

# More on the Continuity of Real Functions<sup>1</sup>

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**Summary.** In this article we demonstrate basic properties of the continuous functions from  $\mathbb{R}$  to  $\mathcal{R}^n$  which correspond to state space equations in control engineering.

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The terminology and notation used here have been introduced in the following articles: [3], [7], [17], [2], [4], [12], [13], [14], [16], [1], [5], [9], [15], [18], [10], [8], [20], [21], [19], [11], [22], and [6].

For simplicity, we use the following convention:  $n, i$  denote elements of  $\mathbb{N}$ ,  $X, X_1$  denote sets,  $r, p, s, x_0, x_1, x_2$  denote real numbers,  $f, f_1, f_2$  denote partial functions from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and  $h$  denotes a partial function from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ .

Let us consider  $n, f, x_0$ . We say that  $f$  is continuous in  $x_0$  if and only if:

(Def. 1) There exists a partial function  $g$  from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $f = g$  and  $g$  is continuous in  $x_0$ .

We now state four propositions:

- (1) If  $h = f$ , then  $f$  is continuous in  $x_0$  iff  $h$  is continuous in  $x_0$ .
- (2) If  $x_0 \in X$  and  $f$  is continuous in  $x_0$ , then  $f|X$  is continuous in  $x_0$ .

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- (3)  $f$  is continuous in  $x_0$  if and only if the following conditions are satisfied:
- (i)  $x_0 \in \text{dom } f$ , and
  - (ii) for every  $r$  such that  $0 < r$  there exists  $s$  such that  $0 < s$  and for every  $x_1$  such that  $x_1 \in \text{dom } f$  and  $|x_1 - x_0| < s$  holds  $|f_{x_1} - f_{x_0}| < r$ .
- (4) Let  $r$  be a real number,  $z$  be an element of  $\mathcal{R}^n$ , and  $w$  be a point of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $z = w$ . Then  $\{y \in \mathcal{R}^n: |y - z| < r\} = \{y; y \text{ ranges over points of } \langle \mathcal{E}^n, \|\cdot\| \rangle: \|y - w\| < r\}$ .

Let  $n$  be an element of  $\mathbb{N}$ , let  $Z$  be a set, and let  $f$  be a partial function from  $Z$  to  $\mathcal{R}^n$ . The functor  $|f|$  yielding a partial function from  $Z$  to  $\mathbb{R}$  is defined by:

(Def. 2)  $\text{dom } |f| = \text{dom } f$  and for every set  $x$  such that  $x \in \text{dom } |f|$  holds  $|f|_x = |f_x|$ .

Let  $n$  be an element of  $\mathbb{N}$ , let  $Z$  be a non empty set, and let  $f$  be a partial function from  $Z$  to  $\mathcal{R}^n$ . The functor  $-f$  yields a partial function from  $Z$  to  $\mathcal{R}^n$  and is defined by:

(Def. 3)  $\text{dom}(-f) = \text{dom } f$  and for every set  $c$  such that  $c \in \text{dom}(-f)$  holds  $(-f)_c = -f_c$ .

One can prove the following propositions:

- (5) Let  $f_1, f_2$  be partial functions from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $g_1, g_2$  be partial functions from  $\mathbb{R}$  to  $\mathcal{R}^n$ . If  $f_1 = g_1$  and  $f_2 = g_2$ , then  $f_1 + f_2 = g_1 + g_2$ .
- (6) Let  $f_1$  be a partial function from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $g_1$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and  $a$  be a real number. If  $f_1 = g_1$ , then  $a \cdot f_1 = a \cdot g_1$ .
- (7) For every partial function  $f_1$  from  $\mathbb{R}$  to  $\mathcal{R}^n$  holds  $(-1) \cdot f_1 = -f_1$ .
- (8) Let  $f_1$  be a partial function from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $g_1$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . If  $f_1 = g_1$ , then  $-f_1 = -g_1$ .
- (9) Let  $f_1$  be a partial function from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $g_1$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . If  $f_1 = g_1$ , then  $\|f_1\| = \|g_1\|$ .
- (10) Let  $f_1, f_2$  be partial functions from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $g_1, g_2$  be partial functions from  $\mathbb{R}$  to  $\mathcal{R}^n$ . If  $f_1 = g_1$  and  $f_2 = g_2$ , then  $f_1 - f_2 = g_1 - g_2$ .
- (11)  $f$  is continuous in  $x_0$  if and only if the following conditions are satisfied:
  - (i)  $x_0 \in \text{dom } f$ , and
  - (ii) for every subset  $N_1$  of  $\mathcal{R}^n$  such that there exists a real number  $r$  such that  $0 < r$  and  $\{y \in \mathcal{R}^n: |y - f_{x_0}| < r\} = N_1$  there exists a neighbourhood  $N$  of  $x_0$  such that for every  $x_1$  such that  $x_1 \in \text{dom } f$  and  $x_1 \in N$  holds  $f_{x_1} \in N_1$ .
- (12)  $f$  is continuous in  $x_0$  if and only if the following conditions are satisfied:
  - (i)  $x_0 \in \text{dom } f$ , and

- (ii) for every subset  $N_1$  of  $\mathcal{R}^n$  such that there exists a real number  $r$  such that  $0 < r$  and  $\{y \in \mathcal{R}^n: |y - f_{x_0}| < r\} = N_1$  there exists a neighbourhood  $N$  of  $x_0$  such that  $f \circ N \subseteq N_1$ .
- (13) If there exists a neighbourhood  $N$  of  $x_0$  such that  $\text{dom } f \cap N = \{x_0\}$ , then  $f$  is continuous in  $x_0$ .
- (14) If  $x_0 \in \text{dom } f_1 \cap \text{dom } f_2$  and  $f_1$  is continuous in  $x_0$  and  $f_2$  is continuous in  $x_0$ , then  $f_1 + f_2$  is continuous in  $x_0$ .
- (15) If  $x_0 \in \text{dom } f_1 \cap \text{dom } f_2$  and  $f_1$  is continuous in  $x_0$  and  $f_2$  is continuous in  $x_0$ , then  $f_1 - f_2$  is continuous in  $x_0$ .
- (16) If  $f$  is continuous in  $x_0$ , then  $r \cdot f$  is continuous in  $x_0$ .
- (17) If  $x_0 \in \text{dom } f$  and  $f$  is continuous in  $x_0$ , then  $|f|$  is continuous in  $x_0$ .
- (18) If  $x_0 \in \text{dom } f$  and  $f$  is continuous in  $x_0$ , then  $-f$  is continuous in  $x_0$ .
- (19) Let  $S$  be a real normed space,  $z$  be a point of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ ,  $f_1$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and  $f_2$  be a partial function from the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  to the carrier of  $S$ . Suppose  $x_0 \in \text{dom}(f_2 \cdot f_1)$  and  $f_1$  is continuous in  $x_0$  and  $z = (f_1)_{x_0}$  and  $f_2$  is continuous in  $z$ . Then  $f_2 \cdot f_1$  is continuous in  $x_0$ .
- (20) Let  $S$  be a real normed space,  $f_1$  be a partial function from  $\mathbb{R}$  to the carrier of  $S$ , and  $f_2$  be a partial function from the carrier of  $S$  to  $\mathbb{R}$ . Suppose  $x_0 \in \text{dom}(f_2 \cdot f_1)$  and  $f_1$  is continuous in  $x_0$  and  $f_2$  is continuous in  $(f_1)_{x_0}$ . Then  $f_2 \cdot f_1$  is continuous in  $x_0$ .

Let us consider  $n$ , let  $f$  be a partial function from  $\mathcal{R}^n$  to  $\mathbb{R}$ , and let  $x_0$  be an element of  $\mathcal{R}^n$ . We say that  $f$  is continuous in  $x_0$  if and only if the condition (Def. 4) is satisfied.

- (Def. 4) There exists a point  $y_0$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and there exists a partial function  $g$  from the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  to  $\mathbb{R}$  such that  $x_0 = y_0$  and  $f = g$  and  $g$  is continuous in  $y_0$ .

One can prove the following two propositions:

- (21) Let  $f$  be a partial function from  $\mathcal{R}^n$  to  $\mathbb{R}$ ,  $h$  be a partial function from the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  to  $\mathbb{R}$ ,  $x_0$  be an element of  $\mathcal{R}^n$ , and  $y_0$  be a point of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $f = h$  and  $x_0 = y_0$ . Then  $f$  is continuous in  $x_0$  if and only if  $h$  is continuous in  $y_0$ .
- (22) Let  $f_1$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  and  $f_2$  be a partial function from  $\mathcal{R}^n$  to  $\mathbb{R}$ . Suppose  $x_0 \in \text{dom}(f_2 \cdot f_1)$  and  $f_1$  is continuous in  $x_0$  and  $f_2$  is continuous in  $(f_1)_{x_0}$ . Then  $f_2 \cdot f_1$  is continuous in  $x_0$ .

Let us consider  $n$ ,  $f$ . We say that  $f$  is continuous if and only if:

- (Def. 5) For every  $x_0$  such that  $x_0 \in \text{dom } f$  holds  $f$  is continuous in  $x_0$ .

One can prove the following propositions:

- (23) Let  $g$  be a partial function from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  and  $f$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . If  $g = f$ , then  $g$  is continuous iff  $f$  is continuous.
- (24) Suppose  $X \subseteq \text{dom } f$ . Then  $f \upharpoonright X$  is continuous if and only if for all  $x_0, r$  such that  $x_0 \in X$  and  $0 < r$  there exists  $s$  such that  $0 < s$  and for every  $x_1$  such that  $x_1 \in X$  and  $|x_1 - x_0| < s$  holds  $|f_{x_1} - f_{x_0}| < r$ .

Let us consider  $n$ . Observe that every partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  which is constant is also continuous.

Let us consider  $n$ . Observe that there exists a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  which is continuous.

Let us consider  $n$ , let  $f$  be a continuous partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and let  $X$  be a set. One can verify that  $f \upharpoonright X$  is continuous.

One can prove the following proposition

- (25) If  $f \upharpoonright X$  is continuous and  $X_1 \subseteq X$ , then  $f \upharpoonright X_1$  is continuous.

Let us consider  $n$ . Note that every partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  which is empty is also continuous.

Let us consider  $n, f$  and let  $X$  be a trivial set. One can verify that  $f \upharpoonright X$  is continuous.

Let us consider  $n$  and let  $f_1, f_2$  be continuous partial functions from  $\mathbb{R}$  to  $\mathcal{R}^n$ . One can check that  $f_1 + f_2$  is continuous.

The following propositions are true:

- (26) If  $X \subseteq \text{dom } f_1 \cap \text{dom } f_2$  and  $f_1 \upharpoonright X$  is continuous and  $f_2 \upharpoonright X$  is continuous, then  $(f_1 + f_2) \upharpoonright X$  is continuous and  $(f_1 - f_2) \upharpoonright X$  is continuous.
- (27) If  $X \subseteq \text{dom } f_1$  and  $X_1 \subseteq \text{dom } f_2$  and  $f_1 \upharpoonright X$  is continuous and  $f_2 \upharpoonright X_1$  is continuous, then  $(f_1 + f_2) \upharpoonright (X \cap X_1)$  is continuous and  $(f_1 - f_2) \upharpoonright (X \cap X_1)$  is continuous.

Let us consider  $n$ , let  $f$  be a continuous partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and let us consider  $r$ . Observe that  $r \cdot f$  is continuous.

The following propositions are true:

- (28) If  $X \subseteq \text{dom } f$  and  $f \upharpoonright X$  is continuous, then  $(r \cdot f) \upharpoonright X$  is continuous.
- (29) If  $X \subseteq \text{dom } f$  and  $f \upharpoonright X$  is continuous, then  $|f| \upharpoonright X$  is continuous and  $(-f) \upharpoonright X$  is continuous.
- (30) If  $f$  is total and for all  $x_1, x_2$  holds  $f_{x_1+x_2} = f_{x_1} + f_{x_2}$  and there exists  $x_0$  such that  $f$  is continuous in  $x_0$ , then  $f \upharpoonright \mathbb{R}$  is continuous.
- (31) For every subset  $Y$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $\text{dom } f$  is compact and  $f \upharpoonright \text{dom } f$  is continuous and  $Y = \text{rng } f$  holds  $Y$  is compact.
- (32) Let  $Y$  be a subset of  $\mathbb{R}$  and  $Z$  be a subset of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Suppose  $Y \subseteq \text{dom } f$  and  $Z = f^\circ Y$  and  $Y$  is compact and  $f \upharpoonright Y$  is continuous. Then  $Z$  is compact.

Let us consider  $n, f$ . We say that  $f$  is Lipschitzian if and only if:

(Def. 6) There exists a partial function  $g$  from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $g = f$  and  $g$  is Lipschitzian.

The following propositions are true:

- (33)  $f$  is Lipschitzian if and only if there exists a real number  $r$  such that  $0 < r$  and for all  $x_1, x_2$  such that  $x_1, x_2 \in \text{dom } f$  holds  $|f_{x_1} - f_{x_2}| \leq r \cdot |x_1 - x_2|$ .
- (34) If  $f = h$ , then  $f$  is Lipschitzian iff  $h$  is Lipschitzian.
- (35)  $f \upharpoonright X$  is Lipschitzian if and only if there exists a real number  $r$  such that  $0 < r$  and for all  $x_1, x_2$  such that  $x_1, x_2 \in \text{dom}(f \upharpoonright X)$  holds  $|f_{x_1} - f_{x_2}| \leq r \cdot |x_1 - x_2|$ .

Let us consider  $n$ . Note that every partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  which is empty is also Lipschitzian.

Let us consider  $n$ . Note that there exists a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  which is empty.

Let us consider  $n$ , let  $f$  be a Lipschitzian partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and let  $X$  be a set. Note that  $f \upharpoonright X$  is Lipschitzian.

We now state the proposition

- (36) If  $f \upharpoonright X$  is Lipschitzian and  $X_1 \subseteq X$ , then  $f \upharpoonright X_1$  is Lipschitzian.

Let us consider  $n$  and let  $f_1, f_2$  be Lipschitzian partial functions from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Observe that  $f_1 + f_2$  is Lipschitzian and  $f_1 - f_2$  is Lipschitzian.

We now state two propositions:

- (37) If  $f_1 \upharpoonright X$  is Lipschitzian and  $f_2 \upharpoonright X_1$  is Lipschitzian, then  $(f_1 + f_2) \upharpoonright (X \cap X_1)$  is Lipschitzian.
- (38) If  $f_1 \upharpoonright X$  is Lipschitzian and  $f_2 \upharpoonright X_1$  is Lipschitzian, then  $(f_1 - f_2) \upharpoonright (X \cap X_1)$  is Lipschitzian.

Let us consider  $n$ , let  $f$  be a Lipschitzian partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ , and let us consider  $p$ . Observe that  $p \cdot f$  is Lipschitzian.

Next we state the proposition

- (39) If  $f \upharpoonright X$  is Lipschitzian and  $X \subseteq \text{dom } f$ , then  $(p \cdot f) \upharpoonright X$  is Lipschitzian.

Let us consider  $n$  and let  $f$  be a Lipschitzian partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Observe that  $|f|$  is Lipschitzian.

Next we state the proposition

- (40) If  $f \upharpoonright X$  is Lipschitzian, then  $-f \upharpoonright X$  is Lipschitzian and  $|f| \upharpoonright X$  is Lipschitzian and  $(-f) \upharpoonright X$  is Lipschitzian.

Let us consider  $n$ . One can check that every partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  which is constant is also Lipschitzian.

Let us consider  $n$ . One can verify that every partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$  which is Lipschitzian is also continuous.

The following propositions are true:

- (41) For all elements  $r, p$  of  $\mathcal{R}^n$  such that for every  $x_0$  such that  $x_0 \in X$  holds  $f_{x_0} = x_0 \cdot r + p$  holds  $f \upharpoonright X$  is continuous.

- (42) For every element  $x_0$  of  $\mathcal{R}^n$  such that  $1 \leq i \leq n$  holds  $\text{proj}(i, n)$  is continuous in  $x_0$ .
- (43) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $h$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Then  $h$  is continuous in  $x_0$  if and only if the following conditions are satisfied:
- (i)  $x_0 \in \text{dom } h$ , and
  - (ii) for every element  $i$  of  $\mathbb{N}$  such that  $i \in \text{Seg } n$  holds  $\text{proj}(i, n) \cdot h$  is continuous in  $x_0$ .
- (44) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $h$  be a partial function from  $\mathbb{R}$  to  $\mathcal{R}^n$ . Then  $h$  is continuous if and only if for every element  $i$  of  $\mathbb{N}$  such that  $i \in \text{Seg } n$  holds  $\text{proj}(i, n) \cdot h$  is continuous.
- (45) For every point  $x_0$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  such that  $1 \leq i \leq n$  holds  $\text{Proj}(i, n)$  is continuous in  $x_0$ .
- (46) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $h$  be a partial function from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Then  $h$  is continuous in  $x_0$  if and only if for every element  $i$  of  $\mathbb{N}$  such that  $i \in \text{Seg } n$  holds  $\text{Proj}(i, n) \cdot h$  is continuous in  $x_0$ .
- (47) Let  $n$  be a non empty element of  $\mathbb{N}$  and  $h$  be a partial function from  $\mathbb{R}$  to the carrier of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . Then  $h$  is continuous if and only if for every element  $i$  of  $\mathbb{N}$  such that  $i \in \text{Seg } n$  holds  $\text{Proj}(i, n) \cdot h$  is continuous.

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