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Modelling Real World Using Stochastic Processes and Filtration

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Summary. First we give an implementation in Mizar [2] basic important definitions of stochastic finance, i.e. filtration ([9], pp. 183 and 185), adapted stochastic process ([9], p. 185) and predictable stochastic process ([6], p. 224). Second we give some concrete formalization and verification to real world examples.

In article [8] we started to define random variables for a similar presentation to the book [6]. Here we continue this study. Next we define the stochastic process. For further definitions based on stochastic process we implement the definition of filtration.

To get a better understanding we give a real world example and connect the statements to the theorems. Other similar examples are given in [10], pp. 143–159 and in [12], pp. 110–124. First we introduce sets which give informations referring to today (Ω_{now}, Def.6), tomorrow (Ω_{fut_1}, Def.7) and the day after tomorrow (Ω_{fut_2}, Def.8). We give an overview for some events in the σ-algebras $\Omega_{now}, \Omega_{fut1}$ and Ω_{fut2}, see theorems (22) and (23).

The given events are necessary for creating our next functions. The implementations take the form of: $\Omega_{now} \subset \Omega_{fut1} \subset \Omega_{fut2}$ see theorem (24). This tells us growing informations from now to the future 1=now, 2=tomorrow, 3=the day after tomorrow.

We install functions $f : \{1, 2, 3, 4\} \to \mathbb{R}$ as following:
$f_1 : x \to 100, \forall x \in \text{dom } f$, see theorem (36),
$f_2 : x \to 80$, for $x = 1$ or $x = 2$ and
$f_2 : x \to 120$, for $x = 3$ or $x = 4$, see theorem (37),
$f_3 : x \to 60$, for $x = 1$, $f_3 : x \to 80$, for $x = 2$ and
$f_3 : x \to 100$, for $x = 3$, $f_3 : x \to 120$, for $x = 4$ see theorem (38).

These functions are real random variable: f_1 over Ω_{now}, f_2 over Ω_{fut1}, f_3 over Ω_{fut2}, see theorems (46), (43) and (40). We can prove that these functions can be used for giving an example for an adapted stochastic process. See theorem (49).

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We want to give an interpretation to these functions: suppose you have an equity A which has now ($= w_1$) the value 100. Tomorrow A changes depending which scenario occurs − e.g. another marketing strategy. In scenario 1 ($= w_{11}$) it has the value 80, in scenario 2 ($= w_{12}$) it has the value 120. The day after tomorrow A changes again. In scenario 1 ($= w_{111}$) it has the value 60, in scenario 2 ($= w_{112}$) the value 80, in scenario 3 ($= w_{121}$) the value 100 and in scenario 4 ($= w_{122}$) it has the value 120. For a visualization refer to the tree:

Now	*tomorrow*	*the day after tomorrow*
		$w_{111} = \{1\}$
	$w_{11} = \{1, 2\}$ <	
		$w_{112} = \{2\}$
$w_1 = \{1, 2, 3, 4\}$ <		
		$w_{121} = \{3\}$
	$w_{12} = \{3, 4\}$ <	
		$w_{122} = \{4\}$

The sets w_1,w_{11},w_{12},w_{111},w_{112},w_{121},w_{122} which are subsets of $\{1, 2, 3, 4\}$, see (22), tell us which market scenario occurs. The functions tell us the values to the relevant market scenario:

Now	*tomorrow*	*the day after tomorrow*
		$f_3(w_i) = 60,\ w_i$ in w_{111}
	$f_2(w_i) = 80$ <	
	w_i in w_{11}	$f_3(w_i) = 80,\ w_i$ in w_{112}
$f_1(w_i) = 100$ <		
w_i in w_1		$f_3(w_i) = 100,\ w_i$ in w_{121}
	$f_2(w_i) = 120$ <	
	w_i in w_{12}	$f_3(w_i) = 120,\ w_i$ in w_{122}

For a better understanding of the definition of the random variable and the relation to the functions refer to [7], p. 20. For the proof of certain sets as σ-fields refer to [7], pp. 10–11 and [9], pp. 1–2.

This article is the next step to the *arbitrage opportunity*. If you use for example a simple probability measure, refer, for example to literature [3], pp. 28–34, [6], p. 6 and p. 232 you can calculate whether an *arbitrage* exists or not. Note, that the example given in literature [3] needs 8 instead of 4 informations as in our model. If we want to code the first 3 given time points into our model we would have the following graph, see theorems (47), (44) and (41):

Now	*tomorrow*	*the day after tomorrow*
		$f_3(w_i) = 180,\ w_i$ in w_{111}
	$f_2(w_i) = 150$ <	
	w_i in w_{11}	$f_3(w_i) = 120,\ w_i$ in w_{112}
$f_1(w_i) = 125$ <		
w_i in w_1		$f_3(w_i) = 120,\ w_i$ in w_{121}
	$f_2(w_i) = 100$ <	
	w_i in w_{12}	$f_3(w_i) = 80,\ w_i$ in w_{122}

The function for the "Call-Option" is given in literature [3], p. 28. The function is realized in Def.5. As a background, more examples for using the definition of filtration are given in [9], pp. 185–188.

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1. Preliminaries

Now we state the proposition:

(1) Let us consider objects a, b. If $a \neq b$, then $\{a\} \subset \{a, b\}$.

Let I be a non empty subset of $\mathbb{N}$. Observe that $I(\in 2^{\mathbb{R}})$ is non empty.

Let us consider an element T of $\mathbb{N}$. Now we state the propositions:

(2) $\{w$, where w is an element of $\mathbb{N} : w > 0$ and $w \leqslant T\} \subseteq \{w$, where w is an element of $\mathbb{N} : w \leqslant T\}$.

(3) $\{w$, where w is an element of $\mathbb{N} : w \leqslant T\}$ is a non empty subset of $\mathbb{N}$.

(4) If $T > 0$, then $\{w$, where w is an element of $\mathbb{N} : w > 0$ and $w \leqslant T\}$ is a non empty subset of $\mathbb{N}$.

Proof: $\{w$, where w is an element of $\mathbb{N} : w > 0$ and $w \leqslant T\}$ is a subset of $\mathbb{N}$. $1 > 0$ and $1 \leqslant T$ by [1, (24)]. □

Now we state the proposition:

(5) Let us consider a non empty set Ω. Then $\Omega \longmapsto 1$ is a function from Ω into $\mathbb{R}$.

2. Special Random Variables

Now we state the proposition:

(6) Let us consider a natural number d, a sequence φ of real numbers, a non empty set Ω, a σ-field F of subsets of Ω, a non empty set X, a sequence G of X, and an element w of Ω. Then $\{$the portfolio value for future extension of d, φ, F, G and $w\}$ is an event of the Borel sets.

Let d be a natural number, φ be a sequence of real numbers, Ω be a non empty set, F be a σ-field of subsets of Ω, X be a non empty set, G be a sequence of X, and w be an element of Ω. Note that the portfolio value for future extension of d, φ, F, G and w yields an element of $\mathbb{R}$. The $\mathcal{RV}$-portfolio value for future extension of φ, F, G and d yielding a function from Ω into $\mathbb{R}$ is defined by

(Def. 1) for every element w of Ω, $it(w) =$ the portfolio value for future extension of d, φ, F, G and w.

Let us observe that the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and d yields a random variable of F and the Borel sets. Let w be an element of Ω. Let us note that the portfolio value for future of d, φ, F, G and w yields a real number and is defined by the term

(Def. 2) $(\sum_{\alpha=0}^{\kappa}($(the elements of the random variables for the future elements of portfolio value of $(\varphi,F,G,w)) \uparrow 1)(\alpha))_{\kappa \in \mathbb{N}}(d-1)$.

Let us note that the portfolio value for future of d, φ, F, G and w yields an element of $\mathbb{R}$. The $\mathcal{RV}$-portfolio value for future of φ, F, G and d yielding a function from Ω into $\mathbb{R}$ is defined by

(Def. 3) for every element w of Ω, $it(w) =$ the portfolio value for future of $d+1$, φ, F, G and w.

Let us note that the $\mathcal{RV}$-portfolio value for future of φ, F, G and d yields a random variable of F and the Borel sets. Now we state the propositions:

(7) Let us consider a natural number d, a sequence φ of real numbers, a non empty set Ω, a σ-field F of subsets of Ω, a non empty set X, a sequence G of X, and an element w of Ω. Then

(i) the portfolio value for future of $d+1$, φ, F, G and $w =$ (the $\mathcal{RV}$-portfolio value for future of φ, F, G and $d)(w)$, and

(ii) {the portfolio value for future of $d+1$, φ, F, G and w} is an event of the Borel sets.

(8) Let us consider a non empty set Ω, a σ-field F of subsets of Ω, a non empty set X, a sequence G of X, a sequence φ of real numbers, and a natural number d. Then the $\mathcal{RV}$-portfolio value for future extension of φ, F, G and $d+1 =$ (the $\mathcal{RV}$-portfolio value for future of φ, F, G and d) $+$ (the random variables for the future elements of portfolio value of $(\varphi,F,G,0)$).

(9) Let us consider non empty sets Ω, Ω_2, a σ-field Σ of subsets of Ω, a σ-field Σ_2 of subsets of Ω_2, and an element s of Ω_2. Then $\Omega \longmapsto s$ is random variable on Σ and Σ_2.

(10) Let us consider a non empty set Ω, a σ-field Σ of subsets of Ω, a random variable $\mathcal{RV}$ of Σ and the Borel sets, and an element K of $\mathbb{R}$. Then $\mathcal{RV} - (\Omega \longmapsto K)$ is a random variable of Σ and the Borel sets. The theorem is a consequence of (9).

Let Ω be a non empty set, $\mathcal{RV}$ be a function from Ω into $\mathbb{R}$, and w be an element of Ω. The functor Set-Call-Option$(\mathcal{RV}, w)$ yielding an element of $\mathbb{R}$ is defined by the term

(Def. 4) $\begin{cases} \mathcal{RV}(w), & \textbf{if } \mathcal{RV}(w) \geqslant 0, \\ 0, & \textbf{otherwise}. \end{cases}$

Let Σ be a σ-field of subsets of Ω, $\mathcal{RV}$ be a random variable of Σ and the Borel sets, and K be an element of $\mathbb{R}$. The Call-Option on $\mathcal{RV}$ and K yielding a function from Ω into $\mathbb{R}$ is defined by

(Def. 5) for every element w of Ω, if $(\mathcal{RV} - (\Omega \longmapsto K))(w) \geqslant 0$, then $it(w) = (\mathcal{RV} - (\Omega \longmapsto K))(w)$ and if $(\mathcal{RV} - (\Omega \longmapsto K))(w) < 0$, then $it(w) = 0$.

3. SPECIAL σ-FIELDS

Let us consider a sequence A_1 of subsets of $\{1, 2, 3, 4\}$ and a real number w. Now we state the propositions:

(11) Suppose $w = 1$ or $w = 3$. Then suppose for every natural number n, $A_1(n) = \emptyset$ or $A_1(n) = \{1, 2\}$ or $A_1(n) = \{3, 4\}$ or $A_1(n) = \{1, 2, 3, 4\}$. Then $\{w\} \neq \text{Intersection}\, A_1$.

(12) Suppose $w = 2$ or $w = 4$. Then suppose for every natural number n, $A_1(n) = \emptyset$ or $A_1(n) = \{1, 2\}$ or $A_1(n) = \{3, 4\}$ or $A_1(n) = \{1, 2, 3, 4\}$. Then $\{w\} \neq \text{Intersection}\, A_1$.

Now we state the propositions:

(13) Let us consider sets M, A_1, A_2. Suppose $M = \{\emptyset, \{1, 2, 3, 4\}\}$ and A_1, $A_2 \in M$. Then $A_1 \cap A_2 \in M$.

(14) Let us consider a sequence A_1 of subsets of $\{1, 2, 3, 4\}$. Suppose for every natural number n and for every natural number k, $A_1(n) \cap A_1(k) \neq \emptyset$ and $\text{rng}\, A_1 \subseteq \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$. Then

(i) $\text{Intersection}\, A_1 = \emptyset$, or

(ii) $\text{Intersection}\, A_1 = \{1, 2\}$, or

(iii) $\text{Intersection}\, A_1 = \{3, 4\}$, or

(iv) $\text{Intersection}\, A_1 = \{1, 2, 3, 4\}$.

PROOF: For every natural number n, $A_1(n) \in \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ by [1, (20)], [4, (3)]. For every natural number n, $A_1(n) = \emptyset$ or $A_1(n) = \{1, 2\}$ or $A_1(n) = \{3, 4\}$ or $A_1(n) = \{1, 2, 3, 4\}$. $\square$

Let us consider a sequence A_1 of subsets of $\{1, 2, 3, 4\}$ and a set M. Now we state the propositions:

(15) Suppose $M = \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ and $\text{Intersection}\, A_1 = \{1, 2, 3, 4\}$. Then $\text{Intersection}\, A_1 \in M$.

(16) Suppose $M = \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ and $\text{Intersection}\, A_1 = \{3, 4\}$. Then $\text{Intersection}\, A_1 \in M$.

(17) Suppose $M = \{\emptyset, \{1, 2\}, \{3, 4\}, \{1, 2, 3, 4\}\}$ and $\text{Intersection}\, A_1 = \{1, 2\}$. Then $\text{Intersection}\, A_1 \in M$.

(18) Suppose $M = \{\emptyset, \{1,2\}, \{3,4\}, \{1,2,3,4\}\}$ and Intersection $A_1 = \emptyset$. Then Intersection $A_1 \in M$.

Now we state the propositions:

(19) Let us consider a set M, and a sequence A_1 of subsets of $\{1,2,3,4\}$. Suppose $\operatorname{rng} A_1 \subseteq M$ and $M = \{\emptyset, \{1,2\}, \{3,4\}, \{1,2,3,4\}\}$. Then Intersection $A_1 \in M$.
PROOF: Intersection $A_1 \in \{\emptyset, \{1,2\}, \{3,4\}, \{1,2,3,4\}\}$ by [11, (13)], (14). □

(20) Let us consider sets M, M_1, and a sequence A_1 of subsets of M_1. Suppose $M_1 = \{1,2,3,4\}$ and $\operatorname{rng} A_1 \subseteq M$ and $M = \{\emptyset, \{1,2,3,4\}\}$. If Intersection $A_1 \neq \emptyset$, then Intersection $A_1 \in M$.
PROOF: For every natural number n, $A_1(n) = \emptyset$ or $A_1(n) = \{1,2,3,4\}$ by [1, (20)], [4, (3)]. If there exists a natural number n such that $A_1(n) = \emptyset$, then Intersection $A_1 = \emptyset$ by [11, (13)]. Intersection $A_1 = \{1,2,3,4\}$ by [11, (13)]. □

(21) Let us consider sets M, M_1, and a sequence A_1 of subsets of M_1. Suppose $M_1 = \{1,2,3,4\}$ and $\operatorname{rng} A_1 \subseteq M$ and $M = \{\emptyset, \{1,2,3,4\}\}$. Let us consider a natural number k_1, and a natural number k_2. Then $A_1(k_1) \cap A_1(k_2) \in M$.
PROOF: $k_1 \in \operatorname{dom} A_1$ by [1, (20)]. $k_2 \in \operatorname{dom} A_1$ by [1, (20)]. $A_1(k_1) \cap A_1(k_2) \in M$. □

The functor Ω_{now} yielding a σ-field of subsets of $\{1,2,3,4\}$ is defined by the term

(Def. 6) $\{\emptyset, \{1,2,3,4\}\}$.

The functor Ω_{fut1} yielding a σ-field of subsets of $\{1,2,3,4\}$ is defined by the term

(Def. 7) $\{\emptyset, \{1,2\}, \{3,4\}, \{1,2,3,4\}\}$.

The functor Ω_{fut2} yielding a σ-field of subsets of $\{1,2,3,4\}$ is defined by the term

(Def. 8) $2^{\{1,2,3,4\}}$.

Let us consider a set Ω.

Let us assume that $\Omega = \{1,2,3,4\}$. Now we state the propositions:

(22) (i) $\{1\} \subseteq \Omega$, and

(ii) $\{2\} \subseteq \Omega$, and

(iii) $\{3\} \subseteq \Omega$, and

(iv) $\{4\} \subseteq \Omega$, and

(v) $\{1,2\} \subseteq \Omega$, and

(vi) $\{3,4\} \subseteq \Omega$, and

(vii) $\emptyset \subseteq \Omega \subseteq \Omega$.

(23) (i) Ω, $\emptyset \in \Omega_{now}$, and

(ii) $\{1, 2\}$, $\{3, 4\}$, Ω, $\emptyset \in \Omega_{fut1}$, and

(iii) Ω, $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\} \in \Omega_{fut2}$.

Now we state the proposition:

(24) $\Omega_{now} \subset \Omega_{fut1} \subset \Omega_{fut2}$.

4. Construction of Filtration and Examples

Now we state the propositions:

(25) There exists a non empty set Ω and there exist σ-fields F_1, F_2, F_3 of subsets of Ω such that $F_1 \subset F_2 \subset F_3$.

(26) There exist non empty sets Ω_1, Ω_2, Ω_3, Ω_4 such that

(i) $\Omega_1 \subset \Omega_2 \subset \Omega_3 \subset \Omega_4$, and

(ii) there exists a σ-field F_1 of subsets of Ω_1 and there exists a σ-field F_2 of subsets of Ω_2 and there exists a σ-field F_3 of subsets of Ω_3 and there exists a σ-field F_4 of subsets of Ω_4 such that $F_1 \subseteq F_2 \subseteq F_3 \subseteq F_4$.

Let I, Ω be non empty sets, Σ be a σ-field of subsets of Ω, M be a many sorted σ-field over I and Σ, and i be an element of I. The functor $\mathcal{M}_{\sigma\text{-field}}(M, i)$ yielding a σ-field of subsets of Ω is defined by the term

(Def. 9) $M(i)$.

Let Ω be a non empty set and I be a non empty subset of $\mathbb{R}$.

A filtration of I and Σ is a many sorted σ-field over I and Σ and is defined by

(Def. 10) for every elements s, t of I such that $s \leqslant t$ holds $it(s)$ is a subset of $it(t)$ and for every element t of I, $it(t) \subseteq \Sigma$.

Let F be a filtration of I and Σ and i be an element of I. The i-$\mathcal{EF}$ of F yielding a σ-field of subsets of Ω is defined by the term

(Def. 11) $F(i)$.

Let k be an element of $\{1, 2, 3\}$. The functor Select12-σ-field(k) yielding a subset of $2^{\{1,2,3,4\}}$ is defined by the term

(Def. 12) $\begin{cases} \Omega_{now}, & \textbf{if } k = 1, \\ \Omega_{fut1}, & \textbf{otherwise}. \end{cases}$

The functor Select123-σ-field(k) yielding a subset of $2^{\{1,2,3,4\}}$ is defined by the term

(Def. 13) $\begin{cases} \text{Select12-}\sigma\text{-field}(k), & \textbf{if } k \leqslant 2, \\ \Omega_{fut2}, & \textbf{otherwise}. \end{cases}$

Now we state the propositions:

(27) Let us consider a σ-field Σ of subsets of $\{1, 2, 3, 4\}$, and a set I. Suppose $I = \{1, 2, 3\}$ and $\Sigma = 2^{\{1,2,3,4\}}$. Then there exists a many sorted σ-field M over I and Σ such that

(i) $M(1) = \Omega_{now}$, and

(ii) $M(2) = \Omega_{fut1}$, and

(iii) $M(3) = \Omega_{fut2}$.

Proof: Define $\mathcal{U}$(element of $\{1, 2, 3\}$) = Select123-σ-field($\$_1$). Consider f_4 being a function from $\{1, 2, 3\}$ into $2^{2^{\{1,2,3,4\}}}$ such that for every element d of $\{1, 2, 3\}$, $f_4(d) = \mathcal{U}(d)$ from [5, Sch. 4]. For every set i such that $i \in I$ holds $f_4(i)$ is a σ-field of subsets of $\{1, 2, 3, 4\}$. $\square$

(28) Let us consider a non empty set Ω, a σ-field Σ of subsets of Ω, and a non empty subset I of $\mathbb{R}$. Suppose $I = \{1, 2, 3\}$ and $\Sigma = 2^{\{1,2,3,4\}}$ and $\Omega = \{1, 2, 3, 4\}$. Then there exists a many sorted σ-field M over I and Σ such that

(i) $M(1) = \Omega_{now}$, and

(ii) $M(2) = \Omega_{fut1}$, and

(iii) $M(3) = \Omega_{fut2}$, and

(iv) M is a filtration of I and Σ.

The theorem is a consequence of (27).

(29) Let us consider a non empty set Ω, a σ-field Σ of subsets of Ω, and a σ-field Σ_2 of subsets of $\{1\}$. Suppose $\Omega = \{1, 2, 3, 4\}$. Then there exists a function X_1 from Ω into $\{1\}$ such that X_1 is random variable of Ω_{now} and Σ_2, random variable of Ω_{fut1} and Σ_2, and random variable of Ω_{fut2} and Σ_2.

(30) Let us consider a non empty set Ω, a σ-field Σ of subsets of Ω, and a non empty subset I of $\mathbb{R}$. Suppose $I = \{1, 2, 3\}$ and $\Sigma = 2^{\{1,2,3,4\}}$ and $\Omega = \{1, 2, 3, 4\}$. Then there exists a many sorted σ-field M over I and Σ such that

(i) $M(1) = \Omega_{now}$, and

(ii) $M(2) = \Omega_{fut1}$, and

(iii) $M(3) = \Omega_{fut2}$, and

(iv) M is a filtration of I and Σ.

The theorem is a consequence of (27).

(31) There exist non empty sets Ω, Ω_2 and there exists a σ-field Σ of subsets of Ω and there exists a σ-field Σ_2 of subsets of Ω_2 and there exists a non empty subset I of $\mathbb{R}$ and there exists a many sorted σ-field Q over I and Σ such that Q is a filtration of I and Σ and there exists a function $\mathcal{RV}$ from Ω into Ω_2 such that for every element i of I, $\mathcal{RV}$ is a random variable of $\mathcal{M}_{\sigma\text{-field}}(Q,i)$ and Σ_2. The theorem is a consequence of (30) and (29).

(32) Let us consider non empty sets Ω, Ω_2, a σ-field Σ of subsets of Ω, a σ-field Σ_2 of subsets of Ω_2, a non empty subset I of $\mathbb{R}$, and a filtration Q of I and Σ. Then there exists a function $\mathcal{RV}$ from Ω into Ω_2 such that for every element i of I, $\mathcal{RV}$ is a random variable of $\mathcal{M}_{\sigma\text{-field}}(Q,i)$ and Σ_2.
Proof: Consider w being an object such that $w \in \Omega_2$. Set $m_1 = w$. Consider m being a function from Ω into Ω_2 such that $m = \Omega \longmapsto m_1$. For every element i of I, m is a random variable of $\mathcal{M}_{\sigma\text{-field}}(Q,i)$ and Σ_2 by [13, (7)], [11, (5), (4)]. $\square$

5. Stochastic Process: Adapted and Predictable

Now we state the proposition:

(33) Let us consider a non empty set Ω, a σ-field Σ of subsets of Ω, and a σ-field Σ_2 of subsets of Ω. If $\Sigma_2 \subseteq \Sigma$, then every event of Σ_2 is an event of Σ.

Let Ω, Ω_2 be non empty sets, Σ be a σ-field of subsets of Ω, Σ_2 be a σ-field of subsets of Ω_2, I be a non empty subset of $\mathbb{R}$, and P be a probability on Σ.

A stochastic process of I, Σ, Σ_2 and P is a function from I into the set of random variables on Σ and Σ_2 and is defined by

(Def. 14) for every element k of I, there exists a function $\mathcal{RV}$ from Ω into Ω_2 such that $it(k) = \mathcal{RV}$ and $\mathcal{RV}$ is random variable on Σ and Σ_2.

Let S be a stochastic process of I, Σ, Σ_2 and P and k be an element of I. The k-$\mathcal{RV}$ of S yielding a random variable of Σ and Σ_2 is defined by the term

(Def. 15) $S(k)$.

An adapted stochastic process of I, Σ, Σ_2, P and S is a function from I into the set of random variables on Σ and Σ_2 and is defined by

(Def. 16) there exists a filtration k of I and Σ such that for every element i of I, the i-$\mathcal{RV}$ of S is random variable on the i-$\mathcal{EF}$ of k and Σ_2.

Let I be a non empty subset of $\mathbb{N}$, J be a non empty subset of $\mathbb{N}$, and S be a stochastic process of $J(\in 2^{\mathbb{R}})$, Σ, Σ_2 and P.

A predictable stochastic process of I, J, Σ, Σ_2, P and S is a function from J into the set of random variables on Σ and Σ_2 and is defined by

(Def. 17) there exists a filtration k of $I(\in 2^{\mathbb{R}})$ and Σ such that for every element j of $J(\in 2^{\mathbb{R}})$ for every element i of $I(\in 2^{\mathbb{R}})$ such that $j-1=i$ holds the j-$\mathcal{RV}$ of S is random variable on the i-$\mathcal{EF}$ of k and Σ_2.

Let I be a non empty subset of $\mathbb{R}$, M be a filtration of I and Σ, and S be a stochastic process of I, Σ, Σ_2 and P. We say that S is M-stochastic process w.r.t. filtration if and only if

(Def. 18) for every element i of I, the i-$\mathcal{RV}$ of S is random variable on the i-$\mathcal{EF}$ of M and Σ_2.

Now we state the proposition:

(34) Let us consider non empty sets Ω, Ω_2, a σ-field Σ of subsets of Ω, a σ-field Σ_2 of subsets of Ω_2, a non empty subset I of $\mathbb{R}$, a probability P on Σ, a filtration M of I and Σ, and a stochastic process S of I, Σ, Σ_2 and P. Suppose S is M-stochastic process w.r.t. filtration. Then S is an adapted stochastic process of I, Σ, Σ_2, P and S.

6. Example for a Stochastic Process

Let k_1, k_2 be elements of $\mathbb{R}$, Ω be a non empty set, and k be an element of Ω. The functors: Set1-$\mathcal{RV}(k_1,k_2,k)$ and Set4-$\mathcal{RV}(k_1,k_2,k)$ yielding elements of $\mathbb{R}$ are defined by terms

(Def. 19) $\begin{cases} k_1, & \textbf{if } k=1 \text{ or } k=2,\\ k_2, & \textbf{otherwise},\end{cases}$

(Def. 20) $\begin{cases} k_1, & \textbf{if } k=3,\\ k_2, & \textbf{otherwise},\end{cases}$

respectively. Let k_2, k_3, k_4 be elements of $\mathbb{R}$. The functor Set3-$\mathcal{RV}(k_2,k_3,k_4,k)$ yielding an element of $\mathbb{R}$ is defined by the term

(Def. 21) $\begin{cases} k_2, & \textbf{if } k=2,\\ \text{Set4-}\mathcal{RV}(k_3,k_4,k), & \textbf{otherwise}.\end{cases}$

Let k_1, k_2, k_3, k_4 be elements of $\mathbb{R}$. The functor Set2-$\mathcal{RV}(k_1,k_2,k_3,k_4,k)$ yielding an element of $\mathbb{R}$ is defined by the term

(Def. 22) $\begin{cases} k_1, & \textbf{if } k=1,\\ \text{Set3-}\mathcal{RV}(k_2,k_3,k_4,k), & \textbf{otherwise}.\end{cases}$

Now we state the proposition:

(35) Let us consider elements k_1, k_2, k_3, k_4 of $\mathbb{R}$, and a set Ω. Suppose $\Omega = \{1,2,3,4\}$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1)=k_1$, and

(ii) $f(2)=k_2$, and

(iii) $f(3) = k_3$, and

(iv) $f(4) = k_4$.

PROOF: Define $\mathcal{U}$(element of Ω) $=$ Set2-$\mathcal{RV}(k_1, k_2, k_3, k_4, \$_1)$. Consider f being a function from Ω into $\mathbb{R}$ such that for every element d of Ω, $f(d) = \mathcal{U}(d)$ from [5, Sch. 4]. $f(1) = k_1$. $f(2) = k_2$. $f(3) = k_3$. $f(4) = k_4$. □

Let us consider a set Ω.

Let us assume that $\Omega = \{1, 2, 3, 4\}$. Now we state the propositions:

(36) There exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = 100$, and

(ii) $f(2) = 100$, and

(iii) $f(3) = 100$, and

(iv) $f(4) = 100$.

The theorem is a consequence of (35).

(37) There exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = 80$, and

(ii) $f(2) = 80$, and

(iii) $f(3) = 120$, and

(iv) $f(4) = 120$.

The theorem is a consequence of (35).

(38) There exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = 60$, and

(ii) $f(2) = 80$, and

(iii) $f(3) = 100$, and

(iv) $f(4) = 120$.

The theorem is a consequence of (35).

(39) Let us consider elements k_1, k_2, k_3, k_4 of $\mathbb{R}$, and a non empty set Ω. Suppose $\Omega = \{1, 2, 3, 4\}$. Let us consider a σ-field Σ of subsets of Ω, a non empty subset I of $\mathbb{R}$, and a filtration M of I and Σ. Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Let us consider an element k of I. Suppose $k = 3$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = k_1$, and

(ii) $f(2) = k_2$, and

(iii) $f(3) = k_3$, and

(iv) $f(4) = k_4$, and

(v) f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets.

Proof: Consider f being a function from Ω into $\mathbb{R}$ such that $f(1) = k_1$ and $f(2) = k_2$ and $f(3) = k_3$ and $f(4) = k_4$. $1, 2, 3, 4 \in \operatorname{dom} f$. f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets by [4, (1)], [11, (4)]. □

Let us consider a non empty set Ω, a σ-field Σ of subsets of Ω, a non empty subset I of $\mathbb{R}$, a filtration M of I and Σ, and an element k of I.

Let us assume that $\Omega = \{1, 2, 3, 4\}$. Now we state the propositions:

(40) Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Then suppose $k = 3$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = 60$, and

(ii) $f(2) = 80$, and

(iii) $f(3) = 100$, and

(iv) $f(4) = 120$, and

(v) f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets.

The theorem is a consequence of (39).

(41) Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Then suppose $k = 3$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = 180$, and

(ii) $f(2) = 120$, and

(iii) $f(3) = 120$, and

(iv) $f(4) = 80$, and

(v) f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets.

The theorem is a consequence of (39).

(42) Let us consider elements k_1, k_2 of $\mathbb{R}$, and a non empty set Ω. Suppose $\Omega = \{1, 2, 3, 4\}$. Let us consider a σ-field Σ of subsets of Ω, a non empty subset I of $\mathbb{R}$, and a filtration M of I and Σ. Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Let us consider an element k of I. Suppose $k = 2$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = k_1$, and

(ii) $f(2) = k_1$, and

(iii) $f(3) = k_2$, and

(iv) $f(4) = k_2$, and

(v) f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets.

PROOF: Consider f being a function from Ω into $\mathbb{R}$ such that $f(1) = k_1$ and $f(2) = k_1$ and $f(3) = k_2$ and $f(4) = k_2$. Set $i = k$. For every set x, $f^{-1}(x) \in$ the i-$\mathcal{EF}$ of M by [4, (1)]. □

Let us consider a non empty set Ω, a σ-field Σ of subsets of Ω, a non empty subset I of $\mathbb{R}$, a filtration M of I and Σ, and an element k of I.

Let us assume that $\Omega = \{1, 2, 3, 4\}$. Now we state the propositions:

(43) Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Then suppose $k = 2$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = 80$, and

(ii) $f(2) = 80$, and

(iii) $f(3) = 120$, and

(iv) $f(4) = 120$, and

(v) f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets.

The theorem is a consequence of (42).

(44) Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Then suppose $k = 2$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = 150$, and

(ii) $f(2) = 150$, and

(iii) $f(3) = 100$, and

(iv) $f(4) = 100$, and

(v) f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets.

The theorem is a consequence of (42).

(45) Let us consider an element k_1 of $\mathbb{R}$, and a non empty set Ω. Suppose $\Omega = \{1, 2, 3, 4\}$. Let us consider a σ-field Σ of subsets of Ω, a non empty subset I of $\mathbb{R}$, and a filtration M of I and Σ. Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Let us consider an element k of I. Suppose $k = 1$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = k_1$, and

(ii) $f(2) = k_1$, and

(iii) $f(3) = k_1$, and

(iv) $f(4) = k_1$, and

(v) f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets.

PROOF: Consider f being a function from Ω into $\mathbb{R}$ such that $f(1) = k_1$ and $f(2) = k_1$ and $f(3) = k_1$ and $f(4) = k_1$. Set $i = k$. For every set x such that $x \in$ the Borel sets holds $f^{-1}(x) \in$ the i-$\mathcal{EF}$ of M by [4, (1)]. □

Let us consider a non empty set Ω, a σ-field Σ of subsets of Ω, a non empty subset I of $\mathbb{R}$, a filtration M of I and Σ, and an element k of I.

Let us assume that $\Omega = \{1, 2, 3, 4\}$. Now we state the propositions:

(46) Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Then suppose $k = 1$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = 100$, and

(ii) $f(2) = 100$, and

(iii) $f(3) = 100$, and

(iv) $f(4) = 100$, and

(v) f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets.

The theorem is a consequence of (45).

(47) Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Then suppose $k = 1$. Then there exists a function f from Ω into $\mathbb{R}$ such that

(i) $f(1) = 125$, and

(ii) $f(2) = 125$, and

(iii) $f(3) = 125$, and

(iv) $f(4) = 125$, and

(v) f is random variable on the k-$\mathcal{EF}$ of M and the Borel sets.

The theorem is a consequence of (45).

Now we state the proposition:

(48) Let us consider a non empty set Ω. Suppose $\Omega = \{1, 2, 3, 4\}$. Let us consider a σ-field Σ of subsets of Ω, and a non empty subset I of $\mathbb{R}$. Suppose $I = \{1, 2, 3\}$ and $\Sigma = 2^{\{1,2,3,4\}}$. Let us consider a filtration M of I and Σ. Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Let us consider a probability P on Σ, and an element i of I. Then there exists a function $\mathcal{RV}$ from Ω into $\mathbb{R}$ such that $\mathcal{RV}$ is random variable on the i-$\mathcal{EF}$ of M and the Borel sets. The theorem is a consequence of (46), (43), and (40).

Let I be a non empty subset of $\mathbb{R}$. Assume $I = \{1, 2, 3\}$. Let i be an element of I. Assume $i = 2$ or $i = 3$. Let Ω be a non empty set. Assume $\Omega = \{1, 2, 3, 4\}$. Let Σ be a σ-field of subsets of Ω. Assume $\Sigma = 2^{\Omega}$. Let f_1 be a function from Ω into $\mathbb{R}$. Assume $f_1(1) = 60$ and $f_1(2) = 80$ and $f_1(3) = 100$ and $f_1(4) = 120$. Let f_2 be a function from Ω into $\mathbb{R}$. Assume $f_2(1) = 80$ and $f_2(2) = 80$ and $f_2(3) = 120$ and $f_2(4) = 120$. Let f_3 be a function from Ω into $\mathbb{R}$. The functor Select12-$\mathcal{RV}(i, \Sigma, f_1, f_2, f_3)$ yielding an element of the set of random variables on Σ and the Borel sets is defined by the term

(Def. 23) $\begin{cases} f_2, & \textbf{if } i = 2, \\ f_1, & \textbf{otherwise}. \end{cases}$

Assume $I = \{1, 2, 3\}$. Assume $\Omega = \{1, 2, 3, 4\}$. Assume $\Sigma = 2^{\Omega}$. Let f_1, f_2 be functions from Ω into $\mathbb{R}$. Assume $f_3(1) = 100$ and $f_3(2) = 100$ and $f_3(3) = 100$ and $f_3(4) = 100$. The functor Select123-$\mathcal{RV}(i, \Sigma, f_1, f_2, f_3)$ yielding an element of the set of random variables on Σ and the Borel sets is defined by the term

(Def. 24) $\begin{cases} \text{Select12-}\mathcal{RV}(i, \Sigma, f_1, f_2, f_3), & \textbf{if } i = 2 \text{ or } i = 3, \\ f_3, & \textbf{otherwise}. \end{cases}$

Now we state the proposition:

(49) Let us consider non empty sets Ω, Ω_2. Suppose $\Omega = \{1, 2, 3, 4\}$. Let us consider a σ-field Σ of subsets of Ω, and a non empty subset I of $\mathbb{R}$. Suppose $I = \{1, 2, 3\}$ and $\Sigma = 2^{\{1,2,3,4\}}$. Let us consider a probability P on Σ, and a filtration M of I and Σ. Suppose $M(1) = \Omega_{now}$ and $M(2) = \Omega_{fut1}$ and $M(3) = \Omega_{fut2}$. Then there exists a stochastic process S of I, Σ, the Borel sets and P such that

(i) for every element k of I, there exists a function $\mathcal{RV}$ from Ω into $\mathbb{R}$ such that $S(k) = \mathcal{RV}$ and $\mathcal{RV}$ is random variable on Σ and the Borel sets and random variable on the k-$\mathcal{EF}$ of M and the Borel sets and there exists a function f from Ω into $\mathbb{R}$ such that if $k = 1$, then $f(1) = 100$ and $f(2) = 100$ and $f(3) = 100$ and $f(4) = 100$ and $S(k) = f$ and there exists a function f from Ω into $\mathbb{R}$ such that if $k = 2$, then $f(1) = 80$ and $f(2) = 80$ and $f(3) = 120$ and $f(4) = 120$ and $S(k) = f$ and there exists a function f from Ω into $\mathbb{R}$ such that if $k = 3$, then $f(1) = 60$ and $f(2) = 80$ and $f(3) = 100$ and $f(4) = 120$ and $S(k) = f$ and S is M-stochastic process w.r.t. filtration, and

(ii) S is an adapted stochastic process of I, Σ, the Borel sets, P and S.

Proof: Consider f_3 being a function from Ω into $\mathbb{R}$ such that $f_3(1) = 100$ and $f_3(2) = 100$ and $f_3(3) = 100$ and $f_3(4) = 100$. Consider f_2 being a function from Ω into $\mathbb{R}$ such that $f_2(1) = 80$ and $f_2(2) = 80$ and $f_2(3) = 120$ and $f_2(4) = 120$. Consider f_1 being a function from Ω into $\mathbb{R}$ such that $f_1(1) = 60$ and $f_1(2) = 80$ and $f_1(3) = 100$ and $f_1(4) = 120$. Define $\mathcal{U}(\text{element of } I) =$ Select123-$\mathcal{RV}(\$_1, \Sigma, f_1, f_2, f_3)$. Consider f_4 being a function from I into the set of random variables on Σ and the Borel sets such that for every element d of I, $f_4(d) = \mathcal{U}(d)$ from [5, Sch. 4]. For every element k of I, there exists a function $\mathcal{RV}$ from Ω into $\mathbb{R}$ such that $f_4(k) = \mathcal{RV}$ and $\mathcal{RV}$ is random variable on Σ and the Borel sets. For every element k of I, there exists a function $\mathcal{RV}$ from Ω into $\mathbb{R}$ such that $f_4(k) = \mathcal{RV}$ and $\mathcal{RV}$ is random variable on Σ and the Borel sets and random variable on the k-$\mathcal{EF}$ of M and the Borel sets and there exists a function f from Ω into $\mathbb{R}$ such that if $k = 1$, then $f(1) = 100$ and $f(2) = 100$ and $f(3) = 100$ and $f(4) = 100$ and $f_4(k) = f$ and there exists a function f from Ω into $\mathbb{R}$ such that if $k = 2$, then $f(1) = 80$ and $f(2) = 80$ and $f(3) = 120$ and $f(4) = 120$ and $f_4(k) = f$ and there exists a function f from Ω into $\mathbb{R}$ such that if $k = 3$, then $f(1) = 60$ and $f(2) = 80$ and $f(3) = 100$ and $f(4) = 120$ and $f_4(k) = f$ and f_4 is M-stochastic process w.r.t. filtration and adapted stochastic process of I, Σ, the Borel sets, P and f_4. □

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