FORMALIZED MATHEMATICS
Vol. 18, No. 3, Pages 159–169, 2010
DOI: 10.2478/v10037-010-0018-6

On L^p Space Formed by Real-Valued Partial Functions

Yasushige Watase
Graduate School of Science and Technology
Shinshu University
Nagano, Japan

Noboru Endou
Gifu National College of Technology
Japan

Yasunari Shidama
Shinshu University
Nagano, Japan

Summary. This article is the continuation of [31]. We define the set of L^p integrable functions – the set of all partial functions whose absolute value raised to the p-th power is integrable. We show that L^p integrable functions form the L^p space. We also prove Minkowski's inequality, Hölder's inequality and that L^p space is Banach space ([15], [27]).

MML identifier: `LPSPACE2`, version: `7.11.07 4.156.1112`

The notation and terminology used in this paper have been introduced in the following papers: [7], [8], [9], [10], [4], [1], [31], [6], [19], [20], [13], [28], [14], [2], [24], [3], [11], [25], [22], [21], [16], [32], [29], [23], [18], [17], [26], [30], [5], and [12].

1. Preliminaries on Powers of Numbers and Operations on Real Sequences

For simplicity, we follow the rules: X denotes a non empty set, x denotes an element of X, S denotes a σ-field of subsets of X, M denotes a σ-measure on S, f, g, f_1, g_1 denote partial functions from X to $\mathbb{R}$, and a, b, c denote real numbers.

The following propositions are true:

(1) For all positive real numbers m, n such that $\frac{1}{m} + \frac{1}{n} = 1$ holds $m > 1$.

ISSN 1426–2630(p), 1898-9934(e)

(2) Let X be a non empty set, S be a σ-field of subsets of X, M be a σ-measure on S, A be an element of S, and f be a partial function from X to $\overline{\mathbb{R}}$. Suppose $A = \operatorname{dom} f$ and f is measurable on A and f is non-negative. Then $\int f \, \mathrm{d}M \in \mathbb{R}$ if and only if f is integrable on M.

Let r be a real number. We say that r is great or equal to 1 if and only if:

(Def. 1) $1 \leq r$.

Let us note that every real number which is great or equal to 1 is also positive.

One can verify that there exists a real number which is great or equal to 1.

In the sequel k denotes a positive real number.

We now state several propositions:

(3) For all real numbers a, b, p such that $0 < p$ and $0 \leq a < b$ holds $a^p < b^p$.

(4) If $a \geq 0$ and $b > 0$, then $a^b \geq 0$.

(5) If $a \geq 0$ and $b \geq 0$ and $c > 0$, then $(a \cdot b)^c = a^c \cdot b^c$.

(6) For all real numbers a, b and for every f such that f is non-negative and $a > 0$ and $b > 0$ holds $(f^a)^b = f^{a \cdot b}$.

(7) For all real numbers a, b and for every f such that f is non-negative and $a > 0$ and $b > 0$ holds $f^a\, f^b = f^{a+b}$.

(8) $f^1 = f$.

(9) Let s_1, s_2 be sequences of real numbers and k be a positive real number. Suppose that for every element n of $\mathbb{N}$ holds $s_1(n) = s_2(n)^k$ and $s_2(n) \geq 0$. Then s_1 is convergent if and only if s_2 is convergent.

(10) Let s_3 be a sequence of real numbers and n, m be elements of $\mathbb{N}$. If $m \leq n$, then $|(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(m)| \leq (\sum_{\alpha=0}^{\kappa}|s_3|(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}|s_3|(\alpha))_{\kappa\in\mathbb{N}}(m)$ and $|(\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(n) - (\sum_{\alpha=0}^{\kappa}(s_3)(\alpha))_{\kappa\in\mathbb{N}}(m)| \leq (\sum_{\alpha=0}^{\kappa}|s_3|(\alpha))_{\kappa\in\mathbb{N}}(n)$.

(11) Let s_3, s_2 be sequences of real numbers and k be a positive real number. Suppose s_3 is convergent and for every element n of $\mathbb{N}$ holds $s_2(n) = |\lim s_3 - s_3(n)|^k$. Then s_2 is convergent and $\lim s_2 = 0$.

2. Real Linear Space of L^p Integrable Functions

Next we state two propositions:

(12) For every positive real number k and for every non empty set X holds $(X \longmapsto 0)^k = X \longmapsto 0$.

(13) For every partial function f from X to $\mathbb{R}$ and for every set D holds $|f{\restriction}D| = |f|{\restriction}D$.

Let us consider X and let f be a partial function from X to $\mathbb{R}$. Observe that $|f|$ is non-negative.

One can prove the following two propositions:

(14) For every partial function f from X to $\mathbb{R}$ such that f is non-negative holds $|f| = f$.

(15) If $X = \mathrm{dom}\, f$ and for every x such that $x \in \mathrm{dom}\, f$ holds $0 = f(x)$, then f is integrable on M and $\int f \,\mathrm{d}M = 0$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor L^p functions(M, k) yielding a non empty subset of $\mathrm{PFunct}_{\mathrm{RLS}}\, X$ is defined by the condition (Def. 2).

(Def. 2) L^p functions$(M, k) = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $\bigvee_{E_1 : \text{element of } S} (M({E_1}^{\mathrm{c}}) = 0 \ \wedge \ \mathrm{dom}\, f = E_1 \ \wedge \ f$ is measurable on $E_1 \ \wedge \ |f|^k$ is integrable on $M)\}$.

Next we state a number of propositions:

(16) For all real numbers a, b, k such that $k > 0$ holds $|a + b|^k \leq (|a| + |b|)^k$ and $(|a| + |b|)^k \leq (2 \cdot \max(|a|, |b|))^k$ and $|a + b|^k \leq (2 \cdot \max(|a|, |b|))^k$.

(17) For all real numbers a, b, k such that $a \geq 0$ and $b \geq 0$ and $k > 0$ holds $(\max(a, b))^k \leq a^k + b^k$.

(18) For every partial function f from X to $\mathbb{R}$ and for all real numbers a, b such that $b > 0$ holds $|a|^b \, |f|^b = |a\, f|^b$.

(19) Let f be a partial function from X to $\mathbb{R}$ and a, b be real numbers. If $a > 0$ and $b > 0$, then $a^b \, |f|^b = (a \, |f|)^b$.

(20) For every partial function f from X to $\mathbb{R}$ and for every real number k and for every set E holds $(f \restriction E)^k = f^k \restriction E$.

(21) For all real numbers a, b, k such that $k > 0$ holds $|a+b|^k \leq 2^k \cdot (|a|^k + |b|^k)$.

(22) Let k be a positive real number and f, g be partial functions from X to $\mathbb{R}$. Suppose f, $g \in L^p$ functions(M, k). Then $|f|^k$ is integrable on M and $|g|^k$ is integrable on M and $|f|^k + |g|^k$ is integrable on M.

(23) $X \longmapsto 0$ is a partial function from X to $\mathbb{R}$ and $X \longmapsto 0 \in L^p$ functions(M, k).

(24) Let k be a real number. Suppose $k > 0$. Let f, g be partial functions from X to $\mathbb{R}$ and x be an element of X. If $x \in \mathrm{dom}\, f \cap \mathrm{dom}\, g$, then $|f + g|^k(x) \leq (2^k \, (|f|^k + |g|^k))(x)$.

(25) If f, $g \in L^p$ functions(M, k), then $f + g \in L^p$ functions(M, k).

(26) If $f \in L^p$ functions(M, k), then $a\, f \in L^p$ functions(M, k).

(27) If f, $g \in L^p$ functions(M, k), then $f - g \in L^p$ functions(M, k).

(28) If $f \in L^p$ functions(M, k), then $|f| \in L^p$ functions(M, k).

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Note that L^p functions(M, k) is multiplicatively-closed and add closed.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. One can check that $\langle L^p\,\mathrm{functions}(M,k), 0_{\mathrm{PFunct}_{\mathrm{RLS}}\,X}(\in L^p\,\mathrm{functions}(M,k)), \mathrm{add}\,|(L^p\,\mathrm{functions}(M,k), \mathrm{PFunct}_{\mathrm{RLS}}\,X), \cdot_{L^p\,\mathrm{functions}(M,k)}\rangle$ is Abelian, add-associative, and real linear space-like.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor RLSp LpFunct(M,k) yields a strict Abelian add-associative real linear space-like non empty RLS structure and is defined by:

(Def. 3) RLSp LpFunct$(M,k) = \langle L^p\,\mathrm{functions}(M,k), 0_{\mathrm{PFunct}_{\mathrm{RLS}}\,X}(\in L^p\,\mathrm{functions}\,(M,k)), \mathrm{add}\,|(L^p\,\mathrm{functions}(M,k), \mathrm{PFunct}_{\mathrm{RLS}}\,X), \cdot_{L^p\,\mathrm{functions}(M,k)}\rangle$.

3. Preliminaries on Real Normed Space of L^p Integrable Functions

In the sequel v, u are vectors of RLSp LpFunct(M,k).

We now state three propositions:

(29) $(v) + (u) = v + u$.

(30) $a\,(u) = a \cdot u$.

(31) Suppose $f = u$. Then

(i) $u + (-1) \cdot u = (X \longmapsto 0)\restriction \mathrm{dom}\, f$, and

(ii) there exist partial functions v, g from X to $\mathbb{R}$ such that $v, g \in L^p\,\mathrm{functions}(M,k)$ and $v = u + (-1) \cdot u$ and $g = X \longmapsto 0$ and $v =^{M}_{\mathrm{a.e.}} g$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor AlmostZeroLpFunctions(M,k) yielding a non empty subset of RLSp LpFunct(M,k) is defined by:

(Def. 4) AlmostZeroLpFunctions$(M,k) = \{f; f$ ranges over partial functions from X to $\mathbb{R}$: $f \in L^p\,\mathrm{functions}(M,k) \ \wedge\ f =^{M}_{\mathrm{a.e.}} X \longmapsto 0\}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. One can check that AlmostZeroLpFunctions(M,k) is add closed and multiplicatively-closed.

Next we state the proposition

(32) $0_{\mathrm{RLSp\ LpFunct}(M,k)} = X \longmapsto 0$ and $0_{\mathrm{RLSp\ LpFunct}(M,k)} \in$ AlmostZeroLpFunctions(M,k).

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor RLSpAlmostZeroLpFunctions(M,k) yielding a non empty RLS structure is defined by:

(Def. 5) RLSpAlmostZeroLpFunctions$(M,k) = \langle$AlmostZeroLpFunctions(M,k), $0_{\mathrm{RLSp\ LpFunct}(M,k)}(\in$ AlmostZeroLpFunctions$(M,k))$, add $|($AlmostZeroLp

Functions(M, k), RLSp LpFunct(M, k)), $\cdot_{\mathrm{AlmostZeroLpFunctions}(M,k)}\rangle$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Observe that RLSp LpFunct(M, k) is strict, Abelian, add-associative, right zeroed, and real linear space-like.

In the sequel v, u are vectors of RLSpAlmostZeroLpFunctions(M, k).

One can prove the following two propositions:

(33) $(v) + (u) = v + u$.

(34) $a\,(u) = a \cdot u$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, let f be a partial function from X to $\mathbb{R}$, and let k be a positive real number. The functor a.e-eq-class $L^p(f, M, k)$ yields a subset of L^p functions(M, k) and is defined as follows:

(Def. 6) a.e-eq-class $L^p(f, M, k) = \{h; h$ ranges over partial functions from X to $\mathbb{R}$: $h \in L^p$ functions$(M, k) \wedge f =^{M}_{\mathrm{a.e.}} h\}$.

Next we state a number of propositions:

(35) If $f \in L^p$ functions(M, k), then there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and $\mathrm{dom}\, f = E$ and f is measurable on E.

(36) If $g \in L^p$ functions(M, k) and $g =^{M}_{\mathrm{a.e.}} f$, then $g \in$ a.e-eq-class $L^p(f, M, k)$.

(37) Suppose there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and $E = \mathrm{dom}\, f$ and f is measurable on E and $g \in$ a.e-eq-class $L^p(f, M, k)$. Then $g =^{M}_{\mathrm{a.e.}} f$ and $f \in L^p$ functions(M, k).

(38) If $f \in L^p$ functions(M, k), then $f \in$ a.e-eq-class $L^p(f, M, k)$.

(39) Suppose there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and $E = \mathrm{dom}\, g$ and g is measurable on E and a.e-eq-class $L^p(f, M, k) \neq \emptyset$ and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$. Then $f =^{M}_{\mathrm{a.e.}} g$.

(40) Suppose $f \in L^p$ functions(M, k) and there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and $E = \mathrm{dom}\, g$ and g is measurable on E and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$. Then $f =^{M}_{\mathrm{a.e.}} g$.

(41) If $f =^{M}_{\mathrm{a.e.}} g$, then a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$.

(42) If $f =^{M}_{\mathrm{a.e.}} g$, then a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$.

(43) If $f \in L^p$ functions(M, k) and $g \in$ a.e-eq-class $L^p(f, M, k)$, then a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$.

(44) Suppose that there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and $E = \mathrm{dom}\, f$ and f is measurable on E and there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and $E = \mathrm{dom}\, f_1$ and f_1 is measurable on E and there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and $E = \mathrm{dom}\, g$ and g is measurable on E and there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and $E = \mathrm{dom}\, g_1$ and g_1 is measurable on

E and a.e-eq-class $L^p(f, M, k)$ is non empty and a.e-eq-class $L^p(g, M, k)$ is non empty and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(f_1, M, k)$ and a.e-eq-class $L^p(g, M, k) =$ a.e-eq-class $L^p(g_1, M, k)$. Then a.e-eq-class $L^p(f+g, M, k) =$ a.e-eq-class $L^p(f_1 + g_1, M, k)$.

(45) If f, f_1, g, $g_1 \in L^p$ functions(M, k) and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(f_1, M, k)$ and a.e-eq-class $L^p(g, M, k) =$ a.e-eq-class $L^p(g_1, M, k)$, then a.e-eq-class $L^p(f + g, M, k) =$ a.e-eq-class $L^p(f_1 + g_1, M, k)$.

(46) Suppose that

(i) there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and dom $f = E$ and f is measurable on E,

(ii) there exists an element E of S such that $M(E^{\mathrm{c}}) = 0$ and dom $g = E$ and g is measurable on E,

(iii) a.e-eq-class $L^p(f, M, k)$ is non empty, and

(iv) a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$.

Then a.e-eq-class $L^p(a\,f, M, k) =$ a.e-eq-class $L^p(a\,g, M, k)$.

(47) If f, $g \in L^p$ functions(M, k) and a.e-eq-class $L^p(f, M, k) =$ a.e-eq-class $L^p(g, M, k)$, then a.e-eq-class $L^p(a\,f, M, k) =$ a.e-eq-class $L^p(a\,g, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor CosetSet(M, k) yielding a non empty family of subsets of L^p functions(M, k) is defined by:

(Def. 7) CosetSet$(M, k) = \{$a.e-eq-class $L^p(f, M, k)$; f ranges over partial functions from X to $\mathbb{R}$: $f \in L^p$ functions$(M, k)\}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor addCoset(M, k) yields a binary operation on CosetSet(M, k) and is defined by the condition (Def. 8).

(Def. 8) Let A, B be elements of CosetSet(M, k) and a, b be partial functions from X to $\mathbb{R}$. If $a \in A$ and $b \in B$, then (addCoset(M, k))$(A, B) =$ a.e-eq-class $L^p(a + b, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor zeroCoset(M, k) yields an element of CosetSet(M, k) and is defined as follows:

(Def. 9) zeroCoset$(M, k) =$ a.e-eq-class $L^p(X \longmapsto 0, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor lmultCoset(M, k) yielding a function from $\mathbb{R} \times$ CosetSet(M, k) into CosetSet(M, k) is defined by the condition (Def. 10).

(Def. 10) Let z be an element of $\mathbb{R}$, A be an element of CosetSet(M, k), and f be a partial function from X to $\mathbb{R}$. If $f \in A$, then (lmultCoset$(M, k))(z, A) =$ a.e-eq-class $L^p(z\, f, M, k)$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor Pre- L^p -Space(M, k) yielding a strict RLS structure is defined by the conditions (Def. 11).

(Def. 11)(i) The carrier of Pre- L^p -Space$(M, k) =$ CosetSet(M, k),

(ii) the addition of Pre- L^p -Space$(M, k) =$ addCoset(M, k),

(iii) $0_{\text{Pre-}\, L^p\, \text{-Space}(M,k)} =$ zeroCoset(M, k), and

(iv) the external multiplication of Pre- L^p -Space$(M, k) =$ lmultCoset(M, k).

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Observe that Pre- L^p -Space(M, k) is non empty.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. Observe that Pre- L^p -Space(M, k) is Abelian, add-associative, right zeroed, right complementable, and real linear space-like.

4. Real Normed Space of L^p Integrable Functions

The following propositions are true:

(48) If $f, g \in L^p$ functions(M, k) and $f =^{M}_{\text{a.e.}} g$, then $\int |f|^k \, \mathrm{d}M = \int |g|^k \, \mathrm{d}M$.

(49) If $f \in L^p$ functions(M, k), then $\int |f|^k \, \mathrm{d}M \in \mathbb{R}$ and $0 \leq \int |f|^k \, \mathrm{d}M$.

(50) If there exists a vector x of Pre- L^p -Space(M, k) such that $f, g \in x$, then $f =^{M}_{\text{a.e.}} g$ and $f, g \in L^p$ functions(M, k).

(51) Let k be a positive real number. Then there exists a function N_1 from the carrier of Pre- L^p -Space(M, k) into $\mathbb{R}$ such that for every point x of Pre- L^p -Space(M, k) holds there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and there exists a real number r such that $r = \int |f|^k \, \mathrm{d}M$ and $N_1(x) = r^{\frac{1}{k}}$.

In the sequel x denotes a point of Pre- L^p -Space(M, k).

We now state two propositions:

(52) If $f \in x$, then $|f|^k$ is integrable on M and $f \in L^p$ functions(M, k).

(53) If $f, g \in x$, then $f =^{M}_{\text{a.e.}} g$ and $\int |f|^k \, \mathrm{d}M = \int |g|^k \, \mathrm{d}M$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor L^p -Norm(M, k) yielding a function from the carrier of Pre- L^p -Space(M, k) into $\mathbb{R}$ is defined by the condition (Def. 12).

(Def. 12) Let x be a point of Pre- L^p -Space(M, k). Then there exists a partial function f from X to $\mathbb{R}$ such that $f \in x$ and there exists a real number r such that $r = \int |f|^k \, \mathrm{d}M$ and $(L^p\text{-Norm}(M, k))(x) = r^{\frac{1}{k}}$.

Let X be a non empty set, let S be a σ-field of subsets of X, let M be a σ-measure on S, and let k be a positive real number. The functor L^p -Space(M, k) yields a non empty normed structure and is defined by:

(Def. 13) L^p -Space(M, k) $=$ $\langle$the carrier of Pre- L^p -Space(M, k), the zero of Pre- L^p -Space(M, k), the addition of Pre- L^p -Space(M, k), the external multiplication of Pre- L^p -Space(M, k), L^p -Norm$(M, k)\rangle$.

In the sequel x, y denote points of L^p -Space(M, k).

One can prove the following propositions:

(54)(i) There exists a partial function f from X to $\mathbb{R}$ such that $f \in L^p$ functions(M, k) and $x =$ a.e-eq-class $L^p(f, M, k)$, and

(ii) for every partial function f from X to $\mathbb{R}$ such that $f \in x$ there exists a real number r such that $0 \leq r = \int |f|^k \, \mathrm{d}M$ and $\|x\| = r^{\frac{1}{k}}$.

(55) If $f \in x$ and $g \in y$, then $f + g \in x + y$ and if $f \in x$, then $a\, f \in a \cdot x$.

(56) If $f \in x$, then $x =$ a.e-eq-class $L^p(f, M, k)$ and there exists a real number r such that $0 \leq r = \int |f|^k \, \mathrm{d}M$ and $\|x\| = r^{\frac{1}{k}}$.

(57) $X \longmapsto 0 \in$ the L^1 functions of M.

(58) If $f \in L^p$ functions(M, k) and $\int |f|^k \, \mathrm{d}M = 0$, then $f =^{M}_{\text{a.e.}} X \longmapsto 0$.

(59) $\int |X \longmapsto 0|^k \, \mathrm{d}M = 0$.

(60) Let m, n be positive real numbers. Suppose $\frac{1}{m} + \frac{1}{n} = 1$ and $f \in L^p$ functions(M, m) and $g \in L^p$ functions(M, n). Then $f\, g \in$ the L^1 functions of M and $f\, g$ is integrable on M.

(61) Let m, n be positive real numbers. Suppose $\frac{1}{m} + \frac{1}{n} = 1$ and $f \in L^p$ functions(M, m) and $g \in L^p$ functions(M, n). Then there exists a real number r_1 such that $r_1 = \int |f|^m \, \mathrm{d}M$ and there exists a real number r_2 such that $r_2 = \int |g|^n \, \mathrm{d}M$ and $\int |f\, g| \, \mathrm{d}M \leq {r_1}^{\frac{1}{m}} \cdot {r_2}^{\frac{1}{n}}$.

(62) Let m be a positive real number and r_1, r_2, r_3 be elements of $\mathbb{R}$. Suppose $1 \leq m$ and f, $g \in L^p$ functions(M, m) and $r_1 = \int |f|^m \, \mathrm{d}M$ and $r_2 = \int |g|^m \, \mathrm{d}M$ and $r_3 = \int |f + g|^m \, \mathrm{d}M$. Then ${r_3}^{\frac{1}{m}} \leq {r_1}^{\frac{1}{m}} + {r_2}^{\frac{1}{m}}$.

Let k be a great or equal to 1 real number, let X be a non empty set, let S be a σ-field of subsets of X, and let M be a σ-measure on S. Note that L^p -Space(M, k) is reflexive, discernible, real normed space-like, real linear space-like, Abelian, add-associative, right zeroed, and right complementable.

5. Preliminaries on Completeness of L^p Space

The following propositions are true:

(63) Let S_1 be a sequence of L^p-Space(M, k). Then there exists a sequence F_1 of partial functions from X into $\mathbb{R}$ such that for every element n of $\mathbb{N}$ holds
$F_1(n) \in L^p$ functions(M, k) and $F_1(n) \in S_1(n)$ and $S_1(n) =$ a.e-eq-class $L^p(F_1(n), M, k)$ and there exists a real number r such that $r = \int |F_1(n)|^k \,\mathrm{d}M$ and $\|S_1(n)\| = r^{\frac{1}{k}}$.

(64) Let S_1 be a sequence of L^p-Space(M, k). Then there exists a sequence F_1 of partial functions from X into $\mathbb{R}$ with the same dom such that for every element n of $\mathbb{N}$ holds
$F_1(n) \in L^p$ functions(M, k) and $F_1(n) \in S_1(n)$ and $S_1(n) =$ a.e-eq-class $L^p(F_1(n), M, k)$ and there exists a real number r such that $0 \leq r = \int |F_1(n)|^k \,\mathrm{d}M$ and $\|S_1(n)\| = r^{\frac{1}{k}}$.

(65) Let X be a real normed space, S_1 be a sequence of X, and S_0 be a point of X. If $\|S_1 - S_0\|$ is convergent and $\lim \|S_1 - S_0\| = 0$, then S_1 is convergent and $\lim S_1 = S_0$.

(66) Let X be a real normed space and S_1 be a sequence of X. Suppose S_1 is Cauchy sequence by norm. Then there exists an increasing function N from $\mathbb{N}$ into $\mathbb{N}$ such that for all elements i, j of $\mathbb{N}$ if $j \geq N(i)$, then $\|S_1(j) - S_1(N(i))\| < 2^{-i}$.

(67) Let F be a sequence of partial functions from X into $\mathbb{R}$. Suppose that for every natural number m holds $F(m) \in L^p$ functions(M, k). Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m) \in L^p$ functions(M, k).

(68) Let F be a sequence of partial functions from X into $\mathbb{R}$. Suppose that for every natural number m holds $F(m)$ is non-negative. Let m be a natural number. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)$ is non-negative.

(69) Let F be a sequence of partial functions from X into $\mathbb{R}$, x be an element of X, and n, m be natural numbers. Suppose F has the same dom and $x \in \operatorname{dom} F(0)$ and for every natural number k holds $F(k)$ is non-negative and $n \leq m$. Then $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(n)(x) \leq (\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}}(m)(x)$.

(70) For every sequence F of partial functions from X into $\mathbb{R}$ such that F has the same dom holds $|F|$ has the same dom.

(71) Let k be a great or equal to 1 real number and S_1 be a sequence of L^p-Space(M, k). If S_1 is Cauchy sequence by norm, then S_1 is convergent.

Let us consider X, S, M and let k be a great or equal to 1 real number. Observe that L^p-Space(M, k) is complete.

6. Relations between L^1 Space and L^p Space

One can prove the following propositions:

(72) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{CosetSet}\,M = \mathrm{CosetSet}(M, 1)$.

(73) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{addCoset}\,M = \mathrm{addCoset}(M, 1)$.

(74) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{zeroCoset}\,M = \mathrm{zeroCoset}(M, 1)$.

(75) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then $\mathrm{lmultCoset}\,M = \mathrm{lmultCoset}(M, 1)$.

(76) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then pre-L-Space M = Pre- L^p -Space$(M, 1)$.

(77) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then L^1-Norm(M) = L^p -Norm$(M, 1)$.

(78) Let X be a non empty set, S be a σ-field of subsets of X, and M be a σ-measure on S. Then L^1-Space(M) = L^p -Space$(M, 1)$.

References

[1] Jonathan Backer, Piotr Rudnicki, and Christoph Schwarzweller. Ring ideals. *Formalized Mathematics*, 9(**3**):565–582, 2001.
[2] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.
[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(**1**):91–96, 1990.
[4] Grzegorz Bancerek and Andrzej Trybulec. Miscellaneous facts about functions. *Formalized Mathematics*, 5(**4**):485–492, 1996.
[5] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(**1**):173–183, 1991.
[6] Józef Białas. The σ-additive measure theory. *Formalized Mathematics*, 2(**2**):263–270, 1991.
[7] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.
[8] Czesław Byliński. The complex numbers. *Formalized Mathematics*, 1(**3**):507–513, 1990.
[9] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.
[10] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.
[11] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(**2**):357–367, 1990.
[12] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(**1**):47–53, 1990.
[13] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(**2**):53–70, 2006, doi:10.2478/v10037-006-0008-x.
[14] Noboru Endou, Yasunari Shidama, and Keiko Narita. Egoroff's theorem. *Formalized Mathematics*, 16(**1**):57–63, 2008, doi:10.2478/v10037-008-0009-z.
[15] P. R. Halmos. *Measure Theory*. Springer-Verlag, 1987.
[16] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.
[17] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(**2**):273–275, 1990.
[18] Jarosław Kotowicz. Real sequences and basic operations on them. *Formalized Mathematics*, 1(**2**):269–272, 1990.

[19] Keiko Narita, Noboru Endou, and Yasunari Shidama. Integral of complex-valued measurable function. *Formalized Mathematics*, 16(**4**):319–324, 2008, doi:10.2478/v10037-008-0039-6.
[20] Keiko Narita, Noboru Endou, and Yasunari Shidama. Lebesgue's convergence theorem of complex-valued function. *Formalized Mathematics*, 17(**2**):137–145, 2009, doi: 10.2478/v10037-009-0015-9.
[21] Andrzej Nędzusiak. σ-fields and probability. *Formalized Mathematics*, 1(**2**):401–407, 1990.
[22] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(**1**):223–230, 1990.
[23] Beata Perkowska. Functional sequence from a domain to a domain. *Formalized Mathematics*, 3(**1**):17–21, 1992.
[24] Jan Popiołek. Real normed space. *Formalized Mathematics*, 2(**1**):111–115, 1991.
[25] Konrad Raczkowski and Andrzej Nędzusiak. Real exponents and logarithms. *Formalized Mathematics*, 2(**2**):213–216, 1991.
[26] Konrad Raczkowski and Andrzej Nędzusiak. Series. *Formalized Mathematics*, 2(**4**):449–452, 1991.
[27] Walter Rudin. *Real and Complex Analysis*. Mc Graw-Hill, Inc., 1974.
[28] Yasunari Shidama and Noboru Endou. Integral of real-valued measurable function. *Formalized Mathematics*, 14(**4**):143–152, 2006, doi:10.2478/v10037-006-0018-8.
[29] Wojciech A. Trybulec. Vectors in real linear space. *Formalized Mathematics*, 1(**2**):291–296, 1990.
[30] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(**1**):67–71, 1990.
[31] Yasushige Watase, Noboru Endou, and Yasunari Shidama. On L^1 space formed by real-valued partial functions. *Formalized Mathematics*, 16(**4**):361–369, 2008, doi:10.2478/v10037-008-0044-9.
[32] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(**1**):181–186, 1990.

Received February 4, 2010