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Transition of Consistency and Satisfiability under Language Extensions[1]

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Summary. This article is the first in a series of two Mizar articles constituting a formal proof of the Gödel Completeness theorem [17] for uncountably large languages. We follow the proof given in [18]. The present article contains the techniques required to expand formal languages. We prove that consistent or satisfiable theories retain these properties under changes to the language they are formulated in.

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The notation and terminology used in this paper have been introduced in the following papers: [8], [1], [2], [11], [16], [4], [15], [12], [13], [7], [6], [22], [3], [19], [23], [24], [5], [20], [9], [10], [21], and [14].

[1]This article is part of the first author's Bachelor thesis under the supervision of the second author.

1. Language Extensions

For simplicity, we adopt the following rules: A_1 denotes an alphabet, P_1 denotes a consistent subset of CQC-WFF A_1, p, r denote elements of CQC-WFF A_1, A denotes a non empty set, J denotes an interpretation of A_1 and A, v denotes an element of the valuations in A_1 and A, k denotes a natural number, l denotes a CQC-variable list of k and A_1, P denotes a predicate symbol of k and A_1, and x, y denote bound variables of A_1.

Let us consider A_1 and let A_2 be an alphabet. We say that A_2 is A_1-expanding if and only if:

(Def. 1) $A_1 \subseteq A_2$.

Let us consider A_1. Note that there exists an alphabet which is A_1-expanding.

Let A_3, A_4 be countable alphabets. One can check that there exists an alphabet which is countable, A_3-expanding, and A_4-expanding.

Let A_1, A_4 be alphabets and let P be a subset of CQC-WFF A_1. We say that P is A_4-consistent if and only if:

(Def. 2) For every subset S of CQC-WFF A_4 such that $P = S$ holds S is consistent.

Let us consider A_1. One can check that there exists a subset of CQC-WFF A_1 which is non empty and consistent.

Let us consider A_1. One can check that every subset of CQC-WFF A_1 which is consistent is also A_1-consistent and every subset of CQC-WFF A_1 which is A_1-consistent is also consistent.

For simplicity, we follow the rules: A_4 is an A_1-expanding alphabet, J_2 is an interpretation of A_4 and A, J_1 is an interpretation of A_1 and A, v_2 is an element of the valuations in A_4 and A, and v_1 is an element of the valuations in A_1 and A.

Next we state several propositions:

(1) $\mathrm{Arity}(P) = \mathrm{len}\, l$.

(2) $\mathrm{Symb}\, A_1 \subseteq \mathrm{Symb}\, A_4$.

(3) The predicate symbols of $A_1 \subseteq$ the predicate symbols of A_4.

(4) The bound variables of $A_1 \subseteq$ the bound variables of A_4.

(5) For every k holds every l is a CQC-variable list of k and A_4.

(6) P is a predicate symbol of k and A_4.

(7) For every A_1-expanding alphabet A_4 holds every p is an element of CQC-WFF A_4.

Let us consider A_1, let A_4 be an A_1-expanding alphabet, and let p be an element of CQC-WFF A_1. The functor A_4 -Cast p yields an element of CQC-WFF A_4 and is defined by:

(Def. 3) $A_4\text{-Cast}\, p = p$.

Let us consider A_1, let A_4 be an A_1-expanding alphabet, and let x be a bound variable of A_1. The functor A_4-Cast x yields a bound variable of A_4 and is defined as follows:

(Def. 4) A_4-Cast $x = x$.

Let us consider A_1, let A_4 be an A_1-expanding alphabet, let us consider k, and let P be a predicate symbol of k and A_1. The functor A_4-Cast P yielding a predicate symbol of k and A_4 is defined as follows:

(Def. 5) A_4-Cast $P = P$.

Let us consider A_1, let A_4 be an A_1-expanding alphabet, let us consider k, and let l be a CQC-variable list of k and A_1. The functor A_4-Cast l yielding a CQC-variable list of k and A_4 is defined as follows:

(Def. 6) A_4-Cast $l = l$.

Next we state the proposition

(8) Let given p, r, x, P, l and A_4 be an A_1-expanding alphabet. Then A_4-Cast VERUM A_1 = VERUM A_4 and A_4-Cast $P[l] = (A_4\text{-Cast}\,P)[A_4\text{-Cast}\,l]$ and A_4-Cast $\neg p = \neg(A_4\text{-Cast}\,p)$ and $A_4\text{-Cast}(p \wedge r) = (A_4\text{-Cast}\,p) \wedge (A_4\text{-Cast}\,r)$ and $A_4\text{-Cast}\,\forall_x p = \forall_{A_4\text{-Cast}\,x}(A_4\text{-Cast}\,p)$.

2. Downward Transfer of Consistency and Satisfiability

The following propositions are true:

(9) Suppose $J_1 = J_2{\restriction}$the predicate symbols of A_1 and $v_1 = v_2{\restriction}$the bound variables of A_1. Then $J_2 \models_{v_2} A_4\text{-Cast}\,r$ if and only if $J_1 \models_{v_1} r$.

(10) Let A_4 be an A_1-expanding alphabet and T_1 be a subset of CQC-WFF A_4. Suppose $P_1 \subseteq T_1$. Let A_2 be a non empty set, J_2 be an interpretation of A_4 and A_2, and v_2 be an element of the valuations in A_4 and A_2. If $J_2 \models_{v_2} T_1$, then there exist A, J, v such that $J \models_v P_1$.

(11) Let f be a finite sequence of elements of CQC-WFF A_4 and g be a finite sequence of elements of CQC-WFF A_1. If $f = g$, then $\mathrm{Ant}(f) = \mathrm{Ant}(g)$ and $\mathrm{Suc}(f) = \mathrm{Suc}(g)$.

(12) For every p holds the still not bound in p = the still not bound in A_4-Cast p.

(13) Let p_2 be an element of CQC-WFF A_4, S be a substitution of A_1, S_2 be a substitution of A_4, x_2 be a bound variable of A_4, and given x, p. If $p = p_2$ and $S = S_2$ and $x = x_2$, then $\mathrm{RestrictSub}(x, p, S) = \mathrm{RestrictSub}(x_2, p_2, S_2)$.

(14) Let p_2 be an element of CQC-WFF A_4, S be a finite substitution of A_1, S_2 be a finite substitution of A_4, and given p. If $S = S_2$ and $p = p_2$, then $\mathrm{upVar}(S, p) = \mathrm{upVar}(S_2, p_2)$.

(15) Let p_2 be an element of CQC-WFF A_4, S be a substitution of A_1, S_2 be a substitution of A_4, x_2 be a bound variable of A_4, and given x, p. If $p = p_2$ and $S = S_2$ and $x = x_2$, then $\mathrm{ExpandSub}(x, p, \mathrm{RestrictSub}(x, \forall_x p, S)) = \mathrm{ExpandSub}(x_2, p_2, \mathrm{RestrictSub}(x_2, \forall_{x_2} p_2, S_2))$.

(16) Let Z be an element of CQC-Sub-WFF A_1 and Z_2 be an element of CQC-Sub-WFF A_4. Suppose $Z_{\mathbf{1}}$ is universal and $(Z_2)_{\mathbf{1}}$ is universal and $\mathrm{Bound}(Z_{\mathbf{1}}) = \mathrm{Bound}((Z_2)_{\mathbf{1}})$ and $\mathrm{Scope}(Z_{\mathbf{1}}) = \mathrm{Scope}((Z_2)_{\mathbf{1}})$ and $Z = Z_2$. Then $\text{S-Bound}({}^{@}Z) = \text{S-Bound}({}^{@}Z_2)$.

(17) Let p_2 be an element of CQC-WFF A_4, x_2, y_2 be bound variables of A_4, and given p, x, y. If $p = p_2$ and $x = x_2$ and $y = y_2$, then $p(x, y) = p_2(x_2, y_2)$.

(18) For every consistent subset P_1 of CQC-WFF A_4 such that P_1 is a subset of CQC-WFF A_1 holds P_1 is A_1-consistent.

3. Upward Transfer of Consistency and Satisfiability

Next we state two propositions:

(19) For every p there exists a countable alphabet A_3 such that p is an element of CQC-WFF A_3 and A_1 is A_3-expanding.

(20) Let P_1 be a finite subset of CQC-WFF A_1. Then there exists a countable alphabet A_3 such that P_1 is a finite subset of CQC-WFF A_3 and A_1 is A_3-expanding.

Let us consider A_1 and let P_1 be a finite subset of CQC-WFF A_1. Note that the still not bound in P_1 is finite.

Next we state three propositions:

(21) Let T_1 be a subset of CQC-WFF A_4. Suppose $P_1 = T_1$. Let given A, J, v. Suppose $J \models_v P_1$. Then there exists a non empty set A_2 and there exists an interpretation J_2 of A_4 and A_2 and there exists an element v_2 of the valuations in A_4 and A_2 such that $J_2 \models_{v_2} T_1$.

(22) For every subset C_1 of CQC-WFF A_1 such that $C_1 \subseteq P_1$ holds C_1 is consistent.

(23) P_1 is A_4-consistent.

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