FORMALIZED MATHEMATICS
Vol. 20, No. 1, Pages 41–45, 2012
DOI: 10.2478/v10037-012-0006-0

VERSITA
versita.com/fm/

Planes and Spheres as Topological Manifolds. Stereographic Projection

Marco Riccardi
Via del Pero 102
54038 Montignoso
Italy

Summary. The goal of this article is to show some examples of topological manifolds: planes and spheres in Euclidean space. In doing it, the article introduces the stereographic projection [25].

MML identifier: `MFOLD_2`, version: `7.12.01 4.167.1133`

The papers [29], [34], [9], [14], [40], [41], [11], [10], [4], [2], [18], [13], [31], [20], [21], [30], [32], [16], [17], [35], [26], [1], [22], [38], [36], [24], [19], [37], [28], [6], [15], [8], [27], [39], [3], [42], [12], [23], [7], [5], and [33] provide the notation and terminology for this paper.

1. Preliminaries

Let us observe that $\emptyset$ is $\emptyset$-valued and $\emptyset$ is onto.

Next we state three propositions:

(1) For every function f and for every set Y holds $\mathrm{dom}(Y \upharpoonright f) = f^{-1}(Y)$.

(2) For every function f and for all sets Y_1, Y_2 such that $Y_2 \subseteq Y_1$ holds $(Y_1 \upharpoonright f)^{-1}(Y_2) = f^{-1}(Y_2)$.

(3) Let S, T be topological structures and f be a function from S into T. If f is homeomorphism, then f^{-1} is homeomorphism.

Let S, T be topological structures. Let us note that the predicate S and T are homeomorphic is symmetric.

For simplicity, we use the following convention: T_1, T_2, T_3 denote topological spaces, A_1 denotes a subset of T_1, A_2 denotes a subset of T_2, and A_3 denotes a subset of T_3.

ISSN 1426–2630(p), 1898-9934(e)

Next we state several propositions:

(4) Let f be a function from T_1 into T_2. Suppose f is homeomorphism. Let g be a function from $T_1 \restriction f^{-1}(A_2)$ into $T_2 \restriction A_2$. If $g = A_2 \restriction f$, then g is homeomorphism.

(5) For every function f from T_1 into T_2 such that f is homeomorphism holds $f^{-1}(A_2)$ and A_2 are homeomorphic.

(6) If A_1 and A_2 are homeomorphic, then A_2 and A_1 are homeomorphic.

(7) If A_1 and A_2 are homeomorphic, then A_1 is empty iff A_2 is empty.

(8) If A_1 and A_2 are homeomorphic and A_2 and A_3 are homeomorphic, then A_1 and A_3 are homeomorphic.

(9) If T_1 is second-countable and T_1 and T_2 are homeomorphic, then T_2 is second-countable.

In the sequel n, k are natural numbers and M, N are non empty topological spaces.

The following propositions are true:

(10) If M is Hausdorff and M and N are homeomorphic, then N is Hausdorff.

(11) If M is n-locally Euclidean and M and N are homeomorphic, then N is n-locally Euclidean.

(12) If M is n-manifold and M and N are homeomorphic, then N is n-manifold.

(13) Let x_1, x_2 be finite sequences of elements of $\mathbb{R}$ and i be an element of $\mathbb{N}$. If $i \in \mathrm{dom}(x_1 \bullet x_2)$, then $(x_1 \bullet x_2)(i) = (x_1)_i \cdot (x_2)_i$ and $(x_1 \bullet x_2)_i = (x_1)_i \cdot (x_2)_i$.

(14) For all finite sequences x_1, x_2, y_1, y_2 of elements of $\mathbb{R}$ such that $\mathrm{len}\, x_1 = \mathrm{len}\, x_2$ and $\mathrm{len}\, y_1 = \mathrm{len}\, y_2$ holds $x_1 \frown y_1 \bullet x_2 \frown y_2 = (x_1 \bullet x_2) \frown (y_1 \bullet y_2)$.

(15) For all finite sequences x_1, x_2, y_1, y_2 of elements of $\mathbb{R}$ such that $\mathrm{len}\, x_1 = \mathrm{len}\, x_2$ and $\mathrm{len}\, y_1 = \mathrm{len}\, y_2$ holds $|(x_1 \frown y_1, x_2 \frown y_2)| = |(x_1, x_2)| + |(y_1, y_2)|$.

In the sequel p, q, p_1 are points of $\mathcal{E}^n_{\mathrm{T}}$ and r is a real number.

One can prove the following propositions:

(16) If $k \in \mathrm{Seg}\, n$, then $(p_1 + p_2)(k) = p_1(k) + p_2(k)$.

(17) For every set X holds X is a linear combination of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ iff X is a linear combination of $\mathcal{E}^n_{\mathrm{T}}$.

(18) Let F be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$, f_1 be a function from $\mathcal{E}^n_{\mathrm{T}}$ into $\mathbb{R}$, F_1 be a finite sequence of elements of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$, and f_2 be a function from $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ into $\mathbb{R}$. If $f_1 = f_2$ and $F = F_1$, then $f_1 \cdot F = f_2 \cdot F_1$.

(19) Let F be a finite sequence of elements of $\mathcal{E}^n_{\mathrm{T}}$ and F_1 be a finite sequence of elements of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$. If $F_1 = F$, then $\sum F = \sum F_1$.

(20) For every linear combination L_2 of $\mathbb{R}^{\mathrm{Seg}\, n}_{\mathbb{R}}$ and for every linear combination L_1 of $\mathcal{E}^n_{\mathrm{T}}$ such that $L_1 = L_2$ holds $\sum L_1 = \sum L_2$.

(21) Let A_4 be a subset of $\mathbb{R}_{\mathbb{R}}^{\mathrm{Seg}\,n}$ and A_5 be a subset of $\mathcal{E}_{\mathrm{T}}^n$. Suppose $A_4 = A_5$. Then A_4 is linearly independent if and only if A_5 is linearly independent.

(22) For every subset V of $\mathcal{E}_{\mathrm{T}}^n$ such that $V = \mathbb{R}\text{N-Base}\,n$ there exists a linear combination l of V such that $p = \sum l$.

(23) $\mathbb{R}\text{N-Base}\,n$ is a basis of $\mathcal{E}_{\mathrm{T}}^n$.

(24) Let V be a subset of $\mathcal{E}_{\mathrm{T}}^n$. Then $V \in$ the topology of $\mathcal{E}_{\mathrm{T}}^n$ if and only if for every p such that $p \in V$ there exists r such that $r > 0$ and $\mathrm{Ball}(p, r) \subseteq V$.

Let n be a natural number and let p be a point of $\mathcal{E}_{\mathrm{T}}^n$.

The functor $\mathrm{InnerProduct}\,p$ yields a function from $\mathcal{E}_{\mathrm{T}}^n$ into $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def. 1) For every point q of $\mathcal{E}_{\mathrm{T}}^n$ holds $(\mathrm{InnerProduct}\,p)(q) = |(p, q)|$.

Let us consider n, p. Note that $\mathrm{InnerProduct}\,p$ is continuous.

2. PLANES

Let us consider n and let us consider p, q. The functor $\mathrm{Plane}(p, q)$ yielding a subset of $\mathcal{E}_{\mathrm{T}}^n$ is defined as follows:

(Def. 2) $\mathrm{Plane}(p, q) = \{y; y$ ranges over points of $\mathcal{E}_{\mathrm{T}}^n$: $|(p, y - q)| = 0\}$.

The following propositions are true:

(25) $(\mathrm{transl}(p_1, \mathcal{E}_{\mathrm{T}}^n))^\circ\, \mathrm{Plane}(p, p_2) = \mathrm{Plane}(p, p_1 + p_2)$.

(26) If $p \neq 0_{\mathcal{E}_{\mathrm{T}}^n}$, then there exists a linearly independent subset A of $\mathcal{E}_{\mathrm{T}}^n$ such that $\overline{\overline{A}} = n - 1$ and $\Omega_{\mathrm{Lin}(A)} = \mathrm{Plane}(p, 0_{\mathcal{E}_{\mathrm{T}}^n})$.

(27) If $p_1 \neq 0_{\mathcal{E}_{\mathrm{T}}^n}$ and $p_2 \neq 0_{\mathcal{E}_{\mathrm{T}}^n}$, then there exists a function R from $\mathcal{E}_{\mathrm{T}}^n$ into $\mathcal{E}_{\mathrm{T}}^n$ such that R is homeomorphism and $R^\circ\, \mathrm{Plane}(p_1, 0_{\mathcal{E}_{\mathrm{T}}^n}) = \mathrm{Plane}(p_2, 0_{\mathcal{E}_{\mathrm{T}}^n})$.

Let us consider n and let us consider p, q. The functor $\mathrm{TPlane}(p, q)$ yields a non empty subspace of $\mathcal{E}_{\mathrm{T}}^n$ and is defined by:

(Def. 3) $\mathrm{TPlane}(p, q) = \mathcal{E}_{\mathrm{T}}^n \restriction \mathrm{Plane}(p, q)$.

The following three propositions are true:

(28) The base finite sequence of $n + 1$ and $n + 1 = (0_{\mathcal{E}_{\mathrm{T}}^n}) ^\frown \langle 1 \rangle$.

(29) For all points p, q of $\mathcal{E}_{\mathrm{T}}^{n+1}$ such that $p \neq 0_{\mathcal{E}_{\mathrm{T}}^{n+1}}$ holds $\mathcal{E}_{\mathrm{T}}^n$ and $\mathrm{TPlane}(p, q)$ are homeomorphic.

(30) For all points p, q of $\mathcal{E}_{\mathrm{T}}^{n+1}$ such that $p \neq 0_{\mathcal{E}_{\mathrm{T}}^{n+1}}$ holds $\mathrm{TPlane}(p, q)$ is n-manifold.

3. Spheres

Let us consider n. The functor $\mathbb{S}^n$ yields a topological space and is defined by:

(Def. 4) $\mathbb{S}^n = \mathrm{TopUnitCircle}(n+1)$.

Let us consider n. Note that $\mathbb{S}^n$ is non empty.

Let us consider n, p and let S be a subspace of $\mathcal{E}^n_{\mathrm{T}}$. Let us assume that $p \in \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1)$. The functor $\sigma_{S,p}$ yielding a function from S into $\mathrm{TPlane}(p, 0_{\mathcal{E}^n_{\mathrm{T}}})$ is defined as follows:

(Def. 5) For every q such that $q \in S$ holds $(\sigma_{S,p})(q) = \frac{1}{1-|(q,p)|} \cdot (q - |(q,p)| \cdot p)$.

Next we state the proposition

(31) For every subspace S of $\mathcal{E}^n_{\mathrm{T}}$ such that $\Omega_S = \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1) \setminus \{p\}$ and $p \in \mathrm{Sphere}((0_{\mathcal{E}^n_{\mathrm{T}}}), 1)$ holds $\sigma_{S,p}$ is homeomorphism.

Let us consider n. One can verify the following observations:

* $\mathbb{S}^n$ is second-countable,
* $\mathbb{S}^n$ is n-locally Euclidean, and
* $\mathbb{S}^n$ is n-manifold.

References

[1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(**2**):377–382, 1990.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. König's theorem. *Formalized Mathematics*, 1(**3**):589–593, 1990.
[4] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[5] Grzegorz Bancerek. Monoids. *Formalized Mathematics*, 3(**2**):213–225, 1992.
[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.
[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[8] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Formalized Mathematics*, 1(**3**):529–536, 1990.
[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[12] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[13] Czesław Byliński. The sum and product of finite sequences of real numbers. *Formalized Mathematics*, 1(**4**):661–668, 1990.
[14] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Formalized Mathematics*, 1(**2**):257–261, 1990.
[15] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(**1**):165–167, 1990.
[16] Agata Darmochwał. The Euclidean space. *Formalized Mathematics*, 2(**4**):599–603, 1991.
[17] Agata Darmochwał and Yatsuka Nakamura. Metric spaces as topological spaces – fundamental concepts. *Formalized Mathematics*, 2(**4**):605–608, 1991.
[18] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[19] Katarzyna Jankowska. Matrices. Abelian group of matrices. *Formalized Mathematics*, 2(**4**):475–480, 1991.

[20] Stanisława Kanas, Adam Lecko, and Mariusz Startek. Metric spaces. *Formalized Mathematics*, 1(**3**):607–610, 1990.
[21] Artur Korniłowicz and Yasunari Shidama. Intersections of intervals and balls in $\mathcal{E}^n_{\mathrm{T}}$. *Formalized Mathematics*, 12(**3**):301–306, 2004.
[22] Artur Korniłowicz and Yasunari Shidama. Some properties of circles on the plane. *Formalized Mathematics*, 13(**1**):117–124, 2005.
[23] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.
[24] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Formalized Mathematics*, 1(**2**):335–342, 1990.
[25] John M. Lee. *Introduction to Topological Manifolds*. Springer-Verlag, New York Berlin Heidelberg, 2000.
[26] Robert Milewski. Bases of continuous lattices. *Formalized Mathematics*, 7(**2**):285–294, 1998.
[27] Yatsuka Nakamura, Artur Korniłowicz, Nagato Oya, and Yasunari Shidama. The real vector spaces of finite sequences are finite dimensional. *Formalized Mathematics*, 17(**1**):1–9, 2009, doi:10.2478/v10037-009-0001-2.
[28] Henryk Oryszczyszyn and Krzysztof Prażmowski. Real functions spaces. *Formalized Mathematics*, 1(**3**):555–561, 1990.
[29] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(**1**):147–152, 1990.
[30] Beata Padlewska. Locally connected spaces. *Formalized Mathematics*, 2(**1**):93–96, 1991.
[31] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[32] Karol Pąk. Basic properties of metrizable topological spaces. *Formalized Mathematics*, 17(**3**):201–205, 2009, doi: 10.2478/v10037-009-0024-8.
[33] Marco Riccardi. The definition of topological manifolds. *Formalized Mathematics*, 19(**1**):41–44, 2011, doi: 10.2478/v10037-011-0007-4.
[34] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.
[35] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(**4**):341–347, 2003.
[36] Wojciech A. Trybulec. Basis of real linear space. *Formalized Mathematics*, 1(**5**):847–850, 1990.
[37] Wojciech A. Trybulec. Linear combinations in real linear space. *Formalized Mathematics*, 1(**3**):581–588, 1990.
[38] Wojciech A. Trybulec. Subspaces and cosets of subspaces in real linear space. *Formalized Mathematics*, 1(**2**):297–301, 1990.
[39] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[40] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(**1**):73–83, 1990.
[41] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.
[42] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Formalized Mathematics*, 5(**1**):75–77, 1996.

Received June 6, 2011