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Isometric Differentiable Functions on Real Normed Space[1]

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Summary. In this article, we formalize isometric differentiable functions on real normed space [17], and their properties.

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The notation and terminology used in this paper have been introduced in the following articles: [3], [2], [8], [4], [5], [18], [10], [11], [19], [14], [16], [1], [6], [9], [15], [23], [24], [21], [22], [13], [25], and [7].

1. Preliminaries

From now on S, T, W, Y denote real normed spaces, f, f_1, f_2 denote partial functions from S to T, Z denotes a subset of S, and i, n denote natural numbers.

Now we state the propositions:

(1) Let us consider a set X and functions I, f. Then $(f{\restriction}X) \cdot I = (f \cdot I){\restriction}I^{-1}(X)$.

(2) Let us consider real normed spaces S, T, a linear operator L from S into T, and points x, y of S. Then $L(x) - L(y) = L(x - y)$.

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(3) Let us consider real normed spaces X, Y, W, a function I from X into Y, and partial functions f_1, f_2 from Y to W. Then

(i) $(f_1 + f_2) \cdot I = f_1 \cdot I + f_2 \cdot I$, and

(ii) $(f_1 - f_2) \cdot I = f_1 \cdot I - f_2 \cdot I$.

PROOF: Set D_1 = the carrier of X. For every element s of D_1, $s \in \mathrm{dom}((f_1+f_2)\cdot I)$ iff $s \in \mathrm{dom}(f_1\cdot I+f_2\cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \mathrm{dom}((f_1+f_2)\cdot I)$ holds $((f_1+f_2)\cdot I)(z) = (f_1\cdot I+f_2\cdot I)(z)$ by [4, (11), (12)]. For every element s of D_1, $s \in \mathrm{dom}((f_1 - f_2) \cdot I)$ iff $s \in \mathrm{dom}(f_1 \cdot I - f_2 \cdot I)$ by [4, (11)]. For every element z of D_1 such that $z \in \mathrm{dom}((f_1 - f_2) \cdot I)$ holds $((f_1 - f_2) \cdot I)(z) = (f_1 \cdot I - f_2 \cdot I)(z)$ by [4, (11), (12)]. □

(4) Let us consider real normed spaces X, Y, W, a function I from X into Y, a partial function f from Y to W, and a real number r. Then $r \cdot (f \cdot I) = (r \cdot f) \cdot I$. PROOF: Set D_1 = the carrier of X. For every element s of D_1, $s \in \mathrm{dom}((r \cdot f) \cdot I)$ iff $s \in \mathrm{dom}(f \cdot I)$ by [4, (11)]. For every element s of D_1, $s \in \mathrm{dom}((r \cdot f) \cdot I)$ iff $I(s) \in \mathrm{dom}(r \cdot f)$ by [4, (11)]. For every element z of D_1 such that $z \in \mathrm{dom}(r \cdot (f \cdot I))$ holds $(r \cdot (f \cdot I))(z) = ((r \cdot f) \cdot I)(z)$ by [4, (12)]. □

(5) Let us consider a partial function f from T to W, a function g from S into T, and a point x of S. Suppose

(i) $x \in \mathrm{dom}\, g$, and

(ii) $g_x \in \mathrm{dom}\, f$, and

(iii) g is continuous in x, and

(iv) f is continuous in g_x.

Then $f \cdot g$ is continuous in x. PROOF: Set $h = f \cdot g$. For every real number r such that $0 < r$ there exists a real number s such that $0 < s$ and for every point x_1 of S such that $x_1 \in \mathrm{dom}\, h$ and $\|x_1 - x\| < s$ holds $\|h_{x_1} - h_x\| < r$ by [14, (7)], [12, (3), (4)]. □

Let X, Y be real normed spaces and x be an element of $X \times Y$. The functor reproj1(x) yielding a function from X into $X \times Y$ is defined by

(Def. 1) Let us consider an element r of X. Then $it(r) = \langle r, \mathbf{x_2} \rangle$.

The functor reproj2(x) yielding a function from Y into $X \times Y$ is defined by

(Def. 2) Let us consider an element r of Y. Then $it(r) = \langle \mathbf{x_1}, r \rangle$.

2. Isometries

Now we state the propositions:

(6) Let us consider a linear operator I from S into T and a point x of S. If I is isometric, then I is continuous in x.

(7) Let us consider real normed spaces S, T and a linear operator f from S into T. Then f is isometric if and only if for every element x of S, $\|f(x)\| = \|x\|$. The theorem is a consequence of (2).

(8) Let us consider a linear operator I from S into T and a subset Z of S. If I is isometric, then I is continuous on Z. The theorem is a consequence of (6).

(9) Let us consider a linear operator I from S into T. Suppose I is one-to-one, onto, and isometric. Then there exists a linear operator J from T into S such that

(i) $J = I^{-1}$, and

(ii) J is one-to-one, onto, and isometric.

The theorem is a consequence of (7). Proof: Reconsider $J = I^{-1}$ as a function from T into S. For every points v, w of T, $J(v+w) = J(v)+J(w)$ by [5, (113)], [4, (34)]. For every point v of T and for every real number r, $J(r\cdot v) = r\cdot J(v)$ by [5, (113)], [4, (34)]. For every point v of T, $\|J(v)\| = \|v\|$ by [5, (113)], [4, (34)]. □

Let us consider a linear operator I from S into T and a sequence s_1 of S. Now we state the propositions:

(10) If I is isometric and s_1 is convergent, then $I \cdot s_1$ is convergent and $\lim(I \cdot s_1) = I(\lim s_1)$.

(11) If I is one-to-one, onto, and isometric, then s_1 is convergent iff $I \cdot s_1$ is convergent.

Let us consider a linear operator I from S into T and a subset Z of S. Now we state the propositions:

(12) If I is one-to-one, onto, and isometric, then Z is closed iff $I^\circ Z$ is closed.

(13) If I is one-to-one, onto, and isometric, then Z is open iff $I^\circ Z$ is open.

(14) If I is one-to-one, onto, and isometric, then Z is compact iff $I^\circ Z$ is compact.

Now we state the propositions:

(15) Let us consider a partial function f from T to W and a linear operator I from S into T. Suppose I is one-to-one, onto, and isometric. Let us consider a point x of S. Suppose $I(x) \in \operatorname{dom} f$. Then $f \cdot I$ is continuous in x if and only if f is continuous in $I(x)$. The theorem is a consequence of (9), (6), and (5).

(16) Let us consider a partial function f from T to W, a linear operator I from S into T, and a set X. Suppose

(i) $X \subseteq$ the carrier of T, and

(ii) I is one-to-one, onto, and isometric.

Then f is continuous on X if and only if $f \cdot I$ is continuous on $I^{-1}(X)$. The theorem is a consequence of (15) and (1). PROOF: For every point y of T such that $y \in X$ holds $f \restriction X$ is continuous in y by [5, (113)], [23, (57)]. □

Let X, Y be real normed spaces. The functor IsoCPNrSP(X, Y) yielding a linear operator from $X \times Y$ into $\prod\langle X, Y\rangle$ is defined by

(Def. 3) Let us consider a point x of X and a point y of Y. Then $it(x, y) = \langle x, y\rangle$.

Now we state the proposition:

(17) Let us consider real normed spaces X, Y. Then $0_{\prod\langle X,Y\rangle} =$ (IsoCPNrSP$(X, Y))(0_{X\times Y})$.

Let X, Y be real normed spaces. Observe that IsoCPNrSP(X, Y) is one-to-one onto and isometric.

Let us note that there exists a linear operator from $X \times Y$ into $\prod\langle X, Y\rangle$ which is one-to-one, onto, and isometric.

Let f be a one-to-one onto isometric linear operator from $X \times Y$ into $\prod\langle X, Y\rangle$. Let us note that the functor f^{-1} yields a linear operator from $\prod\langle X, Y\rangle$ into $X \times Y$. One can verify that f^{-1} is one-to-one onto and isometric as a linear operator from $\prod\langle X, Y\rangle$ into $X \times Y$.

Observe that there exists a linear operator from $\prod\langle X, Y\rangle$ into $X \times Y$ which is one-to-one, onto, and isometric.

Now we state the propositions:

(18) Let us consider real normed spaces X, Y, a point x of X, and a point y of Y. Then (IsoCPNrSP$(X, Y))^{-1}(\langle x, y\rangle) = \langle x, y\rangle$. PROOF: Set $I =$ IsoCPNrSP(X, Y). Set $J = I^{-1}$. For every point x of X and for every point y of Y, $J(\langle x, y\rangle) = \langle x, y\rangle$ by [4, (34)]. □

(19) Let us consider real normed spaces X, Y. Then (IsoCPNrSP$(X, Y))^{-1}(0_{\prod\langle X,Y\rangle}) = 0_{X\times Y}$. The theorem is a consequence of (17).

(20) Let us consider real normed spaces X, Y and a subset Z of $X \times Y$. Then IsoCPNrSP(X, Y) is continuous on Z.

(21) Let us consider real normed spaces X, Y and a subset Z of $\prod\langle X, Y\rangle$. Then (IsoCPNrSP$(X, Y))^{-1}$ is continuous on Z.

(22) Let us consider real normed spaces S, T, W, a point f of the real norm space of bounded linear operators from S into W, a point g of the real norm space of bounded linear operators from T into W, and a linear operator I from S into T. Suppose

(i) I is one-to-one, onto, and isometric, and

(ii) $f = g \cdot I$.

Then $\|f\| = \|g\|$. The theorem is a consequence of (9) and (7). PROOF: Consider J being a linear operator from T into S such that $J = I^{-1}$ and

J is one-to-one, onto, and isometric. Reconsider $g_0 = g$ as a Lipschitzian linear operator from T into W. Reconsider $g_3 = g \cdot I$ as a Lipschitzian linear operator from S into W. For every element x, $x \in \{\|g_0(t)\|$, where t is a vector of $T : \|t\| \leqslant 1\}$ iff $x \in \{\|g_3(w)\|$, where w is a vector of $S : \|w\| \leqslant 1\}$ by [4, (13), (35)]. □

Let us consider S and T. One can verify that every linear operator from S into T which is isometric is also Lipschitzian.

3. Isometric Differentiable Functions on Real Normed Space

Let us consider a real norm space sequence G, a real normed space F, a set i, partial functions f, g from $\prod G$ to F, and a subset X of $\prod G$. Now we state the propositions:

(23) Suppose X is open and $i \in \operatorname{dom} G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i. Then

(i) $f + g$ is partially differentiable on X w.r.t. i, and

(ii) $(f + g){\restriction}^i X = (f{\restriction}^i X) + (g{\restriction}^i X)$.

(24) Suppose X is open and $i \in \operatorname{dom} G$ and f is partially differentiable on X w.r.t. i and g is partially differentiable on X w.r.t. i. Then

(i) $f - g$ is partially differentiable on X w.r.t. i, and

(ii) $(f - g){\restriction}^i X = (f{\restriction}^i X) - (g{\restriction}^i X)$.

Now we state the propositions:

(25) Let us consider a real norm space sequence G, a real normed space F, a set i, a partial function f from $\prod G$ to F, a real number r, and a subset X of $\prod G$. Suppose

(i) X is open, and

(ii) $i \in \operatorname{dom} G$, and

(iii) f is partially differentiable on X w.r.t. i.

Then

(iv) $r \cdot f$ is partially differentiable on X w.r.t. i, and

(v) $r \cdot f{\restriction}^i X = r \cdot (f{\restriction}^i X)$.

Proof: Set $h = r \cdot f$. For every point x of $\prod G$ such that $x \in X$ holds h is partially differentiable in x w.r.t. i and $\operatorname{partdiff}(h, x, i) = r \cdot \operatorname{partdiff}(f, x, i)$ by [19, (24), (30)]. Set $f_3 = f{\restriction}^i X$. For every point x of $\prod G$ such that $x \in X$ holds $(r \cdot f_3)_x = \operatorname{partdiff}(h, x, i)$. □

(26) Let us consider real normed spaces S, T, a Lipschitzian linear operator L from S into T, and a point x_0 of S. Then

(i) L is differentiable in x_0, and

(ii) $L'(x_0) = L$.

The theorem is a consequence of (2). PROOF: Reconsider $L_0 = L$ as a point of the real norm space of bounded linear operators from S into T. Reconsider $R =$ (the carrier of S) $\longmapsto 0_T$ as a partial function from S to T. Set $N =$ the neighbourhood of x_0. For every point x of S such that $x \in N$ holds $L_{0x} - L_{0x_0} = L(x - x_0) + R_{x-x_0}$ by [20, (7)], [21, (4)]. □

(27) Let us consider a partial function f from T to W, a Lipschitzian linear operator I from S into T, and a point I_0 of the real norm space of bounded linear operators from S into T. Suppose $I_0 = I$. Let us consider a point x of S. Suppose f is differentiable in $I(x)$. Then

(i) $f \cdot I$ is differentiable in x, and

(ii) $(f \cdot I)'(x) = f'(I(x)) \cdot I_0$.

The theorem is a consequence of (26).

(28) Let us consider a partial function f from T to W and a linear operator I from S into T. Suppose

(i) I is one-to-one and onto, and

(ii) I is isometric.

Let us consider a point x of S. Then $f \cdot I$ is differentiable in x if and only if f is differentiable in $I(x)$. The theorem is a consequence of (9), (26), and (27).

(29) Let us consider a partial function f from T to W, a linear operator I from S into T, and a set X. Suppose

(i) $X \subseteq$ the carrier of T, and

(ii) I is one-to-one and onto, and

(iii) I is isometric.

Then f is differentiable on X if and only if $f \cdot I$ is differentiable on $I^{-1}(X)$. The theorem is a consequence of (28) and (1). PROOF: For every point y of T such that $y \in X$ holds $f{\restriction}X$ is differentiable in y by [5, (113)]. □

(30) Let us consider real normed spaces X, Y, a partial function f from $\prod\langle X, Y\rangle$ to W, and a subset D of $\prod\langle X, Y\rangle$. Suppose f is differentiable on D. Let us consider a point z of $\prod\langle X, Y\rangle$. Suppose $z \in \operatorname{dom} f'_{\restriction D}$. Then $f'_{\restriction D}(z) = ((f \cdot \mathrm{IsoCPNrSP}(X, Y))'_{\restriction(\mathrm{IsoCPNrSP}(X,Y))^{-1}(D)})_{(\mathrm{IsoCPNrSP}(X,Y))^{-1}(z)} \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}$. The theorem is a consequence of (17), (29), and (27). PROOF: Set $I = \mathrm{IsoCPNrSP}(X, Y)$. Set $J = (\mathrm{IsoCPNrSP}(X, Y))^{-1}$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $\prod\langle X, Y\rangle$ such that $z \in \operatorname{dom} f'_{\restriction D}$ holds $f'_{\restriction D}(z) = (g'_{\restriction E})_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [23, (36)]. □

(31) Let us consider real normed spaces X, Y, a partial function f from $X \times Y$ to W, and a subset D of $X \times Y$. Suppose f is differentiable on D. Let us consider a point z of $X \times Y$. Suppose $z \in \operatorname{dom} f'_{\restriction D}$. Then $f'_{\restriction D}(z) = ((f \cdot (\text{IsoCPNrSP}(X,Y))^{-1})'_{\restriction((\text{IsoCPNrSP}(X,Y))^{-1})^{-1}(D)})_{(\text{IsoCPNrSP}(X,Y))(z)} \cdot ((\text{IsoCPNrSP}(X,Y))^{-1})^{-1}$. The theorem is a consequence of (18), (19), (17), (29), and (27). Proof: Set $I = (\text{IsoCPNrSP}(X,Y))^{-1}$. Set $J = \text{IsoCPNrSP}(X,Y)$. Set $g = f \cdot I$. Set $E = I^{-1}(D)$. For every point z of $X \times Y$ such that $z \in \operatorname{dom} f'_{\restriction D}$ holds $f'_{\restriction D}(z) = (g'_{\restriction E})_{J(z)} \cdot I^{-1}$ by [10, (31)], [5, (113)], [23, (36)]. □

(32) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then

(i) $\text{reproj1}(z) = (\text{IsoCPNrSP}(X,Y))^{-1} \cdot \text{reproj}(1(\in \operatorname{dom}\langle X,Y\rangle), (\text{IsoCPNrSP}(X,Y))(z))$, and

(ii) $\text{reproj2}(z) = (\text{IsoCPNrSP}(X,Y))^{-1} \cdot \text{reproj}(2(\in \operatorname{dom}\langle X,Y\rangle), (\text{IsoCPNrSP}(X,Y))(z))$.

The theorem is a consequence of (18).

Let X, Y be real normed spaces and z be a point of $X \times Y$. Let us note that the functor $z_{\mathbf{1}}$ yields a point of X. One can verify that the functor $z_{\mathbf{2}}$ yields a point of Y. Let X, Y, W be real normed spaces. Let f be a partial function from $X \times Y$ to W. We say that f is partially differentiable in z w.r.t. 1 if and only if

(Def. 4) $f \cdot \text{reproj1}(z)$ is differentiable in $z_{\mathbf{1}}$.

We say that f is partially differentiable in z w.r.t. 2 if and only if

(Def. 5) $f \cdot \text{reproj2}(z)$ is differentiable in $z_{\mathbf{2}}$.

Now we state the propositions:

(33) Let us consider real normed spaces X, Y and a point z of $X \times Y$. Then

(i) $z_{\mathbf{1}}$ = the projection onto $1(\in \operatorname{dom}\langle X,Y\rangle)((\text{IsoCPNrSP}(X,Y))(z))$, and

(ii) $z_{\mathbf{2}}$ = the projection onto $2(\in \operatorname{dom}\langle X,Y\rangle)((\text{IsoCPNrSP}(X,Y))(z))$.

(34) Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, and a partial function f from $X \times Y$ to W. Then

(i) f is partially differentiable in z w.r.t. 1 iff $f \cdot (\text{IsoCPNrSP}(X,Y))^{-1}$ is partially differentiable in $(\text{IsoCPNrSP}(X,Y))(z)$ w.r.t. 1, and

(ii) f is partially differentiable in z w.r.t. 2 iff $f \cdot (\text{IsoCPNrSP}(X,Y))^{-1}$ is partially differentiable in $(\text{IsoCPNrSP}(X,Y))(z)$ w.r.t. 2.

The theorem is a consequence of (32) and (33).

Let X, Y, W be real normed spaces, z be a point of $X \times Y$, and f be a partial function from $X \times Y$ to W. The functor partdiff(f,z) w.r.t. 1 yielding a point of the real norm space of bounded linear operators from X into W is defined by the term

(Def. 6) $(f \cdot \mathrm{reproj1}(z))'(z_{\mathbf{1}})$.

The functor partdiff(f, z) w.r.t. 2 yielding a point of the real norm space of bounded linear operators from Y into W is defined by the term

(Def. 7) $(f \cdot \mathrm{reproj2}(z))'(z_{\mathbf{2}})$.

Now we state the proposition:

(35) Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, and a partial function f from $X \times Y$ to W. Then

(i) partdiff(f, z) w.r.t. $1 =$ partdiff$(f \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}, (\mathrm{IsoCPNrSP}(X, Y))(z), 1)$, and

(ii) partdiff(f, z) w.r.t. $2 =$ partdiff$(f \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}, (\mathrm{IsoCPNrSP}(X, Y))(z), 2)$.

The theorem is a consequence of (32) and (33).

Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, and partial functions f_1, f_2 from $X \times Y$ to W. Now we state the propositions:

(36) Suppose f_1 is partially differentiable in z w.r.t. 1 and f_2 is partially differentiable in z w.r.t. 1. Then

(i) $f_1 + f_2$ is partially differentiable in z w.r.t. 1, and

(ii) partdiff$(f_1 + f_2, z)$ w.r.t. $1 =$ partdiff(f_1, z) w.r.t. $1 +$ partdiff(f_2, z) w.r.t. 1, and

(iii) $f_1 - f_2$ is partially differentiable in z w.r.t. 1, and

(iv) partdiff$(f_1 - f_2, z)$ w.r.t. $1 =$ partdiff(f_1, z) w.r.t. $1 -$ partdiff(f_2, z) w.r.t. 1.

(37) Suppose f_1 is partially differentiable in z w.r.t. 2 and f_2 is partially differentiable in z w.r.t. 2. Then

(i) $f_1 + f_2$ is partially differentiable in z w.r.t. 2, and

(ii) partdiff$(f_1 + f_2, z)$ w.r.t. $2 =$ partdiff(f_1, z) w.r.t. $2 +$ partdiff(f_2, z) w.r.t. 2, and

(iii) $f_1 - f_2$ is partially differentiable in z w.r.t. 2, and

(iv) partdiff$(f_1 - f_2, z)$ w.r.t. $2 =$ partdiff(f_1, z) w.r.t. $2 -$ partdiff(f_2, z) w.r.t. 2.

Let us consider real normed spaces X, Y, W, a point z of $X \times Y$, a real number r, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(38) Suppose f is partially differentiable in z w.r.t. 1. Then

(i) $r \cdot f$ is partially differentiable in z w.r.t. 1, and

(ii) partdiff$(r \cdot f, z)$ w.r.t. $1 = r \cdot$ partdiff(f, z) w.r.t. 1.

(39) Suppose f is partially differentiable in z w.r.t. 2. Then

(i) $r \cdot f$ is partially differentiable in z w.r.t. 2, and

(ii) $\operatorname{partdiff}(r \cdot f, z)$ w.r.t. $2 = r \cdot \operatorname{partdiff}(f, z)$ w.r.t. 2.

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W. We say that f is partially differentiable on Z w.r.t. 1 if and only if

(Def. 8) (i) $Z \subseteq \operatorname{dom} f$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f{\restriction}Z$ is partially differentiable in z w.r.t. 1.

We say that f is partially differentiable on Z w.r.t. 2 if and only if

(Def. 9) (i) $Z \subseteq \operatorname{dom} f$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $f{\restriction}Z$ is partially differentiable in z w.r.t. 2.

Now we state the proposition:

(40) Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W. Then

(i) f is partially differentiable on Z w.r.t. 1 iff $f \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable on $((\mathrm{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$ w.r.t. 1, and

(ii) f is partially differentiable on Z w.r.t. 2 iff $f \cdot (\mathrm{IsoCPNrSP}(X, Y))^{-1}$ is partially differentiable on $((\mathrm{IsoCPNrSP}(X, Y))^{-1})^{-1}(Z)$ w.r.t. 2.

The theorem is a consequence of (18), (19), (17), (34), and (1). PROOF: Set $I = (\mathrm{IsoCPNrSP}(X, Y))^{-1}$. Set $g = f \cdot I$. Set $E = I^{-1}(Z)$. f is partially differentiable on Z w.r.t. 1 iff g is partially differentiable on E w.r.t. 1 by [5, (113)], [4, (34)], [5, (38)]. f is partially differentiable on Z w.r.t. 2 iff g is partially differentiable on E w.r.t. 2 by [5, (113)], [4, (34)], [5, (38)]. □

Let X, Y, W be real normed spaces, Z be a set, and f be a partial function from $X \times Y$ to W. Assume f is partially differentiable on Z w.r.t. 1. The functor $f \restriction^1 Z$ yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from X into W is defined by

(Def. 10) (i) $\operatorname{dom} it = Z$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \operatorname{partdiff}(f, z)$ w.r.t. 1.

Assume f is partially differentiable on Z w.r.t. 2. The functor $f \restriction^2 Z$ yielding a partial function from $X \times Y$ to the real norm space of bounded linear operators from Y into W is defined by

(Def. 11) (i) $\operatorname{dom} it = Z$, and

(ii) for every point z of $X \times Y$ such that $z \in Z$ holds $it_z = \operatorname{partdiff}(f, z)$ w.r.t. 2.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(41) Suppose f is partially differentiable on Z w.r.t. 1. Then $f \restriction^1 Z = (f \cdot (\mathrm{IsoCPNrSP}(X,Y))^{-1} \restriction^1 ((\mathrm{IsoCPNrSP}(X,Y))^{-1})^{-1}(Z)) \cdot \mathrm{IsoCPNrSP}(X,Y)$.

(42) Suppose f is partially differentiable on Z w.r.t. 2. Then $f \restriction^2 Z = (f \cdot (\mathrm{IsoCPNrSP}(X,Y))^{-1} \restriction^2 ((\mathrm{IsoCPNrSP}(X,Y))^{-1})^{-1}(Z)) \cdot \mathrm{IsoCPNrSP}(X,Y)$.

(43) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 1 if and only if $Z \subseteq \mathrm{dom}\, f$ and for every point x of $X \times Y$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 1.

(44) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 2 if and only if $Z \subseteq \mathrm{dom}\, f$ and for every point x of $X \times Y$ such that $x \in Z$ holds f is partially differentiable in x w.r.t. 2.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and partial functions f, g from $X \times Y$ to W. Now we state the propositions:

(45) Suppose Z is open and f is partially differentiable on Z w.r.t. 1 and g is partially differentiable on Z w.r.t. 1. Then

(i) $f + g$ is partially differentiable on Z w.r.t. 1, and

(ii) $(f + g) \restriction^1 Z = (f \restriction^1 Z) + (g \restriction^1 Z)$.

(46) Suppose Z is open and f is partially differentiable on Z w.r.t. 1 and g is partially differentiable on Z w.r.t. 1. Then

(i) $f - g$ is partially differentiable on Z w.r.t. 1, and

(ii) $(f - g) \restriction^1 Z = (f \restriction^1 Z) - (g \restriction^1 Z)$.

(47) Suppose Z is open and f is partially differentiable on Z w.r.t. 2 and g is partially differentiable on Z w.r.t. 2. Then

(i) $f + g$ is partially differentiable on Z w.r.t. 2, and

(ii) $(f + g) \restriction^2 Z = (f \restriction^2 Z) + (g \restriction^2 Z)$.

(48) Suppose Z is open and f is partially differentiable on Z w.r.t. 2 and g is partially differentiable on Z w.r.t. 2. Then

(i) $f - g$ is partially differentiable on Z w.r.t. 2, and

(ii) $(f - g) \restriction^2 Z = (f \restriction^2 Z) - (g \restriction^2 Z)$.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, a real number r, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(49) Suppose Z is open and f is partially differentiable on Z w.r.t. 1. Then

(i) $r \cdot f$ is partially differentiable on Z w.r.t. 1, and

(ii) $r \cdot f \restriction^1 Z = r \cdot (f \restriction^1 Z)$.

(50) Suppose Z is open and f is partially differentiable on Z w.r.t. 2. Then

(i) $r \cdot f$ is partially differentiable on Z w.r.t. 2, and

(ii) $r \cdot f \restriction^2 Z = r \cdot (f \restriction^2 Z)$.

Let us consider real normed spaces X, Y, W, a subset Z of $X \times Y$, and a partial function f from $X \times Y$ to W. Now we state the propositions:

(51) Suppose f is differentiable on Z. Then $f'_{\restriction Z}$ is continuous on Z if and only if $(f \cdot (\mathrm{IsoCPNrSP}(X,Y))^{-1})'_{\restriction((\mathrm{IsoCPNrSP}(X,Y))^{-1})^{-1}(Z)}$ is continuous on $((\mathrm{IsoCPNrSP}(X,Y))^{-1})^{-1}(Z)$.

(52) Suppose Z is open. Then f is partially differentiable on Z w.r.t. 1 and f is partially differentiable on Z w.r.t. 2 and $f \restriction^1 Z$ is continuous on Z and $f \restriction^2 Z$ is continuous on Z if and only if f is differentiable on Z and $f'_{\restriction Z}$ is continuous on Z.

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