessment $(\mathbf{x}, \mu)$ such that, for each player i, the limit opponent-strategy profile ${}^i x_{-i}$ resides in the scope Y_{-i} and the limit strategy ${}^i x_i$ is consistent with y_i which is a "sequentially" best response at every information set $h \in H_i$. Let $PB^{\mathfrak{X}}(Y, u)$ denote the set of perfectly-rational strategy profiles with respect to $(Y,\ \mathfrak{X})$, and let $SB^{\mathfrak{X}}(Y, u)$ denote the set of sequentially-rational strategy profiles with respect to $(Y,\ \mathfrak{X})$.

We next provide two characterizations of perfect rationality and sequential rationality under a wide range of behavioral assumptions. For an extensive form Γ define

$$\mathcal{R}^{\mathfrak{X}} \equiv \left\{(\mathbf{x}, u, y) \in \mathfrak{X} \times U \times \mathbb{Y} : y_i \in B_i\left({}^i x, u_i\right)\ \forall i \in N\right\}.$$

That is, $(\mathbf{x}, u, y) \in R^{\mathfrak{X}}$ represents a "state" where every player is per-

[6] Player i's expected payoff conditional on h is denoted by $v_i(y_i', \left({}^i x, \mu_i\right), u_i | h) = \Sigma_{z \in Z} u_i(z) \Pr\{z | \left(y_i',\ {}^i x_{-i}\right), \mu_i\}$, where $\Pr\{z | \left(y_i',\ {}^i x_{-i}\right), \mu_i\}$ is the probability that z is reached conditionally on h under $\left(y_i',\ {}^i x_{-i}\right)$ and μ_i.

fectly rational for payoffs $u \in U$ and "belief" $\mathbf{x} \in \mathfrak{X}$. Since Γ is finite, $B_i\left({}^i x, u_i\right)$ is characterized by finitely many polynomial inequalities and thus semi-algebraic. By Tarski-Seidenberg Theorem, $\mathcal{R}^{\mathfrak{X}}$ is a semi-algebraic set whenever $\mathfrak{X}$ is semi-algebraic. Let $cl(\mathcal{R}^{\mathfrak{X}})$ and $vcl_U(\mathcal{R}^{\mathfrak{X}})$ denote the *closure of* $\mathcal{R}^{\mathfrak{X}}$ and *vertical closure of* $\mathcal{R}^{\mathfrak{X}}$ *(on* U*)*, respectively, i.e.,

$$cl(\mathcal{R}^{\mathfrak{X}}) \equiv \left\{(\mathbf{x}, u, y) : \left(\mathbf{x}^t, u^t, y^t\right) \to (\mathbf{x}, u, y) \text{ and } \left(\mathbf{x}^t, u^t, y^t\right) \in \mathcal{R}^{\mathfrak{X}} \text{ for all } t\right\};$$

$$vcl_U(\mathcal{R}^{\mathfrak{X}}) \equiv \left\{(\mathbf{x}, u, y) : \left(\mathbf{x}^t, u, y^t\right) \to (\mathbf{x}, u, y) \text{ and } \left(\mathbf{x}^t, u, y^t\right) \in \mathcal{R}^{\mathfrak{X}} \text{ for all } t\right\}.$$

Call "$\mathbf{x}$ is consistent with (Y, y)" if "for every player i, ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i$". The following proposition states that $PB^{\mathfrak{X}}(Y, u)$ and $SB^{\mathfrak{X}}(Y, u)$ are related to the closure and vertical closure of $\mathcal{R}^{\mathfrak{X}}$ (under the "consistency" requirement), respectively.

Proposition 1. *For any* $Y \subseteq \mathbb{Y}$ *and* $\mathfrak{X} \subseteq [int\,(\mathbb{Y})]^n$, *(a)* $y \in PB^{\mathfrak{X}}(Y, u) \Leftrightarrow \exists\,(\mathbf{x}, u, y) \in vcl_U(\mathcal{R}^{\mathfrak{X}})$ *s.t.* $\mathbf{x}$ *is consistent with* (Y, y)*; (b)* $y \in SB^{\mathfrak{X}}(Y, u) \Leftrightarrow \exists\,(\mathbf{x}, u, y) \in cl(\mathcal{R}^{\mathfrak{X}})$ *s.t.* $\mathbf{x}$ *is consistent with* (Y, y).

To relate to Selten's (1975) perfectness, Kreps and Wilson (1982, Proposition 6) provided a useful characterization of sequential equilibrium in terms of "payoff perturbations"; they relaxed Selten's criterion by allowing some (vanishingly) small uncertainty on the part of players' payoffs. BZ94 offered an alternative characterization of sequential equilibrium in terms of "perturbed games". Proposition 1 provides two fundamental characterizations of perfect rationality and sequential rationality under a broader range of behavioral assumptions; for example, if Y is restricted to a singleton set and $\mathfrak{X}$ is restricted to a "common" trembling-beliefs sequence in $\{\mathbf{x} \in [int(\mathbb{Y})]^n : {}^i x = {}^j x \text{ for all } i \neq j\}$, Proposition 1(b) yields an analogy of Kreps and Wilson's (1982) characterization of sequential equilibrium.

3.4 Generic equivalence theorem

In this section, we establish a general "generic" equivalence between perfect rationality and sequential rationality. Our proof is based on the fundamental structure of semi-algebraic set: each semi-algebraic set has only a finite number of open connected components, and has a well-defined dimension. The following property of semi-algebraic sets is crucial in our paper.

Generic Local Triviality [Hardt (1980); Bochnak, Coste and Roy (1987, Corollary 9.3.2)]. *Let B and U be semi-algebraic sets and let $f : B \to U$ be a continuous, semi-algebraic function. There is a (relatively) closed, lower-dimensional semi-algebraic ("critical") subset $U^0 \subset U$ such that for each of the finite number of (relatively) open connected components U^k of $U \backslash U^0$ there is a semi-algebraic ("fiber") set C^k and a semi-algebraic homeomorphism $\varphi^k : U^k \times C^k \to f^{-1}(U^k)$ such that $f(\varphi^k(u, c)) = u$ for all $u \in U^k$ and $c \in C^k$.*

Generic Local Triviality implies that, for any semi-algebraic set, the closure and vertical closure are almost the same: that is, the difference between the closure and vertical closure of a semi-algebraic set is lower-dimensional. Formally,

Proposition 2. *Let $X \subseteq \mathbb{R}^{n+m}$ be a semi-algebraic set. (a) $cl\,(X)$ and $vcl_{\mathbb{R}^n}\,(X)$ are semi-algebraic. (b) There exists a lower-dimensional semi-algebraic subset $X^0_{\mathbb{R}^n} \subset \mathbb{R}^n$ such that $cl\,(X) \backslash vcl_{\mathbb{R}^n}\,(X) \subseteq X^0_{\mathbb{R}^n} \times \mathbb{R}^m$.*

By Propositions 1 and 2, we obtain the central result of this paper: a general "generic" equivalence theorem between perfect rationality and sequential rationality.

Theorem 1. *Consider an extensive form Γ. For any semi-algebraic set $\mathfrak{X} \subseteq [int(\mathbb{Y})]^n$, there is a (relatively) closed, lower-dimensional semi-algebraic subset $U^0 \subset U$ such that, for all $u \in U \backslash U^0$, $PB^{\mathfrak{X}}(Y, u) = SB^{\mathfrak{X}}(Y, u)\ \forall Y \subseteq \mathbb{Y}$. Furthermore, if $\mathfrak{X} = [int(\mathbb{Y})]^n$, there is a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subset U_i = \mathbb{R}^Z$ such that, for all $u \in \Pi_{i\in N}\,(U_i \backslash V^0)$, $PB^{\mathfrak{X}}(Y, u) = SB^{\mathfrak{X}}(Y, u)\ \forall Y \subseteq \mathbb{Y}$.*

Theorem 1 establishes a fundamental and elementary "generic" equivalence between perfect rationality and sequential rationality. More specifically, the equivalence holds for all payoff vectors outside a lower-dimensional subset $U^0 \subset U = \mathbb{R}^{N\times Z}$; under a belief structure in product form: $\mathfrak{X} = [int(\mathbb{Y})]^n$, the equivalence holds for all assigned payoffs for each player outside a lower-dimensional subset $V^0 \subset U_i = \mathbb{R}^Z$, rather than a lower-dimensional subset $U^0 \subset U$. In this paper, we consider two kinds of belief structures used in extensive games:

1. $\mathfrak{X} = [int(\mathbb{Y})]^n$. Under this belief structure, different players i and j are allowed to have distinct trembling sequences ${}^i x^t \rightsquigarrow {}^i x$ and ${}^j x^t \rightsquigarrow$

${}^{j}x$. For all $Y \subseteq \mathbb{Y}$, we denote $PB(Y,u) \equiv PB^{\mathfrak{X}}(Y,u)$ and $SB(Y,u) \equiv SB^{\mathfrak{X}}(Y,u)$.

2. $\mathfrak{X} \equiv \{\mathbf{x} \in [int(\mathbb{Y})]^n : {}^{i}x = {}^{j}x \text{ for all } i \neq j\}$. Under this belief structure, different players i and j are required to have a common trembling sequence $x^t \rightsquigarrow x$. For all $Y \subseteq \mathbb{Y}$, we denote $PB^*(Y,u) \equiv PB^{\mathfrak{X}}(Y,u)$ and $SB^*(Y,u) \equiv SB^{\mathfrak{X}}(Y,u)$; in particular, we write $PB^*(y,u)$ and $SB^*(y,u)$ respectively for $PB^*(\{y\},u)$ and $SB^*(\{y\},u)$, for simplicity.

Definition 2. In game $\Gamma(u)$, we define

(a) [**Perfect Equilibrium**] A strategy profile y is a *perfect equilibrium* if $y \in PB^*(y,u)$, i.e., there exists a (common) sequence $y^t \rightsquigarrow y$ such that for all $i \in N$ and $h \in H_i$, $y_h \in \arg\max_{y'_h \in \mathbb{Y}_h} v_i((y'_h, y^t_{-h}), u_i)$ $\forall t$.

(b) [**Sequential Equilibrium**] A strategy profile y is a *sequential equilibrium* if $y \in SB^*(y,u)$, i.e., there exists a (common) assessment (y,μ) such that for all $i \in N$ and $h \in H_i$, $y_i \in \arg\max_{y'_i \in \mathbb{Y}_i} v_i(y'_i, (y,\mu), u_i|h)$.

(c) [**Perfect Rationalizability**] A strategy profile y is *perfectly rationalizable* if it is supported by a perfectly rationalizable set $Y \subseteq \mathbb{Y}$, i.e., $y \in Y \subseteq PB(Y,u)$.

(d) [**Sequential Rationalizability**] A strategy profile y is *sequentially rationalizable* if it is supported by a sequentially rationalizable set $Y \subseteq \mathbb{Y}$, i.e., $y \in Y \subseteq SB(Y,u)$.

Definition 2(a) is Selten's (1975) notion of perfect equilibrium. Definition 2(b) is Kreps and Wilson's (1982) notion of sequential equilibrium. Definition 2(c) is a variant of Greenberg et al.'s (2009) notion of null MACA, which, if allows for correlations, is equivalent to Herings and Vannetelbosch's (1999) definition of "weakly perfect rationalizability" in simultaneous-move games. Definition 2(d) is a variant of Dekel et al.'s (1999, 2002) sequential rationalizability (with point beliefs).

Remark 1. For simplicity, we consider only point beliefs over opponents' strategies in the notion of rationalizability (see Bernheim (1984)). Apparently, since every singleton of a perfectly/sequentially rationalizable strategy profile is a "weak" version of perfect/sequential equilibrium (by allowing distinct trembling sequences for different players),[7] every perfect/sequential equilibrium must be perfectly/sequentially rationalizable.

For game $\Gamma(u)$, we need to introduce the following notation:

[7] That is, different players may not necessarily have the same beliefs on how players "tremble". Fudenberg and Tirole (1991, p.341) pointed out, "Why should all players have the same theory to explain deviations that, after all, are either probability-0 events or very unlikely, depending on one's methodological point of view? The standard defense is that this requirement is in the spirit of equilibrium analysis, since equilibrium supposes that all players have common beliefs about the others' strategies. Although this restriction is usually imposed, we are not sure that we find it convincing."

$PE\,(u)$: set of perfect equilibria

$SE\,(u)$: set of sequential equilibria

$WPE(u)$: set of "weakly" perfect equilibria

$WSE(u)$: set of "weakly" sequential equilibria

$PR\,(u)$: set of perfectly rationalizable strategy profiles

$SR\,(u)$: set of sequentially rationalizable strategy profiles

According to Theorem 1, $PB^{\mathfrak{X}}(Y,u)$ and $SB^{\mathfrak{X}}(Y,u)$ generically coincide for any arbitrary $Y \subseteq \mathbb{Y}$. Note that the perfect and sequential notions of equilibrium and rationalizability are based on the basic assumptions of perfect rationality and sequential rationality, we obtain "generic" equivalence results for equilibrium and rationalizability, as immediate corollaries of Theorem 1.

Corollary 1. *Consider an extensive form* Γ*. There is a (relatively) closed, lower-dimensional semi-algebraic subset* $V^0 \subset U_i = \mathbb{R}^Z$ *such that, for all* $u \in \Pi_{i\in N}\,(U_i \backslash V^0)$*,* $Y\,(u)$ *is a sequentially rationalizable set in* $\Gamma\,(u)$ *iff* $Y\,(u)$ *is a perfectly rationalizable set in* $\Gamma\,(u)$*. Moreover, a sequentially-rationalizable-set correspondence* $Y\,(\cdot)$ *(i.e.,* $Y : U \rightrightarrows \mathbb{Y}$ *such that* $Y(u) \subseteq SB\,(Y\,(u)\,,u)\ \forall u \in U$*) is perfectly rationalizable for all* $u \in U$ *at which* $SB\,(Y\,(\cdot)\,,\cdot)$ *is lower hemi-continuous and* $PB\,(Y\,(\cdot)\,,\cdot)$ *is upper hemi-continuous.*

In particular, $PR\,(u) = SR\,(u)$ *and* $WPE\,(u) = WSE\,(u)$ *for all* $u \in \Pi_{i\in N}\,(U_i \backslash V^0)$*. Moreover,* $PR\,(u) = SR\,(u)$ *for all* $u \in U$ *at which correspondence* $SR\,(\cdot)$ *is lower hemi-continuous and* $PR\,(\cdot)$ *is upper hemi-*

continuous; $WPE(u) = WSE(u)$ *for all* $u \in U$ *at which correspondence* $WSE(\cdot)$ *is lower hemi-continuous and* $WPE(\cdot)$ *is upper hemi-continuous.*

Remark 2. The class of symmetric games or zero-sum two-person games has the same dimension of U_i, because payoff vectors are fully determined by a particular player i's payoffs. Consequently, Corollary 1 implies that, in the class of symmetric games or zero-sum two-person games, the equivalence holds for all "generically" assigned payoffs $u_i \in U_i \backslash V^0$ (for the particular player i).

A critial assumption of the beliefs structure in Corollary 1 is: $\mathfrak{X} = [int(\mathbb{Y})]^n$; accordingly, we allow two players i and j to have distinct trembling-beliefs sequences ${}^i x^t \rightsquigarrow {}^i x$ and ${}^j x^t \rightsquigarrow {}^j x$, respectively. If we impose a stronger assumption of the beliefs structure, i.e., $\mathfrak{X}$ is restricted to a "common" trembling-beliefs sequence for all players in $\{\mathbf{x} \in [int(\mathbb{Y})]^n : {}^i x = {}^j x$ for all $i \neq j\}$, Theorem 1 yields BZ94's (Theorem 4) "generic" equivalence result for perfect and sequential equilibria.

Corollary 2. *Consider an extensive form* Γ*. There is a (relatively) closed, lower-dimensional semi-algebraic subset* $U^0 \subset U$ *such that* $PE(u) = SE(u)$ *for all* $u \in U \backslash U^0$*. Moreover,* $PE(u) = SE(u)$ *for all* $u \in U$ *at which correspondence* $SE(\cdot)$ *is lower hemi-continuous and* $PE(\cdot)$ *is upper hemi-continuous.*

Normal forms are a special case of extensive forms with simultaneous moves. Corollary 3 asserts that, in any normal form, iterated elimination of weakly dominated strategies (IEWDS) is generically an order-independent procedure which is equivalent to iterated elimination of strictly dominated strategies (IESDS).

Corollary 3. *Consider a normal form Γ. There exists a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subset U_i = \mathbb{R}^Z$ such that for all $u \in \Pi_{i\in N}(U_i \backslash V^0)$, every IEWDS procedure is an IESDS procedure; hence IEWDS is (generically) an order-independent procedure.*

In the context of extensive games, Greenberg et al. (2009) presented a unified solution concept of "mutually acceptable course of action (MACA)" for situations where "perfectly" rational individuals with different beliefs agree to a shared course of action. We end this section by establishing a "generic" equivalence between perfect-MACA and sequential-MACA, as an immediate corollary of Theorem 1. In doing so, we extend the simple version of point beliefs to a more complicated version of (uncorrelated) beliefs in extensive games. Following Dekel et al. (2002), we say a strategy y_i of player i is in the "extensive-form convex hull" of $Y_i \subseteq \mathbb{Y}_i$, denoted by $co^e(Y_i)$, if there is a finite set $\{y_i^1, ..., y_i^M\} \subseteq Y_i$, with trembling sequences $\left(y_i^{m,t}\right)_{m=1}^M \rightsquigarrow (y_i^m)_{m=1}^M$ and a sequence $(\alpha^{m,t})_{m=1}^M \to \alpha$ of distributions on $\{1, ..., M\}$, such that y_i^t generated by the convex combination $\Sigma_{m=1}^M \alpha^{m,t} y_i^{m,t}$ (in terms of "realization outcomes") converges to y_i. Let $co^e(Y) \equiv \Pi_{i\in N} co^e(Y_i)$. In the spirit of Greenberg et al. (2009), we introduce the "perfect" and "sequential" notions of MACA.[8]

[8]The formulation of an "extensive-form convex hull" purports to deal with the no-

Definition 3. In game $\Gamma(u)$, a course of action $\sigma(u) = (\sigma_h(u))_{h\in H}$, with $\sigma_h(u) \in \mathbb{Y}_h \cup \{\emptyset\}$, is a *perfect-MACA* (or *sequential-MACA*) if there is $Y = \Pi_{i\in N} Y_i \subseteq \mathbb{Y}$ supporting $\sigma(u)$, i.e.,

(i) for all $h \in H$, if $\sigma_h(u) \neq \emptyset$, then $y_h = \sigma_h(u)$ for all $y \in Y$;

(ii) $Y \subseteq PB(co^e(Y), u)$ (or $Y \subseteq SB(co^e(Y), u)$).

The following corollary asserts that the notions of perfect-MACA and sequential-MACA are generically equivalent.

Corollary 4. *Consider an extensive form Γ. There is a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subset U_i = \mathbb{R}^Z$ such that for all $u \in \Pi_{i\in N}(U_i \backslash V^0)$, the set of perfect-MACAs coincides with the set of sequential-MACAs and, moreover, a sequential-MACA $\sigma(u)$ is supported by Y iff $\sigma(u)$ is a perfect-MACA supported by Y.*

Remark 3. Greenberg et al. (2009) demonstrated that, by varying the degree of completeness of the underlying course of action, MACA can be related to many commonly used game-theoretic solutions, such as equilibrium, self-confirming equilibrium, and rationalizability. More specifically,

(i) If $\sigma(u)$ is a "complete" MACA (which satisfies $\sigma_h(u) \neq \varnothing$ $\forall h \in H$),

Corollary 4 yields the "generic" equivalence result between "weakly"

torious problem of imperfection under subjective uncertainty over (behavior) strategies; cf. Dekel et al. (2002) and Greenberg et al. (2009) for more discussions. For the purpose of this paper, we here adopt Dekel et al.'s (2002) definition of "extensive-form convex hull" to define the notions of perfect-MACA and sequential-MACA (within Greenberg et al.'s (2009) framework of MACA).

sequential equilibria and "weakly" perfect equilibria in Corollary 1 (cf. Greenberg et al.'s (2009) Claim 3.1.1),

(ii) If $\sigma(u)$ is a "null" MACA (which satisfies $\sigma_h(u) = \varnothing \; \forall h \in H$), Corollary 4 yields a "generic" equivalence result between Dekel et al.'s (1999, 2002) sequential rationalizability and perfect rationalizability with trembling beliefs in an "extensive-form convex hull", rather than the simple version of point beliefs used in Corollary 1 (cf. Greenberg et al.'s (2009) Claim 3.3.1), and

(iii) If $\sigma(u)$ is a "path" MACA (which satisfies $\sigma_h(u) \neq \varnothing$ whenever h is reached with positive probability under $\sigma(u)$), Corollary 4 yields a "generic" equivalence result between Dekel et al.'s (1999, 2002) sequentially rationalizable self-confirming equilibrium (SRSCE) and Greenberg et al.'s (2009) path MACA.

We would also like to point out that Dekel et al. (1999, Footnote 4) expected this kind of "generic" equivalence, but they offered no formal analysis of this issue. We thereby offer such a formal analysis from this perspective.

3.5 Concluding remarks

Blume and Zame (1994) strengthened Kreps and Wilson's (1982) result and showed that, for almost all assignments of payoffs to the terminal nodes, the sets of sequential and perfect equilibria are identical. In this paper, we have extended BZ94's result to more general settings of strategic interactions. We have formulated and proved a general "generic" equivalence theorem between perfect rationality and sequential rationality in extensive games. More specifically, we have presented a general framework to accommodate many structures of beliefs discussed in the literature and shown that the difference between perfectly and sequentially rational correspondences occurs only in a lower-dimensional payoffs set. We have also demonstrated that we can obtain a variety of generic equivalence results for various kinds of solution concepts such as equilibrium, rationalizability, iterated dominance and MACA, as corollaries of our general "generic" equivalence theorem (Theorem 1). The study of this paper helps deepen our understanding of the relationship between perfectly and sequentially rational strategic behavior with diversiform beliefs.

In this paper, we have followed Dekel et al. (1999, 2002) and Greenberg et al. (2009) to adopt a simple and convenient way of defining perfect/sequential equilibrium and rationalizability by using "trembling conjectures" and present a unified framework for the study of the "generic" relationship between perfectly and sequentially rational strategic behavior. Alternatively, one may follow BZ94's approach to analyze perfectly and sequentially rational strategic behavior by using "perturbed games". However, there is no formal formulation of perfect/sequential rationalizability for extensive games, in terms of "perturbed games", in the literature, although Bernheim (1984, pp.1021-1022) outlined such a notion of perfect rationalizability in normal form games. Herings and Vannetelbosch (1999, Example G7) showed that, unlike the notion of perfect equilibrium, there are different definitions of perfect rationalizability by using "trembling conjectures" or "perturbed games" (cf. also Börgers (1994)). In particular, the alternative definition of perfect/sequential rationalizability by using "perturbed games" may suffer Fudenberg and Tirole's (1991) criticism: it implicitly requires that all players have the same theory to form common "trembling conjectures", as illustrated by the following example.

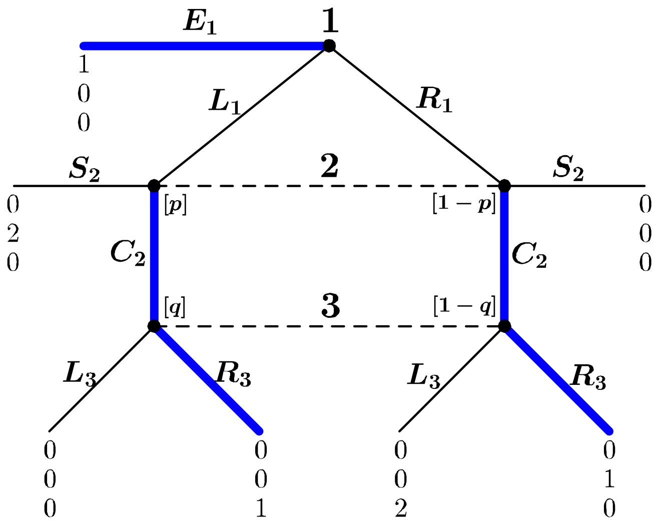

Figure. 2. A three-person game.

It is easy to see that the strategy profile $y = (E_1, C_2, R_3)$, marked by bold lines in Figure 2, is a "weakly" sequential/perfect equilibrium and, by Corollary 1, y is sequentially/perfectly rationalizable for almost all assignments of payoffs to the terminal nodes. But, the profile y is not sequentially/perfectly rationalizable in terms of "perturbed games". To see this, note that (i) in any perturbed game, because E_1 strictly dominates

L_1 and R_1, there is a unique rationalizable strategy for player 1 – i.e., playing L_1 and R_1 with the minimum probabilities specified in the perturbed game, and (ii) player 2 and 3 must hold common "trembling conjectures" in commonly known "perturbed games". But, C_2 is sequentially rational only if $p \leq 1/3$; R_3 is sequentially rational only if $q \geq 2/3$. Subsequently, the profile (C_2, R_3) cannot be sequentially/perfectly rationalizable in terms of "perturbed games". This argument is valid for a neighborhood of the payoffs to the terminal nodes.[9] On the one hand, this kind of implicit requirement of common "trembling conjectures" appears to be less convincing and arguable especially in a non-equilibrium setting. On the other hand, as the example shows, sequential and perfect rationalizability are not equivalent for generic games under such requirement. Therefore, we do not use this alternative way of formulating perfectly/sequentially rational strategic behavior in this paper. Our approach avoids the criticism and successfully extends the generic equivalence to ratinalizability and many other solution concepts.

As we have emphasized, unlike BZ94's approach, our analysis of this paper does not rely directly on semi-algebraic properties of specific solutions concepts (e.g., the semi-algebraic structure of perfect and sequential equilibrium correspondences in BZ94). Instead, our approach of this paper is based upon the primitive set $\mathcal{R}^{\mathfrak{X}}$ of "perfectly-rational" states, which is naturally semi-algebraic with diversiform structures of beliefs, so that

[9] This example also shows that the notions of perfect Bayesian equilibrium and sequential equilibrium are generically different, because (E_1, C_2, R_3) is a perfect Bayesian equilibrium.

it is feasible and applicable to various solution concepts discussed in the literature. We believe that our general "generic" equivalence theorem provides a useful and complementary way for the study of the relationship between perfectly and sequentially rational strategic behavior in complex environments.[10]

Finally, we would like to mention that, in contrast to BZ94's approach to complete-information games through perturbations on payoffs, Weinstein and Yildiz (2007) took a different approach to generic properties of rational strategic behavior and showed, in the framework of incomplete-information games with richness assumption, a generic uniqueness result for the structure of rationalizability by perturbing (in the product topology of the universal type space) the beliefs of the type. It is intriguing to extend the analysis of this paper to a general situation by allowing perturbations both on payoffs and the beliefs of the type. We leave it for future research.

Appendix: Proofs

The following lemma establishes a relationship between perfect rationality and sequential rationality: that is, sequential rationality can be characterized by perfect rationality against payoff perturbations.

Lemma 1. *For any* $Y \subseteq \mathbb{Y}$ *and* $\mathfrak{X} \subseteq [int(\mathbb{Y})]^n$, $y \in SB^{\mathfrak{X}}(Y, u)$ *iff there exist* $u^t \to u$ *and* $\mathbf{x}^t \xrightarrow{\mathfrak{X}} \mathbf{x}$ *such that* $\mathbf{x}$ *is consistent with* (Y, y) *and for*

[10]We note that our approach of this paper is also applicable to the alternative definitions of perfectly and sequentially rational strategic behavior by using "perturbed games". In doing so, we need to consider a more elaborated set of perfectly-rational states with "game perturbations" and then obtain an analogous "generic" equivalence.

each player i, $y_i \in B_i({}^i x^t, u_i^t)$ $\forall t$.

Proof of Lemma 1. "$\Leftarrow$": Let $\mathbf{x}^t \overset{\mathfrak{X}}{\rightarrow} \mathbf{x}$ and $\mathbf{x}$ be consistent with (Y, y). Without loss of generality, assume $(\mathbf{x}^t, \mu^t) \overset{\mathfrak{X}}{\rightarrow} (\mathbf{x}, \mu)$, where μ^t is derived from $\mathbf{x}^t$ using Bayes' rule. Consider player i, suppose that there exist $u_i^t \rightarrow u_i$ such that $y_i \in B_i({}^i x^t, u_i^t)$ for all t. Then, for all $h \in H_i$ and all t, $v_i\left((y_h,\ {}^i x_{-h}^t), u_i^t|h\right) \geq v_i\left((y_h',\ {}^i x_{-h}^t), u_i^t|h\right)$ $\forall y_h' \in \mathbb{Y}_h$.[11] Since $v_i((y_h, \cdot), \cdot|h)$ is continuous, $v_i\left((y_h,\ {}^i x_{-h}), u_i|h\right) \geq v_i\left((y_h',\ {}^i x_{-h}), u_i|h\right)$. By the one deviation property (see, e.g., Osborne and Rubinstein (1994, p.227)), for all $h \in H_i$, $v_i\left((y_i,\ {}^i x_{-i}), u_i|h\right) \geq v_i\left((y_i',\ {}^i x_{-i}), u_i|h\right)$ $\forall y_i' \in \mathbb{Y}_i$. That is, $y_i \in SB_i({}^i x, u_i)$ for all i and thus $y \in SB^{\mathfrak{X}}(Y, u)$.

"$\Rightarrow$": Let $y \in SB^{\mathfrak{X}}(Y, u)$. Then, for each player i there is $(\mathbf{x}^t, \mu^t) \overset{\mathfrak{X}}{\rightarrow} (\mathbf{x}, \mu)$ such that for all i, ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i$ is sequentially optimal to assessment $({}^i x, \mu_i)$. Clearly, ${}^i x^t \rightarrow {}^i x$. We proceed to construct a payoff sequence $u_i^t \rightarrow u_i$ such that $y_i \in B_i({}^i x^t, u_i^t)$ for all t.

Since Γ is finite and perfect recall, we can define a (finite) partition $\{H_i^l\}_{l=1}^L$ of a set H_i as follows: $H_i^l \equiv$

$$\left\{h \in H_i \backslash \cup_{\ell<l} H_i^\ell : \text{no } h' \in \left[H_i \backslash \cup_{\ell<l} H_i^\ell\right] \backslash h \text{ is reached by } h\right\},$$

for all $l \geq 1$. Let $u_i^{t,0} \equiv u_i$. For all t and $l = 1, \cdots, L$, $u_i^{t,l}$ is define recursively as follows:

$$u_i^{t,l}(z) \equiv \begin{cases} u_i^{t,l-1}(z) + \delta_{a_h^*}^t, & \text{if } z \text{ is not precluded by } a_h^* \in \text{support}(y_h) \text{ from } h \\ u_i^{t,l-1}(z), & \text{otherwise} \end{cases},$$

[11] For any $y_i' \in \mathbb{Y}_i$, we define $v_i\left((y_i',\ {}^i x_{-i}^t), u_i^t|h\right) \equiv v_i\left(y_i', ({}^i x^t, \mu_i^t), u_i^t|h\right)$ and $v_i\left((y_i',\ {}^i x_{-i}), u_i|h\right) \equiv v_i\left(y_i', ({}^i x, \mu_i), u_i|h\right)$.

where $h \in H_i^l$, support$(y_h) = \{a_h \in A_h : y_h(a_h) > 0\}$ and

$$\delta^t_{a_h^*} = \max_{a_h \in A_h} v_i\left(\left(a_h,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right) - v_i\left(\left(a_h^*,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right).$$

Therefore, for $l = 1, \cdots, L$, $y_h \in B_h\left({}^i x^t_{-h}, u_i^{t,l}\right)$ $\forall h \in H_i^l$.[12] Since Γ is perfect recall, for all $a_h, a'_h \in A_h$, $v_i\left(\left(a_h,\ {}^i x^t_{-h}\right), u_i^{t,l+1}|h\right) - v_i\left(\left(a'_h,\ {}^i x^t_{-h}\right), u_i^{t,l+1}|h\right) = v_i\left(\left(a_h,\ {}^i x^t_{-h}\right), u_i^{t,l}|h\right) - v_i\left(\left(a'_h,\ {}^i x^t_{-h}\right), u_i^{t,l}|h\right)$. By induction on l, we have $y_h \in B_h\left({}^i x^t_{-h}, u_i^{t,l}\right)$ $\forall h \in \cup_{\ell=1}^l H_i^\ell$. Hence, $y_h \in B_h\left({}^i x^t_{-h}, u_i^{t,L}\right)$ $\forall h \in H_i$, i.e., $y_i \in B_i({}^i x^t, u_i^{t,L})$.

It remains to show $u_i^{t,L} \to u_i^L = u_i$. We prove this by induction on l. Clearly, $u_i^{t,0} \to u_i$ trivially holds. Suppose $u_i^{t,\ell} \to u_i^\ell = u_i$ for $\ell \le l-1$. By construction of $u_i^{t,l}$, it suffices to show $\delta^t_{a_h^*} \to 0$ $\forall h \in H_i^l$. Let $\hat{a}_h \in \arg\max_{a_h \in A_h} v_i\left(\left(a_h,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right)$. Because of the continuity of v_i, for any $\varepsilon > 0$ there is a sufficiently large T such that, for all $t > T$,

$$v_i\left(\left(\hat{a}_h,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right) - v_i\left(\left(\hat{a}_h,\ {}^i x_{-h}\right), u_i^{l-1}|h\right) < \varepsilon;$$
$$v_i\left(\left(a_h^*,\ {}^i x_{-h}\right), u_i^{l-1}|h\right) - v_i\left(\left(a_h^*,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right) < \varepsilon.$$

Since $y_h \in B_h\left({}^i x_{-h}, u_i\right)$ and, by induction assumption, $u_i = u_i^{l-1}$,

$$v_i\left(\left(\hat{a}_h,\ {}^i x_{-h}\right), u_i^{l-1}|h\right) - v_i\left(\left(a_h^*,\ {}^i x_{-h}\right), u_i^{l-1}|h\right) \le 0.$$

Therefore, $v_i\left(\left(\hat{a}_h,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right) - v_i\left(\left(a_h^*,\ {}^i x^t_{-h}\right), u_i^{t,l-1}|h\right) < 2\varepsilon$, i.e., $\delta^t_{a_h^*} \to 0$. $\square$

Proof of Proposition 1. Suppose $y \in PB^{\mathfrak{x}}(Y, u)$. Then, there is $\mathbf{x}^t \xrightarrow{\mathfrak{x}} \mathbf{x}$ such that $(\mathbf{x}^t, u, y) \in \mathcal{R}^{\mathfrak{x}}$ for all t; ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i$ for all i.

[12] For $y_{-h} \in \mathbb{Y}_{-h}$ and $u_i \in U_i$, define $B_h(y_{-h}, u_i) \equiv \{y_h \in \mathbb{Y}_h : v_i((y_h, y_{-h}), u_i) \ge v_i((a_h, y_{-h}), u_i)\ \forall a_h \in A(h)\}$.

Clearly, $(\mathbf{x}^t, u, y) \to (\mathbf{x}, u, y)$. Thus $(\mathbf{x}, u, y) \in vcl_{U_i}\left(\mathcal{R}^{\mathfrak{X}}\right)$. Conversely, suppose $(\mathbf{x}, u, y) \in vcl_{U_i}\left(\mathcal{R}^{\mathfrak{X}}\right)$, ${}^i x_{-i} \in Y_{-i}$ and ${}^i x_i = y_i$ for all i. Then there exists a sequence $(\mathbf{x}^t, u, y^t) \in \mathcal{R}^{\mathfrak{X}}$ converging to $(\mathbf{x}, u, y)$. Since Γ is finite and $y^t \to y$, there is a sufficiently large T such that, for all $t \geq T$, $a_h^* \in$support(y_h) implies $a_h^* \in$support(y_h^t) and $a_h^* \in B_h\left({}^i x^t, u_i\right) \forall h \in H_i$. Therefore, $y_i \in B_i\left({}^i x^t, u_i\right)$ for all i and $t \geq T$. That is, $y \in PB^{\mathfrak{X}}(Y, u)$. Since a similar argument holds true with payoff perturbations, by using Lemma 1, Proposition 1(b) is valid. □

Proof of Proposition 2. (a) $vcl_{\mathbb{R}^n}(X)$ can be rewritten as

$$\left\{(a, b) \in \mathbb{R}^n \times \mathbb{R}^m : \forall \varepsilon > 0, \exists\, (a, b') \in X \text{ s.t. } \|b - b'\| < \varepsilon\right\}.$$

Since X is semi-algebraic, it follows from Tarski-Seidenberg Theorem that $vcl_{\mathbb{R}^n}(X)$ is also semi-algebraic. Similarly, $cl\,(X)$ is semi-algebraic.

(b) Denote $\pi : X \to X_{\mathbb{R}^n}$ as the projection function of X onto the first n coordinates, where $X_{\mathbb{R}^n} = \{a \in \mathbb{R}^n : \exists b \in \mathbb{R}^m \text{ s.t. } (a, b) \in X\}$. X is endowed with the relative topology of the usual product topology $\mathbb{R}^n \times \mathbb{R}^m$. Then π is continuous and its graph is $\{((a, b), a) : (a, b) \in X\}$, which is semi-algebraic. Thus we can apply Generic Local Triviality to π. Denote A^0 as the critical set with $\dim A^0 < n$, $X_{\mathbb{R}^n} \backslash A^0 = \cup_k A^k$ as the decomposition into finitely many connected open components, C^k as the fiber for A^k. Denote φ^k as the semi-algebraic homeomorphism between $A^k \times C^k$ and $\pi^{-1}\left(A^k\right)$. Moreover, all φ^k satisfy the following condition:

$$\forall a \in A^k, \forall c \in C^k, \pi\left(\varphi^k(a, c)\right) = a. \tag{\#}$$

Let $X^0_{\mathbb{R}^n} = \cup_k \left(cl\left(A^k\right)\backslash A^k\right) \cup cl\left(A^0\right)$. Suppose $(a,b) \in cl\,(X)$ and $a \in \mathbb{R}^n \backslash X^0_{\mathbb{R}^n}$. By the decomposition of $X_{\mathbb{R}^n}$, $X = \cup_k \pi^{-1}\left(A^k\right) \cup \pi^{-1}\left(A^0\right)$, then there is a sequence (a^t, b^t) in $\pi^{-1}\left(A^k\right)$ for some component A^k and $(a^t, b^t) \to (a,b)$. Denote $\left(\varphi^k\right)^{-1}(a^t, b^t) = (\hat{a}^t, c^t) \in A^k \times C^k$, then $\pi\left(\varphi^k\left(\hat{a}^t, c^t\right)\right) = \pi\left(a^t, b^t\right) = a^t$. By (#), $a^t = \hat{a}^t$. Since $a \in \mathbb{R}^n \backslash X^0_{\mathbb{R}^n}$, $a \notin cl\left(A^k\right)\backslash A^k$. Since $a^t \to a$, $a \in cl\left(A^k\right)$. Thus $a \in A^k$. Then (a, c^t) is a sequence in $A^k \times C^k$. Denote $\varphi^k(a, c^t) = \left(a, \hat{b}^t\right)$ for all t. Since $\|(a,c^t) - (a^t, c^t)\| \to 0$ and φ^k is continuous, $\left\|\left(a, \hat{b}^t\right) - (a^t, b^t)\right\| = \left\|\varphi^k(a, c^t) - \varphi^k(a^t, c^t)\right\| \to 0$. Therefore, $\left(a, \hat{b}^t\right) \to (a,b)$, i.e., $(a,b) \in vcl_{\mathbb{R}^n}\left(\pi^{-1}\left(A^k\right)\right) \subseteq vcl_{\mathbb{R}^n}(X)$. Since $\dim\left[cl\left(A^0\right)\right] = \dim A^0 < n$ and $\dim\left[cl\left(A^k\right)\backslash A^k\right] < n$, $\dim X^0_{\mathbb{R}^n} < n$.

□

Proof of Theorem 1. Since $\mathfrak{X}$ is semi-algebraic, $\mathcal{R}^{\mathfrak{X}}$ is a semi-algebraic set by Tarski-Seidenberg Theorem. Applying Proposition 2 to $\mathcal{R}^{\mathfrak{X}}$, there exists a closed semi-algebraic subset $U^0 \subset U$ with $\dim U^0 < \dim U$ such that $cl(\mathcal{R}^{\mathfrak{X}}) \backslash vcl_U(\mathcal{R}^{\mathfrak{X}}) \subseteq \mathbb{Y}^n \times U^0 \times \mathbb{Y}$. Therefore, for all $u \in U \backslash U^0$,

$$(\mathbf{x}, u, y) \in cl(\mathcal{R}^{\mathfrak{X}}) \Leftrightarrow (\mathbf{x}, u, y) \in vcl_U(\mathcal{R}^{\mathfrak{X}}). \qquad (*)$$

Consider an arbitrary set $Y \subseteq \mathbb{Y}$. For all $u \in U \backslash U^0$, we have

$$\begin{aligned} y \in SB^{\mathfrak{X}}(Y,u) &\overset{\text{Proposition 1}}{\Longleftrightarrow} \exists\,(\mathbf{x}, u, y) \in cl(\mathcal{R}^{\mathfrak{X}}) \text{ s.t. } {}^i x_{-i} \in Y_{-i} \text{ and } {}^i x_i = y_i\ \forall i \\ &\overset{(*)}{\Longleftrightarrow} \exists\,(\mathbf{x}, u, y) \in vcl_U(\mathcal{R}^{\mathfrak{X}}) \text{ s.t. } {}^i x_{-i} \in Y_{-i} \text{ and } {}^i x_i = y_i\ \forall i \\ &\overset{\text{Proposition 1}}{\Longleftrightarrow} y \in PB^{\mathfrak{X}}(Y,u). \end{aligned}$$

Now let $\mathfrak{X} = [int\,(\mathbb{Y})]^n$. Then, $\mathcal{R}^{\mathfrak{X}} = \Pi_{i \in N} \mathcal{R}^{\mathfrak{X}}_i$ where $\mathcal{R}^{\mathfrak{X}}_i \equiv \{(\,{}^i x, u_i, y_i) \in$

$int\,(\mathbb{Y}) \times U_i \times \mathbb{Y}_i : \ y_i \in B_i\,({}^i x, u_i)\}$ $\forall i \in N$. Applying Proposition 2 to each set $\mathcal{R}_i^{\mathfrak{x}}$, there exists a closed semi-algebraic subset $U_i^0 \subset U_i$ with $\dim U_i^0 < \dim U_i$ such that $cl(\mathcal{R}_i^{\mathfrak{x}}) \backslash vcl_{U_i}(\mathcal{R}_i^{\mathfrak{x}}) \subseteq \mathbb{Y} \times U_i^0 \times \mathbb{Y}_i$. Define $V^0 \equiv \cup_{i \in N} U_i^0$. Therefore, for all $u \in \Pi_{i \in N}\,(U_i \backslash V^0)$, the identity $(*)$ holds. The rest of Theorem 1 follows similarly. □

In order to show Corollaries 1 and 2, we need the following lemma.

Lemma 2. *Let* $Y : U \rightrightarrows \mathbb{Y}$ *and* $Y' : U \rightrightarrows \mathbb{Y}$*. Suppose that* $U^0 \equiv \{u \in U : Y(u) \neq Y'(u)\}$ *is a lower dimensional subset of* U*. Then* $Y(u) \subseteq Y'(u)$ *for all* $u \in U$ *at which* $Y(\cdot)$ *is lower hemi-continuous and* $Y'(\cdot)$ *is upper hemi-continuous.*

Proof of Lemma 2. Since U^0 is lower-dimensional, U^0 contains no open set in U. Let $u \in U$. Therefore, we can find a sequence $\{u^t\}_{t=1}^{\infty}$ in $U \backslash U^0$ such that $u^t \to u$ and $Y(u^t) = Y'(u^t)$ for all t. If $y \in Y(u)$, by lower hemi-continuity of $Y(\cdot)$, there exists a subsequence $u^{t_k} \to u$ such that $y^k \to y$ and $y^k \in Y(u^{t_k}) = Y'(u^{t_k})$. Since correspondence $Y'(\cdot)$ is upper hemi-continuous, $y \in Y'(u)$. That is, $Y(u) \subseteq Y'(u)$. □

Proof of Corollary 1. Let $Y(u)$ be a sequentially rationalizable set in $\Gamma(u)$, i.e., $Y(u) \subseteq SB(Y(u), u)$. By Theorem 1, there exists a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subseteq U_i = \mathbb{R}^Z$ such that $Y(u) \subseteq SB(Y(u), u) = PB(Y(u), u)$ for all $u \in \Pi_{i \in N}\,(U_i \backslash V^0)$. Therefore, for all $u \in \Pi_{i \in N}\,(U_i \backslash V^0)$, sequentially rationalizable sets are precisely perfectly rationalizable sets in $\Gamma(u)$. Now, suppose that $SB(Y(\cdot), \cdot)$ is lower hemi-continuous and $PB(Y(\cdot), \cdot)$ is upper hemi-continuous at u. By

Theorem 1, $\{u \in U : SB(Y(u), u) \neq PB(Y(u), u)\} \subseteq U \backslash [\Pi_{i \in N}(U_i \backslash V^0)]$ is a lower dimensional subset of U. By Lemma 2, $SB(Y(u), u) \subseteq PB(Y(u), u)$. Therefore, $Y(u) \subseteq SB(Y(u), u) \subseteq PB(Y(u), u)$ is a perfectly rationalizable set.

Since $PR(u) = \cup_{Y \subseteq PB(Y,u)} Y$ and $SR(u) = \cup_{Y \subseteq SB(Y,u)} Y$, it follows that $SR(u) = PR(u)$ for all $u \in \Pi_{i \in N}(U_i \backslash V^0)$. Since $WPE(u) = \cup_{Y \subseteq PB(Y,u);\, |Y|=1} Y$ and $WSE(u) = \cup_{Y \subseteq SB(Y,u);\, |Y|=1} Y$,[13] it follows that $WPE(u) = WSE(u)$ for all $u \in \Pi_{i \in N}(U_i \backslash V^0)$. The rest of Corollary 1 follows immediately from Lemma 2. □

Proof of Corollary 2. Let $\mathfrak{X} \equiv \{\mathbf{x} \in [int(\mathbb{Y})]^n : {}^i x = {}^j x \text{ for all } i \neq j\}$. Then, $y \in SE(u)$ iff $y \in SB^{\mathfrak{X}}(y, u)$. Since $int(\mathbb{Y})$ is semi-algebraic in $\mathbb{Y}$, $\mathfrak{X}$ is semi-algebraic in $\mathbb{Y}^n$. By Theorem 1, we can find a semi-algebraic lower-dimensional subset U^0, such that $y \in PB^{\mathfrak{X}}(y, u) = SB^{\mathfrak{X}}(y, u)$ for all $u \in U \backslash U^0$. Note that, for all $u \in U \backslash U^0$, $PB^{\mathfrak{X}}(y, u) = SB^{\mathfrak{X}}(y, u)\ \forall y \in \mathbb{Y}$. Therefore, $PE(u) = \{y : y \in PB^{\mathfrak{X}}(y, u)\} = \{y : y \in SB^{\mathfrak{X}}(y, u)\} = SE(u)$ for all $u \in U \backslash U^0$.

Now, suppose that $SE(\cdot)$ is lower hemi-continuous and $PE(\cdot)$ is upper hemi-continuous at u. Since $\{u \in U : SE(u) \neq PE(u)\} \subseteq U^0$ is a lower dimensional subset, by Lemma 2, $SE(u) \subseteq PE(u)$. Thus, $SE(u) = PE(u)$. □

Proof of Corollary 3. Consider a normal form $\Gamma = (N, \{A_i\}_{i \in N})$. Let $\left(W^k(u)\right)_{k=0}^K$ be an arbitrary (finite) IEWDS procedure in $\Gamma(u)$. Since Γ is a

[13] $|Y| = 1$ means that the cardinality of Y is 1.

normal form, $a \in \mathcal{A}$ is not strictly dominated in $\mathcal{A}$ iff $a \in SB^{\mathfrak{X}(\mathcal{A})}(\Delta(\mathcal{A}), u)$; $a \in \mathcal{A}$ is not weakly dominated iff $a \in PB^{\mathfrak{X}(\mathcal{A})}(\Delta(\mathcal{A}), u)$, where $\mathcal{A} = \Pi_{i \in N}\mathcal{A}_i \subseteq \Pi_{i \in N}A_i$ and $\mathfrak{X}(\mathcal{A}) = [int(\Delta(\mathcal{A}))]^n$. Note that $\mathfrak{X}(\mathcal{A})$ is semi-algebraic and Theorem 1 holds true for all (finitely many) $\mathcal{A}$. Therefore, we can find a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subset U_i = \mathbb{R}^Z$ such that for all $u \in \Pi_{i \in N}(U_i \backslash V^0)$ and $k = 0, 1, ..., K-1$, $a \in W^k(u) \backslash W^{k+1}(u)$ iff a is strictly dominated in $W^k(u)$. That is, $\left(W^k(u)\right)_{k=0}^K$ is an IESDS procedure in $\Gamma(u)$. Since IESDS is order-independent, IEWDS is generically an order-independent procedure. □

Proof of Corollary 4. By Theorem 1, there exists a (relatively) closed, lower-dimensional semi-algebraic subset $V^0 \subset U_i = \mathbb{R}^Z$ such that, for all $u \in \Pi_{i \in N}(U_i \backslash V^0)$, $SB(Y, u) = PB(Y, u)$ $\forall Y \subseteq \mathbb{Y}$. Suppose that a sequential-MACA $\sigma(u)$ is supported by Y. Then, for all $u \in \Pi_{i \in N}(U_i \backslash V^0)$, $Y \subseteq SB(co^e(Y), u) = PB(co^e(Y), u)$. Therefore, $\sigma(u)$ is also a perfect-MACA supported by Y. Since every perfect-MACA is a sequential-MACA, we conclude the proof. □

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