FORMALIZED MATHEMATICS
Vol. 18, No. 1, Pages 77–80, 2010
DOI: 10.2478/v10037-010-0010-1

Representation of the Fibonacci and Lucas Numbers in Terms of Floor and Ceiling

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Summary. In the paper we show how to express the Fibonacci numbers and Lucas numbers using the floor and ceiling operations.

MML identifier: FIB_NUM4, version: 7.11.05 4.134.1080

The notation and terminology used here have been introduced in the following papers: [7], [3], [8], [11], [10], [1], [4], [6], [2], [5], and [9].

1. Preliminaries

One can prove the following propositions:

(1) For all real numbers a, b and for every natural number c holds $(\frac{a}{b})^c = \frac{a^c}{b^c}$.

(2) For every real number a and for all integer numbers b, c such that $a \neq 0$ holds $a^{b+c} = a^b \cdot a^c$.

(3) For every natural number n and for every real number a such that n is even and $a \neq 0$ holds $(-a)^n = a^n$.

(4) For every natural number n and for every real number a such that n is odd and $a \neq 0$ holds $(-a)^n = -a^n$.

(5) $|\overline{\tau}| < 1$.

(6) For every natural number n and for every non empty real number r such that n is even holds $r^n > 0$.

(7) For every natural number n and for every real number r such that n is odd and $r < 0$ holds $r^n < 0$.

ISSN 1426–2630(p), 1898-9934(e)

(8) For every natural number n such that $n \neq 0$ holds $\overline{\tau}^n < \frac{1}{2}$.

(9) For all natural numbers n, m and for every real number r such that m is odd and $n \geq m$ and $r < 0$ and $r > -1$ holds $r^n \geq r^m$.

(10) For all natural numbers n, m such that m is odd and $n \geq m$ holds $\overline{\tau}^n \geq \overline{\tau}^m$.

(11) For all natural numbers n, m such that n is even and m is even and $n \geq m$ holds $\overline{\tau}^n \leq \overline{\tau}^m$.

(12) For all non empty natural numbers m, n such that $m \geq n$ holds $\mathrm{Luc}(m) \geq \mathrm{Luc}(n)$.

(13) For every non empty natural number n holds $\tau^n > \overline{\tau}^n$.

(14) For every natural number n such that $n > 1$ holds $-\frac{1}{2} < \overline{\tau}^n$.

(15) For every natural number n such that $n > 2$ holds $\overline{\tau}^n \geq -\frac{1}{\sqrt{5}}$.

(16) For every natural number n such that $n \geq 2$ holds $\overline{\tau}^n \leq \frac{1}{\sqrt{5}}$.

(17) For every natural number n holds $\frac{\overline{\tau}^n}{\sqrt{5}} + \frac{1}{2} > 0$ and $\frac{\overline{\tau}^n}{\sqrt{5}} + \frac{1}{2} < 1$.

2. Formulas for the Fibonacci Numbers

Next we state two propositions:

(18) For every natural number n holds $\lfloor \frac{\tau^n}{\sqrt{5}} + \frac{1}{2} \rfloor = \mathrm{Fib}(n)$.

(19) For every natural number n such that $n \neq 0$ holds $\lceil \frac{\tau^n}{\sqrt{5}} - \frac{1}{2} \rceil = \mathrm{Fib}(n)$.

We now state a number of propositions:

(20) For every natural number n such that $n \neq 0$ holds $\lfloor \frac{\tau^{2 \cdot n}}{\sqrt{5}} \rfloor = \mathrm{Fib}(2 \cdot n)$.

(21) For every natural number n holds $\lceil \frac{\tau^{2 \cdot n+1}}{\sqrt{5}} \rceil = \mathrm{Fib}(2 \cdot n + 1)$.

(22) For every natural number n such that $n \geq 2$ and n is even holds $\mathrm{Fib}(n+1) = \lfloor \tau \cdot \mathrm{Fib}(n) + 1 \rfloor$.

(23) For every natural number n such that $n \geq 2$ and n is odd holds $\mathrm{Fib}(n+1) = \lceil \tau \cdot \mathrm{Fib}(n) - 1 \rceil$.

(24) For every natural number n such that $n \geq 2$ holds $\mathrm{Fib}(n + 1) = \lfloor \frac{\mathrm{Fib}(n)+\sqrt{5} \cdot \mathrm{Fib}(n)+1}{2} \rfloor$.

(25) For every natural number n such that $n \geq 2$ holds $\mathrm{Fib}(n + 1) = \lceil \frac{(\mathrm{Fib}(n)+\sqrt{5} \cdot \mathrm{Fib}(n))-1}{2} \rceil$.

(26) For every natural number n holds $\mathrm{Fib}(n+1) = \frac{\mathrm{Fib}(n)+\sqrt{5 \cdot \mathrm{Fib}(n)^{\mathbf{2}}+4 \cdot (-1)^n}}{2}$.

(27) For every natural number n such that $n \geq 2$ holds $\mathrm{Fib}(n + 1) = \lfloor \frac{\mathrm{Fib}(n)+1+\sqrt{(5 \cdot \mathrm{Fib}(n)^{\mathbf{2}} - 2 \cdot \mathrm{Fib}(n))+1}}{2} \rfloor$.

(28) For every natural number n such that $n \geq 2$ holds $\mathrm{Fib}(n) = \lfloor \frac{1}{\tau} \cdot (\mathrm{Fib}(n+1) + \frac{1}{2}) \rfloor$.

(29) For all natural numbers n, k such that $n \geq k > 1$ or $k = 1$ and $n > k$ holds $\lfloor \tau^k \cdot \mathrm{Fib}(n) + \frac{1}{2} \rfloor = \mathrm{Fib}(n + k)$.

3. Formulas for the Lucas Numbers

Next we state a number of propositions:

(30) For every natural number n such that $n \geq 2$ holds $\mathrm{Luc}(n) = \lfloor \tau^n + \frac{1}{2} \rfloor$.

(31) For every natural number n such that $n \geq 2$ holds $\mathrm{Luc}(n) = \lceil \tau^n - \frac{1}{2} \rceil$.

(32) For every natural number n such that $n \geq 2$ holds $\mathrm{Luc}(2 \cdot n) = \lceil \tau^{2 \cdot n} \rceil$.

(33) For every natural number n such that $n \geq 2$ holds $\mathrm{Luc}(2 \cdot n + 1) = \lfloor \tau^{2 \cdot n + 1} \rfloor$.

(34) For every natural number n such that $n \geq 2$ and n is odd holds $\mathrm{Luc}(n + 1) = \lfloor \tau \cdot \mathrm{Luc}(n) + 1 \rfloor$.

(35) For every natural number n such that $n \geq 2$ and n is even holds $\mathrm{Luc}(n + 1) = \lceil \tau \cdot \mathrm{Luc}(n) - 1 \rceil$.

(36) For every natural number n such that $n \neq 1$ holds $\mathrm{Luc}(n + 1) = \frac{\mathrm{Luc}(n) + \sqrt{5 \cdot (\mathrm{Luc}(n)^{\mathbf{2}} - 4 \cdot (-1)^n)}}{2}$.

(37) For every natural number n such that $n \geq 4$ holds $\mathrm{Luc}(n + 1) = \lfloor \frac{\mathrm{Luc}(n) + 1 + \sqrt{(5 \cdot \mathrm{Luc}(n)^{\mathbf{2}} - 2 \cdot \mathrm{Luc}(n)) + 1}}{2} \rfloor$.

(38) For every natural number n such that $n > 2$ holds $\mathrm{Luc}(n) = \lfloor \frac{1}{\tau} \cdot (\mathrm{Luc}(n + 1) + \frac{1}{2}) \rfloor$.

(39) For all natural numbers n, k such that $n \geq 4$ and $k \geq 1$ and $n > k$ and n is odd holds $\mathrm{Luc}(n + k) = \lfloor \tau^k \cdot \mathrm{Luc}(n) + 1 \rfloor$.

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Received November 30, 2009