

Riemann Integral of Functions from \mathbb{R} into Real Normed Space

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Summary. In this article, we define the Riemann integral on functions from \mathbb{R} into real normed space and prove the linearity of this operator. As a result, the Riemann integration can be applied to a wider range of functions. The proof method follows the [16].

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The terminology and notation used here have been introduced in the following articles: [2], [3], [4], [5], [7], [10], [8], [9], [1], [14], [6], [13], [15], [11], [19], [17], [12], [18], and [20].

1. PRELIMINARIES

Let X be a real normed space, let A be a closed-interval subset of \mathbb{R} , let f be a function from A into the carrier of X , and let D be a Division of A . A finite sequence of elements of X is said to be a middle volume of f and D if it satisfies the conditions (Def. 1).

(Def. 1)(i) $\text{len } it = \text{len } D$, and

(ii) for every natural number i such that $i \in \text{dom } D$ there exists a point c of X such that $c \in \text{rng}(f \upharpoonright \text{divset}(D, i))$ and $it(i) = \text{vol}(\text{divset}(D, i)) \cdot c$.

Let X be a real normed space, let A be a closed-interval subset of \mathbb{R} , let f be a function from A into the carrier of X , let D be a Division of A , and let F be a middle volume of f and D . The functor $\text{middle sum}(f, F)$ yielding a point of X is defined by:

(Def. 2) $\text{middle sum}(f, F) = \sum F$.

Let X be a real normed space, let A be a closed-interval subset of \mathbb{R} , let f be a function from A into the carrier of X , and let T be a division sequence of A . A function from \mathbb{N} into (the carrier of X)* is said to be a middle volume sequence of f and T if:

(Def. 3) For every element k of \mathbb{N} holds $\text{it}(k)$ is a middle volume of f and $T(k)$.

Let X be a real normed space, let A be a closed-interval subset of \mathbb{R} , let f be a function from A into the carrier of X , let T be a division sequence of A , let S be a middle volume sequence of f and T , and let k be an element of \mathbb{N} . Then $S(k)$ is a middle volume of f and $T(k)$.

Let X be a real normed space, let A be a closed-interval subset of \mathbb{R} , let f be a function from A into the carrier of X , let T be a division sequence of A , and let S be a middle volume sequence of f and T . The functor $\text{middle sum}(f, S)$ yielding a sequence of X is defined as follows:

(Def. 4) For every element i of \mathbb{N} holds
 $(\text{middle sum}(f, S))(i) = \text{middle sum}(f, S(i))$.

2. DEFINITION OF RIEMANN INTEGRAL ON FUNCTIONS FROM \mathbb{R} INTO REAL NORMED SPACE

Let X be a real normed space, let A be a closed-interval subset of \mathbb{R} , and let f be a function from A into the carrier of X . We say that f is integrable if and only if the condition (Def. 5) is satisfied.

(Def. 5) There exists a point I of X such that for every division sequence T of A and for every middle volume sequence S of f and T if δ_T is convergent and $\lim(\delta_T) = 0$, then $\text{middle sum}(f, S)$ is convergent and $\lim \text{middle sum}(f, S) = I$.

We now state three propositions:

- (1) Let X be a real normed space and R_1, R_2, R_3 be finite sequences of elements of X . If $\text{len } R_1 = \text{len } R_2$ and $R_3 = R_1 + R_2$, then $\sum R_3 = \sum R_1 + \sum R_2$.
- (2) Let X be a real normed space and R_1, R_2, R_3 be finite sequences of elements of X . If $\text{len } R_1 = \text{len } R_2$ and $R_3 = R_1 - R_2$, then $\sum R_3 = \sum R_1 - \sum R_2$.
- (3) Let X be a real normed space, R_1, R_2 be finite sequences of elements of X , and a be an element of \mathbb{R} . If $R_2 = a R_1$, then $\sum R_2 = a \cdot \sum R_1$.

Let X be a real normed space, let A be a closed-interval subset of \mathbb{R} , and let f be a function from A into the carrier of X . Let us assume that f is integrable. The functor integral f yields a point of X and is defined by the condition (Def. 6).

(Def. 6) Let T be a division sequence of A and S be a middle volume sequence of f and T . If δ_T is convergent and $\lim(\delta_T) = 0$, then middle sum(f, S) is convergent and \lim middle sum(f, S) = integral f .

We now state four propositions:

- (4) Let X be a real normed space, A be a closed-interval subset of \mathbb{R} , r be a real number, and f, h be functions from A into the carrier of X . If $h = r f$ and f is integrable, then h is integrable and integral $h = r \cdot$ integral f .
- (5) Let X be a real normed space, A be a closed-interval subset of \mathbb{R} , and f, h be functions from A into the carrier of X . If $h = -f$ and f is integrable, then h is integrable and integral $h = -$ integral f .
- (6) Let X be a real normed space, A be a closed-interval subset of \mathbb{R} , and f, g, h be functions from A into the carrier of X . Suppose $h = f + g$ and f is integrable and g is integrable. Then h is integrable and integral $h =$ integral $f +$ integral g .
- (7) Let X be a real normed space, A be a closed-interval subset of \mathbb{R} , and f, g, h be functions from A into the carrier of X . Suppose $h = f - g$ and f is integrable and g is integrable. Then h is integrable and integral $h =$ integral $f -$ integral g .

Let X be a real normed space, let A be a closed-interval subset of \mathbb{R} , and let f be a partial function from \mathbb{R} to the carrier of X . We say that f is integrable on A if and only if:

(Def. 7) There exists a function g from A into the carrier of X such that $g = f \upharpoonright A$ and g is integrable.

Let X be a real normed space, let A be a closed-interval subset of \mathbb{R} , and let f be a partial function from \mathbb{R} to the carrier of X . Let us assume that $A \subseteq \text{dom } f$. The functor $\int_A f(x)dx$ yields an element of X and is defined as follows:

(Def. 8) There exists a function g from A into the carrier of X such that $g = f \upharpoonright A$ and $\int_A f(x)dx =$ integral g .

We now state several propositions:

- (8) Let A be a closed-interval subset of \mathbb{R} , f be a partial function from \mathbb{R} to the carrier of X , and g be a function from A into the carrier of X . Suppose $f \upharpoonright A = g$. Then f is integrable on A if and only if g is integrable.
- (9) Let A be a closed-interval subset of \mathbb{R} , f be a partial function from \mathbb{R} to the carrier of X , and g be a function from A into the carrier of X . If

$A \subseteq \text{dom } f$ and $f \upharpoonright A = g$, then $\int_A f(x)dx = \text{integral } g$.

- (10) Let X, Y be non empty sets, V be a real normed space, g, f be partial functions from X to the carrier of V , and g_1, f_1 be partial functions from Y to the carrier of V . If $g = g_1$ and $f = f_1$, then $g_1 + f_1 = g + f$.
- (11) Let X, Y be non empty sets, V be a real normed space, g, f be partial functions from X to the carrier of V , and g_1, f_1 be partial functions from Y to the carrier of V . If $g = g_1$ and $f = f_1$, then $g_1 - f_1 = g - f$.
- (12) Let r be a real number, X, Y be non empty sets, V be a real normed space, g be a partial function from X to the carrier of V , and g_1 be a partial function from Y to the carrier of V . If $g = g_1$, then $r g_1 = r g$.

3. LINEARITY OF THE INTEGRATION OPERATOR

Next we state three propositions:

- (13) Let r be a real number, A be a closed-interval subset of \mathbb{R} , and f be a partial function from \mathbb{R} to the carrier of X . Suppose $A \subseteq \text{dom } f$ and f is integrable on A . Then $r f$ is integrable on A and $\int_A (r f)(x)dx = r \cdot \int_A f(x)dx$.
- (14) Let A be a closed-interval subset of \mathbb{R} and f_1, f_2 be partial functions from \mathbb{R} to the carrier of X . Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \text{dom } f_1$ and $A \subseteq \text{dom } f_2$. Then $f_1 + f_2$ is integrable on A and $\int_A (f_1 + f_2)(x)dx = \int_A f_1(x)dx + \int_A f_2(x)dx$.
- (15) Let A be a closed-interval subset of \mathbb{R} and f_1, f_2 be partial functions from \mathbb{R} to the carrier of X . Suppose f_1 is integrable on A and f_2 is integrable on A and $A \subseteq \text{dom } f_1$ and $A \subseteq \text{dom } f_2$. Then $f_1 - f_2$ is integrable on A and $\int_A (f_1 - f_2)(x)dx = \int_A f_1(x)dx - \int_A f_2(x)dx$.

Let X be a real normed space, let f be a partial function from \mathbb{R} to the carrier of X , and let a, b be real numbers. The functor $\int_a^b f(x)dx$ yielding an element of X is defined as follows:

$$(\text{Def. 9}) \quad \int_a^b f(x)dx = \begin{cases} \int_{[a,b]} f(x)dx, & \text{if } a \leq b, \\ - \int_{[b,a]} f(x)dx, & \text{otherwise.} \end{cases}$$

One can prove the following propositions:

- (16) Let f be a partial function from \mathbb{R} to the carrier of X , A be a closed-interval subset of \mathbb{R} , and a, b be real numbers. If $A = [a, b]$, then

$$\int_A f(x)dx = \int_a^b f(x)dx.$$

- (17) Let f be a partial function from \mathbb{R} to the carrier of X and A be a closed-interval subset of \mathbb{R} . If $\text{vol}(A) = 0$ and $A \subseteq \text{dom } f$, then f is integrable on A and $\int_A f(x)dx = 0_X$.

- (18) Let f be a partial function from \mathbb{R} to the carrier of X , A be a closed-interval subset of \mathbb{R} , and a, b be real numbers. If $A = [b, a]$ and $A \subseteq \text{dom } f$, then $-\int_A f(x)dx = \int_a^b f(x)dx$.

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