FORMALIZED MATHEMATICS
Vol. 21, No. 1, Pages 25–31, 2013
DOI: 10.2478/forma-2013-0002

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degruyter.com/view/j/forma

The C^k Space[1]

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Summary. In this article, we formalize continuous differentiability of real-valued functions on n-dimensional real normed linear spaces. Next, we give a definition of the C^k space according to [23].

MML identifier: `CKSPACE1`, version: `8.0.01 5.5.1167`

The notation and terminology used in this paper have been introduced in the following articles: [1], [4], [10], [3], [5], [11], [17], [6], [7], [19], [18], [2], [8], [14], [12], [15], [13], [21], [22], [16], [20], and [9].

1. Definition of Continuously Differentiable Functions and Some Properties

Let m be a non zero element of $\mathbb{N}$, f be a partial function from $\mathcal{R}^m$ to $\mathbb{R}$, k be an element of $\mathbb{N}$, and Z be a set. We say that f is continuously differentiable up to order of k and Z if and only if

(Def. 1) (i) $Z \subseteq \operatorname{dom} f$, and

(ii) f is partial differentiable up to order k and Z, and

(iii) for every non empty finite sequence I of elements of $\mathbb{N}$ such that $\operatorname{len} I \leqslant k$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ holds $f{\restriction}^I Z$ is continuous on Z.

Now we state the propositions:

(1) Let us consider a non zero element m of $\mathbb{N}$, a set Z, a non empty finite sequence I of elements of $\mathbb{N}$, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose f is partially differentiable on Z w.r.t. I. Then $\operatorname{dom}(f{\restriction}^I Z) = Z$.

[1]This work was supported by JSPS KAKENHI 22300285.

ISSN 1426–2630(Print), 1898-9934(Online)

(2) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) X is open, and

(ii) $X \subseteq \operatorname{dom} f$.

Then f is continuously differentiable up to order of 1 and X if and only if f is differentiable on X and for every element x_0 of $\mathcal{R}^m$ and for every real number r such that $x_0 \in X$ and $0 < r$ there exists a real number s such that $0 < s$ and for every element x_1 of $\mathcal{R}^m$ such that $x_1 \in X$ and $|x_1 - x_0| < s$ for every element v of $\mathcal{R}^m$, $|f'(x_1)(v) - f'(x_0)(v)| \leqslant r \cdot |v|$.

(3) Let us consider a non zero element m of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) X is open, and

(ii) $X \subseteq \operatorname{dom} f$, and

(iii) f is continuously differentiable up to order of 1 and X.

Then f is continuous on X. The theorem is a consequence of (2).

(4) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, and partial functions f, g from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) f is continuously differentiable up to order of k and X, and

(ii) g is continuously differentiable up to order of k and X, and

(iii) X is open.

Then $f+g$ is continuously differentiable up to order of k and X. The theorem is a consequence of (1). PROOF: For every non empty finite sequence I of elements of $\mathbb{N}$ such that $\operatorname{len} I \leqslant k$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ holds $(f+g){\restriction}^I X$ is continuous on X. □

(5) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, a real number r, and a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) f is continuously differentiable up to order of k and X, and

(ii) X is open.

Then $r \cdot f$ is continuously differentiable up to order of k and X. The theorem is a consequence of (1). PROOF: For every non empty finite sequence I of elements of $\mathbb{N}$ such that $\operatorname{len} I \leqslant k$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ holds $r \cdot f{\restriction}^I X$ is continuous on X. □

(6) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, and partial functions f, g from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) f is continuously differentiable up to order of k and X, and

(ii) g is continuously differentiable up to order of k and X, and

(iii) X is open.

Then $f - g$ is continuously differentiable up to order of k and X. The theorem is a consequence of (1). PROOF: For every non empty finite sequence I of elements of $\mathbb{N}$ such that $\operatorname{len} I \leqslant k$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$ holds $(f-g)\restriction^{I} X$ is continuous on X. $\square$

Let us consider a non zero element m of $\mathbb{N}$, a non empty subset Z of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and non empty finite sequences I, G of elements of $\mathbb{N}$. Now we state the propositions:

(7) $f\restriction^{G \frown I} Z = (f\restriction^{G} Z)\restriction^{I} Z$.

(8) $f\restriction^{G \frown I} Z$ is continuous on Z if and only if $(f\restriction^{G} Z)\restriction^{I} Z$ is continuous on Z.

Now we state the propositions:

(9) Let us consider a non zero element m of $\mathbb{N}$, a non empty subset Z of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, elements i, j of $\mathbb{N}$, and a non empty finite sequence I of elements of $\mathbb{N}$. Suppose

(i) f is continuously differentiable up to order of $i + j$ and Z, and

(ii) $\operatorname{rng} I \subseteq \operatorname{Seg} m$, and

(iii) $\operatorname{len} I = j$.

Then $f\restriction^{I} Z$ is continuously differentiable up to order of i and Z. The theorem is a consequence of (1) and (7).

(10) Let us consider a non zero element m of $\mathbb{N}$, a non empty subset Z of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and elements i, j of $\mathbb{N}$. Suppose

(i) f is continuously differentiable up to order of i and Z, and

(ii) $j \leqslant i$.

Then f is continuously differentiable up to order of j and Z.

(11) Let us consider a non zero element m of $\mathbb{N}$ and a non empty subset Z of $\mathcal{R}^m$. Suppose Z is open. Let us consider an element k of $\mathbb{N}$ and partial functions f, g from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) f is continuously differentiable up to order of k and Z, and

(ii) g is continuously differentiable up to order of k and Z.

Then $f \cdot g$ is continuously differentiable up to order of k and Z. The theorem is a consequence of (10), (1), (3), (9), and (7). PROOF: Define $\mathcal{P}[\text{element of } \mathbb{N}] \equiv$ for every partial functions f, g from $\mathcal{R}^m$ to $\mathbb{R}$ such that f is continuously differentiable up to order of $\$_1$ and Z and g is continuously differentiable up to order of $\$_1$ and Z holds $f \cdot g$ is continuously differentiable up to order of $\$_1$ and Z. Set $Z0 =$ (0 **qua** natural number). $\mathcal{P}[0]$. For every element k of $\mathbb{N}$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. $\square$

(12) Let us consider a non zero element m of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$.

Let us consider an element x of $\mathcal{R}^m$. If $x \in X$, then f is differentiable in x and $f'(x) = \mathcal{R}^m \longmapsto 0$.

(13) Let us consider a non zero element m of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$.

Let us consider an element x_0 of $\mathcal{R}^m$ and a real number r. Suppose

(iii) $x_0 \in X$, and

(iv) $0 < r$.

Then there exists a real number s such that

(v) $0 < s$, and

(vi) for every element x_1 of $\mathcal{R}^m$ such that $x_1 \in X$ and $|x_1 - x_0| < s$ for every element v of $\mathcal{R}^m$, $|f'(x_1)(v) - f'(x_0)(v)| \leqslant r \cdot |v|$.

The theorem is a consequence of (12).

(14) Let us consider a non zero element m of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$.

Then

(iii) f is differentiable on X, and

(iv) $\operatorname{dom} f'_{\restriction X} = X$, and

(v) for every element x of $\mathcal{R}^m$ such that $x \in X$ holds $(f'_{\restriction X})_x = \mathcal{R}^m \longmapsto 0$.

The theorem is a consequence of (12).

(15) Let us consider a non zero element m of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, a real number d, and an element i of $\mathbb{N}$. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$, and

(iii) $1 \leqslant i \leqslant m$.

Then

(iv) f is partially differentiable on X w.r.t. i, and

(v) $f{\restriction^i}X$ is continuous on X.

The theorem is a consequence of (14) and (13).

(16) Let us consider a non zero element m of $\mathbb{N}$, an element i of $\mathbb{N}$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, a non empty subset X of $\mathcal{R}^m$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$, and

(iii) $1 \leqslant i \leqslant m$.

Then $f{\upharpoonright}^i X = X \longmapsto 0$. The theorem is a consequence of (15) and (12).

Let us consider a non zero element m of $\mathbb{N}$, a non empty finite sequence I of elements of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and a real number d. Now we state the propositions:

(17) Suppose X is open and $f = X \longmapsto d$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$. Then

(i) $(\text{PartDiffSeq}(f, X, I))(0) = X \longmapsto d$, and

(ii) for every element i of $\mathbb{N}$ such that $1 \leqslant i \leqslant \operatorname{len} I$ holds $(\text{PartDiffSeq}(f, X, I))(i) = X \longmapsto 0$.

(18) Suppose X is open and $f = X \longmapsto d$ and $\operatorname{rng} I \subseteq \operatorname{Seg} m$. Then

(i) f is partially differentiable on X w.r.t. I, and

(ii) $f{\upharpoonright}^I X$ is continuous on X.

Now we state the proposition:

(19) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty subset X of $\mathcal{R}^m$, a partial function f from $\mathcal{R}^m$ to $\mathbb{R}$, and a real number d. Suppose

(i) X is open, and

(ii) $f = X \longmapsto d$.

Then f is continuously differentiable up to order of k and X. The theorem is a consequence of (18).

Let m be a non zero element of $\mathbb{N}$. Observe that there exists a non empty subset of $\mathcal{R}^m$ which is open.

2. Definition of the C^k Space

Let m be a non zero element of $\mathbb{N}$, k be an element of $\mathbb{N}$, and X be a non empty open subset of $\mathcal{R}^m$. The functor the $\mathbb{C}^k$ functions of k and X yielding a non empty subset of RAlgebra X is defined by the term

(Def. 2) $\{f$ where f is a partial function from $\mathcal{R}^m$ to $\mathbb{R}$: f is continuously differentiable up to order of k and X and $\operatorname{dom} f = X\}$.

Let us note that the $\mathbb{C}^k$ functions of k and X is additively linearly closed and multiplicatively closed.

The functor the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X yielding a subalgebra of RAlgebra X is defined by the term

(Def. 3) $\langle$the $\mathbb{C}^k$ functions of k and X, mult(the $\mathbb{C}^k$ functions of k and X, RAlgebra X), Add(the $\mathbb{C}^k$ functions of k and X, RAlgebra X), Mult(the $\mathbb{C}^k$ functions of k and X, RAlgebra X), One(the $\mathbb{C}^k$ functions of k and X, RAlgebra X), Zero(the $\mathbb{C}^k$ functions of k and X, RAlgebra X)$\rangle$.

Let us note that the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X is Abelian add-associative right zeroed right complementable vector distributive scalar distributive scalar associative scalar unital commutative associative right unital right distributive and vector associative.

Now we state the propositions:

(20) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty open subset X of $\mathcal{R}^m$, vectors F, G, H of the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X, and partial functions f, g, h from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) $f = F$, and

(ii) $g = G$, and

(iii) $h = H$.

Then $H = F + G$ if and only if for every element x of X, $h(x) = f(x) + g(x)$.

(21) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty open subset X of $\mathcal{R}^m$, vectors F, G, H of the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X, partial functions f, g, h from $\mathcal{R}^m$ to $\mathbb{R}$, and a real number a. Suppose

(i) $f = F$, and

(ii) $g = G$.

Then $G = a \cdot F$ if and only if for every element x of X, $g(x) = a \cdot f(x)$.

(22) Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, a non empty open subset X of $\mathcal{R}^m$, vectors F, G, H of the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X, and partial functions f, g, h from $\mathcal{R}^m$ to $\mathbb{R}$. Suppose

(i) $f = F$, and

(ii) $g = G$, and

(iii) $h = H$.

Then $H = F \cdot G$ if and only if for every element x of X, $h(x) = f(x) \cdot g(x)$.

Let us consider a non zero element m of $\mathbb{N}$, an element k of $\mathbb{N}$, and a non empty open subset X of $\mathcal{R}^m$. Now we state the propositions:

(23) $0_\alpha = X \longmapsto 0$, where α is the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X.

(24) $\mathbf{1}_\alpha = X \longmapsto 1$, where α is the $\mathbb{R}$ algebra of $\mathbb{C}^k$ functions of k and X.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.

[2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[4] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.

[5] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.

[6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**): 55–65, 1990.

[7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.

[10] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[11] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.

[12] Noboru Endou, Hiroyuki Okazaki, and Yasunari Shidama. Higher-order partial differentiation. *Formalized Mathematics*, 20(**2**):113–124, 2012. doi:10.2478/v10037-012-0015-z.

[13] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**): 35–40, 1990.

[14] Takao Inoué, Adam Naumowicz, Noboru Endou, and Yasunari Shidama. Partial differentiation of vector-valued functions on n-dimensional real normed linear spaces. *Formalized Mathematics*, 19(**1**):1–9, 2011. doi:10.2478/v10037-011-0001-x.

[15] Andrzej Kondracki. Basic properties of rational numbers. *Formalized Mathematics*, 1(**5**): 841–845, 1990.

[16] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(**1**):17–21, 1992.

[17] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1 (**2**):329–334, 1990.

[18] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**): 341–347, 2003.

[19] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(**3**):501–505, 1990.

[20] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[21] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1 (**1**):73–83, 1990.

[22] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[23] Kosaku Yosida. *Functional Analysis*. Springer Classics in Mathematics, 1996.

Received November 9, 2012