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# Bertrand's Ballot Theorem[1]

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**Summary.** In this article we formalize the Bertrand's Ballot Theorem based on [17]. Suppose that in an election we have two candidates: $A$ that receives $n$ votes and $B$ that receives $k$ votes, and additionally $n \geqslant k$. Then this theorem states that the probability of the situation where $A$ maintains more votes than $B$ throughout the counting of the ballots is equal to $(n-k)/(n+k)$.

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The notation and terminology used in this paper have been introduced in the following articles: [24], [1], [14], [15], [18], [4], [5], [10], [21], [6], [12], [3], [11], [25], [26], [16], [8], [13], [23], and [9].

## 1. Preliminaries

From now on $D$, $D_1$, $D_2$ denote non empty sets, $d$, $d_1$, $d_2$ denote finite 0-sequences of $D$, and $n$, $k$, $i$, $j$ denote natural numbers.

Now we state the propositions:

(1) $\mathrm{XFS2FS}(d \restriction n) = \mathrm{XFS2FS}(d) \restriction n$.

(2) $\mathrm{rng}\, d = \mathrm{rng}\, \mathrm{XFS2FS}(d)$.

(3) Let us consider a finite 0-sequence $d_1$ of $D_1$ and a finite 0-sequence $d_2$ of $D_2$. If $d_1 = d_2$, then $\mathrm{XFS2FS}(d_1) = \mathrm{XFS2FS}(d_2)$.

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(4) If $\mathrm{XFS2FS}(d_1) = \mathrm{XFS2FS}(d_2)$, then $d_1 = d_2$. PROOF: For every $i$ such that $i < \mathrm{len}\, d_1$ holds $d_1(i) = d_2(i)$ by [2, (13), (11)]. □

(5) Let us consider a finite sequence $d$ of elements of $D$. Then $\mathrm{XFS2FS}(\mathrm{FS2XFS}(d)) = d$.

(6) Let us consider a finite sequence $f$ and objects $x$, $y$. Suppose

(i) $\mathrm{rng}\, f \subseteq \{x, y\}$, and

(ii) $x \neq y$.

Then $\overline{\overline{f^{-1}(\{x\})}} + \overline{\overline{f^{-1}(\{y\})}} = \mathrm{len}\, f$.

(7) Let us consider functions $f$, $g$. Suppose $f$ is one-to-one. Let us consider an object $x$. If $x \in \mathrm{dom}\, f$, then $\mathrm{Coim}(f \cdot g, f(x)) = \mathrm{Coim}(g, x)$. PROOF: Set $f_3 = f \cdot g$. $\mathrm{Coim}(f_3, f(x)) \subseteq \mathrm{Coim}(g, x)$ by [6, (11), (12)]. □

(8) Let us consider a real number $r$ and a real-valued finite sequence $f$. Suppose $\mathrm{rng}\, f \subseteq \{0, r\}$. Then $\sum f = r \cdot \overline{\overline{f^{-1}(\{r\})}}$. PROOF: Define $\mathcal{P}$[natural number] $\equiv$ for every real-valued finite sequence $f$ such that $\mathrm{len}\, f = \$_1$ and $\mathrm{rng}\, f \subseteq \{0, r\}$ holds $\sum f = r \cdot \overline{\overline{f^{-1}(\{r\})}}$. $\mathcal{P}[0]$ by [8, (72)]. For every $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [22, (55)], [8, (74)], [25, (70)], [2, (11)]. For every $n$, $\mathcal{P}[n]$ from [2, Sch. 2]. □

## 2. Properties of Elections

In the sequel $A$, $B$ denote objects, $v$ denotes an element of $\{A, B\}^{n+k}$, and $f$, $g$ denote finite sequences.

Let us consider $A$, $n$, $B$, and $k$. The functor $\mathrm{Election}(A, n, B, k)$ yielding a subset of $\{A, B\}^{n+k}$ is defined by

(Def. 1) $v \in \mathit{it}$ if and only if $\overline{\overline{v^{-1}(\{A\})}} = n$.

Let us note that $\mathrm{Election}(A, n, B, k)$ is finite. Now we state the propositions:

(9) $\mathrm{Election}(A, n, A, 0) = \{n \mapsto A\}$. PROOF: $\mathrm{Election}(A, n, A, 0) \subseteq \{n \mapsto A\}$ by [19, (29)], [9, (33)], [21, (9)]. □

(10) If $k > 0$, then $\mathrm{Election}(A, n, A, k)$ is empty.

Let us consider $A$ and $n$. Let $k$ be a non empty natural number. Let us observe that $\mathrm{Election}(A, n, A, k)$ is empty. Now we state the proposition:

(11) $\mathrm{Election}(A, n, B, k) = \mathrm{Choose}(\mathrm{Seg}(n+k), n, A, B)$. PROOF: $\mathrm{Election}(A, n, B, k) \subseteq \mathrm{Choose}(\mathrm{Seg}(n+k), n, A, B)$ by [7, (2)]. □

Let us assume that $A \neq B$. Now we state the propositions:

(12) $v \in \mathrm{Election}(A, n, B, k)$ if and only if $\overline{\overline{v^{-1}(\{B\})}} = k$. The theorem is a consequence of (6).

(13) $\overline{\overline{\mathrm{Election}(A, n, B, k)}} = \binom{n+k}{n}$. The theorem is a consequence of (11).

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## 3. Properties of Dominated Elections

Let us consider $A$, $n$, $B$, and $k$. Let $v$ be a finite sequence. We say that $v$ is an ($A$, $n$, $B$, $k$)-dominated-election if and only if

(Def. 2) (i) $v \in \mathrm{Election}(A,n,B,k)$, and

(ii) for every $i$ such that $i > 0$ holds $\overline{\overline{(v{\restriction}i)^{-1}(\{A\})}} > \overline{\overline{(v{\restriction}i)^{-1}(\{B\})}}$.

Let us assume that $f$ is an ($A$, $n$, $B$, $k$)-dominated-election. Now we state the propositions:

(14) $A \neq B$.

(15) $n > k$. The theorem is a consequence of (14) and (12).

Now we state the propositions:

(16) If $A \neq B$ and $n > 0$, then $n \mapsto A$ is an ($A$, $n$, $B$, 0)-dominated-election.

(17) If $f$ is an ($A$, $n$, $B$, $k$)-dominated-election and $i < n-k$, then $f \frown (i \mapsto B)$ is an ($A$, $n$, $B$, $(k+i)$)-dominated-election. The theorem is a consequence of (14) and (12).

(18) Suppose $f$ is an ($A$, $n$, $B$, $k$)-dominated-election and $g$ is an ($A$, $i$, $B$, $j$)-dominated-election. Then $f \frown g$ is an ($A$, $(n+i)$, $B$, $(k+j)$)-dominated-election. The theorem is a consequence of (14), (12), and (15).

Let us consider $A$, $n$, $B$, and $k$. The functor DominatedElection$(A,n,B,k)$ yielding a subset of Election$(A,n,B,k)$ is defined by

(Def. 3) $f \in it$ if and only if $f$ is an ($A$, $n$, $B$, $k$)-dominated-election.

(19) If $A = B$ or $n \leqslant k$, then DominatedElection$(A,n,B,k)$ is empty. The theorem is a consequence of (14) and (15).

(20) If $n > k$ and $A \neq B$, then $n \mapsto A \frown (k \mapsto B) \in$ DominatedElection$(A,n,B,k)$. The theorem is a consequence of (17) and (16).

(21) If $A \neq B$, then $\overline{\overline{\mathrm{DominatedElection}(A,n,B,k)}} = \overline{\overline{\mathrm{DominatedElection}(0,n,1,k)}}$. PROOF: Set $T = [A \longmapsto 0, B \longmapsto 1]$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every $f$ such that $f = \$_1$ holds $T \cdot f = \$_2$. For every object $x$ such that $x \in$ DominatedElection$(A,n,B,k)$ there exists an object $y$ such that $y \in$ DominatedElection$(0,n,1,k)$ and $\mathcal{P}[x,y]$ by [25, (27), (26)], [5, (92)], (7). Consider $C$ being a function from DominatedElection$(A,n,B,k)$ into DominatedElection$(0,n,1,k)$ such that for every object $x$ such that $x \in$ DominatedElection$(A,n,B,k)$ holds $\mathcal{P}[x, C(x)]$ from [7, Sch. 1]. DominatedElection$(0,n,1,k) \subseteq \mathrm{rng}\, C$ by [25, (27), (26)], [5, (92)], (7). □

(22) Let us consider a finite sequence $p$ of elements of $\mathbb{N}$. Then $p$ is a (0, $n$, 1, $k$)-dominated-election if and only if $p$ is an $(n+k)$-tuple of $\{0,1\}$ and $n > 0$ and $\sum p = k$ and for every $i$ such that $i > 0$ holds $2 \cdot \sum(p{\restriction}i) < i$. PROOF: If $p$ is a (0, $n$, 1, $k$)-dominated-election, then $p$ is an $(n+k)$-tuple of $\{0,1\}$

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and $n > 0$ and $\sum p = k$ and for every $i$ such that $i > 0$ holds $2 \cdot \sum(p \restriction i) < i$ by (8), (12), (15), [25, (70)]. $1 \cdot \overline{\overline{p^{-1}(\{1\})}} = k$. $\overline{\overline{p^{-1}(\{1\})}} + \overline{\overline{p^{-1}(\{0\})}} = \operatorname{len} p$. $1 \cdot \overline{\overline{(p \restriction i)^{-1}(\{1\})}} = \sum(p \restriction i)$. $\overline{\overline{(p \restriction i)^{-1}(\{1\})}} + \overline{\overline{(p \restriction i)^{-1}(\{0\})}} = \operatorname{len}(p \restriction i)$. □

(23) If $f$ is an $(A, n, B, k)$-dominated-election, then $f(1) = A$. The theorem is a consequence of (15).

(24) Let us consider a finite 0-sequence $d$ of $\mathbb{N}$. Then $d \in \mathrm{Domin}_0(n+k, k)$ if and only if $\langle 0 \rangle \frown \mathrm{XFS2FS}(d) \in \mathrm{DominatedElection}(0, n+1, 1, k)$. Proof: Set $X_1 = \mathrm{XFS2FS}(d)$. Set $Z = \langle 0 \rangle$. Set $Z_1 = Z \frown X_1$. Reconsider $D = d$ as a finite 0-sequence of $\mathbb{R}$. $\mathrm{XFS2FS}(d) = \mathrm{XFS2FS}(D)$. If $d \in \mathrm{Domin}_0(n+k, k)$, then $Z_1 \in \mathrm{DominatedElection}(0, n+1, 1, k)$ by [15, (20)], (2), [4, (31), (22)]. $Z_1$ is an $(n+1+k)$-tuple of $\{0, 1\}$. For every $k$ such that $k \leqslant \operatorname{dom} d$ holds $2 \cdot \sum(d \restriction k) \leqslant k$ by [20, (14)], [8, (76)], (1), (3). $d$ is dominated by 0. $\sum d = k$. □

(25) $\overline{\overline{\mathrm{Domin}_0(n+k, k)}} = \overline{\overline{\mathrm{DominatedElection}(0, n+1, 1, k)}}$. Proof: Set $D = \mathrm{Domin}_0(n+k, k)$. Set $B = \mathrm{DominatedElection}(0, n+1, 1, k)$. Set $Z = \langle 0 \rangle$. Define $\mathcal{F}[\text{object}, \text{object}] \equiv$ for every finite 0-sequence $d$ of $\mathbb{N}$ such that $d = \$_1$ holds $\$_2 = Z \frown \mathrm{XFS2FS}(d)$. For every object $x$ such that $x \in D$ there exists an object $y$ such that $y \in B$ and $\mathcal{F}[x, y]$. Consider $f$ being a function from $D$ into $B$ such that for every object $x$ such that $x \in D$ holds $\mathcal{F}[x, f(x)]$ from [7, Sch. 1]. □

(26) $\overline{\overline{\mathrm{Domin}_0(n+k, k)}} = \overline{\overline{\mathrm{DominatedElection}(0, n+1, 1, k)}}$. Proof: Set $D = \mathrm{Domin}_0(n+k, k)$. Set $B = \mathrm{DominatedElection}(0, n+1, 1, k)$. Set $Z = \langle 0 \rangle$. Define $\mathcal{F}[\text{object}, \text{object}] \equiv$ for every finite 0-sequence $d$ of $\mathbb{N}$ such that $d = \$_1$ holds $\$_2 = Z \frown \mathrm{XFS2FS}(d)$. For every object $x$ such that $x \in D$ there exists an object $y$ such that $y \in B$ and $\mathcal{F}[x, y]$. Consider $f$ being a function from $D$ into $B$ such that for every object $x$ such that $x \in D$ holds $\mathcal{F}[x, f(x)]$ from [7, Sch. 1]. □

(27) If $A \neq B$ and $n > k$, then $\overline{\overline{\mathrm{DominatedElection}(A, n, B, k)}} = \frac{n-k}{n+k} \cdot \binom{n+k}{k}$. The theorem is a consequence of (21) and (26).

## 4. Main Theorem

(28) Bertrand's Ballot Theorem:
If $A \neq B$ and $n \geqslant k$, then $\mathrm{P}(\mathrm{DominatedElection}(A, n, B, k)) = \frac{n-k}{n+k}$. The theorem is a consequence of (13), (19), and (27).

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