Second-Order Partial Differentiation of Real Binary Functions

Bing Xie
Qingdao University of Science
and Technology
China

Xiquan Liang
Qingdao University of Science
and Technology
China

Xiuzhuan Shen
Qingdao University of Science
and Technology
China

Summary. In this article we define second-order partial differentiation of real binary functions and discuss the relation of second-order partial derivatives and partial derivatives defined in [17].

The articles [15], [3], [4], [16], [5], [10], [1], [8], [11], [9], [2], [14], [6], [13], [12], [7], and [17] provide the notation and terminology for this paper.

1. Second-Order Partial Derivatives

For simplicity, we adopt the following convention: x, x_0, y, y_0, r are real numbers, z, z_0 are elements of $\mathcal{R}^2$, f, f_1, f_2 are partial functions from $\mathcal{R}^2$ to $\mathbb{R}$, R is a rest, and L is a linear function.

Let us note that every rest is total.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. The functor pdiff1(f, z) yielding a function from $\mathcal{R}^2$ into $\mathbb{R}$ is defined as follows:

(Def. 1)   For every z such that $z \in \mathcal{R}^2$ holds $(\text{pdiff1}(f, z))(z) = \text{partdiff1}(f, z)$.

The functor pdiff2(f, z) yields a function from $\mathcal{R}^2$ into $\mathbb{R}$ and is defined as follows:

(Def. 2)   For every z such that $z \in \mathcal{R}^2$ holds $(\text{pdiff2}(f,z))(z) = \text{partdiff2}(f,z)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. We say that f is partial differentiable on 1st-1st coordinate in z if and only if the condition (Def. 3) is satisfied.

(Def. 3)   There exist real numbers x_0, y_0 such that
   (i) $z = \langle x_0, y_0 \rangle$, and
   (ii)   there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(\text{pdiff1}(f,z),z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\text{SVF1}(\text{pdiff1}(f,z),z))(x) - (\text{SVF1}(\text{pdiff1}(f,z),z))(x_0) = L(x-x_0) + R(x-x_0)$.

We say that f is partial differentiable on 1st-2nd coordinate in z if and only if the condition (Def. 4) is satisfied.

(Def. 4)   There exist real numbers x_0, y_0 such that
   (i) $z = \langle x_0, y_0 \rangle$, and
   (ii)   there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF2}(\text{pdiff1}(f,z),z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\text{SVF2}(\text{pdiff1}(f,z),z))(y) - (\text{SVF2}(\text{pdiff1}(f,z),z))(y_0) = L(y-y_0) + R(y-y_0)$.

We say that f is partial differentiable on 2nd-1st coordinate in z if and only if the condition (Def. 5) is satisfied.

(Def. 5)   There exist real numbers x_0, y_0 such that
   (i) $z = \langle x_0, y_0 \rangle$, and
   (ii)   there exists a neighbourhood N of x_0 such that $N \subseteq \text{dom SVF1}(\text{pdiff2}(f,z),z)$ and there exist L, R such that for every x such that $x \in N$ holds $(\text{SVF1}(\text{pdiff2}(f,z),z))(x) - (\text{SVF1}(\text{pdiff2}(f,z),z))(x_0) = L(x-x_0) + R(x-x_0)$.

We say that f is partial differentiable on 2nd-2nd coordinate in z if and only if the condition (Def. 6) is satisfied.

(Def. 6)   There exist real numbers x_0, y_0 such that
   (i) $z = \langle x_0, y_0 \rangle$, and
   (ii)   there exists a neighbourhood N of y_0 such that $N \subseteq \text{dom SVF2}(\text{pdiff2}(f,z),z)$ and there exist L, R such that for every y such that $y \in N$ holds $(\text{SVF2}(\text{pdiff2}(f,z),z))(y) - (\text{SVF2}(\text{pdiff2}(f,z),z))(y_0) = L(y-y_0) + R(y-y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us assume that f is partial differentiable on 1st-1st coordinate in z. The functor hpartdiff11(f,z) yields a real number and is defined by the condition (Def. 7).

(Def. 7)   There exist real numbers x_0, y_0 such that
   (i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq$ dom $\text{SVF1}(\text{pdiff1}(f, z), z)$ and there exist L, R such that $\text{hpartdiff11}(f, z) = L(1)$ and for every x such that $x \in N$ holds $(\text{SVF1}(\text{pdiff1}(f, z), z))(x) - (\text{SVF1}(\text{pdiff1}(f, z), z))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us assume that f is partial differentiable on 1st-2nd coordinate in z. The functor $\text{hpartdiff12}(f, z)$ yielding a real number is defined by the condition (Def. 8).

(Def. 8) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq$ dom $\text{SVF2}(\text{pdiff1}(f, z), z)$ and there exist L, R such that $\text{hpartdiff12}(f, z) = L(1)$ and for every y such that $y \in N$ holds $(\text{SVF2}(\text{pdiff1}(f, z), z))(y) - (\text{SVF2}(\text{pdiff1}(f, z), z))(y_0) = L(y - y_0) + R(y - y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us assume that f is partial differentiable on 2nd-1st coordinate in z. The functor $\text{hpartdiff21}(f, z)$ yielding a real number is defined by the condition (Def. 9).

(Def. 9) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of x_0 such that $N \subseteq$ dom $\text{SVF1}(\text{pdiff2}(f, z), z)$ and there exist L, R such that $\text{hpartdiff21}(f, z) = L(1)$ and for every x such that $x \in N$ holds $(\text{SVF1}(\text{pdiff2}(f, z), z))(x) - (\text{SVF1}(\text{pdiff2}(f, z), z))(x_0) = L(x - x_0) + R(x - x_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let z be an element of $\mathcal{R}^2$. Let us assume that f is partial differentiable on 2nd-2nd coordinate in z. The functor $\text{hpartdiff22}(f, z)$ yields a real number and is defined by the condition (Def. 10).

(Def. 10) There exist real numbers x_0, y_0 such that

(i) $z = \langle x_0, y_0 \rangle$, and

(ii) there exists a neighbourhood N of y_0 such that $N \subseteq$ dom $\text{SVF2}(\text{pdiff2}(f, z), z)$ and there exist L, R such that $\text{hpartdiff22}(f, z) = L(1)$ and for every y such that $y \in N$ holds $(\text{SVF2}(\text{pdiff2}(f, z), z))(y) - (\text{SVF2}(\text{pdiff2}(f, z), z))(y_0) = L(y - y_0) + R(y - y_0)$.

Next we state several propositions:

(1) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in z, then $\text{SVF1}(\text{pdiff1}(f, z), z)$ is differentiable in x_0.

(2) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in z, then $\text{SVF2}(\text{pdiff1}(f, z), z)$ is differentiable in y_0.

(3) If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in z, then $\text{SVF1}(\text{pdiff2}(f, z), z)$ is differentiable in x_0.

(4)   If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in z, then SVF2(pdiff2(f, z), z) is differentiable in y_0.

(5)   If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-1st coordinate in z, then hpartdiff11(f, z) = (SVF1(pdiff1(f, z), z))$'(x_0)$.

(6)   If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 1st-2nd coordinate in z, then hpartdiff12(f, z) = (SVF2(pdiff1(f, z), z))$'(y_0)$.

(7)   If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-1st coordinate in z, then hpartdiff21(f, z) = (SVF1(pdiff2(f, z), z))$'(x_0)$.

(8)   If $z = \langle x_0, y_0 \rangle$ and f is partial differentiable on 2nd-2nd coordinate in z, then hpartdiff22(f, z) = (SVF2(pdiff2(f, z), z))$'(y_0)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. We say that f is partial differentiable on 1st-1st coordinate on Z if and only if:

(Def. 11) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\upharpoonright}Z$ is partial differentiable on 1st-1st coordinate in z.

We say that f is partial differentiable on 1st-2nd coordinate on Z if and only if:

(Def. 12) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\upharpoonright}Z$ is partial differentiable on 1st-2nd coordinate in z.

We say that f is partial differentiable on 2nd-1st coordinate on Z if and only if:

(Def. 13) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\upharpoonright}Z$ is partial differentiable on 2nd-1st coordinate in z.

We say that f is partial differentiable on 2nd-2nd coordinate on Z if and only if:

(Def. 14) $Z \subseteq \operatorname{dom} f$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f{\upharpoonright}Z$ is partial differentiable on 2nd-2nd coordinate in z.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable on 1st-1st coordinate on Z. The functor $f_{\upharpoonright Z}^{\text{1st}-\text{1st}}$ yields a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and is defined by:

(Def. 15) $\operatorname{dom}(f_{\upharpoonright Z}^{\text{1st}-\text{1st}}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f_{\upharpoonright Z}^{\text{1st}-\text{1st}}(z) = $ hpartdiff11(f, z).

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable on 1st-2nd coordinate on Z. The functor $f_{\upharpoonright Z}^{\text{1st}-\text{2nd}}$ yielding a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ is defined by:

(Def. 16) $\operatorname{dom}(f_{\upharpoonright Z}^{\text{1st}-\text{2nd}}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f_{\upharpoonright Z}^{\text{1st}-\text{2nd}}(z) = $ hpartdiff12(f, z).

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable on 2nd-1st coordinate on Z. The functor $f_{\upharpoonright Z}^{\text{2nd}-\text{1st}}$ yields a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and is defined by:

(Def. 17) $\mathrm{dom}(f_{\restriction Z}^{\mathrm{2nd-1st}}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f_{\restriction Z}^{\mathrm{2nd-1st}}(z) = \mathrm{hpartdiff21}(f, z)$.

Let f be a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and let Z be a set. Let us assume that f is partial differentiable on 2nd-2nd coordinate on Z. The functor $f_{\restriction Z}^{\mathrm{2nd-2nd}}$ yields a partial function from $\mathcal{R}^2$ to $\mathbb{R}$ and is defined by:

(Def. 18) $\mathrm{dom}(f_{\restriction Z}^{\mathrm{2nd-2nd}}) = Z$ and for every element z of $\mathcal{R}^2$ such that $z \in Z$ holds $f_{\restriction Z}^{\mathrm{2nd-2nd}}(z) = \mathrm{hpartdiff22}(f, z)$.

2. Main Properties of Second-Order Partial Derivatives

One can prove the following propositions:

(9) f is partial differentiable on 1st-1st coordinate in z if and only if $\mathrm{pdiff1}(f, z)$ is partial differentiable on 1st coordinate in z.

(10) f is partial differentiable on 1st-2nd coordinate in z if and only if $\mathrm{pdiff1}(f, z)$ is partial differentiable on 2nd coordinate in z.

(11) f is partial differentiable on 2nd-1st coordinate in z if and only if $\mathrm{pdiff2}(f, z)$ is partial differentiable on 1st coordinate in z.

(12) f is partial differentiable on 2nd-2nd coordinate in z if and only if $\mathrm{pdiff2}(f, z)$ is partial differentiable on 2nd coordinate in z.

(13) f is partial differentiable on 1st-1st coordinate in z if and only if $\mathrm{pdiff1}(f, z)$ is partially differentiable in z w.r.t. coordinate 1.

(14) f is partial differentiable on 1st-2nd coordinate in z if and only if $\mathrm{pdiff1}(f, z)$ is partially differentiable in z w.r.t. coordinate 2.

(15) f is partial differentiable on 2nd-1st coordinate in z if and only if $\mathrm{pdiff2}(f, z)$ is partially differentiable in z w.r.t. coordinate 1.

(16) f is partial differentiable on 2nd-2nd coordinate in z if and only if $\mathrm{pdiff2}(f, z)$ is partially differentiable in z w.r.t. coordinate 2.

(17) If f is partial differentiable on 1st-1st coordinate in z, then $\mathrm{hpartdiff11}(f, z) = \mathrm{partdiff1}(\mathrm{pdiff1}(f, z), z)$.

(18) If f is partial differentiable on 1st-2nd coordinate in z, then $\mathrm{hpartdiff12}(f, z) = \mathrm{partdiff2}(\mathrm{pdiff1}(f, z), z)$.

(19) If f is partial differentiable on 2nd-1st coordinate in z, then $\mathrm{hpartdiff21}(f, z) = \mathrm{partdiff1}(\mathrm{pdiff2}(f, z), z)$.

(20) If f is partial differentiable on 2nd-2nd coordinate in z, then $\mathrm{hpartdiff22}(f, z) = \mathrm{partdiff2}(\mathrm{pdiff2}(f, z), z)$.

(21) Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\mathrm{proj}(1, 2))(z_0)$. Suppose f is partial differentiable on 1st-1st coordinate in z_0 and $N \subseteq \mathrm{dom}\,\mathrm{SVF1}(\mathrm{pdiff1}(f, z_0), z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\mathrm{rng}\,c =$

$\{(\text{proj}(1,2))(z_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}(\text{SVF1}(\text{pdiff1}(f, z_0), z_0) \cdot (h+c) - \text{SVF1}(\text{pdiff1}(f, z_0), z_0) \cdot c)$ is convergent and $\text{hpartdiff11}(f, z_0) = \lim(h^{-1}(\text{SVF1}(\text{pdiff1}(f, z_0), z_0) \cdot (h+c) - \text{SVF1}(\text{pdiff1}(f, z_0), z_0) \cdot c))$.

(22)   Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\text{proj}(2,2))(z_0)$. Suppose f is partial differentiable on 1st-2nd coordinate in z_0 and $N \subseteq \text{dom SVF2}(\text{pdiff1}(f, z_0), z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(2,2))(z_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}(\text{SVF2}(\text{pdiff1}(f, z_0), z_0) \cdot (h+c) - \text{SVF2}(\text{pdiff1}(f, z_0), z_0) \cdot c)$ is convergent and $\text{hpartdiff12}(f, z_0) = \lim(h^{-1}(\text{SVF2}(\text{pdiff1}(f, z_0), z_0) \cdot (h+c) - \text{SVF2}(\text{pdiff1}(f, z_0), z_0) \cdot c))$.

(23)   Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\text{proj}(1,2))(z_0)$. Suppose f is partial differentiable on 2nd-1st coordinate in z_0 and $N \subseteq \text{dom SVF1}(\text{pdiff2}(f, z_0), z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(1,2))(z_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}(\text{SVF1}(\text{pdiff2}(f, z_0), z_0) \cdot (h+c) - \text{SVF1}(\text{pdiff2}(f, z_0), z_0) \cdot c)$ is convergent and $\text{hpartdiff21}(f, z_0) = \lim(h^{-1}(\text{SVF1}(\text{pdiff2}(f, z_0), z_0) \cdot (h+c) - \text{SVF1}(\text{pdiff2}(f, z_0), z_0) \cdot c))$.

(24)   Let z_0 be an element of $\mathcal{R}^2$ and N be a neighbourhood of $(\text{proj}(2,2))(z_0)$. Suppose f is partial differentiable on 2nd-2nd coordinate in z_0 and $N \subseteq \text{dom SVF2}(\text{pdiff2}(f, z_0), z_0)$. Let h be a convergent to 0 sequence of real numbers and c be a constant sequence of real numbers. Suppose $\text{rng } c = \{(\text{proj}(2,2))(z_0)\}$ and $\text{rng}(h+c) \subseteq N$. Then $h^{-1}(\text{SVF2}(\text{pdiff2}(f, z_0), z_0) \cdot (h+c) - \text{SVF2}(\text{pdiff2}(f, z_0), z_0) \cdot c)$ is convergent and $\text{hpartdiff22}(f, z_0) = \lim(h^{-1}(\text{SVF2}(\text{pdiff2}(f, z_0), z_0) \cdot (h+c) - \text{SVF2}(\text{pdiff2}(f, z_0), z_0) \cdot c))$.

(25)   Suppose that
  (i) f_1 is partial differentiable on 1st-1st coordinate in z_0, and
  (ii) f_2 is partial differentiable on 1st-1st coordinate in z_0.
    Then $\text{pdiff1}(f_1, z_0) + \text{pdiff1}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0 and $\text{partdiff1}(\text{pdiff1}(f_1, z_0) + \text{pdiff1}(f_2, z_0), z_0) = \text{hpartdiff11}(f_1, z_0) + \text{hpartdiff11}(f_2, z_0)$.

(26)   Suppose that
  (i) f_1 is partial differentiable on 1st-2nd coordinate in z_0, and
  (ii) f_2 is partial differentiable on 1st-2nd coordinate in z_0.
    Then $\text{pdiff1}(f_1, z_0) + \text{pdiff1}(f_2, z_0)$ is partial differentiable on 2nd coordinate in z_0 and $\text{partdiff2}(\text{pdiff1}(f_1, z_0) + \text{pdiff1}(f_2, z_0), z_0) = \text{hpartdiff12}(f_1, z_0) + \text{hpartdiff12}(f_2, z_0)$.

(27)   Suppose that
  (i) f_1 is partial differentiable on 2nd-1st coordinate in z_0, and
  (ii) f_2 is partial differentiable on 2nd-1st coordinate in z_0.
    Then $\text{pdiff2}(f_1, z_0) + \text{pdiff2}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0 and $\text{partdiff1}(\text{pdiff2}(f_1, z_0) + \text{pdiff2}(f_2, z_0), z_0) = \text{hpartdiff21}(f_1, z_0) +$

hpartdiff21(f_2, z_0).

(28) Suppose that

  (i) f_1 is partial differentiable on 2nd-2nd coordinate in z_0, and

  (ii) f_2 is partial differentiable on 2nd-2nd coordinate in z_0.

  Then pdiff2(f_1, z_0) + pdiff2(f_2, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(pdiff2(f_1, z_0) + pdiff2$(f_2, z_0), z_0)$ = hpartdiff22(f_1, z_0) + hpartdiff22(f_2, z_0).

(29) Suppose that

  (i) f_1 is partial differentiable on 1st-1st coordinate in z_0, and

  (ii) f_2 is partial differentiable on 1st-1st coordinate in z_0.

  Then pdiff1(f_1, z_0) − pdiff1(f_2, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(pdiff1(f_1, z_0) − pdiff1$(f_2, z_0), z_0)$ = hpartdiff11(f_1, z_0) − hpartdiff11(f_2, z_0).

(30) Suppose that

  (i) f_1 is partial differentiable on 1st-2nd coordinate in z_0, and

  (ii) f_2 is partial differentiable on 1st-2nd coordinate in z_0.

  Then pdiff1(f_1, z_0) − pdiff1(f_2, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(pdiff1(f_1, z_0) − pdiff1$(f_2, z_0), z_0)$ = hpartdiff12(f_1, z_0) − hpartdiff12(f_2, z_0).

(31) Suppose that

  (i) f_1 is partial differentiable on 2nd-1st coordinate in z_0, and

  (ii) f_2 is partial differentiable on 2nd-1st coordinate in z_0.

  Then pdiff2(f_1, z_0) − pdiff2(f_2, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(pdiff2(f_1, z_0) − pdiff2$(f_2, z_0), z_0)$ = hpartdiff21(f_1, z_0) − hpartdiff21(f_2, z_0).

(32) Suppose that

  (i) f_1 is partial differentiable on 2nd-2nd coordinate in z_0, and

  (ii) f_2 is partial differentiable on 2nd-2nd coordinate in z_0.

  Then pdiff2(f_1, z_0) − pdiff2(f_2, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(pdiff2(f_1, z_0) − pdiff2$(f_2, z_0), z_0)$ = hpartdiff22(f_1, z_0) − hpartdiff22(f_2, z_0).

(33) Suppose f is partial differentiable on 1st-1st coordinate in z_0. Then r pdiff1(f, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(r pdiff1$(f, z_0), z_0)$ = $r \cdot$ hpartdiff11(f, z_0).

(34) Suppose f is partial differentiable on 1st-2nd coordinate in z_0. Then r pdiff1(f, z_0) is partial differentiable on 2nd coordinate in z_0 and partdiff2(r pdiff1$(f, z_0), z_0)$ = $r \cdot$ hpartdiff12(f, z_0).

(35) Suppose f is partial differentiable on 2nd-1st coordinate in z_0. Then r pdiff2(f, z_0) is partial differentiable on 1st coordinate in z_0 and partdiff1(r pdiff2$(f, z_0), z_0)$ = $r \cdot$ hpartdiff21(f, z_0).

(36)   Suppose f is partial differentiable on 2nd-2nd coordinate in z_0. Then $r\,\mathrm{pdiff2}(f, z_0)$ is partial differentiable on 2nd coordinate in z_0 and $\mathrm{partdiff2}(r\,\mathrm{pdiff2}(f, z_0), z_0) = r \cdot \mathrm{hpartdiff22}(f, z_0)$.

(37)   Suppose that
   (i) f_1 is partial differentiable on 1st-1st coordinate in z_0, and
   (ii) f_2 is partial differentiable on 1st-1st coordinate in z_0.
   Then $\mathrm{pdiff1}(f_1, z_0)\,\mathrm{pdiff1}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0.

(38)   Suppose that
   (i) f_1 is partial differentiable on 1st-2nd coordinate in z_0, and
   (ii) f_2 is partial differentiable on 1st-2nd coordinate in z_0.
   Then $\mathrm{pdiff1}(f_1, z_0)\,\mathrm{pdiff1}(f_2, z_0)$ is partial differentiable on 2nd coordinate in z_0.

(39)   Suppose that
   (i) f_1 is partial differentiable on 2nd-1st coordinate in z_0, and
   (ii) f_2 is partial differentiable on 2nd-1st coordinate in z_0.
   Then $\mathrm{pdiff2}(f_1, z_0)\,\mathrm{pdiff2}(f_2, z_0)$ is partial differentiable on 1st coordinate in z_0.

(40)   Suppose that
   (i) f_1 is partial differentiable on 2nd-2nd coordinate in z_0, and
   (ii) f_2 is partial differentiable on 2nd-2nd coordinate in z_0.
   Then $\mathrm{pdiff2}(f_1, z_0)\,\mathrm{pdiff2}(f_2, z_0)$ is partial differentiable on 2nd coordinate in z_0.

(41)   Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 1st-1st coordinate in z_0. Then $\mathrm{SVF1}(\mathrm{pdiff1}(f, z_0), z_0)$ is continuous in $(\mathrm{proj}(1, 2))(z_0)$.

(42)   Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 1st-2nd coordinate in z_0. Then $\mathrm{SVF2}(\mathrm{pdiff1}(f, z_0), z_0)$ is continuous in $(\mathrm{proj}(2, 2))(z_0)$.

(43)   Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 2nd-1st coordinate in z_0. Then $\mathrm{SVF1}(\mathrm{pdiff2}(f, z_0), z_0)$ is continuous in $(\mathrm{proj}(1, 2))(z_0)$.

(44)   Let z_0 be an element of $\mathcal{R}^2$. Suppose f is partial differentiable on 2nd-2nd coordinate in z_0. Then $\mathrm{SVF2}(\mathrm{pdiff2}(f, z_0), z_0)$ is continuous in $(\mathrm{proj}(2, 2))(z_0)$.

(45)   If f is partial differentiable on 1st-1st coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

(46)   If f is partial differentiable on 1st-2nd coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

(47) If f is partial differentiable on 2nd-1st coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

(48) If f is partial differentiable on 2nd-2nd coordinate in z_0, then there exists R such that $R(0) = 0$ and R is continuous in 0.

References

[1] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[4] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[5] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.

[6] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.

[7] Noboru Endou, Yasunari Shidama, and Keiichi Miyajima. Partial differentiation on normed linear spaces $\mathcal{R}^n$. *Formalized Mathematics*, 15(**2**):65–72, 2007, doi:10.2478/v10037-007-0008-5.

[8] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[9] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.

[10] Jarosław Kotowicz. Properties of real functions. *Formalized Mathematics*, 1(**4**):781–786, 1990.

[11] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[12] Konrad Raczkowski and Paweł Sadowski. Real function continuity. *Formalized Mathematics*, 1(**4**):787–791, 1990.

[13] Konrad Raczkowski and Paweł Sadowski. Real function differentiability. *Formalized Mathematics*, 1(**4**):797–801, 1990.

[14] Konrad Raczkowski and Paweł Sadowski. Topological properties of subsets in real numbers. *Formalized Mathematics*, 1(**4**):777–780, 1990.

[15] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.

[16] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

[17] Bing Xie, Xiquan Liang, and Hongwei Li. Partial differentiation of real binary functions. *Formalized Mathematics*, 16(**4**):333–338, 2008, doi:10.2478/v10037-008-0041-z.

———