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Categorical Pullbacks

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Summary. The main purpose of this article is to introduce the categorical concept of pullback in Mizar. In the first part of this article we redefine homsets, monomorphisms, epimorpshisms and isomorphisms [7] within a free-object category [1] and it is shown there that ordinal numbers can be considered as categories. Then the pullback is introduced in terms of its universal property and the Pullback Lemma is formalized [15]. In the last part of the article we formalize the pullback of functors [14] and it is also shown that it is not possible to write an equivalent definition in the context of the previous Mizar formalization of category theory [8].

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The notation and terminology used in this paper have been introduced in the following articles: [2], [8], [17], [18], [6], [13], [9], [10], [3], [11], [20], [21], [16], [19], [4], [5], and [12].

1. Preliminaries

One can verify that every set which is ordinal is also non pair.

Let $\mathscr{C}$ be an empty category structure. Let us note that Mor $\mathscr{C}$ is empty.

Let $\mathscr{C}$ be a non empty category structure. Note that Mor $\mathscr{C}$ is non empty.

Let $\mathscr{C}$ be an empty category structure with identities. Let us note that Ob $\mathscr{C}$ is empty.

Let $\mathscr{C}$ be a non empty category structure with identities. Observe that Ob $\mathscr{C}$ is non empty.

Let $\mathscr{C}$ be category structure with identities and a be an object of $\mathscr{C}$. One can check that id-a is identity.

Now we state the propositions:

(1) Let us consider a category structure $\mathscr{C}$, and a morphism f of $\mathscr{C}$. Suppose $\mathscr{C}$ is not empty. Then $f \in$ the carrier of $\mathscr{C}$.

(2) Let us consider category structure $\mathscr{C}$ with identities, and an object a of $\mathscr{C}$. Suppose $\mathscr{C}$ is not empty. Then $a \in$ the carrier of $\mathscr{C}$.

(3) Let us consider a composable category structure $\mathscr{C}$, and morphisms f_1, f_2, f_3 of $\mathscr{C}$. Suppose $f_1 \triangleright f_2$ and $f_2 \triangleright f_3$ and f_2 is identity. Then $f_1 \triangleright f_3$.

(4) Let us consider a composable category structure $\mathscr{C}$ with identities, and morphisms f_1, f_2 of $\mathscr{C}$. Suppose $f_1 \triangleright f_2$. Then

(i) $\operatorname{dom}(f_1 \circ f_2) = \operatorname{dom} f_2$, and

(ii) $\operatorname{cod}(f_1 \circ f_2) = \operatorname{cod} f_1$.

(5) Let us consider a non empty, composable category structure $\mathscr{C}$ with identities, and morphisms f_1, f_2 of $\mathscr{C}$. Then $f_1 \triangleright f_2$ if and only if $\operatorname{dom} f_1 = \operatorname{cod} f_2$.

(6) Let us consider a composable category structure $\mathscr{C}$ with identities, and a morphism f of $\mathscr{C}$. If f is identity, then $\operatorname{dom} f = f$ and $\operatorname{cod} f = f$.

(7) Let us consider a composable category structure $\mathscr{C}$ with identities, and morphisms f_1, f_2 of $\mathscr{C}$. Suppose $f_1 \triangleright f_2$ and f_1 is identity and f_2 is identity. Then $f_1 = f_2$.

Let us consider a non empty, composable category structure $\mathscr{C}$ with identities and morphisms f_1, f_2 of $\mathscr{C}$. Now we state the propositions:

(8) If $\operatorname{dom} f_1 = f_2$, then $f_1 \triangleright f_2$ and $f_1 \circ f_2 = f_1$.

(9) If $f_1 = \operatorname{cod} f_2$, then $f_1 \triangleright f_2$ and $f_1 \circ f_2 = f_2$.

Now we state the propositions:

(10) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$, a functor $\mathcal{F}$ from $\mathscr{C}_1$ to $\mathscr{C}_2$, a functor $\mathcal{G}$ from $\mathscr{C}_2$ to $\mathscr{C}_3$, and a functor $\mathcal{H}$ from $\mathscr{C}_3$ to $\mathscr{C}_4$. Suppose $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{H}$ is covariant. Then $\mathcal{H} \circ (\mathcal{G} \circ \mathcal{F}) = (\mathcal{H} \circ \mathcal{G}) \circ \mathcal{F}$.

(11) Let us consider categories $\mathscr{C}$, $\mathscr{D}$, and a functor $\mathcal{F}$ from $\mathscr{C}$ to $\mathscr{D}$. Suppose $\mathcal{F}$ is covariant. Then

(i) $\mathcal{F} \circ \mathrm{id}_{\mathscr{C}} = \mathcal{F}$, and

(ii) $\mathrm{id}_{\mathscr{D}} \circ \mathcal{F} = \mathcal{F}$.

(12) Let us consider composable category structures $\mathscr{C}$, $\mathscr{D}$ with identities. Then $\mathscr{C} \cong \mathscr{D}$ if and only if there exists a functor $\mathcal{F}$ from $\mathscr{C}$ to $\mathscr{D}$ such that $\mathcal{F}$ is covariant and bijective. The theorem is a consequence of (5).

(13) Let us consider empty category structures $\mathscr{C}$, $\mathscr{D}$ with identities. Then $\mathscr{C} \cong \mathscr{D}$.

Let us consider category structures $\mathscr{C}$, $\mathscr{D}$ with identities. Now we state the propositions:

(14) Suppose $\mathscr{C} \cong \mathscr{D}$. Then

(i) $\overline{\overline{\mathrm{Mor}\,\mathscr{C}}} = \overline{\overline{\mathrm{Mor}\,\mathscr{D}}}$, and

(ii) $\overline{\overline{\mathrm{Ob}\,\mathscr{C}}} = \overline{\overline{\mathrm{Ob}\,\mathscr{D}}}$.

(15) If $\mathscr{C} \cong \mathscr{D}$ and $\mathscr{C}$ is empty, then $\mathscr{D}$ is empty. The theorem is a consequence of (14).

2. Hom-sets

Let $\mathscr{C}$ be a category structure and a, b be objects of $\mathscr{C}$. The functor $\mathrm{hom}(a, b)$ yielding a subset of $\mathrm{Mor}\,\mathscr{C}$ is defined by the term

(Def. 1) $\{f$, where f is a morphism of $\mathscr{C}$: there exist morphisms f_1, f_2 of $\mathscr{C}$ such that $a = f_1$ and $b = f_2$ and $f \rhd f_1$ and $f_2 \rhd f\}$.

Let $\mathscr{C}$ be a non empty, composable category structure with identities. Observe that the functor $\mathrm{hom}(a, b)$ yields a subset of $\mathrm{Mor}\,\mathscr{C}$ and is defined by the term

(Def. 2) $\{f$, where f is a morphism of $\mathscr{C}$: $\mathrm{dom}\, f = a$ and $\mathrm{cod}\, f = b\}$.

Let $\mathscr{C}$ be a category structure. Assume $\mathrm{hom}(a, b) \neq \emptyset$.

A morphism from a to b is a morphism of $\mathscr{C}$ and is defined by

(Def. 3) $it \in \mathrm{hom}(a, b)$.

Let $\mathscr{C}$ be category structure with identities and a be an object of $\mathscr{C}$. Assume $\mathrm{hom}(a, a) \neq \emptyset$. Observe that the functor id-a yields a morphism from a to a. Let $\mathscr{C}$ be a non empty category structure with identities. Note that $\mathrm{hom}(a, a)$ is non empty.

Let $\mathscr{C}$ be a composable category structure with identities, a, b, c be objects of $\mathscr{C}$, f be a morphism from a to b, and g be a morphism from b to c. Assume $\mathrm{hom}(a, b) \neq \emptyset$ and $\mathrm{hom}(b, c) \neq \emptyset$. The functor $g \cdot f$ yielding a morphism from a to c is defined by the term

(Def. 4) $g \circ f$.

Now we state the propositions:

(16) Let us consider a category structure $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. Suppose $\mathrm{hom}(a, b) \neq \emptyset$. Then there exist morphisms f_1, f_2 of $\mathscr{C}$ such that

(i) $a = f_1$, and

(ii) $b = f_2$, and

(iii) $f \triangleright f_1$, and

(iv) $f_2 \triangleright f$.

(17) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b, c of $\mathscr{C}$, a morphism f_1 from a to b, and a morphism f_2 from b to c. Suppose $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$. Then $f_2 \triangleright f_1$. The theorem is a consequence of (16) and (3).

(18) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f from a to b. Suppose $\hom(a,b) \neq \emptyset$. Then

(i) $f \cdot \text{id-}a = f$, and

(ii) $\text{id-}b \cdot f = f$.

The theorem is a consequence of (17).

(19) Let us consider a non empty, composable category structure $\mathscr{C}$ with identities, and a morphism f of $\mathscr{C}$. Then $f \in \hom(\operatorname{dom} f, \operatorname{cod} f)$.

(20) Let us consider a non empty, composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f of $\mathscr{C}$. Then $f \in \hom(a,b)$ if and only if $\operatorname{dom} f = a$ and $\operatorname{cod} f = b$.

(21) Let us consider a non empty, composable category structure $\mathscr{C}$ with identities, and an object a of $\mathscr{C}$. Then $a \in \hom(a,a)$. The theorem is a consequence of (6).

(22) Let us consider a composable category structure $\mathscr{C}$ with identities, and objects a, b, c of $\mathscr{C}$. Suppose $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$. Then $\hom(a,c) \neq \emptyset$. The theorem is a consequence of (16) and (3).

(23) Let us consider a category $\mathscr{C}$, objects a, b, c, d of $\mathscr{C}$, a morphism f_1 from a to b, a morphism f_2 from b to c, and a morphism f_3 from c to d. Suppose $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$ and $\hom(c,d) \neq \emptyset$. Then $f_3 \cdot (f_2 \cdot f_1) = (f_3 \cdot f_2) \cdot f_1$. The theorem is a consequence of (22) and (17).

(24) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b, c of $\mathscr{C}$, a morphism f_1 from a to b, and a morphism f_2 from b to c. Suppose $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$. Then

(i) if f_1 is identity, then $f_2 \cdot f_1 = f_2$, and

(ii) if f_2 is identity, then $f_2 \cdot f_1 = f_1$.

Proof: $f_2 \triangleright f_1$. If f_1 is identity, then $f_2 \cdot f_1 = f_2$ by [17, (22), (23)]. □

3. Monomorphisms, Epimorphisms and Isomorphisms

Let $\mathscr{C}$ be a composable category structure with identities, a, b be objects of $\mathscr{C}$, and f be a morphism from a to b. We say that f is monomorphic if and only if

(Def. 5) $\hom(a,b) \neq \emptyset$ and for every object c of $\mathscr{C}$ such that $\hom(c,a) \neq \emptyset$ for every morphisms g_1, g_2 from c to a such that $f \cdot g_1 = f \cdot g_2$ holds $g_1 = g_2$.

We say that f is epimorphic if and only if

(Def. 6) $\hom(a,b) \neq \emptyset$ and for every object c of $\mathscr{C}$ such that $\hom(b,c) \neq \emptyset$ for every morphisms g_1, g_2 from b to c such that $g_1 \cdot f = g_2 \cdot f$ holds $g_1 = g_2$.

Now we state the proposition:

(25) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f_1 from a to b. Suppose $\hom(a,b) \neq \emptyset$ and f_1 is identity. Then f_1 is monomorphic. The theorem is a consequence of (24).

Let us consider a category $\mathscr{C}$, objects a, b, c of $\mathscr{C}$, a morphism f_1 from a to b, and a morphism f_2 from b to c. Now we state the propositions:

(26) If f_1 is monomorphic and f_2 is monomorphic, then $f_2 \cdot f_1$ is monomorphic. The theorem is a consequence of (22) and (23).

(27) If $f_2 \cdot f_1$ is monomorphic and $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$, then f_1 is monomorphic. The theorem is a consequence of (23).

Let $\mathscr{C}$ be a composable category structure with identities, a, b be objects of $\mathscr{C}$, and f be a morphism from a to b. We say that f is a section if and only if

(Def. 7) $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and there exists a morphism g from b to a such that $g \cdot f = \text{id-}a$.

We say that f is a retraction if and only if

(Def. 8) $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and there exists a morphism g from b to a such that $f \cdot g = \text{id-}b$.

Now we state the propositions:

(28) Let us consider a category $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. If f is a section, then f is monomorphic. The theorem is a consequence of (23) and (18).

(29) Let us consider a composable category structure $\mathscr{C}$ with identities, objects a, b of $\mathscr{C}$, and a morphism f_1 from a to b. Suppose $\hom(a,b) \neq \emptyset$ and f_1 is identity. Then f_1 is epimorphic. The theorem is a consequence of (24).

Let us consider a category $\mathscr{C}$, objects a, b, c of $\mathscr{C}$, a morphism f_1 from a to b, and a morphism f_2 from b to c. Now we state the propositions:

(30) If f_1 is epimorphic and f_2 is epimorphic, then $f_2 \cdot f_1$ is epimorphic. The theorem is a consequence of (22) and (23).

(31) If $f_2 \cdot f_1$ is epimorphic and $\hom(a,b) \neq \emptyset$ and $\hom(b,c) \neq \emptyset$, then f_2 is epimorphic. The theorem is a consequence of (23).

(32) Let us consider a category $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. If f is a retraction, then f is epimorphic. The theorem is a consequence of (23) and (18).

Let $\mathscr{C}$ be a composable category structure with identities, a, b be objects of $\mathscr{C}$, and f be a morphism from a to b. We say that f is isomorphism if and only if

(Def. 9) $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and there exists a morphism g from b to a such that $g \cdot f = \text{id-}a$ and $f \cdot g = \text{id-}b$.

We say that a and b are isomorphic if and only if

(Def. 10) there exists a morphism f from a to b such that f is isomorphism.

Note that a and b are isomorphic if and only if the condition (Def. 11) is satisfied.

(Def. 11) $\hom(a,b) \neq \emptyset$ and $\hom(b,a) \neq \emptyset$ and there exists a morphism f from a to b and there exists a morphism g from b to a such that $g \cdot f = \text{id-}a$ and $f \cdot g = \text{id-}b$.

Now we state the proposition:

(33) Let us consider a category $\mathscr{C}$, objects a, b of $\mathscr{C}$, and a morphism f from a to b. If f is isomorphism, then f is monomorphic and epimorphic. The theorem is a consequence of (28) and (32).

4. Ordinal Numbers as Categories

Let $\mathscr{C}$ be a category structure. We say that $\mathscr{C}$ is a preorder if and only if

(Def. 12) for every objects a, b of $\mathscr{C}$ and for every morphisms f_1, f_2 of $\mathscr{C}$ such that $f_1, f_2 \in \hom(a,b)$ holds $f_1 = f_2$.

Observe that every category structure which is empty is also a preorder and there exists a category structure which is strict and preorder and every composable category structure with identities which is a preorder is also associative.

Let $\mathscr{C}$ be category structure with identities. The functor $\mathrm{RelOb}\,\mathscr{C}$ yielding a binary relation on $\mathrm{Ob}\,\mathscr{C}$ is defined by the term

(Def. 13) $\{\langle a, b\rangle$, where a, b are objects of $\mathscr{C}$: there exists a morphism f of $\mathscr{C}$ such that $f \in \hom(a,b)\}$.

Let $\mathscr{C}$ be an empty category structure with identities. Let us note that $\mathrm{RelOb}\,\mathscr{C}$ is empty.

Now we state the propositions:

(34) Let us consider a composable category structure $\mathscr{C}$ with identities. Then

(i) $\operatorname{dom}\operatorname{RelOb}\mathscr{C} = \operatorname{Ob}\mathscr{C}$, and

(ii) $\operatorname{rng}\operatorname{RelOb}\mathscr{C} = \operatorname{Ob}\mathscr{C}$.

The theorem is a consequence of (6) and (19).

(35) Let us consider composable category structures $\mathscr{C}_1$, $\mathscr{C}_2$ with identities. Suppose $\mathscr{C}_1 \cong \mathscr{C}_2$. Then $\operatorname{RelOb}\mathscr{C}_1$ and $\operatorname{RelOb}\mathscr{C}_2$ are isomorphic. The theorem is a consequence of (15), (34), and (20).

Let $\mathscr{C}$ be a non empty, composable category structure with identities. One can verify that $\operatorname{RelOb}\mathscr{C}$ is non empty.

Now we state the propositions:

(36) Let us consider preorder, composable category structure $\mathscr{C}$ with identities. Suppose $\mathscr{C}$ is not empty. Then there exists a function $\mathcal{F}$ from $\mathscr{C}$ into $\operatorname{RelOb}\mathscr{C}$ such that

(i) $\mathcal{F}$ is bijective, and

(ii) for every morphism f of $\mathscr{C}$, $\mathcal{F}(f) = \langle \operatorname{dom} f, \operatorname{cod} f \rangle$.

Proof: Reconsider $\mathscr{C}_1 = \mathscr{C}$ as a non empty, composable category structure with identities. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ for every morphism f of $\mathscr{C}_1$ such that $\$_1 = f$ holds $\$_2 = \langle \operatorname{dom} f, \operatorname{cod} f \rangle$. For every element x of the carrier of $\mathscr{C}_1$, there exists an element y of $\operatorname{RelOb}\mathscr{C}_1$ such that $\mathcal{P}[x, y]$. Consider $\mathcal{F}$ being a function from the carrier of $\mathscr{C}_1$ into $\operatorname{RelOb}\mathscr{C}_1$ such that for every element x of the carrier of $\mathscr{C}_1$, $\mathcal{P}[x, \mathcal{F}(x)]$ from [10, Sch. 3]. For every object y such that $y \in \operatorname{RelOb}\mathscr{C}$ holds $y \in \operatorname{rng}\mathcal{F}$ by (20), [9, (3)]. For every objects x_1, x_2 such that $x_1, x_2 \in \operatorname{dom}\mathcal{F}$ and $\mathcal{F}(x_1) = \mathcal{F}(x_2)$ holds $x_1 = x_2$. $\square$

(37) Let us consider an ordinal number O. Then there exists a strict, a preorder category $\mathscr{C}$ such that

(i) $\operatorname{Ob}\mathscr{C} = O$, and

(ii) for every objects o_1, o_2 of $\mathscr{C}$ such that $o_1 \in o_2$ holds $\hom(o_1, o_2) = \{\langle o_1, o_2 \rangle\}$, and

(iii) $\operatorname{RelOb}\mathscr{C} = {\subseteq}_O$, and

(iv) $\operatorname{Mor}\mathscr{C} = O \cup \{\langle o_1, o_2 \rangle$, where o_1, o_2 are elements of $O : o_1 \in o_2\}$.

The theorem is a consequence of (6), (20), and (21).

Let O be an ordinal number and $\mathscr{C}$ be a composable category structure with identities. We say that $\mathscr{C}$ is O-ordered if and only if

(Def. 14) $\operatorname{RelOb}\mathscr{C}$ and ${\subseteq}_O$ are isomorphic.

Let O be a non empty, ordinal number. Let us observe that every composable category structure with identities which is O-ordered is also non empty.

Let O be an ordinal number. Note that there exists a composable category structure with identities which is strict, O-ordered, and preorder.

Let O be an empty, ordinal number. Let us observe that every composable category structure with identities which is O-ordered is also empty.

Now we state the proposition:

(38) Let us consider ordinal numbers O_1, O_2, a O_1-ordered, a preorder category $\mathscr{C}_1$, and a O_2-ordered, a preorder category $\mathscr{C}_2$. Then $O_1 = O_2$ if and only if $\mathscr{C}_1 \cong \mathscr{C}_2$.

PROOF: If $O_1 = O_2$, then $\mathscr{C}_1 \cong \mathscr{C}_2$ by (13), [4, (39), (41)], (36). If $\mathscr{C}_1 \cong \mathscr{C}_2$, then $O_1 = O_2$ by (35), [4, (42), (40)], [5, (10)]. □

Let O be an ordinal number. The functor $\mathbf{O}$ yielding a strict, O-ordered, a preorder category is defined by the term

(Def. 15) the strict, O-ordered, a preorder category.

Now we state the proposition:

(39) There exists a morphism f of $\mathbf{2}$ such that

(i) f is not identity, and

(ii) $\mathrm{Ob}\,\mathbf{2} = \{\mathrm{dom}\,f, \mathrm{cod}\,f\}$, and

(iii) $\mathrm{Mor}\,\mathbf{2} = \{\mathrm{dom}\,f, \mathrm{cod}\,f, f\}$, and

(iv) $\mathrm{dom}\,f$, $\mathrm{cod}\,f$, f are mutually different.

PROOF: Consider $\mathscr{C}$ being a strict, a preorder category such that $\mathrm{Ob}\,\mathscr{C} = 2$ and for every objects o_1, o_2 of $\mathscr{C}$ such that $o_1 \in o_2$ holds $\hom(o_1, o_2) = \{\langle o_1, o_2 \rangle\}$ and $\mathrm{RelOb}\,\mathscr{C} = {\subseteq_2}$ and $\mathrm{Mor}\,\mathscr{C} = 2 \cup \{\langle o_1, o_2 \rangle$, where o_1, o_2 are elements of $2 : o_1 \in o_2\}$. $\mathscr{C} \cong \mathbf{2}$. Consider $\mathcal{F}$ being a functor from $\mathscr{C}$ to $\mathbf{2}$, $\mathcal{G}$ being a functor from $\mathbf{2}$ to $\mathscr{C}$ such that $\mathcal{F}$ is covariant and $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathbf{2}}$. Reconsider $g = \langle 0, 1 \rangle$ as a morphism of $\mathscr{C}$. g is not identity by [17, (22)]. Set $f = \mathcal{F}(g)$. f is not identity by [9, (18)], [17, (34)]. $\overline{\overline{\mathrm{Ob}\,\mathbf{2}}} = \overline{\overline{2}}$. Consider x, y being objects such that $x \neq y$ and $\mathrm{Ob}\,\mathbf{2} = \{x, y\}$. $\mathrm{dom}\,f \neq \mathrm{cod}\,f$. For every object x, $x \in \mathrm{Mor}\,\mathbf{2}$ iff $x \in \{\mathrm{dom}\,f, \mathrm{cod}\,f, f\}$ by [17, (22)], [9, (18)], [17, (34)], [2, (50), (49)]. □

Let $\mathscr{C}$ be a non empty category and f be a morphism of $\mathscr{C}$. The functor $\mathcal{M}_{\mathrm{f}}$ yielding a covariant functor from $\mathbf{2}$ to $\mathscr{C}$ is defined by

(Def. 16) for every morphism g of $\mathbf{2}$ such that g is not identity holds $it(g) = f$.

Now we state the proposition:

(40) Let us consider a non empty category $\mathscr{C}$, and a morphism f of $\mathscr{C}$. Suppose f is identity. Let us consider a morphism g of $\mathbf{2}$. Then $(\mathcal{M}_{\mathrm{f}})(g) = f$.

The theorem is a consequence of (39) and (6).

5. Pullbacks

Let $\mathscr{C}$ be a category, c, c_1, c_2, d be objects of $\mathscr{C}$, and f_1 be a morphism from c_1 to c. Assume $\hom(c_1, c) \neq \emptyset$. Let f_2 be a morphism from c_2 to c. Assume $\hom(c_2, c) \neq \emptyset$. Let p_1 be a morphism from d to c_1. Assume $\hom(d, c_1) \neq \emptyset$. Let p_2 be a morphism from d to c_2. Assume $\hom(d, c_2) \neq \emptyset$. We say that $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 if and only if

(Def. 17) $f_1 \cdot p_1 = f_2 \cdot p_2$ and for every object d_1 of $\mathscr{C}$ and for every morphism g_1 from d_1 to c_1 and for every morphism g_2 from d_1 to c_2 such that $\hom(d_1, c_1) \neq \emptyset$ and $\hom(d_1, c_2) \neq \emptyset$ and $f_1 \cdot g_1 = f_2 \cdot g_2$ holds $\hom(d_1, d) \neq \emptyset$ and there exists a morphism h from d_1 to d such that $p_1 \cdot h = g_1$ and $p_2 \cdot h = g_2$ and for every morphism h_1 from d_1 to d such that $p_1 \cdot h_1 = g_1$ and $p_2 \cdot h_1 = g_2$ holds $h = h_1$.

Now we state the proposition:

(41) Let us consider a category $\mathscr{C}$, objects c, c_1, c_2, d, e of $\mathscr{C}$, a morphism f_1 from c_1 to c, a morphism f_2 from c_2 to c, a morphism p_1 from d to c_1, a morphism p_2 from d to c_2, a morphism q_1 from e to c_1, and a morphism q_2 from e to c_2. Suppose $\hom(c_1, c) \neq \emptyset$ and $\hom(c_2, c) \neq \emptyset$ and $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\hom(e, c_1) \neq \emptyset$ and $\hom(e, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 and $\langle e, q_1, q_2 \rangle$ is a pullback of f_1, f_2. Then d and e are isomorphic. The theorem is a consequence of (23) and (18).

Let us consider a category $\mathscr{C}$, objects c, c_1, c_2, d of $\mathscr{C}$, a morphism f_1 from c_1 to c, a morphism f_2 from c_2 to c, a morphism p_1 from d to c_1, and a morphism p_2 from d to c_2. Now we state the propositions:

(42) Suppose $\hom(c_1, c) \neq \emptyset$ and $\hom(c_2, c) \neq \emptyset$ and $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2.
Then $\langle d, p_2, p_1 \rangle$ is a pullback of f_2, f_1.

(43) Suppose $\hom(c_1, c) \neq \emptyset$ and $\hom(c_2, c) \neq \emptyset$ and $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 and f_1 is monomorphic. Then p_2 is monomorphic. The theorem is a consequence of (22) and (23).

(44) Suppose $\hom(c_1, c) \neq \emptyset$ and $\hom(c_2, c) \neq \emptyset$ and $\hom(d, c_1) \neq \emptyset$ and $\hom(d, c_2) \neq \emptyset$ and $\langle d, p_1, p_2 \rangle$ is a pullback of f_1, f_2 and f_1 is isomorphism. Then p_2 is isomorphism. The theorem is a consequence of (22), (23), and (18).

(45) Let us consider a category $\mathscr{C}$, objects c_1, c_1, c_2, c_3, c_4, c_5, c_6 of $\mathscr{C}$, a morphism f_1 from c_1 to c_2, a morphism f_2 from c_2 to c_3, a morphism f_3 from c_1 to c_4, a morphism f_4 from c_2 to c_5, a morphism f_5 from

c_3 to c_6, a morphism f_6 from c_4 to c_5, and a morphism f_7 from c_5 to c_6. Suppose $\hom(c_1, c_2) \neq \emptyset$ and $\hom(c_2, c_3) \neq \emptyset$ and $\hom(c_1, c_4) \neq \emptyset$ and $\hom(c_2, c_5) \neq \emptyset$ and $\hom(c_3, c_6) \neq \emptyset$ and $\hom(c_4, c_5) \neq \emptyset$ and $\hom(c_5, c_6) \neq \emptyset$ and $\langle c_2, f_2, f_4\rangle$ is a pullback of f_5, f_7. Then $\langle c_1, f_1, f_3\rangle$ is a pullback of f_4, f_6 if and only if $\langle c_1, f_2 \cdot f_1, f_3\rangle$ is a pullback of $f_5, f_7 \cdot f_6$ and $f_4 \cdot f_1 = f_6 \cdot f_3$. The theorem is a consequence of (22) and (23).

6. Pullbacks of Functors

Let $\mathscr{C}$, $\mathscr{D}$ be categories and $\mathcal{F}$ be a functor from $\mathscr{C}$ to $\mathscr{D}$. We say that $\mathcal{F}$ is monomorphic if and only if

(Def. 18) $\mathcal{F}$ is covariant and for every category $\mathscr{B}$ and for every functors $\mathcal{G}_1$, $\mathcal{G}_2$ from $\mathscr{B}$ to $\mathscr{C}$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F} \circ \mathcal{G}_1 = \mathcal{F} \circ \mathcal{G}_2$ holds $\mathcal{G}_1 = \mathcal{G}_2$.

We say that $\mathcal{F}$ is isomorphism if and only if

(Def. 19) $\mathcal{F}$ is covariant and there exists a functor $\mathcal{G}$ from $\mathscr{D}$ to $\mathscr{C}$ such that $\mathcal{G}$ is covariant and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathscr{C}}$ and $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathscr{D}}$.

Let $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{D}$ be categories and $\mathcal{F}_1$ be a functor from $\mathscr{C}_1$ to $\mathscr{C}$. Assume $\mathcal{F}_1$ is covariant. Let $\mathcal{F}_2$ be a functor from $\mathscr{C}_2$ to $\mathscr{C}$. Assume $\mathcal{F}_2$ is covariant. Let $\mathcal{P}_1$ be a functor from $\mathscr{D}$ to $\mathscr{C}_1$. Assume $\mathcal{P}_1$ is covariant. Let $\mathcal{P}_2$ be a functor from $\mathscr{D}$ to $\mathscr{C}_2$. Assume $\mathcal{P}_2$ is covariant. We say that $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2\rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ if and only if

(Def. 20) $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$ and for every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}$ is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}_1$ is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$.

Now we state the proposition:

(46) Let us consider categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{D}$, $\mathscr{E}$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$, a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$, a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$, a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$, a functor $\mathcal{Q}_1$ from $\mathscr{E}$ to $\mathscr{C}_1$, and a functor $\mathcal{Q}_2$ from $\mathscr{E}$ to $\mathscr{C}_2$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\mathcal{Q}_1$ is covariant and $\mathcal{Q}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2\rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ and $\langle \mathscr{E}, \mathcal{Q}_1, \mathcal{Q}_2\rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$. Then $\mathscr{D} \cong \mathscr{E}$.

Proof: There exists a functor $\mathcal{F}_8$ from $\mathscr{D}$ to $\mathscr{E}$ and there exists a functor $\mathcal{G}_3$ from $\mathscr{E}$ to $\mathscr{D}$ such that $\mathcal{F}_8$ is covariant and $\mathcal{G}_3$ is covariant and $\mathcal{G}_3 \circ \mathcal{F}_8 = \mathrm{id}_{\mathscr{D}}$ and $\mathcal{F}_8 \circ \mathcal{G}_3 = \mathrm{id}_{\mathscr{E}}$ by (10), (11), [17, (35)]. □

Let us consider categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{D}$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$, a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$, a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$, and a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$. Now we state the propositions:

(47) Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.
Then $\langle \mathscr{D}, \mathcal{P}_2, \mathcal{P}_1 \rangle$ is a pullback of $\mathcal{F}_2, \mathcal{F}_1$.

(48) Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_1$ is monomorphic. Then $\mathcal{P}_2$ is monomorphic.
Proof: For every category $\mathscr{D}_1$ and for every functors $\mathcal{Q}_1$, $\mathcal{Q}_2$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{Q}_1$ is covariant and $\mathcal{Q}_2$ is covariant and $\mathcal{P}_2 \circ \mathcal{Q}_1 = \mathcal{P}_2 \circ \mathcal{Q}_2$ holds $\mathcal{Q}_1 = \mathcal{Q}_2$ by [17, (35)], (10). □

(49) Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_1$ is isomorphism. Then $\mathcal{P}_2$ is isomorphism. The theorem is a consequence of (10) and (11).

(50) Let us consider categories $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$, $\mathscr{C}_5$, $\mathscr{C}_6$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}_2$, a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}_3$, a functor $\mathcal{F}_3$ from $\mathscr{C}_1$ to $\mathscr{C}_4$, a functor $\mathcal{F}_4$ from $\mathscr{C}_2$ to $\mathscr{C}_5$, a functor $\mathcal{F}_5$ from $\mathscr{C}_3$ to $\mathscr{C}_6$, a functor $\mathcal{F}_6$ from $\mathscr{C}_4$ to $\mathscr{C}_5$, and a functor $\mathcal{F}_7$ from $\mathscr{C}_5$ to $\mathscr{C}_6$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant and $\mathcal{F}_3$ is covariant and $\mathcal{F}_4$ is covariant and $\mathcal{F}_5$ is covariant and $\mathcal{F}_6$ is covariant and $\mathcal{F}_7$ is covariant and $\langle \mathscr{C}_2, \mathcal{F}_2, \mathcal{F}_4 \rangle$ is a pullback of $\mathcal{F}_5, \mathcal{F}_7$. Then $\langle \mathscr{C}_1, \mathcal{F}_1, \mathcal{F}_3 \rangle$ is a pullback of $\mathcal{F}_4, \mathcal{F}_6$ if and only if $\langle \mathscr{C}_1, \mathcal{F}_2 \circ \mathcal{F}_1, \mathcal{F}_3 \rangle$ is a pullback of $\mathcal{F}_5, \mathcal{F}_7 \circ \mathcal{F}_6$ and $\mathcal{F}_4 \circ \mathcal{F}_1 = \mathcal{F}_6 \circ \mathcal{F}_3$.
Proof: For every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_4$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F}_4 \circ \mathcal{G}_1 = \mathcal{F}_6 \circ \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ such that $\mathcal{H}$ is covariant and $\mathcal{F}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{F}_3 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ such that $\mathcal{H}_1$ is covariant and $\mathcal{F}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{F}_3 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$ by [17, (35)], (10). □

(51) Let us consider categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$, and a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$. Suppose $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Then there exists a strict category $\mathscr{D}$ and there exists a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$ and there exists a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$ such that the carrier of $\mathscr{D} = \{\langle f_1, f_2 \rangle$, where f_1 is a morphism of $\mathscr{C}_1, f_2$ is a morphism of $\mathscr{C}_2 : f_1 \in$ the carrier of $\mathscr{C}_1$ and $f_2 \in$ the carrier of $\mathscr{C}_2$ and $\mathcal{F}_1(f_1) = \mathcal{F}_2(f_2)\}$ and the composition of $\mathscr{D} = \{\langle\langle f_1, f_2 \rangle, f_3 \rangle$, where f_1, f_2, f_3 are morphisms of $\mathscr{D} : f_1, f_2, f_3 \in$ the carrier of $\mathscr{D}$ and for every morphisms f_{11}, f_{12}, f_{13} of $\mathscr{C}_1$ and for every morphisms f_{21}, f_{22}, f_{23} of $\mathscr{C}_2$ such that $f_1 = \langle f_{11}, f_{21} \rangle$ and $f_2 = \langle f_{12}, f_{22} \rangle$ and $f_3 = \langle f_{13}, f_{23} \rangle$ holds

$f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}\}$ and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle \mathscr{D}, \mathcal{P}_1, \mathcal{P}_2 \rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.

PROOF: Reconsider $c_7 = \{\langle f_1, f_2 \rangle$, where f_1 is a morphism of $\mathscr{C}_1, f_2$ is a morphism of $\mathscr{C}_2 : f_1 \in$ the carrier of $\mathscr{C}_1$ and $f_2 \in$ the carrier of $\mathscr{C}_2$ and $\mathcal{F}_1(f_1) = \mathcal{F}_2(f_2)\}$ as a set. Set $c_8 = \{\langle\langle x_1, x_2 \rangle, x_3 \rangle$, where x_1, x_2, x_3 are elements of $c_7 : x_1, x_2, x_3 \in c_7$ and for every morphisms f_{11}, f_{12}, f_{13} of $\mathscr{C}_1$ and for every morphisms f_{21}, f_{22}, f_{23} of $\mathscr{C}_2$ such that $x_1 = \langle f_{11}, f_{21} \rangle$ and $x_2 = \langle f_{12}, f_{22} \rangle$ and $x_3 = \langle f_{13}, f_{23} \rangle$ holds $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}\}$. For every object x such that $x \in c_8$ holds $x \in (c_7 \times c_7) \times c_7$. For every objects x, y_1, y_2 such that $\langle x, y_1 \rangle$, $\langle x, y_2 \rangle \in c_8$ holds $y_1 = y_2$. Set $\mathscr{D} = \langle c_7, c_8 \rangle$. For every morphisms g_1, g_2 of $\mathscr{D}$ such that $g_1 \triangleright g_2$ there exist morphisms f_{11}, f_{12}, f_{13} of $\mathscr{C}_1$ and there exist morphisms f_{21}, f_{22}, f_{23} of $\mathscr{C}_2$ such that $g_1 = \langle f_{11}, f_{21} \rangle$ and $g_2 = \langle f_{12}, f_{22} \rangle$ and $\mathcal{F}_1(f_{11}) = \mathcal{F}_2(f_{21})$ and $\mathcal{F}_1(f_{12}) = \mathcal{F}_2(f_{22})$ and $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}$ and $g_1 \circ g_2 = \langle f_{13}, f_{23} \rangle$ by (1), [17, (1)], [9, (1)]. For every morphisms g_1, g_2 of $\mathscr{D}$ such that there exist morphisms f_{11}, f_{12} of $\mathscr{C}_1$ and there exist morphisms f_{21}, f_{22} of $\mathscr{C}_2$ such that $g_1 = \langle f_{11}, f_{21} \rangle$ and $g_2 = \langle f_{12}, f_{22} \rangle$ and $\mathcal{F}_1(f_{11}) = \mathcal{F}_2(f_{21})$ and $\mathcal{F}_1(f_{12}) = \mathcal{F}_2(f_{22})$ and $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ holds $g_1 \triangleright g_2$ by (1), [17, (1)]. For every morphisms g, g_1, g_2 of $\mathscr{D}$ such that $g_1 \triangleright g_2$ holds $g_1 \circ g_2 \triangleright g$ iff $g_2 \triangleright g$. For every morphisms g, g_1, g_2 of $\mathscr{D}$ such that $g_1 \triangleright g_2$ holds $g \triangleright g_1 \circ g_2$ iff $g \triangleright g_1$. For every morphism g_1 of $\mathscr{D}$ such that $g_1 \in$ the carrier of $\mathscr{D}$ there exists a morphism g of $\mathscr{D}$ such that $g \triangleright g_1$ and g is left identity by (2), [17, (31), (32)]. For every morphism g_1 of $\mathscr{D}$ such that $g_1 \in$ the carrier of $\mathscr{D}$ there exists a morphism g of $\mathscr{D}$ such that $g_1 \triangleright g$ and g is right identity by (2), [17, (31), (32)]. For every morphisms g_1, g_2, g_3 of $\mathscr{D}$ such that $g_1 \triangleright g_2$ and $g_2 \triangleright g_3$ and $g_1 \circ g_2 \triangleright g_3$ and $g_1 \triangleright g_2 \circ g_3$ holds $g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$. For every object x, $x \in c_8$ iff $x \in \{\langle\langle f_1, f_2 \rangle, f_3 \rangle$, where f_1, f_2, f_3 are morphisms of $\mathscr{D} : f_1, f_2, f_3 \in$ the carrier of $\mathscr{D}$ and for every morphisms f_{11}, f_{12}, f_{13} of $\mathscr{C}_1$ and for every morphisms f_{21}, f_{22}, f_{23} of $\mathscr{C}_2$ such that $f_1 = \langle f_{11}, f_{21} \rangle$ and $f_2 = \langle f_{12}, f_{22} \rangle$ and $f_3 = \langle f_{13}, f_{23} \rangle$ holds $f_{11} \triangleright f_{12}$ and $f_{21} \triangleright f_{22}$ and $f_{13} = f_{11} \circ f_{12}$ and $f_{23} = f_{21} \circ f_{22}\}$. There exists a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$ and there exists a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$ such that $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$ and for every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}$ is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}_1$ is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$ by [17, (31)], [9, (13)], (1), [17, (32), (34)]. Consider $\mathcal{P}_1$

being a functor from $\mathscr{D}$ to $\mathscr{C}_1$, $\mathcal{P}_2$ being a functor from $\mathscr{D}$ to $\mathscr{C}_2$ such that $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{P}_1 = \mathcal{F}_2 \circ \mathcal{P}_2$ and for every category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{G}_1$ is covariant and $\mathcal{G}_2$ is covariant and $\mathcal{F}_1 \circ \mathcal{G}_1 = \mathcal{F}_2 \circ \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}$ is covariant and $\mathcal{P}_1 \circ \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H} = \mathcal{G}_2$ and for every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{H}_1$ is covariant and $\mathcal{P}_1 \circ \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \circ \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$. $\square$

Let $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$ be categories and $\mathcal{F}_1$ be a functor from $\mathscr{C}_1$ to $\mathscr{C}$. Assume $\mathcal{F}_1$ is covariant. Let $\mathcal{F}_2$ be a functor from $\mathscr{C}_2$ to $\mathscr{C}$. Assume $\mathcal{F}_2$ is covariant.

A pullback of $\mathcal{F}_1$, $\mathcal{F}_2$ is a triple object and is defined by

(Def. 21) there exists a strict category $\mathscr{D}$ and there exists a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$ and there exists a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$ such that *it* $= \langle\mathscr{D}, \mathcal{P}_1, \mathcal{P}_2\rangle$ and $\mathcal{P}_1$ is covariant and $\mathcal{P}_2$ is covariant and $\langle\mathscr{D}, \mathcal{P}_1, \mathcal{P}_2\rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.

Assume $\mathcal{F}_1$ is covariant. Assume $\mathcal{F}_2$ is covariant. The functor $[\![\mathcal{F}_1, \mathcal{F}_2]\!]$ yielding a strict category is defined by the term

(Def. 22) the pullback of $\mathcal{F}_1$, $\mathcal{F}_{2\mathbf{1},3}$.

Assume $\mathcal{F}_1$ is covariant. Assume $\mathcal{F}_2$ is covariant. The functor $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ yielding a functor from $[\![\mathcal{F}_1, \mathcal{F}_2]\!]$ to $\mathscr{C}_1$ is defined by the term

(Def. 23) the pullback of $\mathcal{F}_1$, $\mathcal{F}_{2\mathbf{2},3}$.

The functor $\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ yielding a functor from $[\![\mathcal{F}_1, \mathcal{F}_2]\!]$ to $\mathscr{C}_2$ is defined by the term

(Def. 24) the pullback of $\mathcal{F}_1$, $\mathcal{F}_{2\mathbf{3},3}$.

Let us consider categories $\mathscr{C}, \mathscr{C}_1, \mathscr{C}_2$, a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$, and a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$. Let us assume that $\mathcal{F}_1$ is covariant and $\mathcal{F}_2$ is covariant. Now we state the propositions:

(52) (i) $\pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is covariant, and

(ii) $\pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is covariant, and

(iii) $\langle[\![\mathcal{F}_1, \mathcal{F}_2]\!], \pi_1(\mathcal{F}_1 \boxtimes \mathcal{F}_2), \pi_2(\mathcal{F}_1 \boxtimes \mathcal{F}_2)\rangle$ is a pullback of $\mathcal{F}_1, \mathcal{F}_2$.

(53) $[\![\mathcal{F}_1, \mathcal{F}_2]\!] \cong [\![\mathcal{F}_2, \mathcal{F}_1]\!]$. The theorem is a consequence of (52), (47), and (46).

(54) There exist object-categories $\mathscr{C}$, $\mathscr{C}_1$, $\mathscr{C}_2$ and there exists a functor $\mathcal{F}_1$ from $\mathscr{C}_1$ to $\mathscr{C}$ and there exists a functor $\mathcal{F}_2$ from $\mathscr{C}_2$ to $\mathscr{C}$ such that there exists no object-category $\mathscr{D}$ and there exists a functor $\mathcal{P}_1$ from $\mathscr{D}$ to $\mathscr{C}_1$ and there exists a functor $\mathcal{P}_2$ from $\mathscr{D}$ to $\mathscr{C}_2$ such that $\mathcal{F}_1 \cdot \mathcal{P}_1 = \mathcal{F}_2 \cdot \mathcal{P}_2$ and for every object-category $\mathscr{D}_1$ and for every functor $\mathcal{G}_1$ from $\mathscr{D}_1$ to $\mathscr{C}_1$ and for every functor $\mathcal{G}_2$ from $\mathscr{D}_1$ to $\mathscr{C}_2$ such that $\mathcal{F}_1 \cdot \mathcal{G}_1 = \mathcal{F}_2 \cdot \mathcal{G}_2$ there exists a functor $\mathcal{H}$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{P}_1 \cdot \mathcal{H} = \mathcal{G}_1$ and $\mathcal{P}_2 \cdot \mathcal{H} = \mathcal{G}_2$ and for

every functor $\mathcal{H}_1$ from $\mathscr{D}_1$ to $\mathscr{D}$ such that $\mathcal{P}_1 \cdot \mathcal{H}_1 = \mathcal{G}_1$ and $\mathcal{P}_2 \cdot \mathcal{H}_1 = \mathcal{G}_2$ holds $\mathcal{H} = \mathcal{H}_1$. The theorem is a consequence of (39) and (40).

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