

Differentiability of Polynomials over Reals

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Summary. In this article, we formalize in the Mizar system [3] the notion of the derivative of polynomials over the field of real numbers [4]. To define it, we use the derivative of functions between reals and reals [9].

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1. PRELIMINARIES

From now on c denotes a complex, r denotes a real number, m, n denote natural numbers, and f denotes a complex-valued function.

Now we state the propositions:

- (1) $0 + f = f$.
- (2) $f - 0 = f$.

Let f be a complex-valued function. Observe that $0 + f$ reduces to f and $f - 0$ reduces to f .

Now we state the propositions:

- (3) $c + f = (\text{dom } f \mapsto c) + f$.
- (4) $f - c = f - (\text{dom } f \mapsto c)$.
- (5) $c \cdot f = (\text{dom } f \mapsto c) \cdot f$.
- (6) $f + (\text{dom } f \mapsto 0) = f$. The theorem is a consequence of (3).
- (7) $f - (\text{dom } f \mapsto 0) = f$. The theorem is a consequence of (4).

$$(8) \quad \square^0 = \mathbb{R} \mapsto 1.$$

PROOF: Reconsider $s = 1$ as an element of \mathbb{R} . $\square^0 = \mathbb{R} \mapsto s$ by [8, (34)], [10, (7)]. \square

2. DIFFERENTIABILITY OF REAL FUNCTIONS

One can check that every function from \mathbb{R} into \mathbb{R} which is differentiable is also continuous.

Let f be a differentiable function from \mathbb{R} into \mathbb{R} . The functor f' yielding a function from \mathbb{R} into \mathbb{R} is defined by the term

$$(\text{Def. 1}) \quad f'_{\mathbb{R}}.$$

Now we state the propositions:

(9) Let us consider a function f from \mathbb{R} into \mathbb{R} . Then f is differentiable if and only if for every r , f is differentiable in r .

(10) Let us consider a differentiable function f from \mathbb{R} into \mathbb{R} . Then $f'(r) = f'(r)^1$.

Let f be a function from \mathbb{R} into \mathbb{R} . Observe that f is differentiable if and only if the condition (Def. 2) is satisfied.

$$(\text{Def. 2}) \quad \text{for every } r, f \text{ is differentiable in } r.$$

Let us note that every function from \mathbb{R} into \mathbb{R} which is constant is also differentiable.

Now we state the proposition:

(11) Let us consider a constant function f from \mathbb{R} into \mathbb{R} . Then $f' = \mathbb{R} \mapsto 0$.

PROOF: Reconsider $z = 0$ as an element of \mathbb{R} . $f' = \mathbb{R} \mapsto z$ by [9, (22)], [10, (7)]. \square

One can verify that $\text{id}_{\mathbb{R}}$ is differentiable as a function from \mathbb{R} into \mathbb{R} .

Now we state the proposition:

$$(12) \quad \text{id}'_{\mathbb{R}} = \mathbb{R} \mapsto 1.$$

PROOF: Set $f = \text{id}_{\mathbb{R}}$. Reconsider $z = 1$ as an element of \mathbb{R} . $f' = \mathbb{R} \mapsto z$ by [9, (17)], [10, (7)]. \square

Let us consider n . One can verify that \square^n is differentiable.

Now we state the proposition:

$$(13) \quad (\square^n)' = n \cdot (\square^{n-1}).$$

From now on f, g denote differentiable functions from \mathbb{R} into \mathbb{R} .

¹Left-side $f'(r)$ is the value of the derivative defined in this article for differentiable functions $f : \mathbb{R} \mapsto \mathbb{R}$, and right-side $f'(r)$ is the value of the derivative defined for partial functions in [9].

Let us consider f and g . Let us observe that $f + g$ is differentiable as a function from \mathbb{R} into \mathbb{R} and $f - g$ is differentiable as a function from \mathbb{R} into \mathbb{R} and $f \cdot g$ is differentiable as a function from \mathbb{R} into \mathbb{R} .

Let us consider r . One can verify that $r + f$ is differentiable as a function from \mathbb{R} into \mathbb{R} and $r \cdot f$ is differentiable as a function from \mathbb{R} into \mathbb{R} and $f - r$ is differentiable as a function from \mathbb{R} into \mathbb{R} and $-f$ is differentiable as a function from \mathbb{R} into \mathbb{R} and f^2 is differentiable as a function from \mathbb{R} into \mathbb{R} .

Now we state the propositions:

(14) $(f + g)' = f' + g'$. The theorem is a consequence of (9) and (10).

(15) $(f - g)' = f' - g'$. The theorem is a consequence of (9) and (10).

(16) $(f \cdot g)' = g \cdot f' + f \cdot g'$. The theorem is a consequence of (9) and (10).

(17) $(r + f)' = f'$. The theorem is a consequence of (11), (3), (14), and (6).

(18) $(f - r)' = f'$. The theorem is a consequence of (11), (4), (15), and (7).

(19) $(r \cdot f)' = r \cdot f'$. The theorem is a consequence of (9) and (10).

(20) $(-f)' = -f'$.

3. POLYNOMIALS

In the sequel L denotes a non empty zero structure and x denotes an element of L .

Now we state the proposition:

(21) Let us consider a (the carrier of L)-valued function f , and an object a .

Then $\text{Support}(f + \cdot (a, x)) \subseteq \text{Support } f \cup \{a\}$.

PROOF: $a = z$ or $z \in \text{Support } f$ by [2, (32), (30)]. \square

Let us consider L and x . Let f be a finite-Support sequence of L and a be an object. Observe that $f + \cdot (a, x)$ is finite-Support as a sequence of L .

Now we state the proposition:

(22) Let us consider a polynomial p over L . If $p \neq \mathbf{0}.L$, then $\text{len } p - ' 1 = \text{len } p - 1$.

Let L be a non empty zero structure and x be an element of L . Let us note that $\langle x \rangle$ is constant and $\langle x, 0_L \rangle$ is constant.

Now we state the proposition:

(23) Let us consider a non empty zero structure L , and a constant polynomial p over L . Then

(i) $p = \mathbf{0}.L$, or

(ii) $p = \langle p(0) \rangle$.

Let us consider L , x , and n . The functor $\text{seq}(n, x)$ yielding a sequence of L is defined by the term

(Def. 3) $\mathbf{0}.L + \cdot (n, x)$.

Observe that $\text{seq}(n, x)$ is finite-Support.

Now we state the propositions:

$$(24) \quad (\text{seq}(n, x))(n) = x.$$

$$(25) \quad \text{If } m \neq n, \text{ then } (\text{seq}(n, x))(m) = 0_L.$$

$$(26) \quad \text{the length of } \text{seq}(n, x) \text{ is at most } n + 1.$$

$$(27) \quad \text{If } x \neq 0_L, \text{ then } \text{len seq}(n, x) = n + 1.$$

PROOF: Set $p = \text{seq}(n, x)$. For every m such that the length of p is at most m holds $n + 1 \leq m$ by (24), [1, (13)]. \square

$$(28) \quad \text{seq}(n, 0_L) = \mathbf{0}.L. \text{ The theorem is a consequence of (24).}$$

$$(29) \quad \text{Let us consider a right zeroed, non empty additive loop structure } L, \text{ and elements } x, y \text{ of } L. \text{ Then } \text{seq}(n, x) + \text{seq}(n, y) = \text{seq}(n, x + y). \text{ The theorem is a consequence of (24) and (25).}$$

$$(30) \quad \text{Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure } L, \text{ and an element } x \text{ of } L. \text{ Then } -\text{seq}(n, x) = \text{seq}(n, -x). \text{ The theorem is a consequence of (24) and (25).}$$

$$(31) \quad \text{Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure } L, \text{ and elements } x, y \text{ of } L. \text{ Then } \text{seq}(n, x) - \text{seq}(n, y) = \text{seq}(n, x - y). \text{ The theorem is a consequence of (30) and (29).}$$

Let L be a non empty zero structure and p be a sequence of L . Let us consider n . The functor $p \upharpoonright n$ yielding a sequence of L is defined by the term

(Def. 4) $p + \cdot (n, 0_L)$.

Let p be a polynomial over L . Let us note that $p \upharpoonright n$ is finite-Support.

Let us consider a non empty zero structure L and a sequence p of L . Now we state the propositions:

$$(32) \quad (p \upharpoonright n)(n) = 0_L.$$

$$(33) \quad \text{If } m \neq n, \text{ then } (p \upharpoonright n)(m) = p(m).$$

Now we state the proposition:

$$(34) \quad \text{Let us consider a non empty zero structure } L. \text{ Then } \mathbf{0}.L \upharpoonright n = \mathbf{0}.L. \text{ The theorem is a consequence of (32).}$$

Let L be a non empty zero structure. Let us consider n . One can verify that $\mathbf{0}.L \upharpoonright n$ reduces to $\mathbf{0}.L$.

Let us consider a non empty zero structure L and a polynomial p over L . Now we state the propositions:

(35) If $n > \text{len } p - 1$, then $p \upharpoonright n = p$. The theorem is a consequence of (32).

(36) If $p \neq \mathbf{0}.L$, then $\text{len}(p \upharpoonright (\text{len } p - 1)) < \text{len } p$.

PROOF: Set $m = \text{len } p - 1$. $m = \text{len } p - 1$. the length of $p \upharpoonright m$ is at most $\text{len } p$ by [2, (32)], [7, (8)]. \square

Now we state the proposition:

(37) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L , and a polynomial p over L . Then $p \upharpoonright (\text{len } p - 1) + \text{Leading-Monomial } p = p$. The theorem is a consequence of (32).

Let L be a non empty zero structure and p be a constant polynomial over L . Observe that $\text{Leading-Monomial } p$ is constant.

Now we state the proposition:

(38) Let us consider an add-associative, right zeroed, right complementable, distributive, unital, non empty double loop structure L , and elements x, y of L . Then $\text{eval}(\text{seq}(n, x), y) = (\text{seq}(n, x))(n) \cdot \text{power}(y, n)$. The theorem is a consequence of (28), (27), and (25).

4. DIFFERENTIABILITY OF POLYNOMIALS OVER REALS

In the sequel p, q denote polynomials over \mathbb{R}_F .

Now we state the propositions:

(39) Let us consider an element r of \mathbb{R}_F . Then $\text{power}(r, n) = r^n$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{power}(r, \$1) = r^{\$1}$. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

(40) $\square^n = \text{FPower}(1_{\mathbb{R}_F}, n)$.

PROOF: Reconsider $f = \text{FPower}(1_{\mathbb{R}_F}, n)$ as a function from \mathbb{R} into \mathbb{R} . $\square^n = f$ by [8, (36)], (39). \square

Let us consider an element r of \mathbb{R}_F . Now we state the propositions:

(41) $\text{FPower}(r, n + 1) = \text{FPower}(r, n) \cdot \text{id}_{\mathbb{R}}$.

(42) $\text{FPower}(r, n)$ is a differentiable function from \mathbb{R} into \mathbb{R} .

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \text{FPower}(r, \$1)$ is a differentiable function from \mathbb{R} into \mathbb{R} . $\mathcal{P}[0]$ by [6, (66)]. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

(43) $\text{power}(r, n) = (\square^n)(r)$. The theorem is a consequence of (40).

Let us consider p . The functor p' yielding a sequence of \mathbb{R}_F is defined by

(Def. 5) for every natural number n , $it(n) = p(n + 1) \cdot (n + 1)$.

Note that p' is finite-Support.

Now we state the propositions:

(44) If $p \neq \mathbf{0}.\mathbb{R}_F$, then $\text{len } p' = \text{len } p - 1$.

PROOF: Set $x = \text{len } p - 1$. Set $d = p'$. the length of d is at most x by [7, (8)]. For every n such that the length of d is at most n holds $x \leq n$ by [11, (7)], [7, (10)], [1, (21)]. \square

(45) If $p \neq \mathbf{0}.\mathbb{R}_F$, then $\text{len } p = \text{len } p' + 1$. The theorem is a consequence of (44).

(46) Let us consider a constant polynomial p over \mathbb{R}_F . Then $p' = \mathbf{0}.\mathbb{R}_F$. The theorem is a consequence of (45).

(47) $(p + q)' = p' + q'$.

(48) $(-p)' = -p'$.

(49) $(p - q)' = p' - q'$. The theorem is a consequence of (47) and (48).

(50) Leading-Monomial $p' = \mathbf{0}.\mathbb{R}_F + (\text{len } p - 1) \cdot p(\text{len } p - 1)$.

PROOF: Set $l = \text{Leading-Monomial } p$. Set $m = \text{len } p - 1$. Set $k = \text{len } p - 2$. Reconsider $a = p(m) \cdot m$ as an element of \mathbb{R}_F . Set $f = \mathbf{0}.F + (k, a)$. $l' = f$ by [1, (53)], [2, (31), (32)], [10, (7)]. \square

(51) Let us consider elements r, s of \mathbb{R}_F . Then $\langle r, s \rangle' = \langle s \rangle$.

Let us consider p . The functor $\text{Eval}(p)$ yielding a function from \mathbb{R} into \mathbb{R} is defined by the term

(Def. 6) Polynomial-Function(\mathbb{R}_F, p).

Let us note that $\text{Eval}(p)$ is differentiable.

Now we state the propositions:

(52) $\text{Eval}(\mathbf{0}.\mathbb{R}_F) = \mathbb{R} \mapsto 0$.

PROOF: $\text{Eval}(\mathbf{0}.F) = \mathbb{R} \mapsto 0 (\in \mathbb{R})$ by [5, (17)], [10, (7)]. \square

(53) Let us consider an element r of \mathbb{R}_F . Then $\text{Eval}(\langle r \rangle) = \mathbb{R} \mapsto r$.

PROOF: $\text{Eval}(\langle r \rangle) = \mathbb{R} \mapsto r (\in \mathbb{R})$ by [6, (37)], [10, (7)]. \square

(54) If p is constant, then $\text{Eval}(p)' = \mathbb{R} \mapsto 0$. The theorem is a consequence of (23), (52), and (11).

(55) $\text{Eval}(p + q) = \text{Eval}(p) + \text{Eval}(q)$.

(56) $\text{Eval}(-p) = -\text{Eval}(p)$.

(57) $\text{Eval}(p - q) = \text{Eval}(p) - \text{Eval}(q)$. The theorem is a consequence of (55) and (56).

(58) $\text{Eval}(\text{Leading-Monomial } p) = \text{FPower}(p(\text{len } p - 1), \text{len } p - 1)$.

PROOF: Set $l = \text{Leading-Monomial } p$. Set $m = \text{len } p - 1$. Reconsider $f = \text{FPower}(p(m), m)$ as a function from \mathbb{R} into \mathbb{R} . $\text{Eval}(l) = f$ by [5, (22)]. \square

(59) $\text{Eval}(\text{Leading-Monomial } p) = p(\text{len } p - 1) \cdot (\square^{\text{len } p - 1})$.

PROOF: Set $l = \text{Leading-Monomial } p$. Set $m = \text{len } p - 1$. Set $f = p(m) \cdot (\square^m)$. $\text{Eval}(l) = f$ by (39), [8, (36)], [5, (22)]. \square

(60) Let us consider an element r of \mathbb{R}_F . Then $\text{Eval}(\text{seq}(n, r)) = r \cdot (\square^n)$. The theorem is a consequence of (24), (43), and (38).

(61) $\text{Eval}(p)' = \text{Eval}(p')$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every p such that $\text{len } p \leq \$_1$ holds $\text{Eval}(p)' = \text{Eval}(p')$. $\mathcal{P}[0]$ by [5, (5)], (46), (52), (54). If $\mathcal{P}[n]$, then $\mathcal{P}[n+1]$ by (36), [5, (3)], [1, (13)], (37). $\mathcal{P}[n]$ from [1, Sch. 2]. \square

Let us consider p . Let us observe that $\text{Eval}(p)'$ is differentiable.

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