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# Lattice of $\mathbb{Z}$-module

Yuichi Futa
Japan Advanced Institute
of Science and Technology
Ishikawa, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

**Summary.** In this article, we formalize the definition of lattice of $\mathbb{Z}$-module and its properties in the Mizar system [5]. We formally prove that scalar products in lattices are bilinear forms over the field of real numbers $\mathbb{R}$. We also formalize the definitions of positive definite and integral lattices and their properties. Lattice of $\mathbb{Z}$-module is necessary for lattice problems, LLL (Lenstra, Lenstra and Lovász) base reduction algorithm [14], and cryptographic systems with lattices [15] and coding theory [9].

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## 1. Definition of Lattices of $\mathbb{Z}$-module

Now we state the proposition:

(1) Let us consider non empty sets $D$, $E$, natural numbers $n$, $m$, and a matrix $M$ over $D$ of dimension $n \times m$. Suppose for every natural numbers $i$, $j$ such that $\langle i, j \rangle \in$ the indices of $M$ holds $M_{i,j} \in E$. Then $M$ is a matrix over $E$ of dimension $n \times m$.

Let $a$, $b$ be elements of $\mathbb{F}_{\mathbb{Q}}$ and $x$, $y$ be rational numbers. We identify $x + y$ with $a + b$ and $x \cdot y$ with $a \cdot b$. Let $F$ be a 1-sorted structure. We consider structures of $\mathbb{Z}$-lattice over $F$ which extend vector space structures over $F$ and are systems

$\langle$a carrier, an addition, a zero, a left multiplication, a scalar product⟩

where the carrier is a set, the addition is a binary operation on the carrier, the zero is an element of the carrier, the left multiplication is a function from (the carrier of $F$) $\times$ (the carrier) into the carrier, the scalar product is a function from (the carrier) $\times$ (the carrier) into the carrier of $\mathbb{R}_{\rm F}$.

Note that there exists a structure of $\mathbb{Z}$-lattice over $F$ which is strict and non empty.

Let $D$ be a non empty set, $Z$ be an element of $D$, $a$ be a binary operation on $D$, $m$ be a function from (the carrier of $F$) $\times D$ into $D$, and $s$ be a function from $D \times D$ into the carrier of $\mathbb{R}_{\rm F}$. One can check that $\langle D, a, Z, m, s\rangle$ is non empty.

Let $X$ be a non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\rm R}$ and $x$, $y$ be vectors of $X$. The functor $\langle x,\, y\rangle$ yielding an element of $\mathbb{R}_{\rm F}$ is defined by the term

(Def. 1) (the scalar product of $X$)($\langle x,\, y\rangle$).

Let $x$ be a vector of $X$. The functor $\|x\|$ yielding an element of $\mathbb{R}_{\rm F}$ is defined by the term

(Def. 2) $\langle x,\, x\rangle$.

Let $X$ be a non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\rm R}$. We say that $X$ is $\mathbb{Z}$-lattice-like if and only if

(Def. 3) for every vector $x$ of $X$ such that for every vector $y$ of $X$, $\langle x, y\rangle = 0$ holds $x = 0_X$ and for every vectors $x$, $y$ of $X$, $\langle x, y\rangle = \langle y, x\rangle$ and for every vectors $x$, $y$, $z$ of $X$ and for every element $a$ of $\mathbb{Z}^{\rm R}$, $\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle a \cdot x, y\rangle = a \cdot \langle x, y\rangle$.

Let $V$ be a $\mathbb{Z}$-module and $s$ be a function from (the carrier of $V$) $\times$ (the carrier of $V$) into the carrier of $\mathbb{R}_{\rm F}$. The functor $\mathrm{GenLat}(V, s)$ yielding a non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\rm R}$ is defined by the term

(Def. 4) ⟨the carrier of $V$, the addition of $V$, $0_V$, the left multiplication of $V$, $s$⟩.

Let us note that there exists a non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\rm R}$ which is vector distributive, scalar distributive, scalar associative, scalar unital, Abelian, add-associative, right zeroed, right complementable, and strict.

Let $V$ be a $\mathbb{Z}$-module and $s$ be a function from (the carrier of $V$) $\times$ (the carrier of $V$) into the carrier of $\mathbb{R}_{\rm F}$. One can verify that $\mathrm{GenLat}(V, s)$ is Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, and scalar unital.

Let us consider a $\mathbb{Z}$-module $V$ and a function $s$ from (the carrier of $V$) $\times$ (the carrier of $V$) into the carrier of $\mathbb{R}_{\rm F}$. Now we state the propositions:

(2) $\mathrm{GenLat}(V, s)$ is a submodule of $V$.

(3) $V$ is a submodule of $\mathrm{GenLat}(V, s)$.

Note that there exists an Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, scalar unital, non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\mathrm{R}}$ which is free.

Let $V$ be a free $\mathbb{Z}$-module and $s$ be a function from (the carrier of $V$) $\times$ (the carrier of $V$) into the carrier of $\mathbb{R}_{\mathrm{F}}$. Let us observe that $\mathrm{GenLat}(V, s)$ is free and there exists an Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, scalar unital, non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\mathrm{R}}$ which is torsion-free.

Now we state the proposition:

(4) Let us consider a finite rank, free $\mathbb{Z}$-module $V$, and a function $s$ from (the carrier of $V$) $\times$ (the carrier of $V$) into the carrier of $\mathbb{R}_{\mathrm{F}}$.
Then $\mathrm{GenLat}(V, s)$ is finite rank. The theorem is a consequence of (2).

Let us note that there exists a free, Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, scalar unital, non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\mathrm{R}}$ which is finite rank.

Let $V$ be a finite rank, free $\mathbb{Z}$-module and $s$ be a function from (the carrier of $V$) $\times$ (the carrier of $V$) into the carrier of $\mathbb{R}_{\mathrm{F}}$. Let us note that $\mathrm{GenLat}(V, s)$ is finite rank.

Now we state the proposition:

(5) Let us consider a finite rank, free $\mathbb{Z}$-module $V$, and a function $f$ from (the carrier of $\mathbf{0}_V$) $\times$ (the carrier of $\mathbf{0}_V$) into the carrier of $\mathbb{R}_{\mathrm{F}}$. Suppose $f =$ (the carrier of $\mathbf{0}_V$) $\times$ (the carrier of $\mathbf{0}_V$) $\longmapsto 0_{\mathbb{R}_{\mathrm{F}}}$. Then $\mathrm{GenLat}(\mathbf{0}_V, f)$ is $\mathbb{Z}$-lattice-like.
PROOF: Set $X = \mathrm{GenLat}(\mathbf{0}_V, f)$. For every vector $x$ of $X$ such that for every vector $y$ of $X$, $\langle x, y\rangle = 0$ holds $x = 0_X$ by [10, (26)]. For every vectors $x$, $y$, $z$ of $X$ and for every element $a$ of $\mathbb{Z}^{\mathrm{R}}$, $\langle x, y\rangle = \langle y, x\rangle$ and $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle a \cdot x, y\rangle = a \cdot \langle x, y\rangle$ by [16, (7)], [8, (87)]. □

Note that there exists a non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\mathrm{R}}$ which is $\mathbb{Z}$-lattice-like and there exists a finite rank, free, Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, scalar unital, non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\mathrm{R}}$ which is $\mathbb{Z}$-lattice-like.

There exists a finite rank, free, $\mathbb{Z}$-lattice-like, Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, scalar unital, non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\mathrm{R}}$ which is strict.

A $\mathbb{Z}$-lattice is a finite rank, free, $\mathbb{Z}$-lattice-like, Abelian, add-associative, right zeroed, right complementable, scalar distributive, vector distributive, scalar associative, scalar unital, non empty structure of $\mathbb{Z}$-lattice over $\mathbb{Z}^{\mathrm{R}}$. Now we state the proposition:

(6) Let us consider a non trivial, torsion-free $\mathbb{Z}$-module $V$, a submodule $Z$ of $V$, a non zero vector $v$ of $V$, and a function $f$ from (the carrier of $Z$) $\times$ (the carrier of $Z$) into the carrier of $\mathbb{R}_{\mathrm{F}}$. Suppose $Z = \mathrm{Lin}(\{v\})$ and for every vectors $v_1$, $v_2$ of $Z$ and for every elements $a$, $b$ of $\mathbb{Z}^{\mathrm{R}}$ such that $v_1 = a \cdot v$ and $v_2 = b \cdot v$ holds $f(v_1, v_2) = a \cdot b$. Then $\mathrm{GenLat}(Z, f)$ is $\mathbb{Z}$-lattice-like.
Proof: Set $L = \mathrm{GenLat}(Z, f)$. $L$ is $\mathbb{Z}$-lattice-like by [10, (26)], [12, (19)], [10, (1)], [12, (21)]. □

Observe that there exists a $\mathbb{Z}$-lattice which is non trivial.

Let $V$ be a torsion-free $\mathbb{Z}$-module. Let us observe that $\mathbb{Z}$-MQVectSp($V$) is scalar distributive, vector distributive, scalar associative, scalar unital, add-associative, right zeroed, right complementable, and Abelian as a non empty vector space structure over $\mathbb{F}_{\mathbb{Q}}$.

Now we state the propositions:

(7) Let us consider a $\mathbb{Z}$-lattice $L$, and vectors $v$, $u$ of $L$. Then

(i) $\langle v, -u \rangle = -\langle v, u \rangle$, and

(ii) $\langle -v, u \rangle = -\langle v, u \rangle$.

(8) Let us consider a $\mathbb{Z}$-lattice $L$, and vectors $v$, $u$, $w$ of $L$. Then $\langle v, u+w \rangle = \langle v, u \rangle + \langle v, w \rangle$.

(9) Let us consider a $\mathbb{Z}$-lattice $L$, vectors $v$, $u$ of $L$, and an element $a$ of $\mathbb{Z}^{\mathrm{R}}$. Then $\langle v, a \cdot u \rangle = a \cdot \langle v, u \rangle$.

(10) Let us consider a $\mathbb{Z}$-lattice $L$, vectors $v$, $u$, $w$ of $L$, and elements $a$, $b$ of $\mathbb{Z}^{\mathrm{R}}$. Then

(i) $\langle a \cdot v + b \cdot u, w \rangle = a \cdot \langle v, w \rangle + b \cdot \langle u, w \rangle$, and

(ii) $\langle v, a \cdot u + b \cdot w \rangle = a \cdot \langle v, u \rangle + b \cdot \langle v, w \rangle$.

The theorem is a consequence of (8) and (9).

(11) Let us consider a $\mathbb{Z}$-lattice $L$, and vectors $v$, $u$, $w$ of $L$. Then

(i) $\langle v - u, w \rangle = \langle v, w \rangle - \langle u, w \rangle$, and

(ii) $\langle v, u - w \rangle = \langle v, u \rangle - \langle v, w \rangle$.

The theorem is a consequence of (8) and (9).

(12) Let us consider a $\mathbb{Z}$-lattice $L$, and a vector $v$ of $L$. Then

(i) $\langle v, 0_L \rangle = 0$, and

(ii) $\langle 0_L, v \rangle = 0$.

The theorem is a consequence of (11).

Let $X$ be a $\mathbb{Z}$-lattice. We say that $X$ is integral if and only if

(Def. 5) for every vectors $v$, $u$ of $X$, $\langle v, u\rangle \in \mathbb{Z}$.

Observe that there exists a $\mathbb{Z}$-lattice which is integral.

Let $L$ be an integral $\mathbb{Z}$-lattice and $v$, $u$ be vectors of $L$. Let us observe that $\langle v,\ u\rangle$is integer.

Let $v$ be a vector of $L$. Let us note that $\|v\|$ is integer.

Now we state the propositions:

(13) Let us consider a $\mathbb{Z}$-lattice $L$, a finite subset $I$ of $L$, and a vector $u$ of $L$. Suppose for every vector $v$ of $L$ such that $v \in I$ holds $\langle v, u\rangle \in \mathbb{Z}$. Let us consider a vector $v$ of $L$. If $v \in \mathrm{Lin}(I)$, then $\langle v, u\rangle \in \mathbb{Z}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite subset $I$ of $L$ such that $\overline{\overline{I}} = \$_1$ and for every vector $v$ of $L$ such that $v \in I$ holds $\langle v, u\rangle \in \mathbb{Z}$ for every vector $v$ of $L$ such that $v \in \mathrm{Lin}(I)$ holds $\langle v, u\rangle \in \mathbb{Z}$. $\mathcal{P}[0]$ by [11, (67)], (12). For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [8, (40)], [11, (72)], [1, (44)], [8, (31)]. For every natural number $n$, $\mathcal{P}[n]$ from [3, Sch. 2]. □

(14) Let us consider a $\mathbb{Z}$-lattice $L$, and a basis $I$ of $L$. Suppose for every vectors $v$, $u$ of $L$ such that $v$, $u \in I$ holds $\langle v, u\rangle \in \mathbb{Z}$. Let us consider vectors $v$, $u$ of $L$. Then $\langle v, u\rangle \in \mathbb{Z}$.

Proof: Define $\mathcal{P}$[natural number] $\equiv$ for every finite subset $I$ of $L$ such that $\overline{\overline{I}} = \$_1$ and for every vectors $v$, $u$ of $L$ such that $v$, $u \in I$ holds $\langle v, u\rangle \in \mathbb{Z}$ for every vectors $v$, $u$ of $L$ such that $v$, $u \in \mathrm{Lin}(I)$ holds $\langle v, u\rangle \in \mathbb{Z}$. $\mathcal{P}[0]$ by [11, (67)], (12). For every natural number $n$ such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [8, (40)], [11, (72)], [1, (44)], [8, (31)]. For every natural number $n$, $\mathcal{P}[n]$ from [3, Sch. 2]. □

(15) Let us consider a $\mathbb{Z}$-lattice $L$, and a basis $I$ of $L$. Suppose for every vectors $v$, $u$ of $L$ such that $v$, $u \in I$ holds $\langle v, u\rangle \in \mathbb{Z}$. Then $L$ is integral.

Let $X$ be a $\mathbb{Z}$-lattice. We say that $X$ is positive definite if and only if

(Def. 6) for every vector $v$ of $X$ such that $v \neq 0_X$ holds $\|v\| > 0$.

Let us observe that there exists a $\mathbb{Z}$-lattice which is non trivial, integral, and positive definite.

Let us consider a positive definite $\mathbb{Z}$-lattice $L$ and a vector $v$ of $L$. Now we state the propositions:

(16) $\|v\| = 0$ if and only if $v = 0_L$.

(17) $\|v\| \geqslant 0$. The theorem is a consequence of (12).

Let $X$ be an integral $\mathbb{Z}$-lattice. We say that $X$ is even if and only if

(Def. 7) for every vector $v$ of $X$, $\|v\|$ is even.

One can verify that there exists an integral $\mathbb{Z}$-lattice which is even.

Let $L$ be a $\mathbb{Z}$-lattice. We introduce the notation $\dim(L)$ as a synonym of $\operatorname{rank} L$.

Let $v$, $u$ be vectors of $L$. We say that $v$, $u$ are orthogonal if and only if

(Def. 8) $\langle v, u \rangle = 0$.

Let us note that the predicate is symmetric.

Let us consider a $\mathbb{Z}$-lattice $L$ and vectors $v$, $u$ of $L$.

Let us assume that $v$, $u$ are orthogonal. Now we state the propositions:

(18) (i) $v$, $-u$ are orthogonal, and

(ii) $-v$, $u$ are orthogonal, and

(iii) $-v$, $-u$ are orthogonal.

The theorem is a consequence of (7).

(19) $\|v + u\| = \|v\| + \|u\|$. The theorem is a consequence of (8).

(20) $\|v - u\| = \|v\| + \|u\|$. The theorem is a consequence of (11).

Let $L$ be a $\mathbb{Z}$-lattice.

A $\mathbb{Z}$-sublattice of $L$ is a $\mathbb{Z}$-lattice and is defined by

(Def. 9) the carrier of *it* $\subseteq$ the carrier of $L$ and $0_{it} = 0_L$ and the addition of *it* = (the addition of $L$) $\upharpoonright$ (the carrier of *it*) and the left multiplication of *it* = (the left multiplication of $L$)$\upharpoonright$((the carrier of $\mathbb{Z}^{\mathrm{R}}$) $\times$ (the carrier of *it*)) and the scalar product of *it* = (the scalar product of $L$) $\upharpoonright$ (the carrier of *it*).

Now we state the propositions:

(21) Let us consider a $\mathbb{Z}$-lattice $L$. Then every $\mathbb{Z}$-sublattice of $L$ is a submodule of $L$.

(22) Let us consider an object $x$, a $\mathbb{Z}$-lattice $L$, and $\mathbb{Z}$-sublattices $L_1$, $L_2$ of $L$. Suppose $x \in L_1$ and $L_1$ is a $\mathbb{Z}$-sublattice of $L_2$. Then $x \in L_2$. The theorem is a consequence of (21).

(23) Let us consider an object $x$, a $\mathbb{Z}$-lattice $L$, and a $\mathbb{Z}$-sublattice $L_1$ of $L$. If $x \in L_1$, then $x \in L$. The theorem is a consequence of (21).

(24) Let us consider a $\mathbb{Z}$-lattice $L$, and a $\mathbb{Z}$-sublattice $L_1$ of $L$. Then every vector of $L_1$ is a vector of $L$. The theorem is a consequence of (21).

(25) Let us consider a $\mathbb{Z}$-lattice $L$, and $\mathbb{Z}$-sublattices $L_1$, $L_2$ of $L$. Then $0_{L_1} = 0_{L_2}$.

(26) Let us consider a $\mathbb{Z}$-lattice $L$, a $\mathbb{Z}$-sublattice $L_1$ of $L$, vectors $v_1$, $v_2$ of $L$, and vectors $w_1$, $w_2$ of $L_1$. If $w_1 = v_1$ and $w_2 = v_2$, then $w_1 + w_2 = v_1 + v_2$. The theorem is a consequence of (21).

(27) Let us consider a $\mathbb{Z}$-lattice $L$, a $\mathbb{Z}$-sublattice $L_1$ of $L$, a vector $v$ of $L$, a vector $w$ of $L_1$, and an element $a$ of $\mathbb{Z}^{\mathrm{R}}$. If $w = v$, then $a \cdot w = a \cdot v$. The theorem is a consequence of (21).

(28) Let us consider a $\mathbb{Z}$-lattice $L$, a $\mathbb{Z}$-sublattice $L_1$ of $L$, a vector $v$ of $L$, and a vector $w$ of $L_1$. If $w = v$, then $-w = -v$. The theorem is a consequence of (21).

(29) Let us consider a $\mathbb{Z}$-lattice $L$, a $\mathbb{Z}$-sublattice $L_1$ of $L$, vectors $v_1$, $v_2$ of $L$, and vectors $w_1$, $w_2$ of $L_1$. If $w_1 = v_1$ and $w_2 = v_2$, then $w_1 - w_2 = v_1 - v_2$. The theorem is a consequence of (21).

(30) Let us consider a $\mathbb{Z}$-lattice $L$, and a $\mathbb{Z}$-sublattice $L_1$ of $L$. Then $0_L \in L_1$. The theorem is a consequence of (21).

(31) Let us consider a $\mathbb{Z}$-lattice $L$, and $\mathbb{Z}$-sublattices $L_1$, $L_2$ of $L$. Then $0_{L_1} \in L_2$. The theorem is a consequence of (21).

(32) Let us consider a $\mathbb{Z}$-lattice $L$, and a $\mathbb{Z}$-sublattice $L_1$ of $L$. Then $0_{L_1} \in L$. The theorem is a consequence of (21).

(33) Let us consider a $\mathbb{Z}$-lattice $L$, a $\mathbb{Z}$-sublattice $L_1$ of $L$, and vectors $v_1$, $v_2$ of $L$. If $v_1$, $v_2 \in L_1$, then $v_1 + v_2 \in L_1$. The theorem is a consequence of (21).

(34) Let us consider a $\mathbb{Z}$-lattice $L$, a $\mathbb{Z}$-sublattice $L_1$ of $L$, a vector $v$ of $L$, and an element $a$ of $\mathbb{Z}^{\mathrm{R}}$. If $v \in L_1$, then $a \cdot v \in L_1$. The theorem is a consequence of (21).

(35) Let us consider a $\mathbb{Z}$-lattice $L$, a $\mathbb{Z}$-sublattice $L_1$ of $L$, and a vector $v$ of $L$. If $v \in L_1$, then $-v \in L_1$. The theorem is a consequence of (21).

(36) Let us consider a $\mathbb{Z}$-lattice $L$, a $\mathbb{Z}$-sublattice $L_1$ of $L$, and vectors $v_1$, $v_2$ of $L$. If $v_1$, $v_2 \in L_1$, then $v_1 - v_2 \in L_1$. The theorem is a consequence of (21).

(37) Let us consider a positive definite $\mathbb{Z}$-lattice $L$, a non empty set $A$, an element $z$ of $A$, a binary operation $a$ on $A$, a function $m$ from (the carrier of $\mathbb{Z}^{\mathrm{R}}) \times A$ into $A$, and a function $s$ from $A \times A$ into the carrier of $\mathbb{R}_{\mathrm{F}}$. Suppose $A$ is a linearly closed subset of $L$ and $z = 0_L$ and $a =$ (the addition of $L) \restriction A$ and $m =$ (the left multiplication of $L) \restriction (($the carrier of $\mathbb{Z}^{\mathrm{R}}) \times A)$ and $s =$ (the scalar product of $L) \restriction A$. Then $\langle A, a, z, m, s \rangle$ is a $\mathbb{Z}$-sublattice of $L$.

Proof: Set $L_1 = \langle A, a, z, m, s \rangle$. Set $V_1 = \langle A, a, z, m \rangle$. $L_1$ is a submodule of $V_1$. $L_1$ is $\mathbb{Z}$-lattice-like by [10, (25)], [7, (49)], [10, (28), (29)]. □

(38) Let us consider a $\mathbb{Z}$-lattice $L$, a $\mathbb{Z}$-sublattice $L_1$ of $L$, vectors $w_1$, $w_2$ of $L_1$, and vectors $v_1$, $v_2$ of $L$. Suppose $w_1 = v_1$ and $w_2 = v_2$. Then $\langle w_1, w_2 \rangle = \langle v_1, v_2 \rangle$.

Let $L$ be an integral $\mathbb{Z}$-lattice. Note that every $\mathbb{Z}$-sublattice of $L$ is integral.

Let $L$ be a positive definite $\mathbb{Z}$-lattice. Let us observe that every $\mathbb{Z}$-sublattice of $L$ is positive definite.

Let $V$, $W$ be vector space structures over $\mathbb{Z}^{\mathrm{R}}$.

An $\mathbb{R}$-form of $V$ and $W$ is a function from (the carrier of $V$) $\times$ (the carrier of $W$) into the carrier of $\mathbb{R}_{\mathrm{F}}$. The functor NulFrForm$(V,W)$ yielding an $\mathbb{R}$-form of $V$ and $W$ is defined by the term

(Def. 10) (the carrier of $V$) $\times$ (the carrier of $W$) $\longmapsto 0_{\mathbb{R}_{\mathrm{F}}}$.

Let $V$, $W$ be non empty vector space structures over $\mathbb{Z}^{\mathrm{R}}$ and $f$, $g$ be $\mathbb{R}$-forms of $V$ and $W$. The functor $f+g$ yielding an $\mathbb{R}$-form of $V$ and $W$ is defined by

(Def. 11) for every vector $v$ of $V$ and for every vector $w$ of $W$, $it(v,w) = f(v,w) + g(v,w)$.

Let $f$ be an $\mathbb{R}$-form of $V$ and $W$ and $a$ be an element of $\mathbb{R}_{\mathrm{F}}$. The functor $a \cdot f$ yielding an $\mathbb{R}$-form of $V$ and $W$ is defined by

(Def. 12) for every vector $v$ of $V$ and for every vector $w$ of $W$, $it(v,w) = a \cdot f(v,w)$.

The functor $-f$ yielding an $\mathbb{R}$-form of $V$ and $W$ is defined by

(Def. 13) for every vector $v$ of $V$ and for every vector $w$ of $W$, $it(v,w) = -f(v,w)$.

One can verify that the functor $-f$ is defined by the term

(Def. 14) $(-1_{\mathbb{R}_{\mathrm{F}}}) \cdot f$.

Let $f$, $g$ be $\mathbb{R}$-forms of $V$ and $W$. The functor $f-g$ yielding an $\mathbb{R}$-form of $V$ and $W$ is defined by the term

(Def. 15) $f + -g$.

Observe that the functor $f-g$ is defined by

(Def. 16) for every vector $v$ of $V$ and for every vector $w$ of $W$, $it(v,w) = f(v,w) - g(v,w)$.

Let us note that the functor $f+g$ is commutative.

Now we state the propositions:

(39) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, and an $\mathbb{R}$-form $f$ of $V$ and $W$. Then $f + \mathrm{NulFrForm}(V,W) = f$.

(40) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, and $\mathbb{R}$-forms $f$, $g$, $h$ of $V$ and $W$. Then $(f+g)+h = f+(g+h)$.

(41) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, and an $\mathbb{R}$-form $f$ of $V$ and $W$. Then $f - f = \mathrm{NulFrForm}(V,W)$.

(42) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, an element $a$ of $\mathbb{R}_{\mathrm{F}}$, and $\mathbb{R}$-forms $f$, $g$ of $V$ and $W$. Then $a \cdot (f+g) = a \cdot f + a \cdot g$.

Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, elements $a$, $b$ of $\mathbb{R}_{\mathrm{F}}$, and an $\mathbb{R}$-form $f$ of $V$ and $W$. Now we state the propositions:

(43) $(a+b) \cdot f = a \cdot f + b \cdot f$.

(44) $(a \cdot b) \cdot f = a \cdot (b \cdot f)$.

(45) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, and an $\mathbb{R}$-form $f$ of $V$ and $W$. Then $1_{\mathbb{R}_{\mathrm{F}}} \cdot f = f$.

Let $V$ be a vector space structure over $\mathbb{Z}^{\mathrm{R}}$.

An $\mathbb{R}$-functional of $V$ is a function from the carrier of $V$ into the carrier of $\mathbb{R}_{\mathrm{F}}$. Let $V$ be a non empty vector space structure over $\mathbb{Z}^{\mathrm{R}}$ and $f$, $g$ be $\mathbb{R}$-functionals of $V$. The functor $f+g$ yielding an $\mathbb{R}$-functional of $V$ is defined by

(Def. 17) for every element $x$ of $V$, $it(x) = f(x) + g(x)$.

Let $f$ be an $\mathbb{R}$-functional of $V$. The functor $-f$ yielding an $\mathbb{R}$-functional of $V$ is defined by

(Def. 18) for every element $x$ of $V$, $it(x) = -f(x)$.

Let $f$, $g$ be $\mathbb{R}$-functionals of $V$. The functor $f-g$ yielding an $\mathbb{R}$-functional of $V$ is defined by the term

(Def. 19) $f + -g$.

Let $v$ be an element of $\mathbb{R}_{\mathrm{F}}$ and $f$ be an $\mathbb{R}$-functional of $V$. The functor $v \cdot f$ yielding an $\mathbb{R}$-functional of $V$ is defined by

(Def. 20) for every element $x$ of $V$, $it(x) = v \cdot f(x)$.

Let $V$ be a vector space structure over $\mathbb{Z}^{\mathrm{R}}$. The functor 0FrFunctional($V$) yielding an $\mathbb{R}$-functional of $V$ is defined by the term

(Def. 21) $\Omega_V \longmapsto 0_{\mathbb{R}_{\mathrm{F}}}$.

Let $V$ be a non empty vector space structure over $\mathbb{Z}^{\mathrm{R}}$ and $F$ be an $\mathbb{R}$-functional of $V$. We say that $F$ is homogeneous if and only if

(Def. 22) for every vector $x$ of $V$ and for every scalar $r$ of $V$, $F(r \cdot x) = r \cdot F(x)$.

We say that $F$ is 0-preserving if and only if

(Def. 23) $F(0_V) = 0_{\mathbb{R}_{\mathrm{F}}}$.

Let $V$ be a $\mathbb{Z}$-module. Note that every $\mathbb{R}$-functional of $V$ which is homogeneous is also 0-preserving.

Let $V$ be a non empty vector space structure over $\mathbb{Z}^{\mathrm{R}}$. One can verify that 0FrFunctional($V$) is additive and 0FrFunctional($V$) is homogeneous and 0FrFunctional($V$) is 0-preserving and there exists an $\mathbb{R}$-functional of $V$ which is additive, homogeneous, and 0-preserving.

Now we state the propositions:

(46) Let us consider a non empty vector space structure $V$ over $\mathbb{Z}^{\mathrm{R}}$, and $\mathbb{R}$-functionals $f$, $g$ of $V$. Then $f+g = g+f$.

(47) Let us consider a non empty vector space structure $V$ over $\mathbb{Z}^{\mathrm{R}}$, and $\mathbb{R}$-functionals $f$, $g$, $h$ of $V$. Then $(f+g)+h = f+(g+h)$.

(48) Let us consider a non empty vector space structure $V$ over $\mathbb{Z}^{\mathrm{R}}$, and an element $x$ of $V$. Then $(0\mathrm{FrFunctional}(V))(x) = 0_{\mathbb{R}_{\mathrm{F}}}$.

Let us consider a non empty vector space structure $V$ over $\mathbb{Z}^{\mathrm{R}}$ and an $\mathbb{R}$-functional $f$ of $V$. Now we state the propositions:

(49) $f + 0\mathrm{FrFunctional}(V) = f$.

(50) $f - f = 0\mathrm{FrFunctional}(V)$.

(51) Let us consider a non empty vector space structure $V$ over $\mathbb{Z}^{\mathrm{R}}$, an element $r$ of $\mathbb{R}_{\mathrm{F}}$, and $\mathbb{R}$-functionals $f$, $g$ of $V$. Then $r \cdot (f+g) = r \cdot f + r \cdot g$.

Let us consider a non empty vector space structure $V$ over $\mathbb{Z}^{\mathrm{R}}$, elements $r$, $s$ of $\mathbb{R}_{\mathrm{F}}$, and an $\mathbb{R}$-functional $f$ of $V$. Now we state the propositions:

(52) $(r+s) \cdot f = r \cdot f + s \cdot f$.

(53) $(r \cdot s) \cdot f = r \cdot (s \cdot f)$.

(54) Let us consider a non empty vector space structure $V$ over $\mathbb{Z}^{\mathrm{R}}$, and an $\mathbb{R}$-functional $f$ of $V$. Then $1_{\mathbb{R}_{\mathrm{F}}} \cdot f = f$.

Let $V$ be a non empty vector space structure over $\mathbb{Z}^{\mathrm{R}}$ and $f$, $g$ be additive $\mathbb{R}$-functionals of $V$. Observe that $f+g$ is additive.

Let $f$ be an additive $\mathbb{R}$-functional of $V$. One can check that $-f$ is additive.

Let $v$ be an element of $\mathbb{R}_{\mathrm{F}}$. Let us note that $v \cdot f$ is additive.

Let $f$, $g$ be homogeneous $\mathbb{R}$-functionals of $V$. Let us observe that $f+g$ is homogeneous.

Let $f$ be a homogeneous $\mathbb{R}$-functional of $V$. Note that $-f$ is homogeneous.

Let $v$ be an element of $\mathbb{R}_{\mathrm{F}}$. Observe that $v \cdot f$ is homogeneous.

Let $V$, $W$ be non empty vector space structures over $\mathbb{Z}^{\mathrm{R}}$, $f$ be an $\mathbb{R}$-form of $V$ and $W$, and $v$ be a vector of $V$. The functor FrFunctionalFAF$(f, v)$ yielding an $\mathbb{R}$-functional of $W$ is defined by the term

(Def. 24) $(\mathrm{curry}\, f)(v)$.

Let $w$ be a vector of $W$. The functor FrFunctionalSAF$(f, w)$ yielding an $\mathbb{R}$-functional of $V$ is defined by the term

(Def. 25) $(\mathrm{curry}'\, f)(w)$.

Now we state the propositions:

(55) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, an $\mathbb{R}$-form $f$ of $V$ and $W$, and a vector $v$ of $V$. Then

(i) $\mathrm{dom}\, \mathrm{FrFunctionalFAF}(f, v) =$ the carrier of $W$, and

(ii) $\mathrm{rng}\, \mathrm{FrFunctionalFAF}(f, v) \subseteq$ the carrier of $\mathbb{R}_{\mathrm{F}}$, and

(iii) for every vector $w$ of $W$, $(\mathrm{FrFunctionalFAF}(f, v))(w) = f(v, w)$.

(56) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, an $\mathbb{R}$-form $f$ of $V$ and $W$, and a vector $w$ of $W$. Then

(i) dom FrFunctionalSAF$(f, w) =$ the carrier of $V$, and

(ii) rng FrFunctionalSAF$(f, w) \subseteq$ the carrier of $\mathbb{R}_{\rm F}$, and

(iii) for every vector $v$ of $V$, (FrFunctionalSAF$(f, w))(v) = f(v, w)$.

(57) Let us consider a non empty vector space structure $V$ over $\mathbb{Z}^{\rm R}$, and an element $x$ of $V$. Then (0FrFunctional$(V))(x) = 0_{\mathbb{R}_{\rm F}}$.

(58) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, and a vector $v$ of $V$. Then FrFunctionalFAF(NulFrForm$(V, W), v) =$ 0FrFunctional$(W)$. The theorem is a consequence of (55).

(59) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, and a vector $w$ of $W$. Then FrFunctionalSAF(NulFrForm$(V, W), w) =$ 0FrFunctional$(V)$. The theorem is a consequence of (56).

(60) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, $\mathbb{R}$-forms $f$, $g$ of $V$ and $W$, and a vector $w$ of $W$. Then FrFunctionalSAF$(f + g, w) =$ FrFunctionalSAF$(f, w) +$ FrFunctionalSAF$(g, w)$. The theorem is a consequence of (56).

(61) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, $\mathbb{R}$-forms $f$, $g$ of $V$ and $W$, and a vector $v$ of $V$. Then FrFunctionalFAF$(f + g, v) =$ FrFunctionalFAF$(f, v) +$ FrFunctionalFAF$(g, v)$. The theorem is a consequence of (55).

(62) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, an $\mathbb{R}$-form $f$ of $V$ and $W$, an element $a$ of $\mathbb{R}_{\rm F}$, and a vector $w$ of $W$. Then FrFunctionalSAF$(a \cdot f, w) = a \cdot$ FrFunctionalSAF$(f, w)$. The theorem is a consequence of (56).

(63) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, an $\mathbb{R}$-form $f$ of $V$ and $W$, an element $a$ of $\mathbb{R}_{\rm F}$, and a vector $v$ of $V$. Then FrFunctionalFAF$(a \cdot f, v) = a \cdot$ FrFunctionalFAF$(f, v)$. The theorem is a consequence of (55).

(64) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, an $\mathbb{R}$-form $f$ of $V$ and $W$, and a vector $w$ of $W$. Then FrFunctionalSAF$(-f, w) =$ $-$FrFunctionalSAF$(f, w)$. The theorem is a consequence of (56).

(65) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, an $\mathbb{R}$-form $f$ of $V$ and $W$, and a vector $v$ of $V$. Then FrFunctionalFAF$(-f, v) =$ $-$FrFunctionalFAF$(f, v)$. The theorem is a consequence of (55).

(66) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, $\mathbb{R}$-forms $f$, $g$ of $V$ and $W$, and a vector $w$ of $W$. Then FrFunctionalSAF$(f -$

$g, w) = \mathrm{FrFunctionalSAF}(f, w) - \mathrm{FrFunctionalSAF}(g, w)$. The theorem is a consequence of (56).

(67) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, $\mathbb{R}$-forms $f$, $g$ of $V$ and $W$, and a vector $v$ of $V$. Then $\mathrm{FrFunctionalFAF}(f - g, v) = \mathrm{FrFunctionalFAF}(f, v) - \mathrm{FrFunctionalFAF}(g, v)$. The theorem is a consequence of (55).

Let $V$, $W$ be non empty vector space structures over $\mathbb{Z}^{\mathrm{R}}$, $f$ be an $\mathbb{R}$-functional of $V$, and $g$ be an $\mathbb{R}$-functional of $W$. The functor $\mathrm{FrFormFunctional}(f, g)$ yielding an $\mathbb{R}$-form of $V$ and $W$ is defined by

(Def. 26) for every vector $v$ of $V$ and for every vector $w$ of $W$, $it(v, w) = f(v) \cdot g(w)$.

(68) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, an $\mathbb{R}$-functional $f$ of $V$, a vector $v$ of $V$, and a vector $w$ of $W$. Then $(\mathrm{FrFormFunctional}(f, 0\mathrm{FrFunctional}(W)))(v, w) = 0_{\mathbb{Z}^{\mathrm{R}}}$.

(69) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, an $\mathbb{R}$-functional $g$ of $W$, a vector $v$ of $V$, and a vector $w$ of $W$. Then $(\mathrm{FrFormFunctional}(0\mathrm{FrFunctional}(V), g))(v, w) = 0_{\mathbb{Z}^{\mathrm{R}}}$.

(70) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, and an $\mathbb{R}$-functional $f$ of $V$. Then $\mathrm{FrFormFunctional}(f, 0\mathrm{FrFunctional}(W)) = \mathrm{NulFrForm}(V, W)$. The theorem is a consequence of (68).

(71) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, and an $\mathbb{R}$-functional $g$ of $W$. Then $\mathrm{FrFormFunctional}(0\mathrm{FrFunctional}(V), g) = \mathrm{NulFrForm}(V, W)$. The theorem is a consequence of (69).

(72) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, an $\mathbb{R}$-functional $f$ of $V$, an $\mathbb{R}$-functional $g$ of $W$, and a vector $v$ of $V$. Then $\mathrm{FrFunctionalFAF}(\mathrm{FrFormFunctional}(f, g), v) = f(v) \cdot g$. The theorem is a consequence of (55).

(73) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, an $\mathbb{R}$-functional $f$ of $V$, an $\mathbb{R}$-functional $g$ of $W$, and a vector $w$ of $W$. Then $\mathrm{FrFunctionalSAF}(\mathrm{FrFormFunctional}(f, g), w) = g(w) \cdot f$. The theorem is a consequence of (56).

## 2. Bilinear Forms over Field of Reals and Their Properties

Let $V$, $W$ be non empty vector space structures over $\mathbb{Z}^{\mathrm{R}}$ and $f$ be an $\mathbb{R}$-form of $V$ and $W$. We say that $f$ is additive w.r.t. second argument if and only if

(Def. 27) for every vector $v$ of $V$, $\mathrm{FrFunctionalFAF}(f, v)$ is additive.

We say that $f$ is additive w.r.t. first argument if and only if

(Def. 28) for every vector $w$ of $W$, $\mathrm{FrFunctionalSAF}(f, w)$ is additive.

We say that $f$ is homogeneous w.r.t. second argument if and only if

(Def. 29) for every vector $v$ of $V$, FrFunctionalFAF$(f, v)$ is homogeneous.

We say that $f$ is homogeneous w.r.t. first argument if and only if

(Def. 30) for every vector $w$ of $W$, FrFunctionalSAF$(f, w)$ is homogeneous.

Observe that NulFrForm$(V, W)$ is additive w.r.t. second argument and

NulFrForm$(V, W)$ is additive w.r.t. first argument and there exists an $\mathbb{R}$-form of $V$ and $W$ which is additive w.r.t. second argument and additive w.r.t. first argument and NulFrForm$(V, W)$ is homogeneous w.r.t. second argument and NulFrForm$(V, W)$ is homogeneous w.r.t. first argument.

There exists an $\mathbb{R}$-form of $V$ and $W$ which is additive w.r.t. second argument, homogeneous w.r.t. second argument, additive w.r.t. first argument, and homogeneous w.r.t. first argument.

An $\mathbb{R}$-bilinear form of $V$ and $W$ is an additive w.r.t. first argument, homogeneous w.r.t. first argument, additive w.r.t. second argument, homogeneous w.r.t. second argument $\mathbb{R}$-form of $V$ and $W$. Let $f$ be an additive w.r.t. second argument $\mathbb{R}$-form of $V$ and $W$ and $v$ be a vector of $V$. One can check that FrFunctionalFAF$(f, v)$ is additive.

Let $f$ be an additive w.r.t. first argument $\mathbb{R}$-form of $V$ and $W$ and $w$ be a vector of $W$. Observe that FrFunctionalSAF$(f, w)$ is additive.

Let $f$ be a homogeneous w.r.t. second argument $\mathbb{R}$-form of $V$ and $W$ and $v$ be a vector of $V$. One can check that FrFunctionalFAF$(f, v)$ is homogeneous.

Let $f$ be a homogeneous w.r.t. first argument $\mathbb{R}$-form of $V$ and $W$ and $w$ be a vector of $W$. Observe that FrFunctionalSAF$(f, w)$ is homogeneous.

Let $f$ be an $\mathbb{R}$-functional of $V$ and $g$ be an additive $\mathbb{R}$-functional of $W$. Observe that FrFormFunctional$(f, g)$ is additive w.r.t. second argument.

Let $f$ be an additive $\mathbb{R}$-functional of $V$ and $g$ be an $\mathbb{R}$-functional of $W$. One can check that FrFormFunctional$(f, g)$ is additive w.r.t. first argument.

Let $f$ be an $\mathbb{R}$-functional of $V$ and $g$ be a homogeneous $\mathbb{R}$-functional of $W$. Observe that FrFormFunctional$(f, g)$ is homogeneous w.r.t. second argument.

Let $f$ be a homogeneous $\mathbb{R}$-functional of $V$ and $g$ be an $\mathbb{R}$-functional of $W$. One can check that FrFormFunctional$(f, g)$ is homogeneous w.r.t. first argument.

Let $V$ be a non trivial vector space structure over $\mathbb{Z}^{\mathrm{R}}$, $W$ be a non empty vector space structure over $\mathbb{Z}^{\mathrm{R}}$, and $f$ be an $\mathbb{R}$-functional of $V$. One can verify that FrFormFunctional$(f, g)$ is non trivial and FrFormFunctional$(f, g)$ is non trivial.

Let $F$ be an $\mathbb{R}$-functional of $V$. We say that $F$ is 0-preserving if and only if

(Def. 31) $F(0_V) = 0_{\mathbb{R}_{\mathrm{F}}}$.

Let $V$ be a $\mathbb{Z}$-module. One can check that every $\mathbb{R}$-functional of $V$ which is homogeneous is also 0-preserving.

Let $V$ be a non empty vector space structure over $\mathbb{Z}^{\rm R}$. Let us observe that $0{\rm FrFunctional}(V)$ is 0-preserving and there exists an $\mathbb{R}$-functional of $V$ which is additive, homogeneous, and 0-preserving.

Let $V$ be a non trivial, free $\mathbb{Z}$-module. Note that there exists an $\mathbb{R}$-functional of $V$ which is additive, homogeneous, non constant, and non trivial.

(74) Let us consider a non trivial, free $\mathbb{Z}$-module $V$, and a non constant, 0-preserving $\mathbb{R}$-functional $f$ of $V$. Then there exists a vector $v$ of $V$ such that

(i) $v \neq 0_V$, and

(ii) $f(v) \neq 0_{\mathbb{R}_{\rm F}}$.

Let $V$, $W$ be non trivial, free $\mathbb{Z}$-modules, $f$ be a non constant, 0-preserving $\mathbb{R}$-functional of $V$, and $g$ be a non constant, 0-preserving $\mathbb{R}$-functional of $W$. Note that ${\rm FrFormFunctional}(f, g)$ is non constant.

Let $V$ be a non empty vector space structure over $\mathbb{Z}^{\rm R}$.

An $\mathbb{R}$-linear functional of $V$ is an additive, homogeneous $\mathbb{R}$-functional of $V$. Let $V$, $W$ be non trivial, free $\mathbb{Z}$-modules. Observe that there exists an $\mathbb{R}$-form of $V$ and $W$ which is non trivial, non constant, additive w.r.t. second argument, homogeneous w.r.t. second argument, additive w.r.t. first argument, and homogeneous w.r.t. first argument.

Let $V$, $W$ be non empty vector space structures over $\mathbb{Z}^{\rm R}$ and $f$, $g$ be additive w.r.t. first argument $\mathbb{R}$-forms of $V$ and $W$. Let us observe that $f+g$ is additive w.r.t. first argument. Let $f$, $g$ be additive w.r.t. second argument $\mathbb{R}$-forms of $V$ and $W$. One can check that $f+g$ is additive w.r.t. second argument.

Let $f$ be an additive w.r.t. first argument $\mathbb{R}$-form of $V$ and $W$ and $a$ be an element of $\mathbb{R}_{\rm F}$. Let us observe that $a \cdot f$ is additive w.r.t. first argument.

Let $f$ be an additive w.r.t. second argument $\mathbb{R}$-form of $V$ and $W$. Note that $a \cdot f$ is additive w.r.t. second argument.

Let $f$ be an additive w.r.t. first argument $\mathbb{R}$-form of $V$ and $W$. Let us observe that $-f$ is additive w.r.t. first argument.

Let $f$ be an additive w.r.t. second argument $\mathbb{R}$-form of $V$ and $W$. Let us observe that $-f$ is additive w.r.t. second argument.

Let $f$, $g$ be additive w.r.t. first argument $\mathbb{R}$-forms of $V$ and $W$. Observe that $f-g$ is additive w.r.t. first argument.

Let $f$, $g$ be additive w.r.t. second argument $\mathbb{R}$-forms of $V$ and $W$. One can check that $f-g$ is additive w.r.t. second argument.

Let $f$, $g$ be homogeneous w.r.t. first argument $\mathbb{R}$-forms of $V$ and $W$. Observe that $f+g$ is homogeneous w.r.t. first argument.

Let $f$, $g$ be homogeneous w.r.t. second argument $\mathbb{R}$-forms of $V$ and $W$. One can verify that $f+g$ is homogeneous w.r.t. second argument.

Let $f$ be a homogeneous w.r.t. first argument $\mathbb{R}$-form of $V$ and $W$ and $a$ be an element of $\mathbb{R}_{\rm F}$. Observe that $a \cdot f$ is homogeneous w.r.t. first argument.

Let $f$ be a homogeneous w.r.t. second argument $\mathbb{R}$-form of $V$ and $W$. One can check that $a \cdot f$ is homogeneous w.r.t. second argument.

Let $f$ be a homogeneous w.r.t. first argument $\mathbb{R}$-form of $V$ and $W$. Observe that $-f$ is homogeneous w.r.t. first argument. Let $f$ be a homogeneous w.r.t. second argument $\mathbb{R}$-form of $V$ and $W$. Observe that $-f$ is homogeneous w.r.t. second argument.

Let $f$, $g$ be homogeneous w.r.t. first argument $\mathbb{R}$-forms of $V$ and $W$. Let us note that $f-g$ is homogeneous w.r.t. first argument.

Let $f$, $g$ be homogeneous w.r.t. second argument $\mathbb{R}$-forms of $V$ and $W$. One can verify that $f-g$ is homogeneous w.r.t. second argument.

(75) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, vectors $v$, $u$ of $V$, a vector $w$ of $W$, and an $\mathbb{R}$-form $f$ of $V$ and $W$. If $f$ is additive w.r.t. first argument, then $f(v+u,w)=f(v,w)+f(u,w)$. The theorem is a consequence of (56).

(76) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, a vector $v$ of $V$, vectors $u$, $w$ of $W$, and an $\mathbb{R}$-form $f$ of $V$ and $W$. If $f$ is additive w.r.t. second argument, then $f(v,u+w)=f(v,u)+f(v,w)$. The theorem is a consequence of (55).

(77) Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, vectors $v$, $u$ of $V$, vectors $w$, $t$ of $W$, and an additive w.r.t. first argument, additive w.r.t. second argument $\mathbb{R}$-form $f$ of $V$ and $W$. Then $f(v+u,w+t)=f(v,w)+f(v,t)+(f(u,w)+f(u,t))$. The theorem is a consequence of (75) and (76).

(78) Let us consider right zeroed, non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, an additive w.r.t. second argument $\mathbb{R}$-form $f$ of $V$ and $W$, and a vector $v$ of $V$. Then $f(v,0_W)=0_{\mathbb{Z}^{\rm R}}$. The theorem is a consequence of (76).

(79) Let us consider right zeroed, non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, an additive w.r.t. first argument $\mathbb{R}$-form $f$ of $V$ and $W$, and a vector $w$ of $W$. Then $f(0_V,w)=0_{\mathbb{Z}^{\rm R}}$. The theorem is a consequence of (75).

Let us consider non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\rm R}$, a vector $v$ of $V$, a vector $w$ of $W$, an element $a$ of $\mathbb{Z}^{\rm R}$, and an $\mathbb{R}$-form $f$ of $V$ and $W$. Now we state the propositions:

(80) If $f$ is homogeneous w.r.t. first argument, then $f(a\cdot v,w)=a\cdot f(v,w)$.

The theorem is a consequence of (56).

(81) If $f$ is homogeneous w.r.t. second argument, then $f(v, a \cdot w) = a \cdot f(v, w)$. The theorem is a consequence of (55).

(82) Let us consider add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, a homogeneous w.r.t. first argument $\mathbb{R}$-form $f$ of $V$ and $W$, and a vector $w$ of $W$. Then $f(0_V, w) = 0_{\mathbb{R}_{\mathrm{F}}}$. The theorem is a consequence of (80).

(83) Let us consider add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, a homogeneous w.r.t. second argument $\mathbb{R}$-form $f$ of $V$ and $W$, and a vector $v$ of $V$. Then $f(v, 0_W) = 0_{\mathbb{R}_{\mathrm{F}}}$. The theorem is a consequence of (81).

(84) Let us consider $\mathbb{Z}$-modules $V$, $W$, vectors $v$, $u$ of $V$, a vector $w$ of $W$, and an additive w.r.t. first argument, homogeneous w.r.t. first argument $\mathbb{R}$-form $f$ of $V$ and $W$. Then $f(v-u, w) = f(v, w) - f(u, w)$. The theorem is a consequence of (75) and (80).

(85) Let us consider $\mathbb{Z}$-modules $V$, $W$, a vector $v$ of $V$, vectors $w$, $t$ of $W$, and an additive w.r.t. second argument, homogeneous w.r.t. second argument $\mathbb{R}$-form $f$ of $V$ and $W$. Then $f(v, w-t) = f(v, w) - f(v, t)$. The theorem is a consequence of (76) and (81).

(86) Let us consider $\mathbb{Z}$-modules $V$, $W$, vectors $v$, $u$ of $V$, vectors $w$, $t$ of $W$, and an $\mathbb{R}$-bilinear form $f$ of $V$ and $W$. Then $f(v-u, w-t) = f(v, w) - f(v, t) - (f(u, w) - f(u, t))$. The theorem is a consequence of (84) and (85).

(87) Let us consider add-associative, right zeroed, right complementable, vector distributive, scalar distributive, scalar associative, scalar unital, non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, vectors $v$, $u$ of $V$, vectors $w$, $t$ of $W$, elements $a$, $b$ of $\mathbb{Z}^{\mathrm{R}}$, and an $\mathbb{R}$-bilinear form $f$ of $V$ and $W$. Then $f(v+a \cdot u, w+b \cdot t) = f(v, w) + b \cdot f(v, t) + (a \cdot f(u, w) + a \cdot (b \cdot f(u, t)))$. The theorem is a consequence of (77), (81), and (80).

(88) Let us consider $\mathbb{Z}$-modules $V$, $W$, vectors $v$, $u$ of $V$, vectors $w$, $t$ of $W$, elements $a$, $b$ of $\mathbb{Z}^{\mathrm{R}}$, and an $\mathbb{R}$-bilinear form $f$ of $V$ and $W$. Then $f(v - a \cdot u, w - b \cdot t) = f(v, w) - b \cdot f(v, t) - (a \cdot f(u, w) - a \cdot (b \cdot f(u, t)))$. The theorem is a consequence of (86), (81), and (80).

(89) Let us consider right zeroed, non empty vector space structures $V$, $W$ over $\mathbb{Z}^{\mathrm{R}}$, and an $\mathbb{R}$-form $f$ of $V$ and $W$. Suppose $f$ is additive w.r.t. second argument or additive w.r.t. first argument. Then $f$ is constant if and only if for every vector $v$ of $V$ and for every vector $w$ of $W$, $f(v, w) = 0_{\mathbb{Z}^{\mathrm{R}}}$. The theorem is a consequence of (78) and (79).

## 3. Matrices of Bilinear Form over Field of Real Numbers

Let $V_1$, $V_2$ be finite rank, free $\mathbb{Z}$-modules, $b_1$ be an ordered basis of $V_1$, $b_2$ be an ordered basis of $V_2$, and $f$ be an $\mathbb{R}$-bilinear form of $V_1$ and $V_2$. The functor Bilinear$(f, b_1, b_2)$ yielding a matrix over $\mathbb{R}_{\rm F}$ of dimension $\operatorname{len} b_1 \times \operatorname{len} b_2$ is defined by

(Def. 32) for every natural numbers $i$, $j$ such that $i \in \operatorname{dom} b_1$ and $j \in \operatorname{dom} b_2$ holds $it_{i,j} = f({b_1}_i, {b_2}_j)$.

Now we state the propositions:

(90) Let us consider a finite rank, free $\mathbb{Z}$-module $V$, an $\mathbb{R}$-linear functional $F$ of $V$, a finite sequence $y$ of elements of $V$, a finite sequence $x$ of elements of $\mathbb{Z}^{\rm R}$, and finite sequences $X$, $Y$ of elements of $\mathbb{R}_{\rm F}$. Suppose $X = x$ and $\operatorname{len} y = \operatorname{len} x$ and $\operatorname{len} X = \operatorname{len} Y$ and for every natural number $k$ such that $k \in \operatorname{Seg} \operatorname{len} x$ holds $Y(k) = F(y_k)$. Then $X \cdot Y = F(\sum \operatorname{lmlt}(x, y))$.

Proof: Define $\mathcal{P}$[finite sequence of elements of $V$] $\equiv$ for every finite sequence $x$ of elements of $\mathbb{Z}^{\rm R}$ for every finite sequences $X$, $Y$ of elements of $\mathbb{R}_{\rm F}$ such that $X = x$ and $\operatorname{len} \$_1 = \operatorname{len} x$ and $\operatorname{len} X = \operatorname{len} Y$ and for every natural number $k$ such that $k \in \operatorname{Seg} \operatorname{len} x$ holds $Y(k) = F(\$_{1k})$ holds $X \cdot Y = F(\sum \operatorname{lmlt}(x, \$_1))$. For every finite sequence $y$ of elements of $V$ and for every element $w$ of $V$ such that $\mathcal{P}[y]$ holds $\mathcal{P}[y \frown \langle w \rangle]$ by [4, (22), (39), (59)], [3, (11)]. $\mathcal{P}[\varepsilon_\alpha]$, where $\alpha$ is the carrier of $V$ by [17, (43)]. For every finite sequence $p$ of elements of $V$, $\mathcal{P}[p]$ from [6, Sch. 2]. $\square$

(91) Let us consider finite rank, free $\mathbb{Z}$-modules $V_1$, $V_2$, an ordered basis $b_2$ of $V_2$, an ordered basis $b_3$ of $V_2$, an $\mathbb{R}$-bilinear form $f$ of $V_1$ and $V_2$, a vector $v_1$ of $V_1$, a vector $v_2$ of $V_2$, and finite sequences $X$, $Y$ of elements of $\mathbb{R}_{\rm F}$. Suppose $\operatorname{len} X = \operatorname{len} b_2$ and $\operatorname{len} Y = \operatorname{len} b_2$ and for every natural number $k$ such that $k \in \operatorname{Seg} \operatorname{len} b_2$ holds $Y(k) = f(v_1, {b_2}_k)$ and $X = v_2 \rightarrow b_2$. Then $Y \cdot X = f(v_1, v_2)$. The theorem is a consequence of (55) and (90).

(92) Let us consider finite rank, free $\mathbb{Z}$-modules $V_1$, $V_2$, an ordered basis $b_1$ of $V_1$, an $\mathbb{R}$-bilinear form $f$ of $V_1$ and $V_2$, a vector $v_1$ of $V_1$, a vector $v_2$ of $V_2$, and finite sequences $X$, $Y$ of elements of $\mathbb{R}_{\rm F}$. Suppose $\operatorname{len} X = \operatorname{len} b_1$ and $\operatorname{len} Y = \operatorname{len} b_1$ and for every natural number $k$ such that $k \in \operatorname{Seg} \operatorname{len} b_1$ holds $Y(k) = f({b_1}_k, v_2)$ and $X = v_1 \rightarrow b_1$. Then $X \cdot Y = f(v_1, v_2)$. The theorem is a consequence of (56) and (90).

(93) Every matrix over $\mathbb{Z}^{\rm R}$ is a matrix over $\mathbb{R}_{\rm F}$.

Let $M$ be a matrix over $\mathbb{Z}^{\rm R}$. The functor $\mathbb{Z}2\mathbb{R}(M)$ yielding a matrix over $\mathbb{R}_{\rm F}$ is defined by the term

(Def. 33) $M$.

Let $n$, $m$ be natural numbers and $M$ be a matrix over $\mathbb{Z}^{\rm R}$ of dimension $n\times m$. Note that the functor $\mathbb{Z}2\mathbb{R}(M)$ yields a matrix over $\mathbb{R}_{\rm F}$ of dimension $n\times m$. Let $n$ be a natural number and $M$ be a square matrix over $\mathbb{Z}^{\rm R}$ of dimension $n$. Let us note that the functor $\mathbb{Z}2\mathbb{R}(M)$ yields a square matrix over $\mathbb{R}_{\rm F}$ of dimension $n$. Now we state the propositions:

(94) Let us consider natural numbers $m$, $l$, $n$, a matrix $S$ over $\mathbb{Z}^{\rm R}$ of dimension $l\times m$, a matrix $T$ over $\mathbb{Z}^{\rm R}$ of dimension $m\times n$, a matrix $S_1$ over $\mathbb{R}_{\rm F}$ of dimension $l\times m$, and a matrix $T_1$ over $\mathbb{R}_{\rm F}$ of dimension $m\times n$. If $S = S_1$ and $T = T_1$ and $0 < l$ and $0 < m$, then $S \cdot T = S_1 \cdot T_1$.
Proof: Reconsider $S_3 = S \cdot T$ as a matrix over $\mathbb{Z}^{\rm R}$ of dimension $l\times n$. Reconsider $S_2 = S_1 \cdot T_1$ as a matrix over $\mathbb{R}_{\rm F}$ of dimension $l\times n$. For every natural numbers $i$, $j$ such that $\langle i, j\rangle \in$ the indices of $S_3$ holds $S_{3i,j} = S_{2i,j}$ by [8, (87)], [13, (2), (3), (37)]. $\square$

(95) Let us consider a natural number $n$. Then $I^{n\times n}_{\mathbb{Z}^{\rm R}} = I^{n\times n}_{\mathbb{R}_{\rm F}}$.

(96) Let us consider finite rank, free $\mathbb{Z}$-modules $V_1$, $V_2$, an ordered basis $b_1$ of $V_1$, an ordered basis $b_2$ of $V_2$, an ordered basis $b_3$ of $V_2$, and an $\mathbb{R}$-bilinear form $f$ of $V_1$ and $V_2$. Suppose $0 < \operatorname{rank} V_1$. Then $\operatorname{Bilinear}(f, b_1, b_3) = \operatorname{Bilinear}(f, b_1, b_2) \cdot (\mathbb{Z}2\mathbb{R}(\operatorname{AutMt}(\operatorname{id}_{V_2}, b_3, b_2)))^{\rm T}$.
Proof: Set $n = \operatorname{len} b_2$. Reconsider $I_2 = \operatorname{AutMt}(\operatorname{id}_{V_2}, b_3, b_2)$ as a square matrix over $\mathbb{Z}^{\rm R}$ of dimension $n$. Reconsider $M_1 = \mathbb{Z}2\mathbb{R}({I_2}^{\rm T})$ as a square matrix over $\mathbb{R}_{\rm F}$ of dimension $n$. Set $M_2 = \operatorname{Bilinear}(f, b_1, b_2) \cdot M_1$. For every natural numbers $i$, $j$ such that $\langle i, j\rangle \in$ the indices of $\operatorname{Bilinear}(f, b_1, b_3)$ holds $(\operatorname{Bilinear}(f, b_1, b_3))_{i,j} = M_{2i,j}$ by [8, (87)], [13, (1)], (91). $\square$

(97) Let us consider finite rank, free $\mathbb{Z}$-modules $V_1$, $V_2$, an ordered basis $b_1$ of $V_1$, an ordered basis $b_2$ of $V_2$, an ordered basis $b_3$ of $V_1$, and an $\mathbb{R}$-bilinear form $f$ of $V_1$ and $V_2$. Suppose $0 < \operatorname{rank} V_1$. Then $\operatorname{Bilinear}(f, b_3, b_2) = \mathbb{Z}2\mathbb{R}(\operatorname{AutMt}(\operatorname{id}_{V_1}, b_3, b_1)) \cdot \operatorname{Bilinear}(f, b_1, b_2)$.
Proof: Set $n = \operatorname{len} b_3$. Reconsider $I_2 = \operatorname{AutMt}(\operatorname{id}_{V_1}, b_3, b_1)$ as a square matrix over $\mathbb{Z}^{\rm R}$ of dimension $n$. Reconsider $M_1 = \mathbb{Z}2\mathbb{R}(I_2)$ as a square matrix over $\mathbb{R}_{\rm F}$ of dimension $n$. Set $M_2 = M_1 \cdot \operatorname{Bilinear}(f, b_1, b_2)$. For every natural numbers $i$, $j$ such that $\langle i, j\rangle \in$ the indices of $\operatorname{Bilinear}(f, b_3, b_2)$ holds $(\operatorname{Bilinear}(f, b_3, b_2))_{i,j} = M_{2i,j}$ by [8, (87)], [4, (1)], [13, (1)], (92). $\square$

(98) Let us consider a finite rank, free $\mathbb{Z}$-module $V$, ordered bases $b_1$, $b_2$ of $V$, and an $\mathbb{R}$-bilinear form $f$ of $V$ and $V$. Suppose $0 < \operatorname{rank} V$. Then $\operatorname{Bilinear}(f, b_2, b_2) = \mathbb{Z}2\mathbb{R}(\operatorname{AutMt}(\operatorname{id}_V, b_2, b_1)) \cdot \operatorname{Bilinear}(f, b_1, b_1) \cdot (\mathbb{Z}2\mathbb{R}(\operatorname{AutMt}(\operatorname{id}_V, b_2, b_1)))^{\rm T}$. The theorem is a consequence of (97) and (96).

Let us consider a finite rank, free $\mathbb{Z}$-module $V$, ordered bases $b_1$, $b_2$ of $V$, and a square matrix $M$ over $\mathbb{R}_{\rm F}$ of dimension $\operatorname{rank} V$.

Let us assume that $M = \operatorname{AutMt}(\operatorname{id}_V, b_1, b_2)$. Now we state the propositions:

(99) (i) $\operatorname{Det} M = 1$ and $\operatorname{Det} M^{\mathrm{T}} = 1$, or

(ii) $\operatorname{Det} M = -1$ and $\operatorname{Det} M^{\mathrm{T}} = -1$.

The theorem is a consequence of (94) and (95).

(100) $|\operatorname{Det} M| = 1$. The theorem is a consequence of (99).

Let us consider a finite rank, free $\mathbb{Z}$-module $V$, ordered bases $b_1$, $b_2$ of $V$, and an $\mathbb{R}$-bilinear form $f$ of $V$ and $V$. Now we state the propositions:

(101) $\operatorname{Det} \operatorname{Bilinear}(f, b_2, b_2) = \operatorname{Det} \operatorname{Bilinear}(f, b_1, b_1)$. The theorem is a consequence of (98) and (99).

(102) $|\operatorname{Det} \operatorname{Bilinear}(f, b_2, b_2)| = |\operatorname{Det} \operatorname{Bilinear}(f, b_1, b_1)|$.

Let $V$ be a finite rank, free $\mathbb{Z}$-module, $f$ be an $\mathbb{R}$-bilinear form of $V$ and $V$, and $b$ be an ordered basis of $V$. The functor $\operatorname{GramMatrix}(f, b)$ yielding a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $\operatorname{rank} V$ is defined by the term

(Def. 34) $\operatorname{Bilinear}(f, b, b)$.

The functor $\operatorname{GramDet}(f)$ yielding an element of $\mathbb{R}_{\mathrm{F}}$ is defined by

(Def. 35) for every ordered basis $b$ of $V$, $it = \operatorname{Det} \operatorname{GramMatrix}(f, b)$.

Let $L$ be a $\mathbb{Z}$-lattice. The functor $\operatorname{InnerProduct} L$ yielding an $\mathbb{R}$-form of $L$ and $L$ is defined by the term

(Def. 36) the scalar product of $L$.

One can check that $\operatorname{InnerProduct} L$ is additive w.r.t. first argument, homogeneous w.r.t. first argument, additive w.r.t. second argument, and homogeneous w.r.t. second argument.

Let $b$ be an ordered basis of $L$. The functor $\operatorname{GramMatrix}(b)$ yielding a square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension $\dim(L)$ is defined by the term

(Def. 37) $\operatorname{GramMatrix}(\operatorname{InnerProduct} L, b)$.

The functor $\operatorname{GramDet}(L)$ yielding an element of $\mathbb{R}_{\mathrm{F}}$ is defined by the term

(Def. 38) $\operatorname{GramDet}(\operatorname{InnerProduct} L)$.

(103) Let us consider an integral $\mathbb{Z}$-lattice $L$. Then $\operatorname{InnerProduct} L$ is a bilinear form of $L$, $L$.

Proof: For every object $z$ such that $z \in$ (the carrier of $L$) $\times$ (the carrier of $L$) holds $(\operatorname{InnerProduct} L)(z) \in$ the carrier of $\mathbb{Z}^{\mathrm{R}}$. Reconsider $f = \operatorname{InnerProduct} L$ as a form of $L$, $L$. For every vector $v$ of $L$, $f(\cdot, v)$ is additive by [2, (70)], (8). For every vector $v$ of $L$, $f(\cdot, v)$ is homogeneous by [2, (70)], (9). For every vector $v$ of $L$, $f(v, \cdot)$ is additive by [2, (69)], (8). For every vector $v$ of $L$, $f(v, \cdot)$ is homogeneous by [2, (69)], (9). □

(104) Let us consider an integral $\mathbb{Z}$-lattice $L$, and an ordered basis $b$ of $L$. Then $\operatorname{GramMatrix}(b)$ is a square matrix over $\mathbb{Z}^{\mathrm{R}}$ of dimension $\dim(L)$.

Proof: For every natural numbers $i$, $j$ such that $\langle i, j\rangle \in$ the indices of GramMatrix($b$) holds $(\mathrm{GramMatrix}(b))_{i,j} \in$ the carrier of $\mathbb{Z}^{\mathrm{R}}$ by [8, (87)]. □

Let $L$ be an integral $\mathbb{Z}$-lattice. Note that GramDet($L$) is integer.

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