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Probability Measure on Discrete Spaces and Algebra of Real-Valued Random Variables

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. In this article we continue formalizing probability and randomness started in [13], where we formalized some theorems concerning the probability and real-valued random variables. In this paper we formalize the variance of a random variable and prove Chebyshev's inequality. Next we formalize the product probability measure on the Cartesian product of discrete spaces. In the final part of this article we define the algebra of real-valued random variables.

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The notation and terminology used here have been introduced in the following papers: [21], [3], [16], [1], [9], [17], [14], [4], [5], [11], [15], [6], [12], [22], [13], [19], [20], [8], [10], [18], [2], and [7].

1. Variance

In this paper O_1 denotes a non empty set, r denotes a real number, S_1 denotes a σ-field of subsets of O_1, and P denotes a probability on S_1.

One can prove the following two propositions:

(1) For every one-to-one function f and for all subsets A, B of $\operatorname{dom} f$ such that A misses B holds $\operatorname{rng}(f \restriction A)$ misses $\operatorname{rng}(f \restriction B)$.

(2) For all functions f, g holds $\operatorname{rng}(f \cdot g) \subseteq \operatorname{rng}(f \restriction \operatorname{rng} g)$.

Let us consider O_1, S_1. Observe that there exists a real-valued random variable of S_1 which is non-negative.

Let us consider O_1, S_1 and let X be a real-valued random variable of S_1. Note that $|X|$ is non-negative.

The following propositions are true:

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(3) $O_1 \longmapsto 1 = \chi_{(O_1),O_1}$.

(4) $O_1 \longmapsto r$ is a real-valued random variable of S_1.

(5) For every non empty set X and for every partial function f from X to $\mathbb{R}$ holds $f^2 = (-f)^2$ and $f^2 = |f|^2$.

(6) Let X be a non empty set and f, g be partial functions from X to $\mathbb{R}$. Then $(f+g)^2 = f^2 + 2\,(f\,g) + g^2$ and $(f-g)^2 = (f^2 - 2\,(f\,g)) + g^2$.

Let us consider O_1, S_1, P and let X be a real-valued random variable of S_1. Let us assume that X is integrable on P and $|X|^2$ is integrable on P2M P. The functor variance(X, P) yielding an element of $\mathbb{R}$ is defined by the condition (Def. 1).

(Def. 1) There exists a real-valued random variable Y of S_1 and there exists a real-valued random variable E of S_1 such that $E = O_1 \longmapsto E_P\{X\}$ and $Y = X - E$ and Y is integrable on P and $|Y|^2$ is integrable on P2M P and variance$(X, P) = \int |Y|^2 \, \mathrm{d}\, \mathrm{P2M}\, P$.

2. Chebyshev's Inequality

One can prove the following proposition

(7) Let given O_1, S_1, P, r and X be a real-valued random variable of S_1. Suppose $0 < r$ and X is non-negative and X is integrable on P and $|X|^2$ is integrable on P2M P. Then $P(\{t \in O_1 \colon r \leq |X(t) - E_P\{X\}|\}) \leq \frac{\mathrm{variance}(X,P)}{r^2}$.

3. Product Probability Measure

The following propositions are true:

(8) Let O_1 be a non empty finite set, f be a function from O_1 into $\mathbb{R}$, and P be a function from 2^{O_1} into $\mathbb{R}$. Suppose that

(i) for every set x such that $x \subseteq O_1$ holds $0 \leq P(x) \leq 1$,

(ii) $P(O_1) = 1$, and

(iii) for every finite subset z of O_1 holds $P(z) = \mathrm{setopfunc}(z, O_1, \mathbb{R}, f, +_{\mathbb{R}})$.

Then P is a probability on the trivial σ-field of O_1.

(9) Let D_1 be a non empty set, F be a function from D_1 into $\mathbb{R}$, and Y be a finite subset of D_1. Then there exists a finite sequence p of elements of D_1 such that p is one-to-one and $\mathrm{rng}\, p = Y$ and $\mathrm{setopfunc}(Y, D_1, \mathbb{R}, F, +_{\mathbb{R}}) = \sum \mathrm{FuncSeq}(F, p)$.

(10) Let D_1 be a non empty set, F be a function from D_1 into $\mathbb{R}$, Y be a finite subset of D_1, and p be a finite sequence of elements of D_1. If p is one-to-one and $\mathrm{rng}\, p = Y$, then $\mathrm{setopfunc}(Y, D_1, \mathbb{R}, F, +_{\mathbb{R}}) = \sum \mathrm{FuncSeq}(F, p)$.

(11) Let D_2, D_3 be non empty sets, F_1 be a function from D_2 into $\mathbb{R}$, F_2 be a function from D_3 into $\mathbb{R}$, G be a function from $D_2 \times D_3$ into $\mathbb{R}$, Y_1 be a non empty finite subset of D_2, and p_1 be a finite sequence of elements of D_2. Suppose p_1 is one-to-one and $\operatorname{rng} p_1 = Y_1$. Let p_2 be a finite sequence of elements of D_3, p_3 be a finite sequence of elements of $D_2 \times D_3$, Y_2 be a non empty finite subset of D_3, and Y_3 be a finite subset of $D_2 \times D_3$. Suppose that

(i) p_2 is one-to-one,

(ii) $\operatorname{rng} p_2 = Y_2$,

(iii) p_3 is one-to-one,

(iv) $\operatorname{rng} p_3 = Y_3$,

(v) $Y_3 = Y_1 \times Y_2$, and

(vi) for all sets x, y such that $x \in Y_1$ and $y \in Y_2$ holds $G(x, y) = F_1(x) \cdot F_2(y)$.

Then $\sum \mathrm{FuncSeq}(G, p_3) = \sum \mathrm{FuncSeq}(F_1, p_1) \cdot \sum \mathrm{FuncSeq}(F_2, p_2)$.

(12) Let D_2, D_3 be non empty sets, F_1 be a function from D_2 into $\mathbb{R}$, F_2 be a function from D_3 into $\mathbb{R}$, G be a function from $D_2 \times D_3$ into $\mathbb{R}$, Y_1 be a non empty finite subset of D_2, Y_2 be a non empty finite subset of D_3, and Y_3 be a finite subset of $D_2 \times D_3$. Suppose $Y_3 = Y_1 \times Y_2$ and for all sets x, y such that $x \in Y_1$ and $y \in Y_2$ holds $G(x, y) = F_1(x) \cdot F_2(y)$. Then $\mathrm{setopfunc}(Y_3, D_2 \times D_3, \mathbb{R}, G, +_{\mathbb{R}}) = \mathrm{setopfunc}(Y_1, D_2, \mathbb{R}, F_1, +_{\mathbb{R}}) \cdot \mathrm{setopfunc}(Y_2, D_3, \mathbb{R}, F_2, +_{\mathbb{R}})$.

(13) Let D_1 be a non empty set, F be a function from D_1 into $\mathbb{R}$, and Y be a finite subset of D_1. If for every set x such that $x \in Y$ holds $0 \leq F(x)$, then $0 \leq \mathrm{setopfunc}(Y, D_1, \mathbb{R}, F, +_{\mathbb{R}})$.

(14) Let D_1 be a non empty set, F be a function from D_1 into $\mathbb{R}$, and Y_1, Y_2 be finite subsets of D_1. Suppose $Y_1 \subseteq Y_2$ and for every set x such that $x \in Y_2$ holds $0 \leq F(x)$. Then $\mathrm{setopfunc}(Y_1, D_1, \mathbb{R}, F, +_{\mathbb{R}}) \leq \mathrm{setopfunc}(Y_2, D_1, \mathbb{R}, F, +_{\mathbb{R}})$.

(15) Let O_1 be a non empty finite set, P be a probability on the trivial σ-field of O_1, Y be a non empty finite subset of O_1, and f be a function from O_1 into $\mathbb{R}$. Then there exists a finite sequence G of elements of $\mathbb{R}$ and there exists a finite sequence s of elements of Y such that

(i) $\operatorname{len} G = \overline{\overline{Y}}$,

(ii) s is one-to-one,

(iii) $\operatorname{rng} s = Y$,

(iv) $\operatorname{len} s = \overline{\overline{Y}}$,

(v) for every natural number n such that $n \in \operatorname{dom} G$ holds $G(n) = f(s(n)) \cdot P(\{s(n)\})$, and

(vi) $\int f{\restriction}Y \,\mathrm{d}\,\mathrm{P2M}\,P = \sum G$.

Let O_2, O_3 be non empty finite sets, let P_1 be a probability on the trivial

σ-field of O_2, and let P_2 be a probability on the trivial σ-field of O_3. The functor Product-Probability(O_2, O_3, P_1, P_2) yielding a probability on the trivial σ-field of $O_2 \times O_3$ is defined by the condition (Def. 2).

(Def. 2) There exists a function Q from $O_2 \times O_3$ into $\mathbb{R}$ such that

(i) for all sets x, y such that $x \in O_2$ and $y \in O_3$ holds $Q(x, y) = P_1(\{x\}) \cdot P_2(\{y\})$, and

(ii) for every finite subset z of $O_2 \times O_3$ holds (Product-Probability$(O_2, O_3, P_1, P_2))(z) = \mathrm{setopfunc}(z, O_2 \times O_3, \mathbb{R}, Q, +_{\mathbb{R}})$.

Next we state two propositions:

(16) Let O_2, O_3 be non empty finite sets, P_1 be a probability on the trivial σ-field of O_2, P_2 be a probability on the trivial σ-field of O_3, Y_1 be a non empty finite subset of O_2, and Y_2 be a non empty finite subset of O_3. Then (Product-Probability$(O_2, O_3, P_1, P_2))(Y_1 \times Y_2) = P_1(Y_1) \cdot P_2(Y_2)$.

(17) Let O_2, O_3 be non empty finite sets, P_1 be a probability on the trivial σ-field of O_2, P_2 be a probability on the trivial σ-field of O_3, and y_1, y_2 be sets. If $y_1 \in O_2$ and $y_2 \in O_3$, then (Product-Probability$(O_2, O_3, P_1, P_2))(\{\langle y_1, y_2 \rangle\}) = P_1(\{y_1\}) \cdot P_2(\{y_2\})$.

4. Algebra of Real-valued Random Variables

Let O_1 be a non empty set and let S_1 be a σ-field of subsets of O_1. The $\mathbb{R}$-valued random variables set of S_1 yields a non empty subset of RAlgebra O_1 and is defined as follows:

(Def. 3) The $\mathbb{R}$-valued random variables set of $S_1 = \{f : f$ ranges over real-valued random variables of $S_1\}$.

Let us consider O_1, S_1. Note that the $\mathbb{R}$-valued random variables set of S_1 is additively-linearly-closed and multiplicatively-closed.

Let us consider O_1, S_1. The $\mathbb{R}$ algebra of real-valued-random-variables of S_1 yielding an algebra is defined by the condition (Def. 4).

(Def. 4) The $\mathbb{R}$ algebra of real-valued-random-variables of S_1 = $\langle$the $\mathbb{R}$-valued random variables set of S_1, mult(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1), Add(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1), Mult(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1), One(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1), Zero(the $\mathbb{R}$-valued random variables set of S_1, RAlgebra O_1)$\rangle$.

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