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Extended Real-Valued Double Sequence and Its Convergence[1]

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Summary. In this article we introduce the convergence of extended real-valued double sequences [16], [17]. It is similar to our previous articles [15], [10]. In addition, we also prove Fatou's lemma and the monotone convergence theorem for double sequences.

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The notation and terminology used in this paper have been introduced in the following articles: [5], [21], [15], [10], [12], [6], [7], [22], [13], [11], [14], [1], [2], [8], [18], [24], [25], [26], [20], [23], [3], [4], and [9].

1. Preliminaries

Let X be a non empty set. One can verify that there exists a function from X into $\mathbb{R}$ which is non-negative and non-positive and there exists a function from X into $\overline{\mathbb{R}}$ which is without $-\infty$, without $+\infty$, non-negative, and non-positive and every function from X into $\overline{\mathbb{R}}$ which is non-negative is also without $-\infty$ and every function from X into $\overline{\mathbb{R}}$ which is non-positive is also without $+\infty$ and there exists a without $+\infty$ function from X into $\overline{\mathbb{R}}$ which is without $-\infty$.

Let f be a function from X into $\overline{\mathbb{R}}$. Let us observe that the functor $-f$ yields a function from X into $\overline{\mathbb{R}}$. Let f be a without $-\infty$ function from X into $\overline{\mathbb{R}}$. Note that $-f$ is without $+\infty$.

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Let f be a without $+\infty$ function from X into $\overline{\mathbb{R}}$. Let us observe that $-f$ is without $-\infty$.

Let f be a non-negative function from X into $\overline{\mathbb{R}}$. Note that $-f$ is non-positive.

Let f be a non-positive function from X into $\overline{\mathbb{R}}$. Let us observe that $-f$ is non-negative.

Let A, B be non empty sets and f be a without $-\infty$ function from $A \times B$ into $\overline{\mathbb{R}}$. Let us observe that f^{T} is without $-\infty$.

Let f be a without $+\infty$ function from $A \times B$ into $\overline{\mathbb{R}}$. One can verify that f^{T} is without $+\infty$.

Let f be a non-negative function from $A \times B$ into $\overline{\mathbb{R}}$. One can check that f^{T} is non-negative.

Let f be a non-positive function from $A \times B$ into $\overline{\mathbb{R}}$. Note that f^{T} is non-positive.

Now we state the propositions:

(1) Let us consider a sequence s of extended reals. Then $(\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa\in\mathbb{N}} = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$.

PROOF: Define $\mathcal{Q}[\text{natural number}] \equiv (-(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})(\$_1) = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$. For every natural number n, $\mathcal{Q}[n]$ from [1, Sch. 2]. Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa}(-s)(\alpha))_{\kappa\in\mathbb{N}}(\$_1) = (-(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}})(\$_1)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(2) Let us consider a non empty set X, and a partial function f from X to $\overline{\mathbb{R}}$. Then $--f = f$.

(3) Let us consider non empty sets X, Y, and a function f from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(-f)^{\mathrm{T}} = -f^{\mathrm{T}}$.

Let s be a non-negative sequence of extended reals. One can verify that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is non-negative.

Let s be a non-positive sequence of extended reals. Let us observe that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}$ is non-positive.

Now we state the propositions:

(4) Let us consider a non-negative sequence s of extended reals, and a natural number m. Then $s(m) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv s(\$_1) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(5) Let us consider a non-positive sequence s of extended reals, and a natural number m. Then $s(m) \geqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa\in\mathbb{N}}(m)$. The theorem is a consequence of (4), (1), and (2).

(6) Let us consider a non empty set X. Then every without $-\infty$, without $+\infty$ function from X into $\overline{\mathbb{R}}$ is a function from X into $\mathbb{R}$.

Let X be a non empty set and f_1, f_2 be without $-\infty$ functions from X into $\overline{\mathbb{R}}$. One can verify that the functor $f_1 + f_2$ yields a without $-\infty$ function from X into $\overline{\mathbb{R}}$. Let f_1, f_2 be without $+\infty$ functions from X into $\overline{\mathbb{R}}$. One can verify that the functor $f_1 + f_2$ yields a without $+\infty$ function from X into $\overline{\mathbb{R}}$. Let f_1 be a without $-\infty$ function from X into $\overline{\mathbb{R}}$ and f_2 be a without $+\infty$ function from X into $\overline{\mathbb{R}}$. Let us observe that the functor $f_1 - f_2$ yields a without $-\infty$ function from X into $\overline{\mathbb{R}}$. Let f_1 be a without $+\infty$ function from X into $\overline{\mathbb{R}}$ and f_2 be a without $-\infty$ function from X into $\overline{\mathbb{R}}$. Observe that the functor $f_1 - f_2$ yields a without $+\infty$ function from X into $\overline{\mathbb{R}}$. Now we state the propositions:

(7) Let us consider a non empty set X, an element x of X, and functions f_1, f_2 from X into $\overline{\mathbb{R}}$. Then

(i) if f_1 is without $-\infty$ and f_2 is without $-\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and

(ii) if f_1 is without $+\infty$ and f_2 is without $+\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and

(iii) if f_1 is without $-\infty$ and f_2 is without $+\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$, and

(iv) if f_1 is without $+\infty$ and f_2 is without $-\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$.

(8) Let us consider a non empty set X, and without $-\infty$ functions f_1, f_2 from X into $\overline{\mathbb{R}}$. Then

(i) $f_1 + f_2 = f_1 - -f_2$, and

(ii) $-(f_1 + f_2) = -f_1 - f_2$.

The theorem is a consequence of (7).

(9) Let us consider a non empty set X, and without $+\infty$ functions f_1, f_2 from X into $\overline{\mathbb{R}}$. Then

(i) $f_1 + f_2 = f_1 - -f_2$, and

(ii) $-(f_1 + f_2) = -f_1 - f_2$.

The theorem is a consequence of (7).

(10) Let us consider a non empty set X, a without $-\infty$ function f_1 from X into $\overline{\mathbb{R}}$, and a without $+\infty$ function f_2 from X into $\overline{\mathbb{R}}$. Then

(i) $f_1 - f_2 = f_1 + -f_2$, and

(ii) $f_2 - f_1 = f_2 + -f_1$, and

(iii) $-(f_1 - f_2) = -f_1 + f_2$, and

(iv) $-(f_2 - f_1) = -f_2 + f_1$.

The theorem is a consequence of (8), (2), and (9).

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and n, m be natural numbers. One can check that the functor $f(n, m)$ yields an element of $\overline{\mathbb{R}}$. Now we state the propositions:

(11) Let us consider without $-\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m. Then $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$. The theorem is a consequence of (7).

(12) Let us consider without $+\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m. Then $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$. The theorem is a consequence of (7).

(13) Let us consider a without $-\infty$ function f_1 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a without $+\infty$ function f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m. Then

(i) $(f_1 - f_2)(n, m) = f_1(n, m) - f_2(n, m)$, and

(ii) $(f_2 - f_1)(n, m) = f_2(n, m) - f_1(n, m)$.

The theorem is a consequence of (7).

(14) Let us consider non empty sets X, Y, and without $-\infty$ functions f_1, f_2 from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(f_1 + f_2)^{\mathrm{T}} = {f_1}^{\mathrm{T}} + {f_2}^{\mathrm{T}}$. The theorem is a consequence of (7).

(15) Let us consider non empty sets X, Y, and without $+\infty$ functions f_1, f_2 from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(f_1 + f_2)^{\mathrm{T}} = {f_1}^{\mathrm{T}} + {f_2}^{\mathrm{T}}$. The theorem is a consequence of (7).

(16) Let us consider non empty sets X, Y, a without $-\infty$ function f_1 from $X \times Y$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function f_2 from $X \times Y$ into $\overline{\mathbb{R}}$. Then

(i) $(f_1 - f_2)^{\mathrm{T}} = {f_1}^{\mathrm{T}} - {f_2}^{\mathrm{T}}$, and

(ii) $(f_2 - f_1)^{\mathrm{T}} = {f_2}^{\mathrm{T}} - {f_1}^{\mathrm{T}}$.

The theorem is a consequence of (7).

One can verify that every sequence of extended reals which is convergent to $+\infty$ is also convergent and every sequence of extended reals which is convergent to $-\infty$ is also convergent and every sequence of extended reals which is convergent to a finite limit is also convergent and there exists a sequence of extended reals which is convergent and there exists a without $-\infty$ sequence of extended reals which is convergent and there exists a without $+\infty$ sequence of extended reals which is convergent.

Now we state the proposition:

(17) Let us consider a convergent sequence s of extended reals. Then

(i) s is convergent to a finite limit iff $-s$ is convergent to a finite limit, and

(ii) s is convergent to $+\infty$ iff $-s$ is convergent to $-\infty$, and

(iii) s is convergent to $-\infty$ iff $-s$ is convergent to $+\infty$, and

(iv) $-s$ is convergent, and

(v) $\lim(-s) = -\lim s$.

The theorem is a consequence of (2).

Let us consider without $-\infty$ sequences s_1, s_2 of extended reals. Now we state the propositions:

(18) Suppose s_1 is convergent to $+\infty$ and s_2 is convergent to $+\infty$. Then

(i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

(19) Suppose s_1 is convergent to $+\infty$ and s_2 is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

Now we state the proposition:

(20) Let us consider without $+\infty$ sequences s_1, s_2 of extended reals. Suppose s_1 is convergent to $+\infty$ and s_2 is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

Let us consider without $-\infty$ sequences s_1, s_2 of extended reals. Now we state the propositions:

(21) Suppose s_1 is convergent to $-\infty$ and s_2 is convergent to $-\infty$. Then

(i) $s_1 + s_2$ is convergent to $-\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = -\infty$.

The theorem is a consequence of (7).

(22) Suppose s_1 is convergent to $-\infty$ and s_2 is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to $-\infty$ and convergent, and

(ii) $\lim(s_1 + s_2) = -\infty$.

The theorem is a consequence of (7).

(23) Suppose s_1 is convergent to a finite limit and s_2 is convergent to a finite limit. Then

(i) $s_1 + s_2$ is convergent to a finite limit and convergent, and

(ii) $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$.

The theorem is a consequence of (7).

Now we state the propositions:

(24) Let us consider without $+\infty$ sequences s_1, s_2 of extended reals. Then

(i) if s_1 is convergent to $+\infty$ and s_2 is convergent to $+\infty$, then $s_1 + s_2$ is convergent to $+\infty$ and convergent and $\lim(s_1 + s_2) = +\infty$, and

(ii) if s_1 is convergent to $+\infty$ and s_2 is convergent to a finite limit, then $s_1 + s_2$ is convergent to $+\infty$ and convergent and $\lim(s_1 + s_2) = +\infty$, and

(iii) if s_1 is convergent to $-\infty$ and s_2 is convergent to $-\infty$, then $s_1 + s_2$ is convergent to $-\infty$ and convergent and $\lim(s_1 + s_2) = -\infty$, and

(iv) if s_1 is convergent to $-\infty$ and s_2 is convergent to a finite limit, then $s_1 + s_2$ is convergent to $-\infty$ and convergent and $\lim(s_1 + s_2) = -\infty$, and

(v) if s_1 is convergent to a finite limit and s_2 is convergent to a finite limit, then $s_1 + s_2$ is convergent to a finite limit and convergent and $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$.

The theorem is a consequence of (17), (21), (10), (9), (2), (22), (18), (19), and (23).

(25) Let us consider a without $-\infty$ sequence s_1 of extended reals, and a without $+\infty$ sequence s_2 of extended reals. Then

(i) if s_1 is convergent to $+\infty$ and s_2 is convergent to $-\infty$, then $s_1 - s_2$ is convergent to $+\infty$ and convergent and $s_2 - s_1$ is convergent to $-\infty$ and convergent and $\lim(s_1 - s_2) = +\infty$ and $\lim(s_2 - s_1) = -\infty$, and

(ii) if s_1 is convergent to $+\infty$ and s_2 is convergent to a finite limit, then $s_1 - s_2$ is convergent to $+\infty$ and convergent and $s_2 - s_1$ is convergent to $-\infty$ and convergent and $\lim(s_1 - s_2) = +\infty$ and $\lim(s_2 - s_1) = -\infty$, and

(iii) if s_1 is convergent to $-\infty$ and s_2 is convergent to a finite limit, then $s_1 - s_2$ is convergent to $-\infty$ and convergent and $s_2 - s_1$ is convergent to $+\infty$ and convergent and $\lim(s_1 - s_2) = -\infty$ and $\lim(s_2 - s_1) = +\infty$, and

(iv) if s_1 is convergent to a finite limit and s_2 is convergent to a finite limit, then $s_1 - s_2$ is convergent to a finite limit and convergent and $s_2 - s_1$ is convergent to a finite limit and convergent and $\lim(s_1 - s_2) = \lim s_1 - \lim s_2$ and $\lim(s_2 - s_1) = \lim s_2 - \lim s_1$.

The theorem is a consequence of (17), (24), (18), (10), (19), (22), (23), and (2).

2. Subsequences of Convergent Extended Real-Valued Sequences

Let us consider sequences s_1, s_2 of extended reals. Now we state the propositions:

(26) Suppose s_2 is a subsequence of s_1 and s_1 is convergent to a finite limit. Then

(i) s_2 is convergent to a finite limit, and

(ii) $\lim s_1 = \lim s_2$.

Proof: Consider g being a real number such that $\lim s_1 = g$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s_1(m) - \lim s_1| < p$ and s_1 is convergent to a finite limit. Reconsider $L = \lim s_1$ as an extended real number. There exists a real number g such that for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|(s_2(m) - g$ **qua** extended real$)| < p$ by [19, (14)], [7, (15)]. For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|s_2(m) - L| < p$ by [19, (14)], [7, (15)]. □

(27) Suppose s_2 is a subsequence of s_1 and s_1 is convergent to $+\infty$. Then

(i) s_2 is convergent to $+\infty$, and

(ii) $\lim s_2 = +\infty$.

(28) Suppose s_2 is a subsequence of s_1 and s_1 is convergent to $-\infty$. Then

(i) s_2 is convergent to $-\infty$, and

(ii) $\lim s_2 = -\infty$.

3. Convergency for Extended Real-Valued Double Sequences

Let us consider a function R from $\mathbb{N} \times \mathbb{N}$ into $\mathbb{R}$. Now we state the propositions:

(29) Suppose the lim in the first coordinate of R is convergent. Then the first coordinate major iterated lim of $R =$ lim(the lim in the first coordinate of R).

(30) Suppose the lim in the second coordinate of R is convergent. Then the second coordinate major iterated lim of $R =$ lim(the lim in the second coordinate of R).

Let E be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that E is P-convergent to a finite limit if and only if

(Def. 1) there exists a real number p such that for every real number e such that $0 < e$ there exists a natural number N such that for every natural numbers n, m such that $n \geqslant N$ and $m \geqslant N$ holds $|E(n, m) - (p$ **qua** extended real$)| < e$.

We say that E is P-convergent to $+\infty$ if and only if

(Def. 2) for every real number g such that $0 < g$ there exists a natural number N such that for every natural numbers n, m such that $n \geqslant N$ and $m \geqslant N$ holds $g \leqslant E(n, m)$.

We say that E is P-convergent to $-\infty$ if and only if

(Def. 3) for every real number g such that $g < 0$ there exists a natural number N such that for every natural numbers n, m such that $n \geqslant N$ and $m \geqslant N$ holds $E(n, m) \leqslant g$.

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that f is convergent in the first coordinate to $+\infty$ if and only if

(Def. 4) for every element m of $\mathbb{N}$, $\text{curry}'(f, m)$ is convergent to $+\infty$.

We say that f is convergent in the first coordinate to $-\infty$ if and only if

(Def. 5) for every element m of $\mathbb{N}$, $\text{curry}'(f, m)$ is convergent to $-\infty$.

We say that f is convergent in the first coordinate to a finite limit if and only if

(Def. 6) for every element m of $\mathbb{N}$, $\text{curry}'(f, m)$ is convergent to a finite limit.

We say that f is convergent in the first coordinate if and only if

(Def. 7) for every element m of $\mathbb{N}$, $\text{curry}'(f, m)$ is convergent.

We say that f is convergent in the second coordinate to $+\infty$ if and only if

(Def. 8) for every element m of $\mathbb{N}$, $\text{curry}(f, m)$ is convergent to $+\infty$.

We say that f is convergent in the second coordinate to $-\infty$ if and only if

(Def. 9) for every element m of $\mathbb{N}$, $\text{curry}(f, m)$ is convergent to $-\infty$.

We say that f is convergent in the second coordinate to a finite limit if and only if

(Def. 10) for every element m of $\mathbb{N}$, curry(f, m) is convergent to a finite limit.

We say that f is convergent in the second coordinate if and only if

(Def. 11) for every element m of $\mathbb{N}$, curry(f, m) is convergent.

Now we state the propositions:

(31) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) if f is convergent in the first coordinate to $+\infty$ or convergent in the first coordinate to $-\infty$ or convergent in the first coordinate to a finite limit, then f is convergent in the first coordinate, and

(ii) if f is convergent in the second coordinate to $+\infty$ or convergent in the second coordinate to $-\infty$ or convergent in the second coordinate to a finite limit, then f is convergent in the second coordinate.

(32) Let us consider non empty sets X, Y, Z, a function F from $X \times Y$ into Z, and an element x of X. Then curry$(F, x) =$ curry$'(F^{\mathrm{T}}, x)$.

(33) Let us consider non empty sets X, Y, Z, a function F from $X \times Y$ into Z, and an element y of Y. Then curry$'(F, y) =$ curry(F^{T}, y).

(34) Let us consider non empty sets X, Y, a function F from $X \times Y$ into $\overline{\mathbb{R}}$, and an element x of X. Then curry$(-F, x) = -$curry(F, x).

(35) Let us consider non empty sets X, Y, a function F from $X \times Y$ into $\overline{\mathbb{R}}$, and an element y of Y. Then curry$'(-F, y) = -$curry$'(F, y)$.

Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(36) (i) f is convergent in the first coordinate to $+\infty$ iff f^{T} is convergent in the second coordinate to $+\infty$, and

(ii) f is convergent in the second coordinate to $+\infty$ iff f^{T} is convergent in the first coordinate to $+\infty$, and

(iii) f is convergent in the first coordinate to $-\infty$ iff f^{T} is convergent in the second coordinate to $-\infty$, and

(iv) f is convergent in the second coordinate to $-\infty$ iff f^{T} is convergent in the first coordinate to $-\infty$, and

(v) f is convergent in the first coordinate to a finite limit iff f^{T} is convergent in the second coordinate to a finite limit, and

(vi) f is convergent in the second coordinate to a finite limit iff f^{T} is convergent in the first coordinate to a finite limit.

The theorem is a consequence of (33) and (32).

(37) (i) f is convergent in the first coordinate to $+\infty$ iff $-f$ is convergent in the first coordinate to $-\infty$, and

(ii) f is convergent in the first coordinate to $-\infty$ iff $-f$ is convergent in the first coordinate to $+\infty$, and

(iii) f is convergent in the first coordinate to a finite limit iff $-f$ is convergent in the first coordinate to a finite limit, and

(iv) f is convergent in the second coordinate to $+\infty$ iff $-f$ is convergent in the second coordinate to $-\infty$, and

(v) f is convergent in the second coordinate to $-\infty$ iff $-f$ is convergent in the second coordinate to $+\infty$, and

(vi) f is convergent in the second coordinate to a finite limit iff $-f$ is convergent in the second coordinate to a finite limit.

The theorem is a consequence of (35), (17), (2), and (34).

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The functors: the lim in the first coordinate of f and the lim in the second coordinate of f yielding sequences of extended reals are defined by conditions

(Def. 12) for every element m of $\mathbb{N}$, the lim in the first coordinate of $f(m) = \lim \mathrm{curry}'(f, m)$,

(Def. 13) for every element n of $\mathbb{N}$, the lim in the second coordinate of $f(n) = \lim \mathrm{curry}(f, n)$,

respectively. Now we state the proposition:

(38) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the lim in the first coordinate of $f =$ the lim in the second coordinate of f^{T}, and

(ii) the lim in the second coordinate of $f =$ the lim in the first coordinate of f^{T}.

The theorem is a consequence of (33) and (32).

Let X, Y be non empty sets, F be a without $+\infty$ function from $X \times Y$ into $\overline{\mathbb{R}}$, and x be an element of X. Let us observe that $\mathrm{curry}(F, x)$ is without $+\infty$.

Let y be an element of Y. One can verify that $\mathrm{curry}'(F, y)$ is without $+\infty$.

Let F be a without $-\infty$ function from $X \times Y$ into $\overline{\mathbb{R}}$ and x be an element of X. Let us note that $\mathrm{curry}(F, x)$ is without $-\infty$.

Let y be an element of Y. Observe that $\mathrm{curry}'(F, y)$ is without $-\infty$.

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The partial sums in the second coordinate of f yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by

(Def. 14) for every natural numbers n, m, $it(n, 0) = f(n, 0)$ and $it(n, m + 1) = it(n, m) + f(n, m + 1)$.

The partial sums in the first coordinate of f yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by

(Def. 15) for every natural numbers n, m, $it(0, m) = f(0, m)$ and $it(n + 1, m) = it(n, m) + f(n + 1, m)$.

Let f be a without $-\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us note that the partial sums in the second coordinate of f is without $-\infty$.

Let f be a without $+\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Observe that the partial sums in the second coordinate of f is without $+\infty$.

Let f be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us observe that the partial sums in the second coordinate of f is non-negative as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let f be a non-positive function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. One can check that the partial sums in the second coordinate of f is non-positive as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let f be a without $-\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us note that the partial sums in the first coordinate of f is without $-\infty$.

Let f be a without $+\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Observe that the partial sums in the first coordinate of f is without $+\infty$.

Let f be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us observe that the partial sums in the first coordinate of f is non-negative as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let f be a non-positive function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. One can check that the partial sums in the first coordinate of f is non-positive as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The functor $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by the term

(Def. 16) the partial sums in the second coordinate of the partial sums in the first coordinate of f.

Now we state the propositions:

(39) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m. Then

(i) (the partial sums in the first coordinate of f)(n, m) = (the partial sums in the second coordinate of f^{T})(m, n), and

(ii) (the partial sums in the second coordinate of f)(n, m) = (the partial sums in the first coordinate of f^{T})(m, n).

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the first coordinate of f)$(\$_1, m)$ = (the partial sums in the second coordinate of f^{T})$(m, \$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ (the partial sums in the second coordinate of f)$(n, \$_1)$ = (the partial sums in the first coordinate of $f^{\mathrm{T}})(\$_1, n)$. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number k, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

(40) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) (the partial sums in the first coordinate of $f)^{\mathrm{T}}$ = the partial sums in the second coordinate of f^{T}, and

(ii) (the partial sums in the second coordinate of $f)^{\mathrm{T}}$ = the partial sums in the first coordinate of f^{T}.

The theorem is a consequence of (39).

(41) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, an extended real-valued function g, and a natural number n. Suppose for every natural number k, (the partial sums in the first coordinate of $f)(n, k) = g(k)$. Then

(i) for every natural number k, $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, k) = (\sum_{\alpha=0}^{\kappa} g(\alpha))_{\kappa \in \mathbb{N}}(k)$, and

(ii) (the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}})(n) = \sum g$.

(42) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $-f$ = $-$(the partial sums in the second coordinate of f), and

(ii) the partial sums in the first coordinate of $-f$ = $-$(the partial sums in the first coordinate of f).

Proof: For every element z of $\mathbb{N} \times \mathbb{N}$, ($-$(the partial sums in the second coordinate of $f))(z)$ = (the partial sums in the second coordinate of $-f)(z)$ by [9, (87)]. For every element z of $\mathbb{N} \times \mathbb{N}$, ($-$(the partial sums in the first coordinate of $f))(z)$ = (the partial sums in the first coordinate of $-f)(z)$ by [9, (87)]. □

(43) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and elements m, n of $\mathbb{N}$. Then

(i) (the partial sums in the first coordinate of $f)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(m)$, and

(ii) (the partial sums in the second coordinate of $f)(m, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the first coordinate of $f)(\$_1, n) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f)(m, \$_1) = (\sum_{\alpha=0}^{\kappa}(\mathrm{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number k, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

(44) Let us consider without $-\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $f_1 + f_2 =$ (the partial sums in the second coordinate of f_1)+(the partial sums in the second coordinate of f_2), and

(ii) the partial sums in the first coordinate of $f_1 + f_2 =$ (the partial sums in the first coordinate of f_1) + (the partial sums in the first coordinate of f_2).

The theorem is a consequence of (11).

(45) Let us consider without $+\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $f_1 + f_2 =$ (the partial sums in the second coordinate of f_1)+(the partial sums in the second coordinate of f_2), and

(ii) the partial sums in the first coordinate of $f_1 + f_2 =$ (the partial sums in the first coordinate of f_1) + (the partial sums in the first coordinate of f_2).

The theorem is a consequence of (10), (9), (2), (42), (44), and (8).

(46) Let us consider a without $-\infty$ function f_1 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) the partial sums in the second coordinate of $f_1 - f_2 =$ (the partial sums in the second coordinate of f_1)−(the partial sums in the second coordinate of f_2), and

(ii) the partial sums in the first coordinate of $f_1 - f_2 =$ (the partial sums in the first coordinate of f_1) − (the partial sums in the first coordinate of f_2), and

(iii) the partial sums in the second coordinate of $f_2 - f_1 =$ (the partial sums in the second coordinate of f_2)−(the partial sums in the second coordinate of f_1), and

(iv) the partial sums in the first coordinate of $f_2 - f_1 =$ (the partial sums in the first coordinate of f_2) − (the partial sums in the first coordinate of f_1).

The theorem is a consequence of (10), (44), (42), and (45).

(47) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m. Then

(i) $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) =$ (the partial sums in the second coordinate of f)$(n+1, m) + (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, m)$, and

(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of f)$(n, m+1)$ = (the partial sums in the first coordinate of f)$(n, m+1)$ + (the partial sums in the first coordinate of the partial sums in the second coordinate of f)(n, m).

PROOF: Set $R_1 = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$. Set C_1 = the partial sums in the first coordinate of the partial sums in the second coordinate of f. Set R_2 = the partial sums in the first coordinate of f. Set C_2 = the partial sums in the second coordinate of f. Define $\mathcal{P}[\text{natural number}] \equiv R_1(n+1, \$_1) = C_2(n+1, \$_1) + R_1(n, \$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv C_1(\$_1, m+1) = R_2(\$_1, m+1) + C_1(\$_1, m)$. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$. For every natural number k, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

(48) Let us consider a without $+\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m. Then

(i) $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n+1, m)$ = (the partial sums in the second coordinate of f)$(n+1, m) + (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, m)$, and

(ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of f)$(n, m+1)$ = (the partial sums in the first coordinate of f)$(n, m+1)$ + (the partial sums in the first coordinate of the partial sums in the second coordinate of f)(n, m).

The theorem is a consequence of (2), (42), and (47).

(49) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ or without $+\infty$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ = the partial sums in the first coordinate of the partial sums in the second coordinate of f.

(50) Let us consider without $-\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa}(f_1 + f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}$. The theorem is a consequence of (44).

(51) Let us consider without $+\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa}(f_1 + f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}$. The theorem is a consequence of (45).

(52) Let us consider a without $-\infty$ function f_1 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) $(\sum_{\alpha=0}^{\kappa}(f_1 - f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}$, and

(ii) $(\sum_{\alpha=0}^{\kappa}(f_2 - f_1)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}}$.

The theorem is a consequence of (46).

(53) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an element k of $\mathbb{N}$. Then

(i) curry'(the partial sums in the first coordinate of f, k) $=$ $(\sum_{\alpha=0}^{\kappa}(\text{curry}'(f, k))(\alpha))_{\kappa \in \mathbb{N}}$, and

(ii) curry(the partial sums in the second coordinate of f, k) $=$ $(\sum_{\alpha=0}^{\kappa}(\text{curry}(f, k))(\alpha))_{\kappa \in \mathbb{N}}$.

The theorem is a consequence of (43).

(54) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ or without $+\infty$. Then

(i) $\text{curry}((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}, 0)$ $=$ curry(the partial sums in the second coordinate of $f, 0$), and

(ii) $\text{curry}'((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}, 0) =$ curry'(the partial sums in the first coordinate of $f, 0$).

(55) Let us consider non empty sets C, D, without $-\infty$ functions F_1, F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element c of C. Then $\text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c)$. The theorem is a consequence of (7).

(56) Let us consider non empty sets C, D, without $-\infty$ functions F_1, F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element d of D. Then $\text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d)$. The theorem is a consequence of (7).

(57) Let us consider non empty sets C, D, without $+\infty$ functions F_1, F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element c of C. Then $\text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c)$. The theorem is a consequence of (7).

(58) Let us consider non empty sets C, D, without $+\infty$ functions F_1, F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element d of D. Then $\text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d)$. The theorem is a consequence of (7).

(59) Let us consider non empty sets C, D, a without $-\infty$ function F_1 from $C \times D$ into $\overline{\mathbb{R}}$, a without $+\infty$ function F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element c of C. Then

(i) $\text{curry}(F_1 - F_2, c) = \text{curry}(F_1, c) - \text{curry}(F_2, c)$, and

(ii) $\text{curry}(F_2 - F_1, c) = \text{curry}(F_2, c) - \text{curry}(F_1, c)$.

The theorem is a consequence of (7).

(60) Let us consider non empty sets C, D, a without $-\infty$ function F_1 from $C \times D$ into $\overline{\mathbb{R}}$, a without $+\infty$ function F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element d of D. Then

(i) $\text{curry}'(F_1 - F_2, d) = \text{curry}'(F_1, d) - \text{curry}'(F_2, d)$, and

(ii) $\text{curry}'(F_2 - F_1, d) = \text{curry}'(F_2, d) - \text{curry}'(F_1, d)$.

The theorem is a consequence of (7).

4. Non-Negative Extended Real-Valued Double Sequences

Now we state the propositions:

(61) Let us consider a non-negative sequence s of extended reals. Suppose $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is not convergent to $+\infty$. Let us consider a natural number n. Then $s(n)$ is a real number.

(62) Let us consider a non-negative sequence s of extended reals. Suppose s is non-decreasing. Then s is convergent to $+\infty$ or convergent to a finite limit.

Let f be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and n be an element of $\mathbb{N}$. Let us observe that curry(f, n) is non-negative and curry$'(f, n)$ is non-negative.

Now we state the propositions:

(63) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an element m of $\mathbb{N}$. Then curry(the partial sums in the second coordinate of f, m) is non-decreasing.

Proof: Set $P =$ curry(the partial sums in the second coordinate of f, m). For every natural numbers n, j such that $j \leqslant n$ holds $P(j) \leqslant P(n)$ by [4, (51)], [1, (13), (20)]. $\square$

(64) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an element n of $\mathbb{N}$. Then curry$'$(the partial sums in the first coordinate of f, n) is non-decreasing. The theorem is a consequence of (63), (40), and (33).

Let f be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and m be an element of $\mathbb{N}$. One can check that curry(the partial sums in the second coordinate of f, m) is non-decreasing and curry$'$(the partial sums in the first coordinate of f, m) is non-decreasing.

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(65) (i) if f is convergent in the first coordinate, then the lim in the first coordinate of f is non-negative, and

(ii) if f is convergent in the second coordinate, then the lim in the second coordinate of f is non-negative.

(66) (i) the partial sums in the first coordinate of f is convergent in the first coordinate, and

(ii) the partial sums in the second coordinate of f is convergent in the second coordinate.

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, an element m of $\mathbb{N}$, and a natural number n.

Let us assume that curry$'$(the partial sums in the first coordinate of f, m) is not convergent to $+\infty$. Now we state the propositions:

(67) $f(n, m)$ is a real number.

(68) $f(m, n)$ is a real number.

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and natural numbers n, m. Now we state the propositions:

(69) Suppose for every natural number i such that $i \leqslant n$ holds $f(i, m)$ is a real number. Then (the partial sums in the first coordinate of f)$(n, m) < +\infty$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then (the partial sums in the first coordinate of f)$(\$_1, m) < +\infty$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)], [1, (13)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. □

(70) Suppose for every natural number i such that $i \leqslant m$ holds $f(n, i)$ is a real number. Then (the partial sums in the second coordinate of f)$(n, m) < +\infty$.
PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant m$, then (the partial sums in the second coordinate of f)$(n, \$_1) < +\infty$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)], [1, (13)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. □

Now we state the proposition:

(71) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to $-\infty$. Then there exists an element m of $\mathbb{N}$ such that curry$'$(the partial sums in the first coordinate of f, m) is convergent to $-\infty$. The theorem is a consequence of (54).

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a natural number m. Now we state the propositions:

(72) for every element k of $\mathbb{N}$ such that $k \leqslant m$ holds curry(the partial sums in the second coordinate of f, k) is not convergent to $+\infty$ if and only if for every element k of $\mathbb{N}$ such that $k \leqslant m$ holds lim curry(the partial sums in the second coordinate of f, k) $< +\infty$. The theorem is a consequence of (62).

(73) for every element k of $\mathbb{N}$ such that $k \leqslant m$ holds curry$'$(the partial sums in the first coordinate of f, k) is not convergent to $+\infty$ if and only if for every element k of $\mathbb{N}$ such that $k \leqslant m$ holds lim curry$'$(the partial sums in the first coordinate of f, k) $< +\infty$. The theorem is a consequence of (62).

(74) $(\sum_{\alpha=0}^{\kappa}$(the lim in the second coordinate of the partial sums in the second coordinate of f)$(\alpha))_{\kappa \in \mathbb{N}}(m) = +\infty$ if and only if there exists an element k of $\mathbb{N}$ such that $k \leqslant m$ and curry(the partial sums in the second coordinate

of $f, k)$ is convergent to $+\infty$. The theorem is a consequence of (72), (65), and (4).

(75) $(\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) = +\infty$ if and only if there exists an element k of $\mathbb{N}$ such that $k \leqslant m$ and curry$'$(the partial sums in the first coordinate of $f, k)$ is convergent to $+\infty$. The theorem is a consequence of (38), (40), (74), and (32).

Now we state the proposition:

(76) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m. Then

(i) (the partial sums in the first coordinate of $f)(n, m) \geqslant f(n, m)$, and

(ii) (the partial sums in the second coordinate of $f)(n, m) \geqslant f(n, m)$.

PROOF: Define $\mathcal{P}$[natural number] $\equiv$ if $\$_1 \leqslant n$, then (the partial sums in the first coordinate of $f)(\$_1, m) \geqslant f(\$_1, m)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)]. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}$[natural number] $\equiv$ if $\$_1 \leqslant m$, then (the partial sums in the second coordinate of $f)(n, \$_1) \geqslant f(n, \$_1)$. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k + 1]$ by [4, (51)]. For every natural number k, $\mathcal{Q}[k]$ from [1, Sch. 2]. □

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and an element m of $\mathbb{N}$. Now we state the propositions:

(77) Suppose there exists an element k of $\mathbb{N}$ such that $k \leqslant m$ and curry(the partial sums in the second coordinate of $f, k)$ is convergent to $+\infty$. Then

(i) curry(the partial sums in the second coordinate of the partial sums in the first coordinate of $f, m)$ is convergent to $+\infty$, and

(ii) lim curry(the partial sums in the second coordinate of the partial sums in the first coordinate of $f, m) = +\infty$.

PROOF: For every real number g such that $0 < g$ there exists a natural number N such that for every natural number n such that $N \leqslant n$ holds $g \leqslant$ (curry(the par- tial sums in the second coordinate of the partial sums in the first coordinate of $f, m))(n)$ by [26, (7)], (76). □

(78) Suppose there exists an element k of $\mathbb{N}$ such that $k \leqslant m$ and curry$'$(the partial sums in the first coordinate of $f, k)$ is convergent to $+\infty$. Then

(i) curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, m)$ is convergent to $+\infty$, and

(ii) lim curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of $f, m) = +\infty$.

The theorem is a consequence of (40), (32), and (77).

Now we state the propositions:

(79) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit if and only if the partial sums in the first coordinate of f is convergent in the first coordinate to a finite limit. The theorem is a consequence of (54), (47), (7), and (23).

(80) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit. Let us consider an element m of $\mathbb{N}$. Then $(\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) =$ lim curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of f, m).

Proof: The partial sums in the first coordinate of f is convergent in the first coordinate to a finite limit. Define $\mathcal{P}$[natural number] $\equiv$ for every element k of $\mathbb{N}$ such that $k \leqslant \$_1$ holds $(\sum_{\alpha=0}^{\kappa}$(the lim in the first coordinate of the partial sums in the first coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(k) =$ lim curry$'$(the partial sums in the first coordinate of the partial sums in the second coordinate of f, k). For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [1, (13)], [14, (7)], (47), [4, (51)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(81) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit if and only if the partial sums in the second coordinate of f is convergent in the second coordinate to a finite limit. The theorem is a consequence of (36), (40), and (79).

(82) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit. Let us consider an element m of $\mathbb{N}$. Then $(\sum_{\alpha=0}^{\kappa}$(the lim in the second coordinate of the partial sums in the second coordinate of $f)(\alpha))_{\kappa \in \mathbb{N}}(m) =$ lim curry(the partial sums in the second coordinate of the partial sums in the first coordinate of f, m). The theorem is a consequence of (36), (40), (38), (80), and (32).

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a sequence s of extended reals. Now we state the propositions:

(83) Suppose for every element m of $\mathbb{N}$, $s(m) = \liminf \text{curry}'(f, m)$. Then $\sum s \leqslant \liminf$(the lim in the second coordinate of the partial sums in the second coordinate of f).

Proof: For every element m of $\mathbb{N}$ and for every elements N, n of $\mathbb{N}$

such that $n \geqslant N$ holds (the inferior realsequence curry$'(f, m))(N) \leqslant f(n, m)$ by [26, (7), (8)]. Define $\mathcal{F}$(element of $\mathbb{N}$) = the inferior realsequence curry$'(f, \$_1)$. Define $\mathcal{G}$(element of $\mathbb{N}$, element of $\mathbb{N}$) = (the inferior realsequence curry$'(f, \$_2))(\$_1)$. Consider g being a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$ and for every element m of $\mathbb{N}$, $g(n, m) = \mathcal{G}(n, m)$ from [5, Sch. 4]. For every element m of $\mathbb{N}$ and for every elements N, n of $\mathbb{N}$ such that $n \geqslant N$ holds (the partial sums in the second coordinate of $g)(N, m) \leqslant$ (the partial sums in the second coordinate of $f)(n, m)$. For every element m of $\mathbb{N}$ and for every elements N, n of $\mathbb{N}$ such that $n \geqslant N$ holds (the partial sums in the second coordinate of $g)(N, m) \leqslant$ (the inferior realsequence the lim in the second coordinate of the partial sums in the second coordinate of $f)(n)$ by [26, (37), (23)]. Define $\mathcal{Q}$[natural number] $\equiv$ for every element m of $\mathbb{N}$ such that $m = \$_1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) = \lim$ curry$'$(the partial sums in the second coordinate of g, m). For every element m of $\mathbb{N}$, curry$'$(the partial sums in the second coordinate of g, m) is convergent by [26, (7), (37)]. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$ by [26, (37)], [4, (51), (52)], [14, (11)]. For every natural number k, $\mathcal{Q}[k]$ from [1, Sch. 2]. For every natural number m, $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) \leqslant \liminf$(the lim in the second coordinate of the partial sums in the second coordinate of f) by [26, (37), (38)]. For every object m such that $m \in \operatorname{dom} s$ holds $0 \leqslant s(m)$ by [4, (51), (52)], [26, (23)]. $\square$

(84) Suppose for every element m of $\mathbb{N}$, $s(m) = \liminf \text{curry}(f, m)$. Then $\sum s \leqslant \liminf$(the lim in the first coordinate of the partial sums in the first coordinate of f). The theorem is a consequence of (32), (83), (38), and (40).

Now we state the proposition:

(85) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a sequence s of extended reals, and natural numbers n, m. Then

(i) if for every natural numbers i, j, $f(i, j) \leqslant s(i)$, then (the partial sums in the first coordinate of $f)(n, m) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)$, and

(ii) if for every natural numbers i, j, $f(i, j) \leqslant s(j)$, then (the partial sums in the second coordinate of $f)(n, m) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ (the partial sums in the second coordinate of $f)(n, \$_1) \leqslant (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k, $\mathcal{P}[k]$ from [1, Sch. 2]. $\square$

Let us consider a sequence s of extended reals and an extended real number r. Now we state the propositions:

(86) If for every natural number n, $s(n) \leqslant r$, then $\limsup s \leqslant r$.
PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = r$. Consider f being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $f(n) = \mathcal{F}(n)$ from [7, Sch. 4]. For every natural number n, $f(n) = r$. For every natural number n, $s(n) \leqslant f(n)$. □

(87) If for every natural number n, $r \leqslant s(n)$, then $r \leqslant \liminf s$.
PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = r$. Consider f being a function from $\mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $f(n) = \mathcal{F}(n)$ from [7, Sch. 4]. For every natural number n, $f(n) = r$. For every natural number n, $f(n) \leqslant s(n)$. □

Now we state the proposition:

(88) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) for every natural numbers i_1, i_2, j such that $i_1 \leqslant i_2$ holds (the partial sums in the first coordinate of $f)(i_1, j) \leqslant$ (the partial sums in the first coordinate of $f)(i_2, j)$, and

(ii) for every natural numbers i, j_1, j_2 such that $j_1 \leqslant j_2$ holds (the partial sums in the second coordinate of $f)(i, j_1) \leqslant$ (the partial sums in the second coordinate of $f)(i, j_2)$.

Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and natural numbers i, j, k. Now we state the propositions:

(89) Suppose for every element m of $\mathbb{N}$, $\text{curry}'(f, m)$ is non-decreasing and $i \leqslant j$. Then (the partial sums in the second coordinate of $f)(i, k) \leqslant$ (the partial sums in the second coordinate of $f)(j, k)$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ (the partial sums in the second coordinate of $f)(i, \$_1) \leqslant$ (the partial sums in the second coordinate of $f)(j, \$_1)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [26, (7)]. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(90) Suppose for every element n of $\mathbb{N}$, $\text{curry}(f, n)$ is non-decreasing and $i \leqslant j$. Then (the partial sums in the first coordinate of $f)(k, i) \leqslant$ (the partial sums in the first coordinate of $f)(k, j)$. The theorem is a consequence of (32), (89), and (39).

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a sequence s of extended reals. Now we state the propositions:

(91) Suppose for every element m of $\mathbb{N}$, $\text{curry}'(f, m)$ is non-decreasing and $s(m) = \lim \text{curry}'(f, m)$. Then

(i) the lim in the second coordinate of the partial sums in the second coordinate of f is non-decreasing, and

(ii) $\sum s = \lim$(the lim in the second coordinate of the partial sums in the second coordinate of f).

PROOF: $\sum s \leqslant \liminf$(the lim in the second coordinate of the partial sums in the second coordinate of f). For every natural numbers n, m, $f(n,m) \leqslant s(m)$ by [26, (37)], [6, (3)]. For every natural numbers n, m such that $m \leqslant n$ holds (the lim in the second coordinate of the partial sums in the second coordinate of $f)(m) \leqslant$ (the lim in the second coordinate of the partial sums in the second coordinate of $f)(n)$ by [26, (37)], (89), [26, (38)]. For every natural number n, (the lim in the second coordinate of the partial sums in the second coordinate of $f)(n) \leqslant \sum s$ by [26, (37)], [4, (39)], (87), [26, (41)]. $\limsup$(the lim in the second coordinate of the partial sums in the second coordinate of f) $\leqslant \sum s$. □

(92) Suppose for every element m of $\mathbb{N}$, curry(f, m) is non-decreasing and $s(m) = \lim \mathrm{curry}(f, m)$. Then

(i) the lim in the first coordinate of the partial sums in the first coordinate of f is non-decreasing, and

(ii) $\sum s = \lim$(the lim in the first coordinate of the partial sums in the first coordinate of f).

The theorem is a consequence of (32), (91), (33), and (40).

5. Pringsheim Sense Convergence for Extended Real-Valued Double Sequences

Let us consider a function f from $\mathbb{N}\times\mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(93) If f is P-convergent to $+\infty$, then f is not P-convergent to $-\infty$ and f is not P-convergent to a finite limit.

(94) If f is P-convergent to $-\infty$, then f is not P-convergent to $+\infty$ and f is not P-convergent to a finite limit.

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that f is P-convergent if and only if

(Def. 17) f is P-convergent to a finite limit or P-convergent to $+\infty$ or P-convergent to $-\infty$.

Assume f is P-convergent. The functor P-lim f yielding an extended real is defined by

(Def. 18) there exists a real number p such that $it = p$ and for every real number e such that $0 < e$ there exists a natural number N such that for every natural numbers n, m such that $n \geqslant N$ and $m \geqslant N$ holds $|f(n,m) - it| < e$

and f is P-convergent to a finite limit or $it = +\infty$ and f is P-convergent to $+\infty$ or $it = -\infty$ and f is P-convergent to $-\infty$.

Now we state the propositions:

(95) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a real number r. Suppose for every natural numbers n, m, $f(n, m) = r$. Then

(i) f is P-convergent to a finite limit, and

(ii) P-lim $f = r$.

(96) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f(n_1, m_1) \leqslant f(n_2, m_2)$. Then

(i) f is P-convergent, and

(ii) P-lim $f = \sup \operatorname{rng} f$.

(97) Let us consider functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers n, m, $f_1(n, m) \leqslant f_2(n, m)$. Then $\sup \operatorname{rng} f_1 \leqslant \sup \operatorname{rng} f_2$.

(98) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m. Then $f(n, m) \leqslant \sup \operatorname{rng} f$.

Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and an extended real number K. Now we state the propositions:

(99) If for every natural numbers n, m, $f(n, m) \leqslant K$, then $\sup \operatorname{rng} f \leqslant K$.

(100) If $K \neq +\infty$ and for every natural numbers n, m, $f(n, m) \leqslant K$, then $\sup \operatorname{rng} f < +\infty$.

Now we state the propositions:

(101) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\sup \operatorname{rng} f \neq +\infty$ if and only if there exists a real number K such that $0 < K$ and for every natural numbers n, m, $f(n, m) \leqslant K$.

(102) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an extended real c. Suppose for every natural numbers n, m, $f(n, m) = c$. Then

(i) f is P-convergent, and

(ii) P-lim $f = c$, and

(iii) P-lim $f = \sup \operatorname{rng} f$.

(103) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and without $-\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_1(n_1, m_1) \leqslant f_1(n_2, m_2)$ and for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_2(n_1, m_1) \leqslant f_2(n_2, m_2)$ and for every natural numbers n, m, $f_1(n, m) + f_2(n, m) = f(n, m)$. Then

(i) f is P-convergent, and

(ii) P-lim $f = \sup \operatorname{rng} f$, and

(iii) P-lim $f =$ P-lim $f_1 +$ P-lim f_2, and

(iv) $\sup \operatorname{rng} f = \sup \operatorname{rng} f_1 + \sup \operatorname{rng} f_2$.

The theorem is a consequence of (96) and (101).

Let us consider a without $-\infty$ function f_1 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a function f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a real number c. Now we state the propositions:

(104) Suppose $0 \leqslant c$ and for every natural numbers n, m, $f_2(n,m) = c \cdot f_1(n,m)$. Then

(i) $\sup \operatorname{rng} f_2 = c \cdot \sup \operatorname{rng} f_1$, and

(ii) f_2 is without $-\infty$.

The theorem is a consequence of (102) and (101).

(105) Suppose $0 \leqslant c$ and for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_1(n_1, m_1) \leqslant f_1(n_2, m_2)$ and for every natural numbers n, m, $f_2(n,m) = c \cdot f_1(n,m)$. Then

(i) for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leqslant n_2$ and $m_1 \leqslant m_2$ holds $f_2(n_1, m_1) \leqslant f_2(n_2, m_2)$, and

(ii) f_2 is without $-\infty$ and P-convergent, and

(iii) P-lim $f_2 = \sup \operatorname{rng} f_2$, and

(iv) P-lim $f_2 = c \cdot$ P-lim f_1.

The theorem is a consequence of (96) and (104).

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