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The Differentiable Functions from $\mathbb{R}$ into $\mathcal{R}^n$

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Summary. In control engineering, differentiable partial functions from $\mathbb{R}$ into $\mathcal{R}^n$ play a very important role. In this article, we formalized basic properties of such functions.

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The notation and terminology used in this paper are introduced in the following articles: [25], [26], [6], [2], [27], [8], [7], [24], [1], [4], [3], [5], [9], [22], [20], [28], [21], [10], [23], [17], [13], [11], [12], [15], [19], [18], [16], and [14].

Let us observe that there exists a sequence of real numbers which is convergent to 0 and non-zero.

For simplicity, we adopt the following convention: x_0, r denote real numbers, i, m denote elements of $\mathbb{N}$, n denotes a non empty element of $\mathbb{N}$, Y denotes a subset of $\mathbb{R}$, Z denotes an open subset of $\mathbb{R}$, and f_1, f_2 denote partial functions from $\mathbb{R}$ to $\mathcal{R}^n$.

The following proposition is true

(1) For all partial functions f_1, f_2 from $\mathbb{R}$ to $\mathcal{R}^m$ holds $f_1 - f_2 = f_1 + -f_2$.

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Let n be a non empty element of $\mathbb{N}$, let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let x be a real number. We say that f is differentiable in x if and only if:

(Def. 1) There exists a partial function g from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $f = g$ and g is differentiable in x.

One can prove the following proposition

(2) Let n be a non empty element of $\mathbb{N}$, f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, h be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and x be a real number. Suppose $h = f$. Then f is differentiable in x if and only if h is differentiable in x.

Let n be a non empty element of $\mathbb{N}$, let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, and let x be a real number. The functor $f'(x)$ yields an element of $\mathcal{R}^n$ and is defined as follows:

(Def. 2) There exists a partial function g from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ such that $f = g$ and $f'(x) = g'(x)$.

One can prove the following proposition

(3) Let n be a non empty element of $\mathbb{N}$, f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, h be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and x be a real number. If $h = f$, then $f'(x) = h'(x)$.

Let us consider n, f, X. We say that f is differentiable on X if and only if:

(Def. 3) $X \subseteq \operatorname{dom} f$ and for every x such that $x \in X$ holds $f{\restriction}X$ is differentiable in x.

The following propositions are true:

(4) If f is differentiable on X, then X is a subset of $\mathbb{R}$.

(5) f is differentiable on Z iff $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds f is differentiable in x.

(6) If f is differentiable on Y, then Y is open.

Let us consider n, f, X. Let us assume that f is differentiable on X. The functor $f'_{\restriction X}$ yields a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and is defined by:

(Def. 4) $\operatorname{dom}(f'_{\restriction X}) = X$ and for every x such that $x \in X$ holds $f'_{\restriction X}(x) = f'(x)$.

One can prove the following propositions:

(7) Suppose $Z \subseteq \operatorname{dom} f$ and there exists an element r of $\mathcal{R}^n$ such that $\operatorname{rng} f = \{r\}$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $(f'_{\restriction Z})_x = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(8) Let x_0 be a real number, f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$, g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and N be a neighbourhood of x_0. Suppose $f = g$ and f is differentiable in x_0 and $N \subseteq \operatorname{dom} f$. Let given h, c.

Suppose $\operatorname{rng} c = \{x_0\}$ and $\operatorname{rng}(h+c) \subseteq N$. Then $h^{-1} \cdot ((g_*(h+c)) - (g_*c))$ is convergent and $f'(x_0) = \lim(h^{-1} \cdot ((g_*(h+c)) - (g_*c)))$.

(9) If f is differentiable in x_0, then $r \cdot f$ is differentiable in x_0 and $(r \cdot f)'(x_0) = r \cdot f'(x_0)$.

(10) If f is differentiable in x_0, then $-f$ is differentiable in x_0 and $(-f)'(x_0) = -f'(x_0)$.

(11) If f_1 is differentiable in x_0 and f_2 is differentiable in x_0, then $f_1 + f_2$ is differentiable in x_0 and $(f_1 + f_2)'(x_0) = {f_1}'(x_0) + {f_2}'(x_0)$.

(12) If f_1 is differentiable in x_0 and f_2 is differentiable in x_0, then $f_1 - f_2$ is differentiable in x_0 and $(f_1 - f_2)'(x_0) = {f_1}'(x_0) - {f_2}'(x_0)$.

(13) Suppose $Z \subseteq \operatorname{dom} f$ and f is differentiable on Z. Then $r \cdot f$ is differentiable on Z and for every x such that $x \in Z$ holds $(r \cdot f)'_{\restriction Z}(x) = r \cdot f'(x)$.

(14) If $Z \subseteq \operatorname{dom} f$ and f is differentiable on Z, then $-f$ is differentiable on Z and for every x such that $x \in Z$ holds $(-f)'_{\restriction Z}(x) = -f'(x)$.

(15) Suppose $Z \subseteq \operatorname{dom}(f_1 + f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 + f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 + f_2)'_{\restriction Z}(x) = {f_1}'(x) + {f_2}'(x)$.

(16) Suppose $Z \subseteq \operatorname{dom}(f_1 - f_2)$ and f_1 is differentiable on Z and f_2 is differentiable on Z. Then $f_1 - f_2$ is differentiable on Z and for every x such that $x \in Z$ holds $(f_1 - f_2)'_{\restriction Z}(x) = {f_1}'(x) - {f_2}'(x)$.

(17) If $Z \subseteq \operatorname{dom} f$ and $f \restriction Z$ is constant, then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = \langle \underbrace{0, \ldots, 0}_{n} \rangle$.

(18) Let r, p be elements of $\mathcal{R}^n$. Suppose $Z \subseteq \operatorname{dom} f$ and for every x such that $x \in Z$ holds $f_x = x \cdot r + p$. Then f is differentiable on Z and for every x such that $x \in Z$ holds $f'_{\restriction Z}(x) = r$.

(19) For every real number x_0 such that f is differentiable in x_0 holds f is continuous in x_0.

(20) If f is differentiable on X, then $f \restriction X$ is continuous.

(21) If f is differentiable on X and $Z \subseteq X$, then f is differentiable on Z.

Let n be a non empty element of $\mathbb{N}$ and let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. We say that f is differentiable if and only if:

(Def. 5) f is differentiable on $\operatorname{dom} f$.

Let us consider n. One can check that $\mathbb{R} \longmapsto \langle \underbrace{0, \ldots, 0}_{n} \rangle$ is differentiable.

Let us consider n. Note that there exists a function from $\mathbb{R}$ into $\mathcal{R}^n$ which is differentiable.

One can prove the following proposition

(22) For every differentiable partial function f from $\mathbb{R}$ to $\mathcal{R}^n$ such that $Z \subseteq \operatorname{dom} f$ holds f is differentiable on Z.

In the sequel G_1, R are rests of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and D_1, L are linears of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Next we state a number of propositions:

(23) Let R be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose R is total. Then R is rest-like if and only if for every real number r such that $r > 0$ there exists a real number d such that $d > 0$ and for every real number z such that $z \neq 0$ and $|z| < d$ holds $|z|^{-1} \cdot \|R_z\| < r$.

(24) Let g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and x_0 be a real number. Suppose $1 \leq i \leq n$ and g is differentiable in x_0. Then $\mathrm{Proj}(i, n) \cdot g$ is differentiable in x_0 and $(\mathrm{Proj}(i, n))(g'(x_0)) = (\mathrm{Proj}(i, n) \cdot g)'(x_0)$.

(25) Let g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and x_0 be a real number. Then g is differentiable in x_0 if and only if for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\mathrm{Proj}(i, n) \cdot g$ is differentiable in x_0.

(26) Let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and x_0 be a real number. Suppose $1 \leq i \leq n$ and f is differentiable in x_0. Then $\mathrm{Proj}(i, n) \cdot f$ is differentiable in x_0 and $(\mathrm{Proj}(i, n))(f'(x_0)) = (\mathrm{Proj}(i, n) \cdot f)'(x_0)$.

(27) Let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$ and x_0 be a real number. Then f is differentiable in x_0 if and only if for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\mathrm{Proj}(i, n) \cdot f$ is differentiable in x_0.

(28) Let g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $1 \leq i \leq n$ and g is differentiable on X. Then $\mathrm{Proj}(i, n) \cdot g$ is differentiable on X and $\mathrm{Proj}(i, n) \cdot g'_{\restriction X} = (\mathrm{Proj}(i, n) \cdot g)'_{\restriction X}$.

(29) Let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Suppose $1 \leq i \leq n$ and f is differentiable on X. Then $\mathrm{Proj}(i, n) \cdot f$ is differentiable on X and $\mathrm{Proj}(i, n) \cdot f'_{\restriction X} = (\mathrm{Proj}(i, n) \cdot f)'_{\restriction X}$.

(30) Let g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Then g is differentiable on X if and only if for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\mathrm{Proj}(i, n) \cdot g$ is differentiable on X.

(31) Let f be a partial function from $\mathbb{R}$ to $\mathcal{R}^n$. Then f is differentiable on X if and only if for every element i of $\mathbb{N}$ such that $1 \leq i \leq n$ holds $\mathrm{Proj}(i, n) \cdot f$ is differentiable on X.

(32) For every function J from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$ and for every point x_0 of $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $J = \mathrm{proj}(1, 1)$ holds J is continuous in x_0.

(33) For every function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $I = \mathrm{proj}(1, 1)^{-1}$ holds I is continuous in x_0.

(34) Let S, T be real normed spaces, f_1 be a partial function from S to $\mathbb{R}$, f_2 be a partial function from $\mathbb{R}$ to T, and x_0 be a point of S. Suppose $x_0 \in \mathrm{dom}(f_2 \cdot f_1)$ and f_1 is continuous in x_0 and f_2 is continuous in $(f_1)_{x_0}$. Then $f_2 \cdot f_1$ is continuous in x_0.

(35) Let J be a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$, x_0 be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, y_0 be an element of $\mathbb{R}$, g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and f

be a partial function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $J = \mathrm{proj}(1,1)$ and $x_0 \in \mathrm{dom}\, f$ and $y_0 \in \mathrm{dom}\, g$ and $x_0 = \langle y_0 \rangle$ and $f = g \cdot J$. Then f is continuous in x_0 if and only if g is continuous in y_0.

(36) Let I be a function from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, x_0 be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, y_0 be an element of $\mathbb{R}$, g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and f be a partial function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $I = \mathrm{proj}(1,1)^{-1}$ and $x_0 \in \mathrm{dom}\, f$ and $y_0 \in \mathrm{dom}\, g$ and $x_0 = \langle y_0 \rangle$ and $f \cdot I = g$. Then f is continuous in x_0 if and only if g is continuous in y_0.

(37) For every function I from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$ such that $I = \mathrm{proj}(1,1)^{-1}$ holds I is differentiable in x_0 and $I'(x_0) = \langle 1 \rangle$.

Let n be a non empty element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\mathbb{R}$, and let x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. We say that f is differentiable in x if and only if the condition (Def. 6) is satisfied.

(Def. 6) There exists a partial function g from $\mathcal{R}^n$ to $\mathbb{R}$ and there exists an element y of $\mathcal{R}^n$ such that $f = g$ and $x = y$ and g is differentiable in y.

Let n be a non empty element of $\mathbb{N}$, let f be a partial function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\mathbb{R}$, and let x be a point of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. The functor $f'(x)$ yields a function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\mathbb{R}$ and is defined by:

(Def. 7) There exists a partial function g from $\mathcal{R}^n$ to $\mathbb{R}$ and there exists an element y of $\mathcal{R}^n$ such that $f = g$ and $x = y$ and $f'(x) = g'(y)$.

We now state several propositions:

(38) Let J be a function from $\mathcal{R}^1$ into $\mathbb{R}$ and x_0 be an element of $\mathcal{R}^1$. If $J = \mathrm{proj}(1,1)$, then J is differentiable in x_0 and $J'(x_0) = J$.

(39) Let J be a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$ and x_0 be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$. If $J = \mathrm{proj}(1,1)$, then J is differentiable in x_0 and $J'(x_0) = J$.

(40) Let I be a function from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$. Suppose $I = \mathrm{proj}(1,1)^{-1}$. Then

(i) for every rest R of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $R \cdot I$ is a rest of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and

(ii) for every linear operator L from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $L \cdot I$ is a linear of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

(41) Let J be a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$. Suppose $J = \mathrm{proj}(1,1)$. Then

(i) for every rest R of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $R \cdot J$ is a rest of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and

(ii) for every linear L of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ holds $L \cdot J$ is a bounded linear operator from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

(42) Let I be a function from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, x_0 be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, y_0 be an element of $\mathbb{R}$, g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and f be a partial function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $I = \mathrm{proj}(1,1)^{-1}$ and $x_0 \in \mathrm{dom}\, f$ and $y_0 \in \mathrm{dom}\, g$ and $x_0 = \langle y_0 \rangle$ and $f \cdot I = g$ and f is

differentiable in x_0. Then g is differentiable in y_0 and $g'(y_0) = f'(x_0)(\langle 1 \rangle)$ and for every element r of $\mathbb{R}$ holds $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$.

(43) Let I be a function from $\mathbb{R}$ into $\langle \mathcal{E}^1, \|\cdot\| \rangle$, x_0 be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, y_0 be an element of $\mathbb{R}$, g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and f be a partial function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $I = \mathrm{proj}(1,1)^{-1}$ and $x_0 \in \mathrm{dom}\, f$ and $y_0 \in \mathrm{dom}\, g$ and $x_0 = \langle y_0 \rangle$ and $f \cdot I = g$. Then f is differentiable in x_0 if and only if g is differentiable in y_0.

(44) Let J be a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$, x_0 be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, y_0 be an element of $\mathbb{R}$, g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and f be a partial function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $J = \mathrm{proj}(1,1)$ and $x_0 \in \mathrm{dom}\, f$ and $y_0 \in \mathrm{dom}\, g$ and $x_0 = \langle y_0 \rangle$ and $f = g \cdot J$. Then f is differentiable in x_0 if and only if g is differentiable in y_0.

(45) Let J be a function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ into $\mathbb{R}$, x_0 be a point of $\langle \mathcal{E}^1, \|\cdot\| \rangle$, y_0 be an element of $\mathbb{R}$, g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and f be a partial function from $\langle \mathcal{E}^1, \|\cdot\| \rangle$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $J = \mathrm{proj}(1,1)$ and $x_0 \in \mathrm{dom}\, f$ and $y_0 \in \mathrm{dom}\, g$ and $x_0 = \langle y_0 \rangle$ and $f = g \cdot J$ and g is differentiable in y_0. Then f is differentiable in x_0 and $g'(y_0) = f'(x_0)(\langle 1 \rangle)$ and for every element r of $\mathbb{R}$ holds $f'(x_0)(\langle r \rangle) = r \cdot g'(y_0)$.

(46) Let R be a rest of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $R_0 = 0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}$. Let e be a real number. Suppose $e > 0$. Then there exists a real number d such that $d > 0$ and for every real number h such that $|h| < d$ holds $\|R_h\| \leq e \cdot |h|$.

In the sequel m, n denote non empty elements of $\mathbb{N}$.

One can prove the following propositions:

(47) For every rest R of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and for every bounded linear operator L from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^m, \|\cdot\| \rangle$ holds $L \cdot R$ is a rest of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

(48) Let R_1 be a rest of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $(R_1)_0 = 0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}$. Let R_2 be a rest of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $(R_2)_{0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}} = 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}$. Let L be a linear of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Then $R_2 \cdot (L + R_1)$ is a rest of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

(49) Let R_1 be a rest of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose $(R_1)_0 = 0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}$. Let R_2 be a rest of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose $(R_2)_{0_{\langle \mathcal{E}^n, \|\cdot\| \rangle}} = 0_{\langle \mathcal{E}^m, \|\cdot\| \rangle}$. Let L_1 be a linear of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ and L_2 be a bounded linear operator from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ into $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Then $L_2 \cdot R_1 + R_2 \cdot (L_1 + R_1)$ is a rest of $\langle \mathcal{E}^m, \|\cdot\| \rangle$.

(50) Let x_0 be an element of $\mathbb{R}$ and g be a partial function from $\mathbb{R}$ to $\langle \mathcal{E}^n, \|\cdot\| \rangle$. Suppose g is differentiable in x_0. Let f be a partial function from $\langle \mathcal{E}^n, \|\cdot\| \rangle$ to $\langle \mathcal{E}^m, \|\cdot\| \rangle$. Suppose f is differentiable in g_{x_0}. Then $f \cdot g$ is differentiable in x_0 and $(f \cdot g)'(x_0) = f'(g_{x_0})(g'(x_0))$.

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