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Probability on Finite Set and Real-Valued Random Variables

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Summary. In the various branches of science, probability and randomness provide us with useful theoretical frameworks. The *Formalized Mathematics* has already published some articles concerning the probability: [23], [24], [25], and [30]. In order to apply those articles, we shall give some theorems concerning the probability and the real-valued random variables to prepare for further studies.

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The articles [12], [28], [3], [14], [1], [18], [27], [9], [29], [11], [4], [21], [10], [2], [5], [6], [20], [25], [24], [30], [7], [16], [17], [19], [8], [15], [26], [13], and [22] provide the notation and terminology for this paper.

1. Probability on Finite Set

One can prove the following four propositions:

(1) Let X be a non empty set, S_1 be a σ-field of subsets of X, M be a σ-measure on S_1, f be a partial function from X to $\overline{\mathbb{R}}$, E be an element of S_1, and a be a real number. Suppose f is integrable on M and $E \subseteq \mathrm{dom}\, f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $a \leq f(x)$. Then $\overline{\mathbb{R}}(a) \cdot M(E) \leq \int f{\restriction}E \,\mathrm{d}M$.

(2) Let X be a non empty set, S_1 be a σ-field of subsets of X, M be a σ-measure on S_1, f be a partial function from X to $\mathbb{R}$, E be an element of S_1, and a be a real number. Suppose f is integrable on M and $E \subseteq \mathrm{dom}\, f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $a \leq f(x)$. Then $\overline{\mathbb{R}}(a) \cdot M(E) \leq \int f{\restriction}E \,\mathrm{d}M$.

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(3) Let X be a non empty set, S_1 be a σ-field of subsets of X, M be a σ-measure on S_1, f be a partial function from X to $\overline{\mathbb{R}}$, E be an element of S_1, and a be a real number. Suppose f is integrable on M and $E \subseteq \operatorname{dom} f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $f(x) \leq a$. Then $\int f \restriction E \, \mathrm{d}M \leq \overline{\mathbb{R}}(a) \cdot M(E)$.

(4) Let X be a non empty set, S_1 be a σ-field of subsets of X, M be a σ-measure on S_1, f be a partial function from X to $\mathbb{R}$, E be an element of S_1, and a be a real number. Suppose f is integrable on M and $E \subseteq \operatorname{dom} f$ and $M(E) < +\infty$ and for every element x of X such that $x \in E$ holds $f(x) \leq a$. Then $\int f \restriction E \, \mathrm{d}M \leq \overline{\mathbb{R}}(a) \cdot M(E)$.

2. Random Variables

For simplicity, we follow the rules: O is a non empty set, r is a real number, S is a σ-field of subsets of O, P is a probability on S, and E is a finite non empty set.

Let E be a non empty set. We introduce the trivial σ-field of E as a synonym of 2^E. Then the trivial σ-field of E is a σ-field of subsets of E.

Next we state a number of propositions:

(5) Let O be a non empty finite set and f be a partial function from O to $\mathbb{R}$. Then there exists a finite sequence F of separated subsets of the trivial σ-field of O and there exists a finite sequence s of elements of $\operatorname{dom} f$ such that
$\operatorname{dom} f = \bigcup \operatorname{rng} F$ and $\operatorname{dom} F = \operatorname{dom} s$ and s is one-to-one and $\operatorname{rng} s = \operatorname{dom} f$ and $\operatorname{len} s = \overline{\overline{\operatorname{dom} f}}$ and for every natural number k such that $k \in \operatorname{dom} F$ holds $F(k) = \{s(k)\}$ and for every natural number n and for all elements x, y of O such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $f(x) = f(y)$.

(6) Let O be a non empty finite set and f be a partial function from O to $\mathbb{R}$. Then

(i) f is simple function in the trivial σ-field of O, and

(ii) $\operatorname{dom} f$ is an element of the trivial σ-field of O.

(7) Let O be a non empty finite set, M be a σ-measure on the trivial σ-field of O, and f be a partial function from O to $\mathbb{R}$. If $\operatorname{dom} f \neq \emptyset$ and $M(\operatorname{dom} f) < +\infty$, then f is integrable on M.

(8) Let O be a non empty finite set and f be a partial function from O to $\mathbb{R}$. Then there exists an element X of the trivial σ-field of O such that $\operatorname{dom} f = X$ and f is measurable on X.

(9) Let O be a non empty finite set, M be a σ-measure on the trivial σ-field of O, f be a function from O into $\mathbb{R}$, x be a finite sequence of elements of $\overline{\mathbb{R}}$, and s be a finite sequence of elements of O. Suppose $M(O) < +\infty$ and

s is one-to-one and $\operatorname{rng} s = O$ and $\operatorname{len} s = \overline{\overline{O}}$. Then there exists a finite sequence F of separated subsets of the trivial σ-field of O and there exists a finite sequence a of elements of $\mathbb{R}$ such that

(i) $\operatorname{dom} f = \bigcup \operatorname{rng} F$,

(ii) $\operatorname{dom} a = \operatorname{dom} s$,

(iii) $\operatorname{dom} F = \operatorname{dom} s$,

(iv) for every natural number k such that $k \in \operatorname{dom} F$ holds $F(k) = \{s(k)\}$, and

(v) for every natural number n and for all elements x, y of O such that $n \in \operatorname{dom} F$ and $x, y \in F(n)$ holds $f(x) = f(y)$.

(10) Let O be a non empty finite set, M be a σ-measure on the trivial σ-field of O, f be a function from O into $\mathbb{R}$, x be a finite sequence of elements of $\overline{\mathbb{R}}$, and s be a finite sequence of elements of O. Suppose that

(i) $M(O) < +\infty$,

(ii) $\operatorname{len} x = \overline{\overline{O}}$,

(iii) s is one-to-one,

(iv) $\operatorname{rng} s = O$,

(v) $\operatorname{len} s = \overline{\overline{O}}$, and

(vi) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = \overline{\mathbb{R}}(f(s(n))) \cdot M(\{s(n)\})$.

Then $\int f \, \mathrm{d}M = \sum x$.

(11) Let O be a non empty finite set, M be a σ-measure on the trivial σ-field of O, and f be a function from O into $\mathbb{R}$. Suppose $M(O) < +\infty$. Then there exists a finite sequence x of elements of $\overline{\mathbb{R}}$ and there exists a finite sequence s of elements of O such that

(i) $\operatorname{len} x = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = \overline{\mathbb{R}}(f(s(n))) \cdot M(\{s(n)\})$, and

(vi) $\int f \, \mathrm{d}M = \sum x$.

(12) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, f be a function from O into $\mathbb{R}$, x be a finite sequence of elements of $\mathbb{R}$, and s be a finite sequence of elements of O. Suppose that

(i) $\operatorname{len} x = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$, and

(v) for every natural number n such that $n \in \operatorname{dom} x$ holds $x(n) = f(s(n)) \cdot P(\{s(n)\})$.

Then $\int f \, \mathrm{d}\, \mathrm{P2M}\, P = \sum x$.

(13) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, and f be a function from O into $\mathbb{R}$. Then there exists a finite sequence F of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of O such that

(i) $\mathrm{len}\, F = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\mathrm{rng}\, s = O$,

(iv) $\mathrm{len}\, s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \mathrm{dom}\, F$ holds $F(n) = f(s(n)) \cdot P(\{s(n)\})$, and

(vi) $\int f \, \mathrm{d}\, \mathrm{P2M}\, P = \sum F$.

(14) Let E be a finite non empty set and A be a sequence of subsets of E. Suppose A is non-increasing. Then there exists an element N of $\mathbb{N}$ such that for every element m of $\mathbb{N}$ such that $N \leq m$ holds $A(N) = A(m)$.

(15) Let E be a finite non empty set and A be a sequence of subsets of E. Suppose A is non-increasing. Then there exists an element N of $\mathbb{N}$ such that for every element m of $\mathbb{N}$ such that $N \leq m$ holds $\mathrm{Intersection}\, A = A(m)$.

(16) Let E be a finite non empty set and A be a sequence of subsets of E. Suppose A is non-decreasing. Then there exists an element N of $\mathbb{N}$ such that for every element m of $\mathbb{N}$ such that $N \leq m$ holds $A(N) = A(m)$.

(17) Let E be a finite non empty set and A be a sequence of subsets of E. Suppose A is non-decreasing. Then there exists a natural number N such that for every natural number m such that $N \leq m$ holds $\bigcup A = A(m)$.

Let us consider E. The trivial probability of E yielding a probability on the trivial σ-field of E is defined as follows:

(Def. 1) For every event A_1 of E holds (the trivial probability of E)$(A_1) = \mathrm{P}(A_1)$.

Let us consider O, S. A function from O into $\mathbb{R}$ is said to be a real-valued random variable of S if:

(Def. 2) There exists an element X of S such that $X = O$ and it is measurable on X.

In the sequel f, g are real-valued random variables of S.

Next we state the proposition

(18) $f + g$ is a real-valued random variable of S.

Let us consider O, S, f, g. Then $f + g$ is a real-valued random variable of S.

We now state the proposition

(19) $f - g$ is a real-valued random variable of S.

Let us consider O, S, f, g. Then $f - g$ is a real-valued random variable of S.

Next we state the proposition

(20) For every real number r holds $r\, f$ is a real-valued random variable of S.

Let us consider O, S, f and let r be a real number. Then $r\,f$ is a real-valued random variable of S.

Next we state two propositions:

(21) For all partial functions f, g from O to $\mathbb{R}$ holds $\overline{\mathbb{R}}(f)\,\overline{\mathbb{R}}(g) = \overline{\mathbb{R}}(f\,g)$.

(22) $f\,g$ is a real-valued random variable of S.

Let us consider O, S, f, g. Then $f\,g$ is a real-valued random variable of S.

Next we state two propositions:

(23) For every real number r such that $0 \leq r$ and f is non-negative holds f^r is a real-valued random variable of S.

(24) $|f|$ is a real-valued random variable of S.

Let us consider O, S, f. Then $|f|$ is a real-valued random variable of S.

We now state the proposition

(25) For every real number r such that $0 \leq r$ holds $|f|^r$ is a real-valued random variable of S.

Let us consider O, S, f, P. We say that f is integrable on P if and only if:

(Def. 3) f is integrable on P2M P.

Let us consider O, S, P and let f be a real-valued random variable of S. Let us assume that f is integrable on P. The functor $E_P\{f\}$ yielding an element of $\mathbb{R}$ is defined as follows:

(Def. 4) $E_P\{f\} = \int f \,\mathrm{d}\,\mathrm{P2M}\,P$.

One can prove the following propositions:

(26) If f is integrable on P and g is integrable on P, then $E_P\{f+g\} = E_P\{f\} + E_P\{g\}$.

(27) If f is integrable on P, then $E_P\{r\,f\} = r \cdot E_P\{f\}$.

(28) If f is integrable on P and g is integrable on P, then $E_P\{f-g\} = E_P\{f\} - E_P\{g\}$.

(29) For every non empty finite set O holds every function from O into $\mathbb{R}$ is a real-valued random variable of the trivial σ-field of O.

(30) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, and X be a real-valued random variable of the trivial σ-field of O. Then X is integrable on P.

(31) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, X be a real-valued random variable of the trivial σ-field of O, F be a finite sequence of elements of $\mathbb{R}$, and s be a finite sequence of elements of O. Suppose that

(i) $\operatorname{len} F = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$, and

(v) for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = X(s(n)) \cdot P(\{s(n)\})$.

Then $E_P\{X\} = \sum F$.

(32) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, and X be a real-valued random variable of the trivial σ-field of O. Then there exists a finite sequence F of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of O such that

(i) $\operatorname{len} F = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = X(s(n)) \cdot P(\{s(n)\})$, and

(vi) $E_P\{X\} = \sum F$.

(33) Let O be a non empty finite set, P be a probability on the trivial σ-field of O, and X be a real-valued random variable of the trivial σ-field of O. Then there exists a finite sequence F of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of O such that

(i) $\operatorname{len} F = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \operatorname{dom} F$ holds $F(n) = X(s(n)) \cdot P(\{s(n)\})$, and

(vi) $E_P\{X\} = \sum F$.

(34) Let O be a non empty finite set, X be a real-valued random variable of the trivial σ-field of O, G be a finite sequence of elements of $\mathbb{R}$, and s be a finite sequence of elements of O. Suppose $\operatorname{len} G = \overline{\overline{O}}$ and s is one-to-one and $\operatorname{rng} s = O$ and $\operatorname{len} s = \overline{\overline{O}}$ and for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = X(s(n))$. Then $E_{\text{the trivial probability of } O}\{X\} = \frac{\sum G}{\overline{\overline{O}}}$.

(35) Let O be a non empty finite set and X be a real-valued random variable of the trivial σ-field of O. Then there exists a finite sequence G of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of O such that

(i) $\operatorname{len} G = \overline{\overline{O}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = O$,

(iv) $\operatorname{len} s = \overline{\overline{O}}$,

(v) for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = X(s(n))$, and

(vi) $E_{\text{the trivial probability of } O}\{X\} = \frac{\sum G}{\overline{\overline{O}}}$.

(36) Let X be a real-valued random variable of S. Suppose $0 < r$ and X is non-negative and X is integrable on P. Then $P(\{t \in O\colon r \leq X(t)\}) \leq \frac{E_P\{X\}}{r}$.

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