

Random Variables and Product of Probability Spaces¹

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Summary. We have been working on the formalization of the probability and the randomness. In [15] and [16], we formalized some theorems concerning the real-valued random variables and the product of two probability spaces. In this article, we present the generalized formalization of [15] and [16]. First, we formalize the random variables of arbitrary set and prove the equivalence between random variable on Σ , Borel sets and a real-valued random variable on Σ . Next, we formalize the product of countably infinite probability spaces.

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The notation and terminology used in this paper have been introduced in the following articles: [1], [14], [12], [4], [11], [18], [7], [8], [5], [2], [3], [9], [13], [22], [15], [16], [20], [21], [17], [19], [6], and [10].

1. RANDOM VARIABLES

In this paper Ω , Ω_1 , Ω_2 denote non empty sets, Σ denotes a σ -field of subsets of Ω , S_1 denotes a σ -field of subsets of Ω_1 , and S_2 denotes a σ -field of subsets of Ω_2 .

Now we state the proposition:

- (1) Let us consider a non empty set B and a function f . Then $f^{-1}(\cup B) = \cup\{f^{-1}(Y) \text{ where } Y \text{ is an element of } B : \text{not contradiction}\}$.

Let us consider a function f from Ω_1 into Ω_2 , a sequence B of subsets of Ω_2 , and a sequence D of subsets of Ω_1 . Now we state the propositions:

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- (2) If for every element n of \mathbb{N} , $D(n) = f^{-1}(B(n))$, then $f^{-1}(\cup B) = \cup D$.
- (3) If for every element n of \mathbb{N} , $D(n) = f^{-1}(B(n))$, then $f^{-1}(\text{Intersection } B) = \text{Intersection } D$.

Now we state the propositions:

- (4) Let us consider a function F from Ω into \mathbb{R} and a real number r . Suppose F is a real-valued random variable on Σ . Then $F^{-1}(]-\infty, r[) \in \Sigma$. PROOF: Consider X being an element of Σ such that $X = \Omega$ and F is measurable on X . For every element z , $z \in F^{-1}(]-\infty, r[)$ iff $z \in \Omega_{\Sigma} \cap \text{LE-dom}(F, r)$. \square
- (5) Let us consider a function F from Ω into \mathbb{R} . Suppose F is a real-valued random variable on Σ . Then $\{x \text{ where } x \text{ is an element of the Borel sets : } F^{-1}(x) \text{ is element of } \Sigma\}$ is a σ -field of subsets of \mathbb{R} . The theorem is a consequence of (4) and (3). PROOF: Set $S = \{x \text{ where } x \text{ is an element of the Borel sets : } F^{-1}(x) \text{ is an element of } \Sigma\}$. For every element x such that $x \in S$ holds $x \in$ the Borel sets. Set $r_0 =$ the element of \mathbb{R} . Reconsider $y_0 = \text{halfline}(r_0)$ as an element of the Borel sets. For every subset A of \mathbb{R} such that $A \in S$ holds $A^c \in S$. For every sequence A_1 of subsets of \mathbb{R} such that $\text{rng } A_1 \subseteq S$ holds $\text{Intersection } A_1 \in S$. \square

Let us consider a function f from Ω into \mathbb{R} . Now we state the propositions:

- (6) Suppose f is a real-valued random variable on Σ . Then $\{x \text{ where } x \text{ is an element of the Borel sets : } f^{-1}(x) \text{ is an element of } \Sigma\} =$ the Borel sets.
- (7) f is random variable on Σ and the Borel sets if and only if f is a real-valued random variable on Σ .
- (8) The set of random variables on Σ and the Borel sets = the real-valued random variables set on Σ .

Let us consider Ω_1 , Ω_2 , S_1 , and S_2 . Let F be a function from Ω_1 into Ω_2 . We say that F is (S_1, S_2) -random variable-like if and only if

(Def. 1) F is random variable on S_1 and S_2 .

Observe that there exists a function from Ω_1 into Ω_2 which is (S_1, S_2) -random variable-like.

A random variable of S_1 and S_2 is an (S_1, S_2) -random variable-like function from Ω_1 into Ω_2 . Now we state the proposition:

- (9) Let us consider a function f from Ω into \mathbb{R} . Then f is a random variable of Σ and the Borel sets if and only if f is a real-valued random variable on Σ .

Let F be a function. We say that F is random variable family-like if and only if

(Def. 2) Let us consider a set x . Suppose $x \in \text{dom } F$. Then there exist non empty sets Ω_1, Ω_2 and there exists a σ -field S_1 of subsets of Ω_1 and there exists

a σ -field S_2 of subsets of Ω_2 and there exists a random variable f of S_1 and S_2 such that $F(x) = f$.

One can verify that there exists a function which is random variable family-like.

A random variable family is a random variable family-like function. In this paper F denotes a random variable of S_1 and S_2 .

Let Y be a non empty set, S be a σ -field of subsets of Y , and F be a function.

We say that F is S -measure valued if and only if

(Def. 3) Let us consider a set x . If $x \in \text{dom } F$, then there exists a σ -measure M on S such that $F(x) = M$.

Note that there exists a function which is S -measure valued.

Let F be a function. We say that F is S -probability valued if and only if

(Def. 4) Let us consider a set x . If $x \in \text{dom } F$, then there exists a probability P on S such that $F(x) = P$.

Let us note that there exists a function which is S -probability valued.

Let X, Y be non empty sets. One can verify that there exists an S -probability valued function which is X -defined.

One can verify that there exists an X -defined S -probability valued function which is total.

Let Y be a non empty set. Let us note that every function which is S -probability valued is also S -measure valued.

Let F be a function. We say that F is S -random variable family if and only if

(Def. 5) Let us consider a set x . Suppose $x \in \text{dom } F$. Then there exists a real-valued random variable Z on S such that $F(x) = Z$.

Observe that there exists a function which is S -random variable family.

Now we state the propositions:

(10) Let us consider an element y of S_2 . Suppose $y \neq \emptyset$. Then $\{z \text{ where } z \text{ is an element of } \Omega_1 : F(z) \text{ is an element of } y\} = F^{-1}(y)$. PROOF: Set $D = \{z \text{ where } z \text{ is an element of } \Omega_1 : F(z) \text{ is an element of } y\}$. For every element x , $x \in D$ iff $x \in F^{-1}(y)$. \square

(11) Let us consider a random variable F of S_1 and S_2 . Then

(i) $\{x \text{ where } x \text{ is a subset of } \Omega_1 : \text{there exists an element } y \text{ of } S_2 \text{ such that } x = F^{-1}(y)\} \subseteq S_1$, and

(ii) $\{x \text{ where } x \text{ is a subset of } \Omega_1 : \text{there exists an element } y \text{ of } S_2 \text{ such that } x = F^{-1}(y)\}$ is a σ -field of subsets of Ω_1 .

The theorem is a consequence of (3). PROOF: Set $S = \{x \text{ where } x \text{ is a subset of } \Omega_1 : \text{there exists an element } y \text{ of } S_2 \text{ such that } x = F^{-1}(y)\}$. For every element x such that $x \in S$ holds $x \in S_1$. For every subset A of

Ω_1 such that $A \in S$ holds $A^c \in S$. For every sequence A_1 of subsets of Ω_1 such that $\text{rng } A_1 \subseteq S$ holds $\text{Intersection } A_1 \in S$. \square

Let us consider Ω_1, Ω_2, S_1 , and S_2 . Let M be a measure on S_1 and F be a random variable of S_1 and S_2 . The functor the image measure of F and M yielding a measure on S_2 is defined by

(Def. 6) Let us consider an element y of S_2 . Then $it(y) = M(F^{-1}(y))$.

Let M be a σ -measure on S_1 . Note that the image measure of F and M is σ -additive.

Now we state the proposition:

(12) Let us consider a probability P on S_1 and a random variable F of S_1 and S_2 . Then (the image measure of F and $\text{P2M } P$)(Ω_2) = 1.

Let us consider Ω_1, Ω_2, S_1 , and S_2 . Let P be a probability on S_1 and F be a random variable of S_1 and S_2 . The functor $\text{probability}(F, P)$ yielding a probability on S_2 is defined by the term

(Def. 7) M2P the image measure of F and $\text{P2M } P$.

Now we state the propositions:

(13) Let us consider a probability P on S_1 and a random variable F of S_1 and S_2 . Then $\text{probability}(F, P)$ = the image measure of F and $\text{P2M } P$. The theorem is a consequence of (12).

(14) Let us consider a probability P on S_1 , a random variable F of S_1 and S_2 , and a set y . If $y \in S_2$, then $(\text{probability}(F, P))(y) = P(F^{-1}(y))$. The theorem is a consequence of (13).

(15) Every function from Ω_1 into Ω_2 is a random variable of the trivial σ -field of Ω_1 and the trivial σ -field of Ω_2 .

(16) Let us consider a non empty set S . Then every non empty finite sequence of elements of S is a random variable of the trivial σ -field of $\text{Seg len } F$ and the trivial σ -field of S . The theorem is a consequence of (15).

(17) Let us consider finite non empty sets V, S , a random variable G of the trivial σ -field of V and the trivial σ -field of S , and a set y . Suppose $y \in$ the trivial σ -field of S . Then $(\text{probability}(G, \text{the trivial probability of } V))(y) = \frac{\overline{G^{-1}(y)}}{\overline{V}}$. The theorem is a consequence of (14).

(18) Let us consider a finite non empty set S , a non empty finite sequence s of elements of S , and a set x . Suppose $x \in S$. Then there exists a random variable G of the trivial σ -field of $\text{Seg len } s$ and the trivial σ -field of S such that

(i) $G = s$, and

(ii) $(\text{probability}(G, \text{the trivial probability of } \text{Seg len } s))(\{x\}) = \text{Prob}_D(x, s)$.

The theorem is a consequence of (16) and (17).

2. PRODUCT OF PROBABILITY SPACES

Let D be a non-empty many sorted set indexed by \mathbb{N} and n be a natural number. One can check that $D(n)$ is non empty.

Let S, F be many sorted sets indexed by \mathbb{N} . We say that F is σ -field S -sequence-like if and only if

(Def. 8) Let us consider a natural number n . Then $F(n)$ is a σ -field of subsets of $S(n)$.

Let S be a many sorted set indexed by \mathbb{N} . Let us observe that there exists a many sorted set indexed by \mathbb{N} which is σ -field S -sequence-like.

Let D be a many sorted set indexed by \mathbb{N} . A σ -field sequence of D is a σ -field D -sequence-like many sorted set indexed by \mathbb{N} . Let S be a σ -field sequence of D and n be a natural number. Note that the functor $S(n)$ yields a σ -field of subsets of $D(n)$. Let D be a non-empty many sorted set indexed by \mathbb{N} . Let M be a many sorted set indexed by \mathbb{N} . We say that M is S -probability sequence-like if and only if

(Def. 9) Let us consider a natural number n . Then $M(n)$ is a probability on $S(n)$.

Observe that there exists a many sorted set indexed by \mathbb{N} which is S -probability sequence-like.

A probability sequence of S is an S -probability sequence-like many sorted set indexed by \mathbb{N} . Let P be a probability sequence of S and n be a natural number. One can verify that the functor $P(n)$ yields a probability on $S(n)$. Let D be a many sorted set indexed by \mathbb{N} . The functor the product domain D yielding a many sorted set indexed by \mathbb{N} is defined by

(Def. 10) (i) $it(0) = D(0)$, and

(ii) for every natural number i , $it(i + 1) = it(i) \times D(i + 1)$.

Now we state the proposition:

(19) Let us consider a many sorted set D indexed by \mathbb{N} . Then

(i) (the product domain D)(0) = $D(0)$, and

(ii) (the product domain D)(1) = $D(0) \times D(1)$, and

(iii) (the product domain D)(2) = $D(0) \times D(1) \times D(2)$, and

(iv) (the product domain D)(3) = $D(0) \times D(1) \times D(2) \times D(3)$.

Let D be a non-empty many sorted set indexed by \mathbb{N} . Let us note that the product domain D is non-empty.

Let D be a finite-yielding many sorted set indexed by \mathbb{N} . One can check that the product domain D is finite-yielding.

Let us consider Ω and Σ . Let P be a set. Assume P is a probability on Σ . The functor $\text{modetrans}(P, \Sigma)$ yielding a probability on Σ is defined by the term

(Def. 11) P .

Let D be a finite-yielding non-empty many sorted set indexed by \mathbb{N} . The functor the trivial σ -field sequence D yielding a σ -field sequence of D is defined by

(Def. 12) Let us consider a natural number n . Then $it(n) =$ the trivial σ -field of $D(n)$.

Let P be a probability sequence of the trivial σ -field sequence D and n be a natural number. One can check that the functor $P(n)$ yields a probability on the trivial σ -field of $D(n)$. The functor $\text{ProductProbability}(P, D)$ yielding a many sorted set indexed by \mathbb{N} is defined by

(Def. 13) (i) $it(0) = P(0)$, and
(ii) for every natural number i , $it(i + 1) =$
 $\text{Product-Probability}((\text{the product domain } D)(i), D(i + 1), \text{modetrans}$
 $(it(i), \text{the trivial } \sigma\text{-field of } (\text{the product domain } D)(i)), P(i + 1))$.

Let us consider a finite-yielding non-empty many sorted set D indexed by \mathbb{N} , a probability sequence P of the trivial σ -field sequence D , and a natural number n . Now we state the propositions:

(20) $(\text{ProductProbability}(P, D))(n)$ is a probability on the trivial σ -field of $(\text{the product domain } D)(n)$.

(21) There exists a probability P_4 on the trivial σ -field of $(\text{the product domain } D)(n)$ such that

- (i) $P_4 = (\text{ProductProbability}(P, D))(n)$, and
- (ii) $(\text{ProductProbability}(P, D))(n + 1) = \text{Product-Probability}((\text{the product domain } D)(n), D(n + 1), P_4, P(n + 1))$.

Now we state the proposition:

(22) Let us consider a finite-yielding non-empty many sorted set D indexed by \mathbb{N} and a probability sequence P of the trivial σ -field sequence D . Then

- (i) $(\text{ProductProbability}(P, D))(0) = P(0)$, and
- (ii) $(\text{ProductProbability}(P, D))(1) =$
 $\text{Product-Probability}(D(0), D(1), P(0), P(1))$, and
- (iii) there exists a probability P_1 on the trivial σ -field of $D(0) \times D(1)$ such that $P_1 = (\text{ProductProbability}(P, D))(1)$ and $(\text{ProductProbability}(P, D))(2) = \text{Product-Probability}(D(0) \times D(1), D(2), P_1, P(2))$, and
- (iv) there exists a probability P_2 on the trivial σ -field of $D(0) \times D(1) \times D(2)$ such that $P_2 = (\text{ProductProbability}(P, D))(2)$ and $(\text{ProductProbability}(P, D))(3) = \text{Product-Probability}(D(0) \times D(1) \times D(2), D(3), P_2, P(3))$, and
- (v) there exists a probability P_3 on the trivial σ -field of $D(0) \times D(1) \times D(2) \times D(3)$ such that $P_3 = (\text{ProductProbability}(P, D))(3)$ and

$$(\text{ProductProbability}(P, D))(4) = \text{Product-Probability}(D(0) \times D(1) \times D(2) \times D(3), D(4), P_3, P(4)).$$

The theorem is a consequence of (19) and (21).

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