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Constructing Binary Huffman Tree[1]

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Summary. Huffman coding is one of a most famous entropy encoding methods for lossless data compression [16]. JPEG and ZIP formats employ variants of Huffman encoding as lossless compression algorithms. Huffman coding is a bijective map from source letters into leaves of the Huffman tree constructed by the algorithm. In this article we formalize an algorithm constructing a binary code tree, Huffman tree.

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The notation and terminology used in this paper have been introduced in the following articles: [9], [1], [20], [8], [14], [11], [12], [23], [22], [2], [3], [18], [19], [17], [25], [26], [24], [4], [5], [6], [7], and [13].

1. Constructing Binary Decoded Trees

Let D be a non empty set and x be an element of D. Observe that the root tree of x is binary as a decorated tree.

The functor $\mathbb{R}_{\mathbb{N}}$ yielding a set is defined by the term

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(Def. 1) $\mathbb{N} \times \mathbb{R}$.

Note that $\mathbb{R}_{\mathbb{N}}$ is non empty.

Let D be a non empty set. The binary finite trees of D yielding a set of trees decorated with elements of D is defined by

(Def. 2) Let us consider a tree T decorated with elements of D. Then $\mathrm{dom}\, T$ is finite and T is binary if and only if $T \in it$.

The Boolean binary finite trees of D yielding a non empty subset of $2^{\text{the binary finite trees of } D}$ is defined by the term

(Def. 3) $\{x$, where x is an element of $2^{\alpha} : x$ is finite and $x \neq \emptyset\}$, where α is the binary finite trees of D.

In this paper $\mathbb{S}$ denotes a non empty finite set, p denotes a probability on the trivial σ-field of $\mathbb{S}$, T_1 denotes a finite sequence of elements of the Boolean binary finite trees of $\mathbb{R}_{\mathbb{N}}$, and q denotes a finite sequence of elements of $\mathbb{N}$.

Let us consider $\mathbb{S}$ and p. The functor $\mathrm{InitTrees}\, p$ yielding a non empty finite subset of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ is defined by the term

(Def. 4) $\{T$, where T is an element of $\mathrm{FinTrees}(\mathbb{R}_{\mathbb{N}}) : T$ is a finite binary tree decorated with elements of $\mathbb{R}_{\mathbb{N}}$ and there exists an element x of $\mathbb{S}$ such that $T =$ the root tree of $\langle (\mathrm{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle\}$.

Let p be a tree decorated with elements of $\mathbb{R}_{\mathbb{N}}$. The value of root from right of p yielding a real number is defined by the term

(Def. 5) $p(\emptyset)_{\mathbf{2}}$.

The value of root from left of p yielding a natural number is defined by the term

(Def. 6) $p(\emptyset)_{\mathbf{1}}$.

Let T be a finite binary tree decorated with elements of $\mathbb{R}_{\mathbb{N}}$ and p be an element of $\mathrm{dom}\, T$. The value of tree of p yielding a real number is defined by the term

(Def. 7) $T(p)_{\mathbf{2}}$.

Let p, q be finite binary trees decorated with elements of $\mathbb{R}_{\mathbb{N}}$ and k be a natural number. The functor $\mathrm{MakeTree}(p, q, k)$ yielding a finite binary tree decorated with elements of $\mathbb{R}_{\mathbb{N}}$ is defined by the term

(Def. 8) $\langle k$, (the value of root from right of p) $+$ (the value of root from right of q)$\rangle$-tree(p, q).

Let X be a non empty finite subset of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$. The maximal value of X yielding a natural number is defined by

(Def. 9) There exists a non empty finite subset L of $\mathbb{N}$ such that

(i) $L = \{$the value of root from left of p, where p is an element of the binary finite trees of $\mathbb{R}_{\mathbb{N}} : p \in X\}$, and

(ii) $it = \max L$.

Now we state the propositions:

(1) Let us consider a non empty finite subset X of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and a finite binary tree w decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Suppose $X = \{w\}$. Then the maximal value of $X =$ the value of root from left of w. PROOF: Consider L being a non empty finite subset of $\mathbb{N}$ such that $L = \{$the value of root from left of p, where p is an element of the binary finite trees of $\mathbb{R}_{\mathbb{N}} : p \in X\}$ and the maximal value of $X = \max L$. For every element n such that $n \in L$ holds $n =$ the value of root from left of w. For every element n such that $n =$ the value of root from left of w holds $n \in L$. □

(2) Let us consider non empty finite subsets X, Y, Z of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$. Suppose $Z = X \cup Y$. Then the maximal value of $Z = \max($the maximal value of X, the maximal value of $Y)$.

(3) Let us consider non empty finite subsets X, Z of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and a set Y. Suppose $Z = X \setminus Y$. Then the maximal value of $Z \leqslant$ the maximal value of X. The theorem is a consequence of (2).

(4) Let us consider a non empty finite subset X of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and an element p of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$. Suppose $p \in X$. Then the value of root from left of $p \leqslant$ the maximal value of X.

Let X be a non empty finite subset of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$. A minimal value tree of X is a finite binary tree decorated with elements of $\mathbb{R}_{\mathbb{N}}$ and is defined by

(Def. 10) (i) *it* $\in X$, and

(ii) for every finite binary tree q decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that $q \in X$ holds the value of root from right of *it* $\leqslant$ the value of root from right of q.

Now we state the propositions:

(5) $\overline{\overline{\text{InitTrees}\,p}} = \overline{\overline{\mathbb{S}}}$. PROOF: Reconsider $f_1 = (\text{CFS}(\mathbb{S}))^{-1}$ as a function from $\mathbb{S}$ into Seg $\overline{\overline{\mathbb{S}}}$. Define $\mathcal{P}[\text{element}, \text{element}] \equiv \$_2 =$ the root tree of $\langle f_1(\$_1)$, $p(\{\$_1\})\rangle$. For every element x such that $x \in \mathbb{S}$ there exists an element y such that $y \in \text{InitTrees}\,p$ and $\mathcal{P}[x, y]$ by [12, (5)], [13, (87)], [7, (3)]. Consider f being a function from $\mathbb{S}$ into InitTrees p such that for every element x such that $x \in \mathbb{S}$ holds $\mathcal{P}[x, f(x)]$ from [12, Sch. 1]. □

(6) Let us consider a non empty finite subset X of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and finite binary trees s, t decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Then $\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)) \notin X$.

Let X be a set. The set of leaves of X yielding a subset of $2^{\mathbb{R}_{\mathbb{N}}}$ is defined by the term

(Def. 11) $\{\text{Leaves}(p)$, where p is an element of the binary finite trees of $\mathbb{R}_{\mathbb{N}} : p \in X\}$.

Now we state the propositions:

(7) Let us consider a finite binary tree X decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Then the set of leaves of $\{X\} = \{\text{Leaves}(X)\}$. PROOF: For every element x, $x \in$ the set of leaves of $\{X\}$ iff $x \in \{\text{Leaves}(X)\}$. □

(8) Let us consider sets X, Y. Then the set of leaves of $X \cup Y =$ (the set of leaves of X) $\cup$ (the set of leaves of Y). PROOF: For every element x, $x \in$ the set of leaves of $X \cup Y$ iff $x \in$ (the set of leaves of X) $\cup$ (the set of leaves of Y). □

(9) Let us consider trees s, t. Then $\emptyset \notin \text{Leaves}(\overbrace{t, s})$. PROOF: For every element p, $p \in \overbrace{t, s}$ iff $p \in$ the elementary tree of 0 by [4, (19), (29)], [10, (130)]. □

(10) Let us consider trees s, t. Then $\text{Leaves}(\overbrace{t, s}) = \{\langle 0 \rangle \frown p$, where p is an element of $t : p \in \text{Leaves}(t)\} \cup \{\langle 1 \rangle \frown p$, where p is an element of $s : p \in \text{Leaves}(s)\}$. The theorem is a consequence of (9). PROOF: Set $L = \{\langle 0 \rangle \frown p$, where p is an element of $t : p \in \text{Leaves}(t)\}$. Set $R = \{\langle 1 \rangle \frown p$, where p is an element of $s : p \in \text{Leaves}(s)\}$. Set $H = \text{Leaves}(\overbrace{t, s})$. For every element x, $x \in H$ iff $x \in L \cup R$ by [2, (23)], [9, (6)]. □

Let us consider decorated trees s, t, an element x, and a finite sequence q of elements of $\mathbb{N}$. Now we state the propositions:

(11) If $q \in \text{dom}\, t$, then $(x\text{-tree}(t, s))(\langle 0 \rangle \frown q) = t(q)$.

(12) If $q \in \text{dom}\, s$, then $(x\text{-tree}(t, s))(\langle 1 \rangle \frown q) = s(q)$.

Now we state the propositions:

(13) Let us consider decorated trees s, t and an element x. Then $\text{Leaves}(x\text{-tree}(t, s)) = \text{Leaves}(t) \cup \text{Leaves}(s)$. The theorem is a consequence of (10), (11), and (12). PROOF: Set $L = \{\langle 0 \rangle \frown p$, where p is an element of $\text{dom}\, t : p \in \text{Leaves}(\text{dom}\, t)\}$. Set $R = \{\langle 1 \rangle \frown p$, where p is an element of $\text{dom}\, s : p \in \text{Leaves}(\text{dom}\, s)\}$. For every element z, $z \in (x\text{-tree}(t, s))^{\circ} L$ iff $z \in t^{\circ}(\text{Leaves}(\text{dom}\, t))$. For every element z, $z \in (x\text{-tree}(t, s))^{\circ} R$ iff $z \in s^{\circ}(\text{Leaves}(\text{dom}\, s))$. □

(14) Let us consider a natural number k and finite binary trees s, t decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Then $\bigcup$ the set of leaves of $\{s, t\} = \bigcup$ the set of leaves of $\{\text{MakeTree}(t, s, k)\}$. The theorem is a consequence of (8), (7), and (13).

(15) Leaves(the elementary tree of 0) = the elementary tree of 0. PROOF: For every element x, $x \in$ Leaves(the elementary tree of 0) iff $x \in$ the elementary tree of 0 by [4, (29), (54)]. □

(16) Let us consider an element x, a non empty set D, and a finite binary tree T decorated with elements of D. Suppose $T =$ the root tree of x. Then $\text{Leaves}(T) = \{x\}$. The theorem is a consequence of (15).

2. BINARY HUFFMAN TREE

Let us consider $\mathbb{S}$, p, T_1, and q. We say that T_1, q, and p are constructing binary Huffman tree if and only if

(Def. 12) (i) $T_1(1) = \mathrm{InitTrees}\, p$, and

(ii) $\mathrm{len}\, T_1 = \overline{\overline{\mathbb{S}}}$, and

(iii) for every natural number i such that $1 \leqslant i < \mathrm{len}\, T_1$ there exist non empty finite subsets X, Y of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and there exists a minimal value tree s of X and there exists a minimal value tree t of Y and there exists a finite binary tree v decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that $T_1(i) = X$ and $Y = X \setminus \{s\}$ and $v \in \{\mathrm{MakeTree}(t, s, ((\text{the maximal value of } X)+1)), \mathrm{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$ and $T_1(i+1) = (X \setminus \{t, s\}) \cup \{v\}$, and

(iv) there exists a finite binary tree T decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that $\{T\} = T_1(\mathrm{len}\, T_1)$, and

(v) $\mathrm{dom}\, q = \mathrm{Seg}\, \overline{\overline{\mathbb{S}}}$, and

(vi) for every natural number k such that $k \in \mathrm{Seg}\, \overline{\overline{\mathbb{S}}}$ holds $q(k) = \overline{\overline{T_1(k)}}$ and $q(k) \neq 0$, and

(vii) for every natural number k such that $k < \overline{\overline{\mathbb{S}}}$ holds $q(k+1) = q(1)-k$, and

(viii) for every natural number k such that $1 \leqslant k < \overline{\overline{\mathbb{S}}}$ holds $2 \leqslant q(k)$.

Now we state the proposition:

(17) There exists T_1 and there exists q such that T_1, q, and p are constructing binary Huffman tree. The theorem is a consequence of (5) and (6). PROOF: Define $\mathcal{A}[\text{natural number}, \text{set}, \text{set}] \equiv$ if there exist elements u, v such that $u \neq v$ and u, $v \in \$_2$, then there exist non empty finite subsets X, Y of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and there exists a minimal value tree s of X and there exists a minimal value tree t of Y and there exists a finite binary tree w decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that $\$_2 = X$ and $Y = X \setminus \{s\}$ and $w \in \{\mathrm{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \mathrm{MakeTree}(s, t, ((\text{the maximal value of } X)+1))\}$ and $\$_3 = (X \setminus \{t, s\}) \cup \{w\}$. For every natural number n such that $1 \leqslant n < \overline{\overline{\mathbb{S}}}$ for every element x of the Boolean binary finite trees of $\mathbb{R}_{\mathbb{N}}$, there exists an element y of the Boolean binary finite trees of $\mathbb{R}_{\mathbb{N}}$ such that $\mathcal{A}[n, x, y]$. Reconsider $I = \mathrm{InitTrees}\, p$ as an element of the Boolean binary finite trees of $\mathbb{R}_{\mathbb{N}}$. Consider T_1 being a finite sequence of elements of the Boolean binary finite trees of $\mathbb{R}_{\mathbb{N}}$ such that $\mathrm{len}\, T_1 = \overline{\overline{\mathbb{S}}}$ and $T_1(1) = I$ or $\overline{\overline{\mathbb{S}}} = 0$ and for every natural number n such that $1 \leqslant n < \overline{\overline{\mathbb{S}}}$ holds $\mathcal{A}[n, T_1(n), T_1(n + 1)]$ from [15, Sch. 4]. Define $\mathcal{B}[\text{element}, \text{element}] \equiv$ there exists a finite set X such that

$T_1(\$_1) = X$ and $\$_2 = \overline{\overline{X}}$ and $\$_2 \neq 0$. For every natural number k such that $k \in \operatorname{Seg} \overline{\overline{\mathbb{S}}}$ there exists an element x of $\mathbb{N}$ such that $\mathcal{B}[k, x]$ by [11, (3)]. Consider q being a finite sequence of elements of $\mathbb{N}$ such that $\operatorname{dom} q = \operatorname{Seg} \overline{\overline{\mathbb{S}}}$ and for every natural number k such that $k \in \operatorname{Seg} \overline{\overline{\mathbb{S}}}$ holds $\mathcal{B}[k, q(k)]$ from [8, Sch. 5]. For every natural number k such that $k \in \operatorname{Seg} \overline{\overline{\mathbb{S}}}$ holds $q(k) = \overline{\overline{T_1(k)}}$ and $q(k) \neq 0$. For every natural number k such that $1 \leqslant k < \overline{\overline{\mathbb{S}}}$ holds if $2 \leqslant q(k)$, then $q(k+1) = q(k) - 1$ by [8, (1)], [2, (11), (13)]. Define $\mathcal{C}[\text{natural number}] \equiv$ if $\$_1 < \overline{\overline{\mathbb{S}}}$, then $q(\$_1 + 1) = q(1) - \$_1$. For every natural number n such that $\mathcal{C}[n]$ holds $\mathcal{C}[n+1]$ by [2, (11)], [8, (1)], [2, (14), (13)]. For every natural number n, $\mathcal{C}[n]$ from [2, Sch. 2]. For every natural number n such that $1 \leqslant n < \overline{\overline{\mathbb{S}}}$ holds $2 \leqslant q(n)$ by [2, (21), (13)]. For every natural number k such that $1 \leqslant k < \operatorname{len} T_1$ there exist non empty finite subsets X, Y of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and there exists a minimal value tree s of X and there exists a minimal value tree t of Y and there exists a finite binary tree w decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that $T_1(k) = X$ and $Y = X \setminus \{s\}$ and $w \in \{\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \text{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$ and $T_1(k+1) = (X \setminus \{t, s\}) \cup \{w\}$ by [8, (1)]. Consider T_2 being a finite set such that $T_1(\overline{\overline{\mathbb{S}}}) = T_2$ and $q(\overline{\overline{\mathbb{S}}}) = \overline{\overline{T_2}}$ and $q(\overline{\overline{\mathbb{S}}}) \neq 0$. Consider u being an element such that $T_2 = \{u\}$. □

Let us consider $\mathbb{S}$ and p. A binary Huffman tree of p is a finite binary tree decorated with elements of $\mathbb{R}_{\mathbb{N}}$ and is defined by

(Def. 13) There exists a finite sequence T_1 of elements of the Boolean binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and there exists a finite sequence q of elements of $\mathbb{N}$ such that T_1, q, and p are constructing binary Huffman tree and $\{it\} = T_1(\operatorname{len} T_1)$.

In this paper T denotes a binary Huffman tree of p.

Now we state the propositions:

(18) $\bigcup$ the set of leaves of $\operatorname{InitTrees} p = \{z$, where z is an element of $\mathbb{N} \times \mathbb{R}$: there exists an element x of $\mathbb{S}$ such that $z = \langle (\operatorname{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle\}$. The theorem is a consequence of (16). Proof: Set $L = \bigcup$ the set of leaves of $\operatorname{InitTrees} p$. Set $R = \{z$, where z is an element of $\mathbb{N} \times \mathbb{R}$: there exists an element x of $\mathbb{S}$ such that $z = \langle (\operatorname{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle\}$. For every element x, $x \in L$ iff $x \in R$ by [13, (87)], [7, (3)]. □

(19) Suppose T_1, q, and p are constructing binary Huffman tree. Let us consider a natural number i. Suppose $1 \leqslant i \leqslant \operatorname{len} T_1$. Then $\bigcup$ the set of leaves of $T_1(i) = \{z$, where z is an element of $\mathbb{N} \times \mathbb{R}$: there exists an element x of $\mathbb{S}$ such that $z = \langle (\operatorname{CFS}(\mathbb{S}))^{-1}(x), p(\{x\}) \rangle\}$. The theorem is a consequence of (18), (8), and (14). Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 < \operatorname{len} T_1$, then $\bigcup$ the set of leaves of $T_1(\$_1 + 1) = \{z$, where z is an element of $\mathbb{N} \times \mathbb{R}$: there exists an element x of $\mathbb{S}$ such that $z = \langle (\operatorname{CFS}(\mathbb{S}))^{-1}(x)$,

$p(\{x\})\rangle\}$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [2, (11)], [13, (78), (32)]. For every natural number k, $\mathcal{P}[k]$ from [2, Sch. 2]. □

(20) $\text{Leaves}(T) = \{z$, where z is an element of $\mathbb{N}\times\mathbb{R}$: there exists an element x of $\mathbb{S}$ such that $z = \langle(\text{CFS}(\mathbb{S}))^{-1}(x), p(\{x\})\rangle\}$. The theorem is a consequence of (19) and (7).

(21) Suppose T_1, q, and p are constructing binary Huffman tree. Let us consider a natural number i, a finite binary tree T decorated with elements of $\mathbb{R}_{\mathbb{N}}$, and elements t, s, r of $\text{dom}\, T$. Suppose

(i) $T \in T_1(i)$, and

(ii) $t \in \text{dom}\, T \setminus \text{Leaves}(\text{dom}\, T)$, and

(iii) $s = t \frown \langle 0\rangle$, and

(iv) $r = t \frown \langle 1\rangle$.

Then the value of tree of $t =$ (the value of tree of s) + (the value of tree of r). The theorem is a consequence of (15), (11), and (12). Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1 \leqslant \text{len}\, T_1$, then for every finite binary tree T decorated with elements of $\mathbb{R}_{\mathbb{N}}$ and for every elements a, b, c of $\text{dom}\, T$ such that $T \in T_1(\$_1)$ and $a \in \text{dom}\, T \setminus \text{Leaves}(\text{dom}\, T)$ and $b = a \frown \langle 0\rangle$ and $c = a \frown \langle 1\rangle$ holds the value of tree of $a =$ (the value of tree of b) + (the value of tree of c). For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [2, (16), (14)], [8, (44)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. □

(22) Let us consider elements t, s, r of $\text{dom}\, T$. Suppose

(i) $t \in \text{dom}\, T \setminus \text{Leaves}(\text{dom}\, T)$, and

(ii) $s = t \frown \langle 0\rangle$, and

(iii) $r = t \frown \langle 1\rangle$.

Then the value of tree of $t =$ (the value of tree of s) + (the value of tree of r). The theorem is a consequence of (21).

(23) Let us consider a non empty finite subset X of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$. Suppose a finite binary tree T decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Suppose $T \in X$. Let us consider an element p of $\text{dom}\, T$ and an element r of $\mathbb{N}$. Suppose $r = T(p)_{\mathbf{1}}$. Then $r \leqslant$ the maximal value of X. Let us consider finite binary trees s, t, w decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Suppose

(i) $s, t \in X$, and

(ii) $w = \text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1))$.

Let us consider an element p of $\text{dom}\, w$ and an element r of $\mathbb{N}$. Suppose $r = w(p)_{\mathbf{1}}$. Then $r \leqslant$ (the maximal value of X) + 1. The theorem is a consequence of (11) and (12). Proof: For every element a such that

$a \in \operatorname{dom} d$ holds $a = \emptyset$ or there exists an element f of $\operatorname{dom} t$ such that $a = \langle 0 \rangle \frown f$ or there exists an element f of $\operatorname{dom} s$ such that $a = \langle 1 \rangle \frown f$ by [2, (23)]. □

(24) Suppose T_1, q, and p are constructing binary Huffman tree. Let us consider a natural number i. Suppose $1 \leqslant i < \operatorname{len} T_1$. Let us consider non empty finite subsets X, Y of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$. Suppose

(i) $X = T_1(i)$, and

(ii) $Y = T_1(i+1)$.

Then the maximal value of $Y =$ (the maximal value of $X) + 1$. Proof: Consider X, Y being non empty finite subsets of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$, s being a minimal value tree of X, t being a minimal value tree of Y, v being a finite binary tree decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that $T_1(i) = X$ and $Y = X \setminus \{s\}$ and $v \in \{\text{MakeTree}(t, s, ((\text{the maximal value of } X) + 1)), \text{MakeTree}(s, t, ((\text{the maximal value of } X) + 1))\}$ and $T_1(i+1) = (X \setminus \{t, s\}) \cup \{v\}$. Consider L_1 being a non empty finite subset of $\mathbb{N}$ such that $L_1 = \{$the value of root from left of p, where p is an element of the binary finite trees of $\mathbb{R}_{\mathbb{N}} : p \in X0\}$ and the maximal value of $X0 = \max L_1$. Consider L_4 being a non empty finite subset of $\mathbb{N}$ such that $L_4 = \{$the value of root from left of p, where p is an element of the binary finite trees of $\mathbb{R}_{\mathbb{N}} : p \in Y0\}$ and the maximal value of $Y0 = \max L_4$. Reconsider $p_1 = v$ as an element of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$. For every extended real x such that $x \in L_4$ holds $x \leqslant$ the value of root from left of p_1 by [2, (16)]. □

Let us consider a natural number i, a non empty finite subset X of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$, a finite binary tree T decorated with elements of $\mathbb{R}_{\mathbb{N}}$, an element p of $\operatorname{dom} T$, and an element r of $\mathbb{N}$. Now we state the propositions:

(25) Suppose T_1, q, and p are constructing binary Huffman tree. Then if $X = T_1(i)$, then if $T \in X$, then if $r = T(p)_{\mathbf{1}}$, then $r \leqslant$ the maximal value of X.

(26) Suppose T_1, q, and p are constructing binary Huffman tree. Then if $X = T_1(i)$, then if $T \in X$, then if $r = T(p)_{\mathbf{1}}$, then $r \leqslant$ the maximal value of X.

Now we state the proposition:

(27) Suppose T_1, q, and p are constructing binary Huffman tree. Let us consider a natural number i, finite binary trees s, t decorated with elements of $\mathbb{R}_{\mathbb{N}}$, and a non empty finite subset X of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$. Suppose

(i) $X = T_1(i)$, and

(ii) $s, t \in X$.

Let us consider a finite binary tree z decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Suppose $z \in X$. Then $\langle$(the maximal value of $X) + 1$, (the value of root from right of t) + (the value of root from right of s)$\rangle \notin \operatorname{rng} z$. The theorem is a consequence of (26).

Let x be an element. Note that the root tree of x is one-to-one.

Now we state the propositions:

(28) Let us consider a non empty finite subset X of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and finite binary trees s, t, w decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Suppose

- (i) for every finite binary tree T decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that $T \in X$ for every element p of $\operatorname{dom} T$ for every element r of $\mathbb{N}$ such that $r = T(p)_{\mathbf{1}}$ holds $r \leqslant$ the maximal value of X, and
- (ii) for every finite binary trees p, q decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that p, $q \in X$ and $p \neq q$ holds $\operatorname{rng} p \cap \operatorname{rng} q = \emptyset$, and
- (iii) s, $t \in X$, and
- (iv) $s \neq t$, and
- (v) $w \in X \setminus \{s, t\}$.

Then rng MakeTree$(t, s, (($the maximal value of $X) + 1)) \cap \operatorname{rng} w = \emptyset$. The theorem is a consequence of (11) and (12). Proof: Set $d =$ MakeTree$(t, s,$ $(($the maximal value of $X) + 1))$. For every element a such that $a \in \operatorname{dom} d$ holds $a = \emptyset$ or there exists an element f of $\operatorname{dom} t$ such that $a = \langle 0 \rangle \frown f$ or there exists an element f of $\operatorname{dom} s$ such that $a = \langle 1 \rangle \frown f$ by [2, (23)]. Consider n_2 being an element such that $n_2 \in \operatorname{rng} d \cap \operatorname{rng} w$. Consider a_1 being an element such that $a_1 \in \operatorname{dom} d$ and $n_2 = d(a_1)$. Consider b_1 being an element such that $b_1 \in \operatorname{dom} w$ and $n_2 = w(b_1)$. $w \in X$ and $w \neq s$ and $w \neq t$. □

(29) Suppose T_1, q, and p are constructing binary Huffman tree. Let us consider a natural number i and finite binary trees T, S decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Suppose

- (i) T, $S \in T_1(i)$, and
- (ii) $T \neq S$.

Then $\operatorname{rng} T \cap \operatorname{rng} S = \emptyset$. The theorem is a consequence of (26) and (28). Proof: Define $\mathcal{P}[$natural number$] \equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} T_1$, then for every finite binary trees T, S decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that T, $S \in T_1(\$_1)$ and $T \neq S$ holds $\operatorname{rng} T \cap \operatorname{rng} S = \emptyset$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [21, (8)], [2, (16), (14)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. □

(30) Let us consider a non empty finite subset X of the binary finite trees of $\mathbb{R}_{\mathbb{N}}$ and finite binary trees s, t decorated with elements of $\mathbb{R}_{\mathbb{N}}$. Suppose

- (i) s is one-to-one, and

(ii) t is one-to-one, and

(iii) t, $s \in X$, and

(iv) $\operatorname{rng} s \cap \operatorname{rng} t = \emptyset$, and

(v) for every finite binary tree z decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that $z \in X$ holds $\langle$(the maximal value of $X) + 1$, (the value of root from right of t) + (the value of root from right of s)$\rangle \notin \operatorname{rng} z$.

Then MakeTree$(t, s,$ ((the maximal value of X) + 1)) is one-to-one. The theorem is a consequence of (11) and (12). Proof: Set $d =$ MakeTree$(t, s,$ ((the maximal value of $X) + 1$)). For every element a such that $a \in \operatorname{dom} d$ holds $a = \emptyset$ or there exists an element f of $\operatorname{dom} t$ such that $a = \langle 0 \rangle ^\frown f$ or there exists an element f of $\operatorname{dom} s$ such that $a = \langle 1 \rangle ^\frown f$ by [2, (23)]. For every element x such that $x \in \operatorname{dom} d$ and $x \neq \emptyset$ holds $d(x) \neq d(\emptyset)$ by [11, (3)]. For every elements x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} d$ and $d(x_1) = d(x_2)$ holds it is not true that there exists an element f of $\operatorname{dom} s$ such that $x_1 = \langle 1 \rangle ^\frown f$ and there exists an element f of $\operatorname{dom} t$ such that $x_2 = \langle 0 \rangle ^\frown f$ by [11, (3)]. For every elements x_1, x_2 such that x_1, $x_2 \in \operatorname{dom} d$ and $d(x_1) = d(x_2)$ holds $x_1 = x_2$. □

(31) Suppose T_1, q, and p are constructing binary Huffman tree. Let us consider a natural number i and a finite binary tree T decorated with elements of $\mathbb{R}_{\mathbb{N}}$. If $T \in T_1(i)$, then T is one-to-one. The theorem is a consequence of (27), (29), and (30). Proof: Define $\mathcal{P}$[natural number] $\equiv$ if $1 \leqslant \$_1 \leqslant \operatorname{len} T_1$, then for every finite binary tree T decorated with elements of $\mathbb{R}_{\mathbb{N}}$ such that $T \in T_1(\$_1)$ holds T is one-to-one. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [2, (16), (14)]. For every natural number i, $\mathcal{P}[i]$ from [2, Sch. 2]. □

Let us consider p.

Now we are at the position where we can present the Main Theorem of the paper: Every binary Huffman tree of p is one-to-one.

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