

Some Properties of p -Groups and Commutative p -Groups

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Summary. This article describes some properties of p -groups and some properties of commutative p -groups.

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The notation and terminology used here have been introduced in the following papers: [7], [4], [8], [6], [10], [9], [11], [5], [1], [3], [2], and [12].

1. p -GROUPS

For simplicity, we use the following convention: G is a group, a, b are elements of G , m, n are natural numbers, and p is a prime natural number.

One can prove the following propositions:

- (1) If for every natural number r holds $n \neq p^r$, then there exists an element s of \mathbb{N} such that s is prime and $s \mid n$ and $s \neq p$.
- (2) For all natural numbers n, m such that $n \mid p^m$ there exists a natural number r such that $n = p^r$ and $r \leq m$.
- (3) If $a^n = \mathbf{1}_G$, then $(a^{-1})^n = \mathbf{1}_G$.
- (4) If $(a^{-1})^n = \mathbf{1}_G$, then $a^n = \mathbf{1}_G$.
- (5) $\text{ord}(a^{-1}) = \text{ord}(a)$.
- (6) $\text{ord}(a^b) = \text{ord}(a)$.
- (7) Let G be a group, N be a subgroup of G , and a, b be elements of G . Suppose N is normal and $b \in N$. Let given n . Then there exists an element g of G such that $g \in N$ and $(a \cdot b)^n = a^n \cdot g$.

- (8) Let G be a group, N be a normal subgroup of G , a be an element of G , and S be an element of G/N . If $S = a \cdot N$, then for every n holds $S^n = a^n \cdot N$.
- (9) Let G be a group, H be a subgroup of G , and a, b be elements of G . If $a \cdot H = b \cdot H$, then there exists an element h of G such that $a = b \cdot h$ and $h \in H$.
- (10) Let G be a finite group and N be a normal subgroup of G . If N is a subgroup of $Z(G)$ and G/N is cyclic, then G is commutative.
- (11) Let G be a finite group and N be a normal subgroup of G . If $N = Z(G)$ and G/N is cyclic, then G is commutative.
- (12) For every finite group G and for every subgroup H of G such that $\overline{H} \neq \overline{G}$ there exists an element a of G such that $a \notin H$.

Let p be a natural number, let G be a group, and let a be an element of G . We say that a is p -power if and only if:

(Def. 1) There exists a natural number r such that $\text{ord}(a) = p^r$.

We now state the proposition

- (13) $\mathbf{1}_G$ is m -power.

Let us consider G, m . One can verify that there exists an element of G which is m -power.

Let us consider p, G and let a be a p -power element of G . Observe that a^{-1} is p -power.

One can prove the following proposition

- (14) If a^b is p -power, then a is p -power.

Let us consider p, G, b and let a be a p -power element of G . One can verify that a^b is p -power.

Let us consider p , let G be a commutative group, and let a, b be p -power elements of G . Observe that $a \cdot b$ is p -power.

Let us consider p and let G be a finite p -group group. One can verify that every element of G is p -power.

The following proposition is true

- (15) Let G be a finite group, H be a subgroup of G , and a be an element of G . If H is p -group and $a \in H$, then a is p -power.

Let us consider p and let G be a finite p -group group. One can verify that every subgroup of G is p -group.

We now state the proposition

- (16) $\{\mathbf{1}\}_G$ is p -group.

Let us consider p and let G be a group. Note that there exists a subgroup of G which is p -group.

Let us consider p , let G be a finite group, let G_1 be a p -group subgroup of G , and let G_2 be a subgroup of G . One can verify that $G_1 \cap G_2$ is p -group and $G_2 \cap G_1$ is p -group.

Next we state the proposition

- (17) For every finite group G such that every element of G is p -power holds G is p -group.

Let us consider p , let G be a finite p -group group, and let N be a normal subgroup of G . Note that G/N is p -group.

The following four propositions are true:

- (18) Let G be a finite group and N be a normal subgroup of G . If N is p -group and G/N is p -group, then G is p -group.
- (19) Let G be a finite commutative group and H, H_1, H_2 be subgroups of G . Suppose H_1 is p -group and H_2 is p -group and the carrier of $H = H_1 \cdot H_2$. Then H is p -group.
- (20) Let G be a finite group and H, N be subgroups of G . Suppose N is a normal subgroup of G and H is p -group and N is p -group. Then there exists a strict subgroup P of G such that the carrier of $P = H \cdot N$ and P is p -group.
- (21) Let G be a finite group and N_1, N_2 be normal subgroups of G . Suppose N_1 is p -group and N_2 is p -group. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and N is p -group.

Let us consider p , let G be a p -group finite group, let H be a finite group, and let g be a homomorphism from G to H . Observe that $\text{Im } g$ is p -group.

The following proposition is true

- (22) For all strict groups G, H such that G and H are isomorphic and G is p -group holds H is p -group.

Let p be a prime natural number and let G be a group. Let us assume that G is p -group. The functor $\text{expon}(G, p)$ yields a natural number and is defined by:

$$\text{(Def. 2)} \quad \overline{G} = p^{\text{expon}(G, p)}.$$

Let p be a prime natural number and let G be a group. Then $\text{expon}(G, p)$ is an element of \mathbb{N} .

Next we state four propositions:

- (23) For every finite group G and for every subgroup H of G such that G is p -group holds $\text{expon}(H, p) \leq \text{expon}(G, p)$.
- (24) For every strict finite group G such that G is p -group and $\text{expon}(G, p) = 0$ holds $G = \{1\}_G$.
- (25) For every strict finite group G such that G is p -group and $\text{expon}(G, p) = 1$ holds G is cyclic.

- (26) Let G be a finite group, p be a prime natural number, and a be an element of G . If G is p -group and $\text{expon}(G, p) = 2$ and $\text{ord}(a) = p^2$, then G is commutative.

2. COMMUTATIVE p -GROUPS

Let p be a natural number and let G be a group. We say that G is p -commutative group-like if and only if:

- (Def. 3) For all elements a, b of G holds $(a \cdot b)^p = a^p \cdot b^p$.

Let p be a natural number and let G be a group. We say that G is p -commutative group if and only if:

- (Def. 4) G is p -group and p -commutative group-like.

Let p be a natural number. Observe that every group which is p -commutative group is also p -group and p -commutative group-like and every group which is p -group and p -commutative group-like is also p -commutative group.

The following proposition is true

- (27) $\{1\}_G$ is p -commutative group-like.

Let us consider p . Note that there exists a group which is p -commutative group, finite, cyclic, and commutative.

Let us consider p and let G be a p -commutative group-like finite group. Note that every subgroup of G is p -commutative group-like.

Let us consider p . Note that every group which is p -group, finite, and commutative is also p -commutative group.

We now state the proposition

- (28) For every strict finite group G such that $\overline{G} = p$ holds G is p -commutative group.

Let us consider p, G . One can check that there exists a subgroup of G which is p -commutative group and finite.

Let us consider p , let G be a finite group, let H_1 be a p -commutative group-like subgroup of G , and let H_2 be a subgroup of G . One can check that $H_1 \cap H_2$ is p -commutative group-like and $H_2 \cap H_1$ is p -commutative group-like.

Let us consider p , let G be a finite p -commutative group-like group, and let N be a normal subgroup of G . One can verify that G/N is p -commutative group-like.

One can prove the following propositions:

- (29) Let G be a finite group and a, b be elements of G . Suppose G is p -commutative group-like. Let given n . Then $(a \cdot b)^{p^n} = a^{p^n} \cdot b^{p^n}$.
- (30) Let G be a finite commutative group and H, H_1, H_2 be subgroups of G . Suppose H_1 is p -commutative group and H_2 is p -commutative group and the carrier of $H = H_1 \cdot H_2$. Then H is p -commutative group.

- (31) Let G be a finite group, H be a subgroup of G , and N be a strict normal subgroup of G . Suppose N is a subgroup of $Z(G)$ and H is p -commutative group and N is p -commutative group. Then there exists a strict subgroup P of G such that the carrier of $P = H \cdot N$ and P is p -commutative group.
- (32) Let G be a finite group and N_1, N_2 be normal subgroups of G . Suppose N_2 is a subgroup of $Z(G)$ and N_1 is p -commutative group and N_2 is p -commutative group. Then there exists a strict normal subgroup N of G such that the carrier of $N = N_1 \cdot N_2$ and N is p -commutative group.
- (33) Let G, H be groups. Suppose G and H are isomorphic and G is p -commutative group-like. Then H is p -commutative group-like.
- (34) Let G, H be strict groups. Suppose G and H are isomorphic and G is p -commutative group. Then H is p -commutative group.

Let us consider p , let G be a p -commutative group-like finite group, let H be a finite group, and let g be a homomorphism from G to H . Observe that $\text{Im } g$ is p -commutative group-like.

The following propositions are true:

- (35) For every strict finite group G such that G is p -group and $\text{expon}(G, p) = 0$ holds G is p -commutative group.
- (36) For every strict finite group G such that G is p -group and $\text{expon}(G, p) = 1$ holds G is p -commutative group.

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