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A NEW THEORY OF LAPSE-RATE*

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ABSTRACT. A new type of vertical motion of air different from penetrative convection is discussed. For small vertical displacements the motion is assumed to be similar to that in long gravitational waves, the volume of an element remaining constant during the displacement. Vertical motion of this type is the result of differences of pressure set up in the vertical direction and the pressures on an element in the other principal directions do not alter during the process of small displacement. As a result of the change in pressure, the molecular energy of the element is altered which results in a change of temperature, and consequently in a lapse-rate.

There is close agreement between the value of the lapse-rate deduced for small displacements without acceleration and the mean lapse-rates in the free atmosphere observed at different stations under normal conditions, so that the theory of mass motion developed may be taken to be a true picture of what is happening in the free atmosphere. If the type of motion described here can be called cumulative convection, the free atmosphere may be said to be normally in 'cumulative equilibrium.'

An explanation is also obtained for the ascent of air masses in a cyclone and their descent in an anticyclone. These movements are due to gradients of pressure in the vertical direction and the condition for motion is that when the lapse-rate is greater than the equilibrium rate, the acceleration is upwards, and when it is less than that, it is downwards.

1. INTRODUCTION

It is found as a result of investigations with sounding balloons, that the rate at which temperature decreases with height is quite uniform in the free atmosphere, from a height of about 3 km. from the ground up to the tropopause. There is fall in temperature below 3 km. also but the lapse-rate is not regular, probably due to local disturbances and fluctuations that arise at the surface of the earth. But in the free atmosphere the lapse-rate is practically constant, the temperature-height curves of any ascent being almost linear, and the value about 6°C. to 7°C. per km.

The decrease of temperature with height in the troposphere is explained to be due to the convection that is taking place in it. Calculating on the basis that the atmosphere is in adiabatic equilibrium, the lapse-rate

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$\left(-\frac{dT}{dy}\right)$ is found to be equal to $g.\frac{A}{C_p}$ for dry air where g is the acceleration due to gravity, A, the reciprocal of the mechanical equivalent of heat and C_p the specific heat at constant pressure. Substituting the values of these constants, the above adiabatic lapse-rate is found to be 9·86°C. per km., which is much higher than the observed value mentioned above. The theoretical value obtained for ascending saturated air is less than the above value, but it is not a constant, as it depends upon the temperature of the air. If the atmosphere be in equilibrium for ascending saturated air, the lapse-rate should vary considerably from place to place and time to time, depending upon the temperature at the surface of the earth; it should also vary with height. But observations, as already stated, are rather different. Besides, the theory of ascending saturated air cannot be applied to regions where air masses are found to be descending as in anticyclones. The stability of air is usually discussed with reference to the adiabatic lapse-rates and the conclusion reached is that a dry atmosphere is in stable, unstable or neutral equilibrium, according as the lapse-rate is less than, greater than, or equal to the dry adiabatic lapse-rate and that saturated air is stable for downward motion when the lapse-rate is less than the dry adiabatic, but unstable for upward displacements unless the lapse-rate is less than the saturated adiabatic.[1] No satisfactory explanation has so far been given as to why the rate should be uniform throughout the free atmosphere up to the tropopause and fairly constant in value irrespective of time and place. There may be several factors like radiation and eddy diffusion, as discussed by Douglas,[2] which tend to produce a constant rate, but up till now, no theoretical value has been obtained which agrees with observations.[3]

Closely connected with lapse-rate are some other phenomena which too have not yet received satisfactory explanation. The centre of a cyclone is found to be a comparatively cold region and yet air masses are found to move upwards in it. An anticyclone may be warm, yet air masses are found to descend downwards in it though the lapse-rate is less than the dry adiabatic. These peculiar phenomena cannot evidently be explained by the hypothesis of penetrative convection. They are therefore attributed to be due to differences of pressure in the vertical direction and the term 'cumulative convection'[4] is applied to the large movements of air that take place in these cases. Yet there has been no theoretical calculation of what the lapse-rate should be in such a type of motion.

An attempt is made in this paper to develop a theory of lapse-rate (which may be said to be the lapse-rate for cumulative convection) for vertical motion of air due to difference of pressure, assuming the motion to be like that in long gravitational waves. The aid of the kinetic theory of gases is also sought for a simple relationship that obtains between pressure exerted by a gas in any direction and the kinetic energy of the molecules of the gas in that direction. The results obtained seem to be in fair agreement with observations.

2. LONG WAVES

The idea of long waves, the wave-length of which is very great as compared to the height of disturbance, is very useful in understanding the picture of air motion presented in this paper. Vertical motion of air in the case of long waves is easily seen to be different from that of penetrative convection. In the latter case, an element of air is considered to be warmer and therefore lighter than its surroundings, to be displaced from one layer to another, and then to expand adiabatically to attain the pressure there. The surroundings are considered to be practically undisturbed except for filling up the space vacated by the element. Motion of air in the case of long waves would be entirely different. Here there is motion on a large scale. Vast sheets of air rise or fall. And as they do so, two neighbouring elements in the same horizontal plane will move together and the difference in their vertical displacements will be negligibly small. The consequence of it is that there is no question of expansion of the element sideways suddenly as it rises. Even if there be a slight difference in the vertical displacements of these two elements, the result will not be (in the first place) expansion sideways, but mass motion, in a horizontal direction, the velocity of motion depending upon the rate of variation of pressure with distance.

This leads to an important assumption that is made in the development of the theory that there is little tendency for an element of air to change its volume as it is displaced slightly from one point to another in the vertical direction. This assumption may be considered to be not unreasonable in the light of what has been stated above, when the displacements are small. All displacements in the vertical direction therefore, according to the theory presented here, take place at constant volume, the displacements of the air elements being small.

3. RELATION BETWEEN MOLECULAR ENERGY AND PRESSURE

There is a very simple relation in the kinetic theory of gases between the pressure exerted by a gas on an element of area and the kinetic energy of the molecules of the gas in a direction normal to the area.

If n be the number of molecules per c.c. of the gas, m, the mass of a molecule, and $\bar{v^2}$ the average of thesquares of the velocity components of the gas in the direction of the Y axis, the pressure exerted on an element of area in the XZ plane (which we may say is the pressure in the Y direction) is

$$P_y = n.m.\bar{v^2} \qquad \dots (1)$$

But if the kinetic energy possessed by the molecules per unit volume of the gas in this direction be called E_y,

$$E_y = \tfrac{1}{2}\, n.m.\overline{v^2} \qquad (2)$$

$$P_y = 2.\ E_y \qquad (2a)$$

or

$$dP_y = 2.\ dE_y \qquad (2b)$$

4. THEORY OF MASS MOTION OF AIR (CUMULATIVE CONVECTION)

Mass motion of air between two points which are close to each other is due to the difference of pressures between the two points in the direction of the line that joins them. The importance of considering direction for the pressure arises from the fact that in the case of vertical motion of an element of air in a gravitational field, the pressure alters very rapidly with height. Let P_x, P_y, P_z, be the pressures at the point A (the co-ordinates of which are x, y, z) in the three principal directions. Let B be a point the co-ordinates of which are (for the sake of simplicity) x, $y+dy$, z. Let the initial pressure at this point in the three directions be P'_x, P'_y, P'_z. If we assume that there is static equilibrium in the beginning, we know that

$$P_x = P_y = P_z,$$

$$P'_x = P'_y = P'_z \text{ and}$$

$$P_y = P'_y + \rho g\, dy \ (\rho \text{, being the density at A}).$$

Let us now examine what takes place if there is a little accumulation and therefore some excess of pressure at A, so that

$$P_y > P'_y + \rho g\, dy.$$

There will be motion upwards, the equation for vertical motion being

$$\frac{Dv_0}{Dt} + g + \frac{1}{\rho}\frac{\partial P_y}{\partial y} = 0 \qquad \ldots (3)$$

where v_0 represents the mass velocity of the gas in the vertical direction and, $\frac{Dv_0}{Dt}$ represents the acceleration of the element (following its motion).

Hence
$$-\frac{1}{\rho}.\frac{\partial P_y}{\partial y} = g + \frac{Dv_0}{Dt}.$$

The value of $-\frac{\partial P_y}{\partial y}$, when there is accumulation below, will be greater than the value it would have in static equilibrium and consequently $\frac{Dv_0}{Dt}$ will be positive, the acceleration being upward. If on the other hand there is accumulation at the higher point B,

$$P_y < P'_y + \rho g\ dy.$$

the accelerated motion is downwards, and in

$$-\frac{\partial P_y}{\partial y} = \rho g + \rho\frac{Dv_0}{Dt}$$

$\frac{Dv_0}{Dt}$ is negative.

We shall now investigate the motion of an element. Let us assume for definiteness that the accumulation is below and that the motion is upwards. Let the element at A reach B in a short time, dt, the whole volume remaining constant in this process. As the element moves upwards, the value of P_y alters at a very rapid rate as it moves against a gravitational field of force, and its molecular energy in the vertical direction also diminishes at the same rate. But there is no such variation of pressure laterally as the surrounding elements also come up with it. Hence by the time the element reaches B, the energy corresponding to one degree of freedom only becomes less while that corresponding to the other degrees of freedom remains practically constant. Due to the law of equipartition of energy, however, there may be a transference of energy taking place between that degree and the other degrees which will tend to make the kinetic energy and hence the pressure equal in all directions. But since any such equalization should be preceded by a decrease in the Y direction, we shall assume that in a very short interval of mass motion of the gas, the pressures and therefore the energies are different in different directions We shall thus assume that for the element in vertical motion as pictured here P_x and P_z do not alter but P_y alone alters during a small displacement of the element.

For a small displacement dy, the change in the value of P_y may be taken to be $dP_y = \frac{\partial P_y}{\partial y} \cdot dy$ ... (4)

By equation (3) $\frac{\partial P_y}{\partial y} = -\rho\left(g + \frac{Dv_0}{Dt}\right)$ ... (5)

It may be noted in this connection that the effect of the rotation of the earth is ignored as the change it produces in the motion is negligible in comparison with g as pointed out by Brunt.[5] It is further assumed that the element has no motion in any other direction than the Y direction, in order to simplify the theory and study the effect of only the vertical motion on the element.

Now from equations (4) and (5) we have

$$dP_y = -\rho\left(g + \frac{Dv_0}{Dt}\right)dy \qquad \dots \quad (6)$$

and consequently from equation (2b)

$$dE_y = -\tfrac{1}{2}\,\rho\left(g + \frac{Dv_0}{Dt}\right)dy \qquad \dots \quad (7)$$

This is the change in the kinetic energy of the molecules of the element in the vertical direction per unit volume due to the displacement dy. Since we have assumed that there is no change in pressures in the other directions, the energies corresponding to all the other degrees of freedom of the molecules are not altered during this displacement. The change in the total kinetic energy per unit volume of the molecules of the element dE will be therefore equal to dE_y, and this does not alter in value due to any subsequent redistribution between all the different degrees of freedom of the molecules.

Therefore the change in the total kinetic energy of the molecules of the gas per unit volume

$$dE = -\tfrac{1}{2}\rho\left(g + \frac{Dv_0}{Dt}\right).\ dy \qquad \dots \quad (8)$$

The corresponding change of temperature dT suffered by the element may be easily computed applying the first law of thermodynamics :

$$dT = \frac{dE}{\rho.J.C_v} \qquad \dots \quad (9)$$

where J is the mechanical equivalent of heat and C_v the specific heat of air at constant volume (the displacement having been considered to take place at constant volume). Therefore from equations (8) and (9) we have for the rate of change of temperature of the element with height

$$\frac{dT}{dy} = -\frac{1}{2.\ J.\ C_v}\left(g + \frac{Dv_0}{Dt}\right) \qquad \dots \quad (10)$$

Let us assume that the acceleration $\frac{Dv_0}{Dt}$ is zero when the atmosphere is in equilibrium. We shall call the corresponding rate at which an element falls in temperature with height the equilibrium lapse-rate. The equilibrium lapse-rate is thus equal to

$$-\frac{dT}{dy} = \frac{g}{2.\ J.C_v} \qquad \dots \quad (11)$$

Substituting the values, C_v (for dry air) $=.1715$. Cal. per gm. (the value for damp air does not differ materially from this),

$$J=4.185 \times 10^7 \text{ ergs. per cal.}$$

and $$g=981.0 \text{ cm. per sec. per sec.}$$

We have $-\frac{dT}{dy}$ from equation (11) equal to

$$\frac{981}{2\times 4.185\times 10^7\times 0.1715}$$

$$=6^\circ.836\times 10^{+5} \text{ C. per cm.}$$

$$=6^\circ.836 \text{ C. per km.}$$

This is easily seen to be one-fifth of what is usually called the auto-convection gradient.[6]

The value 6·836 C. per km. is the rate at which an element of air would fall in temperature as it is displaced upwards vertically through a short distance as part of a vast sheet or layer of air, the volume of the element remaining constant during the small vertical displacement. Continuous vertical motion of air itself may be considered to take place in successive short steps, each step consisting of two stages:

(*i*) The stage in which an element undergoes a vertical displacement according to the way stated above.

(*ii*) The stage in which adjustments take place in a horizontal level, with expansion or contraction of the element sideways. But these changes in volume cannot be so sudden as to be completely adiabatic, because what may result in a horizontal direction as a consequence of this type of convection is a gradient of pressure which produces a wind. Hence the change of volume which an element undergoes at the new level is very likely to be isothermal. Consequently the lapse rate in an atmosphere in which such motion takes place will not differ appreciably from the value calculated above. Observations of the free atmosphere go to show that this type of (cumulative) convection is playing an important part in the movements of air massses in our atmosphere.

5. COMPARISON WITH OBSERVATIONS IN THE FREE ATMOSPHERE

Individual observations in the troposphere above a height of about 3 km. show that the temperature decreases at a uniform rate with height, the

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temperature-height curves being almost straight lines. This itself is in agreement with the result deduced above. Besides, the mean value of the lapse-rate obtained from a number of observations taken at different stations in the middle latitudes is found to agree quite satisfactorily with the result calculated above. The following table gives the mean lapse-rates observed at different stations in the middle latitudes. It is obtained from a table given by Dines who made use of Gold's values.

TABLE I [7]

(*Mean Lapse-rates*)

Height in km.	Petrograd.	Scotland.	Berlin.	England S E.	Paris	Vienna.	Pavia.
	Deg. A.	Deg. A.	Deg. A.	Deg. A.	Deg. A.	Deg. A.	Deg. A.
13·5	−0·1	−0·2	0·6	−0·2	0·2	0·0	−1·3
12·5	−2·7	−0·2	−1·0	0·1	0·2	−1·3	−0·3
11·5	−0·3	−1·1	0·9	0·8	0·7	0·1	2·4
10·5	1·3	0·7	2·7	2·6	4·1	3·4	4·2
9·5	3·1	3·6	4·9	5·3	5·7	5·1	4·6
8·5	5·4	5·4	6·3	6·1	6·9	6·7	6·6
7·5	7·3	7·8	7·7	7·1	7·4	7·6	7·3
6·5	6·2	7·0	7·1	7·1	7·1	7·6	8·2
5·5	6·5	7·0	6·9	7·0	6·7	6·8	6·8
4·5	5·9	6·4	6·2	6·9	6·2	6·3	6·7
3·5	5·6	5·6	5·9	6·0	5·5	5·7	5·9
2·5	5·4	5·7	4·8	5·5	4·7	5·4	6·3
1·5	4·3	5·0	5·1	4·8	4·0	4·6	5·6

In calculating the mean value of the lapse-rate in the troposphere, we must exclude observations below a height of 3 km., as this region is subject to many local disturbances and fluctuations and the lapse-rates are irregular. Also we should not take the region in which the tropopause rises and falls, as the mean value in it would be obviously misleading. Hence, below are given the mean lapse rates at different stations calculated from the values lying between the thick lines in the above table.

	Free troposphere between.	Mean lapse-rate.
Petrograd ...	3 km.–8 km.	6·30 Deg. A. per km.
Scotland ...	do.	6·96 ,,
Berlin ...	3 km.–9 km.	6·68 ,,
England S.E ...	do.	6·70 ,,
Paris ...	do.	6·63 ,,
Vienna ...	do.	6·78 ,,
Pavia ...	do.	6·92 ,,
	Mean ...	6·71 ,,

The equilibrium lapse rate is theoretically deduced to be 6°·836 C. per km. The agreement between the above mean observed value and the theoretical value is very close and we may therefore conclude that the 'free troposphere' (lying between 3 km. and the tropopause) in the middle latitudes, to which belong the stations in the above table. is normally in equilibrium for mass motion of the type discussed in this paper (*i.e.*, for cumulative convection).

Mean lapse-rates of the free atmosphere obtained at Agra[8] and[9] Poona in India are given below. The mean for Poona is calculated for lapse-rates between 3 and 15 geodynamic kilometres and that for Agra between 3 and 14 gkm. The mean values for the free atmosphere (6°·87 C. and 6°·35·C. respectively per km.) show fair agreement with the theoretical value deduced.

TABLE II.

Height in gkm.	Lapse-rate (Poona) (1928-'31).	Lapse-rate (Agra) (1925-'28).
	eg. C.	Deg. C.
14-15	6·7	...
13-14	8·0	6·3
12-13	8·3	7·1
11-12	8·2	7·0
10-11	8·1	6·9
9-10	7·8	6·7
5-9	7·2	6·7
7-8	6·8	6·3
6-7	5·9	6·1
5-6	5·9	6·2
4-5	5·6	6·1
3-4	5·8	5·9
Mean/gkm.	7·02	6·48
Mean/km.	6·88	6·35

6. LAPSE-RATES IN ASCENDING AND DESCENDING CURRENTS OF AIR (CYCLONIC AND ANTI-CYCLONIC SYSTEMS)

In the discussion made above on the equilibrium lapse-rate it is assumed that $\frac{Dv_0}{Dt}$ is equal to zero. Now we may take up for consideration the cases when it is not equal to zero. If we call $\frac{Dv_0}{Dt}$ the vertical acceleration of the element, we see that there is a possibility for two cases in addition to the case of equilibrium: (*i*) The vertical acceleration may be positive or (*ii*) it may be negative. From equation (10)

$$-\frac{dT}{dy} = \frac{1}{2J.C_v}\left(g+\frac{Dv_0}{Dt}\right).$$

We see that in case (*i*) when $\frac{Dv_0}{Dt}$ is positive (*i.e.*, the velocity increases with increase of height) the quantity within the parentheses will be greater than g and the lapse-rate will be higher in value than the equilibrium lapse-rate. In cases (*ii*) $\frac{Dv_0}{Dt}$ is negative (*i. e.*, the increase of velocity is down-ward) the quantity within parenthes becomes less than g, and the lapse-rate will be less than the equilibrium rate.

Thus we arrive at a new and interesting result that for upward motion of air with increasing velocity, the lapse-rate is higher and for downward motion with increasing velocity it is less than the equilibrium lapse-rate. It has already been stated that the acceleration will be upwards ($\frac{Dv_0}{Dt}$ positive) when there is accumulation below and downwards ($\frac{Dv_0}{Dt}$ negative) when the accumulation is above. These conditions are found to obtain in cyclonic and anticyclonic syetems respectively.

Cyclonic systems :—It is a widely observed fact that in a region of low pressure, there is an upward current of air, the velocity of which increases with height above the ground. The acceleration upward, in cases of strong convection on such occasions, becomes appreciable as compared to g and the lapse-rate should be accordingly affected. An example for upward acceleration quoted by Ramanathan[10] from Brunt, may be given in this connection. " An estimate of upward acceleration in hailstorms has been made by Brunt. He has shown that in order to raise a spherical hail stone of radius 3 cm. in an ascending current, an upward vertical velocity exceeding 55 metres per sec. would be required. If this velocity is developed within a distance of 3 km, the acceleration will be 50 cm/sec^2."

Thus in a region of low pressure the lapse-rate should be, according to the theory developed here, greater than the equilibrium lapse-rate.

Anticyclonic systems :—It is also known widely that in an anticyclone there is a downward motion of air ; but this does not seem to be as vigorous as the upward motion in a cyclone. Still since the downward motion is a result of accumulation in upper levels of air, and when an element of air moves under difference of pressure there is acceleration, it is quite possible that there is downward acceleration (however small) in anticyclonic systems in the free atmosphere. Thus $\frac{Dv_0}{Dt}$ becomes negative in these cases and the lapse rate becomes less than the equilibrium rate. But since the convection in them is not so rapid as in cyclonic systems the difference from the equilibrium value will not be as great in them as in the latter.

These conclusions obtain confirmation from observations made in cyclones and anticyclones. Below is given a table showing the mean temperatures in cyclones and anticyclones. The results are due to Dines.[11]

TABLE III[12]

Height in km.	Cyclone—989 mb. Temperature.	Anticyclone—1026 mb. Temperature.
14	224 Deg. A.	215 Deg. A.
13	25	15
12	25	17
11	24	20
10	25	25
9	26	31
8	28	28
7	34	46
6	42	53
5	49	59
4	56	65
3	63	71
2	70	76
1	276	279

For the cyclone we may take the 'free troposphere' to be between 3. km. and 7 km. and for the anticyclone between 3 km. and 9 km. at the lowest. Then we have from the above table the following mean values for the lapse-rates :

TABLE IV

	Mean lapse-rate	Difference from calculated equilibrium rate (6·836)
Cyclone	7°·25 C.	+0·414° C.
Anticyclone	6°·67 C.	−0·166° C.

Some individual cases may also be quoted in support of this view. There was an anticyclone over Upper India between the 19th and the 26th of December, 1930. The sounding [13] on the 19th Dec., gave a mean lapse-rate (between 4 gkm. and 15 gkm.) of 6°·45 C. per gkm. and that on the 22nd, 6°·66 C. per gkm. The mean of these two values is 6°·56 C. per gkm. or 6°·43 C. per km., a value less than the theoretical value for equilibrium. (6°·836 C. per km.)

A cyclone crossed the East Coast (India) at Nellore on the 17th of November, 1933. Soundings [14] made during the period of the cyclone on 15th, 16th, 17th, 18th and 19th of that month at Madras observatory, 110 miles away, gave the mean lapse-rates 7·32, 7·15, 7·08, 7·64 and 7·47 deg. C. respectively per gkm. (between 3 gkm. and 15 gkm.) of the free atmosphere. The mean value is 7·33 deg. C. per gkm. or 7·19 deg. C. per km., which is clearly higher than the theoretical equilibrium lapse-rate.

Thus the result deduced theoretically that the lapse-rate should be greater in a cyclonic system and less in an anticyclonic system than the equilibrium rate is supported by observations of the upper atmosphere. We have therefore a satisfactory explanation for the ascent of (cold) air in a cyclone and the descent of air (which may be warm) in an anticyclone even when the lapse-rate is less than the adiabatic. These vertical motions may therefore be considered to be due to pressure differences set up in the vertical direction and not to convection of the penetrative type.

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