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Beyond Problem-Solving: Elementary Students' Mathematical Dispositions When Faced With The Challenge Of Unsolved Problems

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BEYOND PROBLEM-SOLVING: ELEMENTARY STUDENTS' MATHEMATICAL DISPOSITIONS WHEN FACED WITH THE CHALLENGE OF UNSOLVED PROBLEMS

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260 Pages

The goal of this study was to document the characteristics of students' dispositions towards mathematics when they engaged in the exploration of parts of unsolved problems: Graceful Tree Conjecture and Collatz Conjecture. Ten students, Grades 4 and 5, from an after-school program in the Midwest participated in the study. I focused on the cognitive, affective, and conative aspects of their mathematical dispositions as they participated in 7 problem-solving sessions and two interviews.

With regard to cognitive aspects of the students' dispositions, I focused on the students attempts to identify and justify patterns for labeling graphs. Overall, the unsolved problems were accessible to the students and they found patterns that enabled them to gracefully label specific classes of graphs for the Graceful Tree Conjecture. With regard to affective aspects of students' dispositions, I found five themes that characterized their beliefs about the nature of mathematics. Also, students exhibited a variety of emotions throughout the study. The two emotions exhibited most frequently were frustration and joy. The third type of disposition that students exhibited was the conative construct of perseverance. This was related to the interplay of frustration and joy and characterized the productive struggle that students experienced throughout the study. To examine students’ dispositions in greater depth, I conducted a case study analysis of the positional identities of two students, which I report in a detailed narrative.

KEYWORDS: Dispositions, Elementary Education, Graceful Tree Conjecture, Mathematics, Problem Solving, Unsolved Problems

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When a child draws a picture, we call her an artist. When a child plays the piano, we call her a musician, and when a child plays a sport, we call her an athlete. When a child does mathematics, why do we not call her a mathematician?

Productive Mathematical Dispositions

The American mathematician William Thurston (1994) claimed that, "mathematics is one of the most intellectually gratifying of human activities" (p. 170). He explained that mathematicians engage in solving problems, and their goal is to progress what humans know about mathematics. Burton (1999) reported that when he interviewed mathematicians, they said they found pleasure in the struggle of doing mathematics. The mathematicians from Burton's interviews described mathematics as "a world of uncertainties and explorations, and the feelings of excitement, frustration and satisfaction, associated with these journeys, but, above all, a world of connections, relationships and linkages" (p. 138). Moreover, according to Burton, "mathematics is no longer seen, by the majority of mathematicians, as an individual activity" (p. 139). Not only do mathematicians believe they should work collaboratively, but Boaler (2016) reported that top mathematicians in the country have commented that solving a problem is not about speed but about working slowly and thinking deeply.

These characterizations reflect a certain disposition toward mathematics. A *mathematical disposition* is "a personal point of view on mathematics that includes what mathematics is about; what it can and should be used for; who does it; and the role it plays, or should play, in one's activities and subcultures" (Gainsburg, 2007, pp. 477–478). It is important that students develop positive dispositions toward mathematics—about what it means to know and learn mathematics—similar to those of mathematicians. In 2000, the National Council of Teachers of

Mathematics (NCTM) presented a vision for mathematics education that is characterized as follows:

> Imagine a classroom, a school, or a school district where all students have access to high-quality, engaging mathematics instruction. There are ambitious expectations for all students, which accommodations for those who need it. Knowledgeable teachers have adequate resources to support their work and are continually growing as professionals. The curriculum is mathematically rich, offering students opportunities to learn important mathematical concepts and procedures with understanding. Technology is an essential component of the environment. Students confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge for a wide variety of mathematical topics, sometimes approaching the same problem for different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. Alone or in groups and with access to technology, they work productively and reflectively, with the skilled guidance of their teachers. Orally and in writing, students communicate their ideas and results effectively. They value mathematics and engage actively in learning it. (p. 3)

To achieve this vision, NCTM published a set of mathematics content and process standards to guide the teaching and learning of mathematics across the grade levels. More recently, the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices [NGA] & Council of Chief State School Officers [CCSSO], 2010) outlines a similar set of standards, including eight standards for mathematical practice that should be the focus of instruction for all students in Kindergarten through Grade 12. The mathematical practices involve (a) making sense of problems and persevering in solving them, (b) reasoning abstractly and quantitatively, (c) constructing viable arguments and critiquing the reasoning of others, (d) modeling with mathematics, (e) using appropriate tools strategically, (f) attending to precision, (g) looking for and making use of structure, and (h) looking for and expressing regularity in repeated reasoning (NGA & CCSSO, 2010, pp. 6–8).

Thus, according to the standards that are informing mathematics curriculum and instruction in the United States, students should be experiencing mathematics in ways that reflect the views expressed by mathematicians about what it means to know and do mathematics. As a result, the dispositions students develop about mathematics should be positive and productive.

Students' Mathematical Dispositions

There are many mathematical dispositions held by students. It has been well documented, however, that many students have negative feelings toward mathematics. Beilock, Gunderson, Ramirez, and Levine (2010) stated that people have fear and anxiety about doing mathematics. "The view most students develop of mathematics is that it is a cut-and-dried, 400-year-old list of theorems and applications. They do not get the sense that mathematics is an exciting, living, and developing subject" (Su, 2010, p. 760). Similarly, Allen (2004) reported that the following quote was characteristic of the way Grade 6 students in her study viewed mathematics: "Maths lessons is all sums and hard stuff isn't it. It's not something you enjoy" (p. 240).

Many students think the role of mathematics in school is to get the right answer to a question, and students tend to think they are good at mathematics if they can solve a problem quickly (Boaler, 2016). Schoenfeld (1989) found that the average amount of time students spend on homework problems was just less than 2 minutes. When the students in his study were asked how long they would work on a problem before knowing it was impossible, the average answer was 12 minutes. More recently, in *The Math Gene,* Delvin (2000) gave a list of beliefs people hold about mathematics, including “To most people, mathematics is calculating with numbers” (p. 71) and “mathematics is predictable. It involves following precise rules…. There is always a right answer. (And it’s in the back of the book)” (p. 72).

The mathematical dispositions of students described above are vastly different from the mathematical dispositions of the mathematicians described by Burton (1999). As Boaler (2016) explained, “When we ask students what math is, they will typically give descriptions that are very different from those given by experts in the field” (p. 21). Mathematicians believe mathematics is about discovering and working slowly through mathematics; students believe that mathematics problems should be solved quickly and that they should be told if their solution is correct. Mathematicians believe it is acceptable to spend years on a single mathematics problem (Singh, 1997); many students think that a problem should be solvable in several minutes. Mathematicians believe mathematics is alive and something to discover; students believe that mathematics is something they need to learn and memorize. Mathematicians believe everyone can be successful at mathematics; students believe certain people either have the mathematics gene or they do not.

The mathematical dispositions students hold may be perpetuated by their experiences in school mathematics. Mathematics classrooms are traditionally set up where teachers explain procedures and rules to students (Fosnot & Dolk, 2002) and mathematics is about answering questions quickly and knowing the answers. These concerns date back to 1990 when Lampert explained that

> these cultural assumptions are shaped by school experience, in which *doing* mathematics means following the rules laid down by the teacher; *knowing* mathematics means remembering and applying the correct rule when the teacher asks a questions; and *mathematical truth is determined* when the answer is ratified by the teacher. (p. 32)

Lampert further explained that mathematics is thought of as a subject where a teacher asks a question, the student solves the problem by applying a rule, and the teacher ratifies the answer. She said, "The teacher and the textbook are the authorities, and mathematics is not a subject to be created or explore" (Lampert, 1990, p. 32).

In 2014, NCTM released *Principles to Action: Ensuring Mathematical Success for All* in which they described current unproductive beliefs about the learning and teaching of mathematics, similar to those described by Lampert in 1990. These unproductive beliefs include the following: basic skills need to be grasped prior to a student learning mathematics, students should memorize steps to solve a problem, "an effective teacher makes the mathematics easy for students by guiding them step by step through problem solving to ensure that they are not frustrated or confused" (National Council of Teachers of Mathematics [NCTM], 2014, p. 11), and a teacher's role is to tell the students what they need to know and demonstrate the process of solving every mathematics problem. Furthermore, according to NCTM, many teachers and parents believe mathematics should be taught the way they learned mathematics, through memorization and algorithms; many teachers believe that if they wander from that method their teaching will not be effective. Thus, for students to experience an instructional environment that supports the development of positive mathematical dispositions, changes in teachers' and parents' beliefs about how mathematics should be taught may be needed.

Changing Students' Dispositions

Many mathematics educators and professional organizations report that problem solving should be a main focus of school mathematics and could have a positive influence on students' mathematical dispositions and identities (e.g., Aguirre, Mayfield-Ingram, & Martin, 2013; Boaler, 2016; NCTM, 2014). *Problem solving* is

> a situation that proposes a mathematical question whose solution is not immediately accessible to the solver, because he [or she] does not have an algorithm for relating the data with the unknown or a process that automatically relates the data with the conclusion. Therefore, he [or she] must search, investigate, establish relationships, involve his [or her] affect, etc., to deal with it. (Callejo & Vila, 2009, p. 112)

Boaler (2016) explained that mathematical problem solving should involve tasks that have multiple methods, pathways, and representations; should be accessible to all students; and should involve students in reasoning about solutions and presenting convincing justifications. Similarly, NCTM (2014) recommended that the problems given to students should have multiple solution approaches and promote mathematical reasoning. Aguiree, Mayfield-Ingram, and Martin (2013) argued that problems should have high cognitive demand and allow students to debate and justify their solution. They also noted that tasks should have different entry points so students of all skill levels can interact with the problem and the problem should encourage collaboration among students.

With the recent introduction of the Common Core State Standards in the United States, problem solving may have become a main point of discussion, but the topic of problem solving in mathematics education has been around since at least the 1980s. It became the theme of school mathematics when NCTM released the *Agenda for Action* in 1980, which stated that the focus of mathematics in schools should be problem solving. In 1989, NCTM recommended that problems be given to students that contain no obvious solution and that could take "hours, days and even weeks to solve" (p. 6). In 1992, Schoenfeld claimed that "solving problems is 'the heart of mathematics'" (p. 339).

The goals and recommendations set forth for problem solving since the 1980s have not been fully realized or widely recognized (NCTM, 2014; Weiss & Pasley, 2004). As described above, many teachers and parents still believe that mathematics should be taught through rules, procedures, and memorization (NCTM, 2014). In a study conducted by Weiss and Pasley (2004), 59% of teachers observed by the researchers used a traditional, low-quality lesson structure of demonstrate, guide practice, assign independent practice, and assess. Although many teachers do present students with cognitively demanding problem-solving tasks (Smith & Stein, 1998), research has shown that teachers have a tendency to implement those tasks in such a way that the cognitive demand is diminished, resulting in students performing procedures rather than solving problems (NCTM, 2014). As a result, during their school years, most students will have experienced thousands of low-level tasks that have shaped their dispositions toward mathematics. As Schoenfeld (1989) reported, "Whether or not the student is conscious of it, this prior experience shapes the amount of time and effort that will be invested in [the problems they encounter]" (p. 341). Yet, students' engagement in tasks of high cognitive demand is "strongly connected with their sense of identity, leading to increased engagement and motivation in mathematics" (NCTM, 2014, p. 17). Therefore, it is critical that students have opportunities to explore high cognitive demand tasks and persevere in solving challenging mathematics problems.

Although there is an abundance of research on mathematical problem solving (e.g., Cai, 2003; Lester 1994; Schoenfeld, 1992), Lesh and Zawojewski (2007) stated that "a fresh perspective of problem solving is needed—one that goes beyond current school curricula and state standards" (p. 780). Unsolved problems have the potential to provide the kind of fresh perspective called for.

There are many problems in mathematics that have yet to be solved (e.g., show that every even integer greater than 2 can be expressed as the sum of two primes). Many of these different unsolved mathematics problems are ones that elementary children can understand and attempt to solve (Pachter, 2015). Schoenfeld (1992) explained that these types of unsolved problems are similar to what a mathematician does on a daily basis but are different only on the scale of the problem; he claimed that students' mathematical experiences should "prepare them for tackling such challenges" (p. 339). He further argued that students should participate in solving real problems and work on problems that have similar difficulty levels to unsolved problems and that are just as complex.

When students are engaged with unsolved problems, they may struggle because there is not a clear path to the solution. However, struggle should not be avoided and can be productive to the learning process. Productive struggle can occur when students investigate problems deeply, engage in experimentation, and have perseverance to make sense of the mathematics at hand, rather than purely pursuing a correct answer. The research literature has reported the benefits of allowing students to struggle productively (e.g., Hiebert & Grouws, 2007; Reinhart, 2000). For example, Kapur (2010) completed a study comparing seventh-grade students who were given the opportunity to struggle productively with students who were not given the opportunity. The students who engaged in productive struggle while solving problems had long-term benefits. They were able to significantly outperform the students who did not have an opportunity to struggle on problem-solving tasks, and they were able to transform their knowledge to new, higher-level concepts in mathematics that had yet to be taught. The benefits of productive struggle for student learning are reflected in NCTM's (2014) claim that "effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships" (p. 48).

Problem Statement and Research Question

Mathematicians view mathematics as a beautiful field of study and discovery. Although we want students to develop views of mathematics that are similar to those of mathematicians, students typically think of mathematics as something mathematicians have created that they now need to learn (Fosnot & Dolk, 2002). Many students think that they do not have the "mathematics gene" and have fear or anxiety about mathematics. However, Lampert (1990) conducted a study in which fifth-grade students engaged in many of the common practices of mathematicians. She found that when students engaged in these practices, they acted as mathematicians and behaved differently than students who did not have this experience. More recently, Boaler (2016) claimed that it could be empowering for students to work like a mathematician. The work of mathematicians involves problems that have yet to be solved.

I became interested in students attempting unsolved mathematics problems when I read *Fermat's Enigma: The Epic Quest to Solve the World's Greatest Mathematical Problem* by Simon Singh (1997). The book discussed the history of the famous mathematics problem, Fermat's Last Theorem, that no three positive integers can satisfy the equation $a^n + b^n = c^n$ for any integer value of *n* that is greater than two. Singh described the excitement people had at attempting to solve the problem. Many people of all ages tried to win the prize money that would be given to the first person to solve the problem. This was a problem that children and adults could understand and work on. One particular child, Andrew Wiles, became interested in this problem before he pursued a career in mathematics. After he became a mathematician, Wiles spent many years attempting to solve this problem and was eventually successful.

As I read Singh's (1997) book, I wondered what would happen if all students were given the opportunity to explore problems in mathematics that have yet to be solved. I wondered whether that experience would change how students view mathematics, and whether they would be empowered and develop dispositions similar to those of mathematicians. With these ideas in mind, I examined the following question:

> What are the characteristics of students' dispositions toward mathematics when they engage in the exploration of unsolved problems?

Conceptual Framework

I believe that learning occurs through social practices and that students construct understanding through active participation in learning communities. Thus, to investigate students' dispositions as they engaged in unsolved problems, I used a conceptual framework that integrated communities of practice (Wenger, 1998), positional identities (Holland, Skinner, Lachicotte, & Cain, 1998), and positioning theory (van Langenhove & Harré, 1999). First, I used communities of practice (Wenger, 1998) as an overarching framework. The ideas of dispositions are very broad and I needed a way to narrow my focus. Thus, I used communities of practice as a way to help focus my study. Identity is a large portion of communities of practice, and I used that idea of narrow my focus. I believe students display and develop their identity through dispositions. Thus, I investigated students' mathematical dispositions and the dispositions they display through their in-the-moment identity. Because of also looking at their identity, positional identities (Holland et al.,1998) and positioning theory (van Langenhove & Harré, 1999) became part of the conceptual framework.

Communities of Practice

The overarching lens of my dissertation study was the social theory of learning created by Wenger (1998) called communities of practice. This theory is based on Lave and Wenger's (1991) theory of legitimate peripheral participation. I used this perspective as a way to create a community of practice among the students that is similar to the practice of mathematicians and to situate the students' engagement with unsolved problems. Lesh and Zawojewski (2007) posited that communities of practice is a useful theoretical perspective for research related to mathematical problem solving.

Wenger (1998) claimed that learning is a fundamental part of our lives. When learning, we are "active participants in the *practices* of social communities and constructing *identities* in relation to these communities" (p. 4). The idea of practices and identities are two major components of communities of practice.

Practice. Practice is "a process by which we can experience the world and our engagement with it as meaningful" (Wenger, 1998, p. 51). The idea of practice is the doing and it is always social. It draws on historical and social situations and gives meaning to the things we do.

> It includes what is said and what is left unsaid; what is represented and what is assumed. It includes the language, tools, documents, images, symbols, well-defined roles, specified criteria, codified procedures, regulations, and contracts that various practices make explicit for a variety of purposes. But it also includes all the implicit relations, tacit

> conventions, subtle cues, untold rules of thumb, recognizable intuitions, specific perceptions, well-tuned sensitivities, embodied understandings, underlying assumptions, and shared world views. (Wenger, 1998, p. 47)

When someone is a member of a community of practice, he or she is a practitioner.

People have different experiences each day in their life. A person may experience patterns in their life, but each experience is different and new and allows a person the process of negotiation of meaning. Negotiation of meaning is the process by which a person views and experiences the world and by which their participation is meaningful. The negotiation of meaning includes participation. Our participation in the world is a "continual process of renewed negotiation" (p. 54). Participation refers to the social experience a person has in their community. It also involves the personal aspect of a person because it relates to a person's feelings and sense of belonging. Participation involves the whole person. According to Wenger, participation shapes the experiences we have and the communities.

The relationship between practice and community defines a community of practice (Wenger, 1998). The three dimensions of a community of practice are mutual engagement, joint enterprise, and shared repertoire. There is mutual engagement because people must negotiate meaning together and it is how the community is defined. Engagement is not mutual simply because members of a community are in the same room or classroom but because "they sustain dense relations of mutual engagement organized around what they are there to do" (Wenger, 1998, p. 74). For significant learning to occur there must be mutual engagement. The members of the community interact and learn from each other and because of this the practices evolve. The participation in learning allows a member or a group to build identity, change perspectives, change the way they participate, and the way they experience life. There is a joint enterprise, which is defined by the participants, that comes from the mutual engagement from the community and is a way to hold participants accountable for their goal. It allows for the engagement to align and a way to have accountability. The shared repertoire of "practice includes routines, words, tools, ways of doing things, stories, gestures, symbols, genres, actions, or concepts that the community has produced or adopted in the course of its existence, and which have become part of its practice" (Wenger, 1998, p. 83). It is a way to renegotiate "the meaning of various elements" (p. 95). When a community of practice forms, there are several indicators, including, (a) sustained mutual relationships, (b) a common way of engaging in a practice, (c) having shared knowledge, (d) understanding what each person can do and how they contribute to the enterprise, (e) shortcuts of ways to communicate with each other, (f) inside jokes or shared stories, and (g) no preambles because the interactions and conversations are more of an ongoing process (Wenger, 1998).

Identities. According to Wenger (2010)—in his social learning theory communities of practice—identity is a reflection of the complex relationship between the personal and the social. With a focus on identity, the framework of the social learning theory is extended because (a) "it narrows the focus onto the person, but from a social perspective, [and] (b) it expands the focus beyond communities of practice, calling attention to broader processes of identification and social structures" (Wenger, 1998, p. 145). In other words, identity allows the framework to swivel between an individual and the social, allowing each to be discussed in terms of the other.

Every act a person exhibits—whether private or public—reflects a person's identity (Wenger, 1998). How a person interprets her or his position, attempts to engage in a solving a problem, what the person knows, or understand what he or she does is part of the person's identity. These identities are “shaped by belonging to a community” (p. 146) and depend on their engagement within the practice.

Wenger (1998) explained different ways identity is in practice, for example, as it relates to one’s negotiated experience and community membership. When identity is looked at as negotiated experience it is defined “by the ways we experience our selves through participation as well as by the ways we and others reify our selves” (p. 149). When a person defines who they are by the unfamiliar and the familiar, they are looking at their identity through their community membership. More specifically, the dimensions of their identify within the community refers to their mutual engagement within the community and their accountability towards the enterprise. A person’s identify is ongoing, progressive, and shaped through effort.

Positional Identities

How Wenger (1998) and Holland, Skinner, Lachicotte, and Cain (1998) defined identity is similar. Wenger defined identity as “a way of talking about how learning changes who we are and creates personal histories of becoming in the context of our communities” (p. 5). Holland et al. defined identity as “a concept that figuratively combines the intimate or personal world with the collective space of cultural forms and social relations” (p. 5). Wenger mentioned how a person acts reflects their identity; this is again similar to Holland et al. However, Holland et al. gave this a name—positioning—and created a theoretical framework based on positioning and identities. Positioning is a linguistic behavior (Holland et al., 1998). For example, if a person says “please” and “thank you” they have positioned herself or himself as a moral person and then also identified as a moral person.

Another part of Holland et al.’s (1998) theory pertains to the construct of figured worlds. Figured worlds are similar in ways to communities of practice. Figured worlds are “developed through participants and their work …[,] are social encounters in which participants' positions matter…. [, and] are socially organized and reproduced" (Holland et al., 1998, p. 41). Schools could be defined as a figured world. The students in my study have only experienced mathematics as the figured world of school mathematics.

Positional identities "have to do with behavior as indexical of claims to social relationships with others. They have to do with how one identifies one's position relative to others" (Holland et al., 1998, p. 127). Holland et al. called people's actions based on events as social positions. The social positions "become dispositions through participation in, identification with, and development of expertise within the figured world" (p. 136). The social position develops into a positional identity over a long period of time. "The long term, however, happens through day-to-day encounters and is built, again and again, by means of artifacts, or indices of positions, that newcomers gradually learn to identify and then possibly to identify themselves" (p. 133). Because of the short duration of my study, I was not able to examine the positional identities that are enacted over a long period of time and there was not time for a new figured world or a community of practice to fully develop. However, I was able to examine the social positions, what I refer to throughout as *in-the-moment identities*, that were enacted day-to-day.

Positional identity is displayed through students' social position in terms of dispositions and in-the-moment identities. These identities come from "a set of dispositions towards themselves in relation to where they can enter, what they can say, what emotions they can have, and what they can do in a given situation" (p. 143).

Positioning Theory

My use of the term in-the-moment identity is similar to Wood's (2013) term *micro-identity*, which is how a student positions himself or herself or someone else positions them in any instant of time. Wood claimed that positioning theory (van Langenhove & Harré, 1999) is a way to understand students' micro-identities because it documents the exact moment when the identity was enacted. She reported that "framing identity as interactional positioning means that a close examination of interactions can reveal how identities are assumed, abandoned, elaborated, and altered over short periods of time and across situations" (Wood, 2013, p. 778). This is because positions are fluid and change based on the different situations people encounter (van Langenhove & Harré, 1999). Thus, I have drawn on positioning theory to inform my examination of students' in-the-moment identities.

Positioning theory is "the study of local moral orders as ever-shifting patterns of mutual and contestable rights and obligations of speaking and acting" (van Langenhove & Harré, 1999, p. 1). It examines the dense-texture of interactions between different people from a person's own point of view as well as others. It is "the discursive construction of personal stories that make a person's actions intelligible and relatively determinate as social acts and within which the members of the conversations have specific locations" (van Langenhove & Harré, 1999, p. 16). People can be positioned by themselves or by others in different ways such as being strong or weak, good at math or bad at math, the boss or the follower, and so on. People are positioned based on conversations and actions. These conversations create different storylines and how the people position themselves become connected to the storylines. For example, a teacher could position herself giving instructions and controlling the activities completed in the class. The conversations or actions of the teacher position the other people in the class as students. These positionings reflect one's personal identity (van Langenhove & Harré, 1999).

Use of Conceptual Framework

Aspects of the theories of communities of practice (Wenger, 1998), positional identities (Holland et al., 1998), and positional theory (van Langenhove, & Harré, 1999) comprised a conceptual framework for this study. I kept these ideas in the forefront throughout the whole study. Communities of practice was an overarching theory. I wanted to have students experience mathematics through a community of practice different from traditional school mathematics, and experience it in a community more similar to that of a mathematician through the use of unsolved mathematics problems. I use Holland et al.'s idea of identity as opposed to Wenger's idea of identity because I found his definition of identity more clear. Positional identities were a way to define and make sense of student's in-the-moment identities that were displayed through their dispositions.

Chapter One Summary

Problem solving has been a major portion of mathematics education research (e.g., Cai, 2003; Lester 1994; Schoenfeld, 1992), but the idea of unsolved mathematics problems has yet to be researched. We also know that students do not have the same dispositions about mathematics that mathematicians have (Boaler, 2016). This study explores students' dispositions while they engage with unsolved problems. Communities of practice (Wenger, 1998), positional identities (Holland et al., 1998), and positional theory (van Langenhove & Harré, 1999) make up the conceptual framework used to explore the study.

The chapters that follow describe the details and results of my dissertation study on elementary students engagement with unsolved mathematics problems. In Chapter II, I summarize the literature on elementary students' problem-solving, dispositions, and in-the-moment identities enacted during problem solving. In Chapter III, I describe the semi-structured, task based interviews used for this qualitative research study, the participants, the after school program demographics, the context of the problem-solving sessions, data sources, and data analysis. In Chapter IV, I share the results from the analysis for the study. More specifically, I discuss the results from the interviews, the dispositions the students displayed while they engaged with unsolved mathematics problems, and two case studies from two students that participated in the study. In Chapter V, I end the dissertation by providing the conclusions, implications, limitations, and recommendations.

CHAPTER II: REVIEW OF LITERATURE

My examination of the current literature includes four main sections: unsolved mathematics problems, problem solving, dispositions towards mathematics, and students' mathematical identities when problem solving. These four topics are pertinent to my study. First, unsolved problems and problem solving are the main focus of this study. The students engaged with unsolved mathematics problems. Second, my research question is based on students' dispositions while they engage with unsolved mathematics problems. This means dispositions was a critical component of the study. Lastly, I explored the different in-the-moment identities and statements of self-concept the students displayed through their dispositions while they were engaged in unsolved problems, creating a need to review the literature on students' mathematical identities during problem solving.

Unsolved Mathematics Problems

Mathematics educators have suggested that students need to experience mathematics similar to how mathematicians experience mathematics (Boaler, 2016; Lampert, 1990). Further, several mathematics educators and mathematicians have suggested that students engage with unsolved mathematics problems (Frenzel, Pekrun, Dicke, & Goetz, 2012; Hamiton & Saarnio, n.d.; Schoenfeld, 1992; Patchter, 2015). Frenzel, Pekrun, Dicke, and Goetz (2012) and Schoenfeld (1992) both have suggested all students should engage in unsolved or unexplored problems as a way to create a positive emotional experience for students.

In 2013, mathematicians gathered with mathematics educators at a conference to discuss and encourage unsolved problems in all grades, kindergarten through twelfth grade (Hamiton & Saarnio, n.d.). At the conference, attendees selected 13 unsolved problems to be the representative problems with one problem being selected per grade level. They also selected extra problems that would be appropriate for different grade levels. The goals of each problem were that the problem be appropriate for the grade level curriculum, be engaging to students and fun, not confuse students, and be easy and cheap for a teacher to implement. The authors did not share why they thought students should engage with unsolved mathematics problems; they only gave their goals for selecting the problems.

Lior Pachter (2015), a mathematician, claimed that "the emphasis on what K–12 students ought to learn about what is known has sidelined an important discussion about what they should learn about what is not known" (Pachter, 2015, para. 2). He reported that students should engage with unsolved mathematics problems and he generated a list of unsolved problems that match the Common Core State Standards for each grade level. His requirements for the problems included that they must be interesting to the students, understandable, and have a balance with different areas of mathematics focus. He stated that by introducing students to unsolved problems they would be stimulated to ask questions in mathematics where the answer is unknown.

Through an Internet search, a blog—that is an anecdotal report only—was found. It was by a mathematics professor at the University of Kentucky, Benjamin Braun (2015). In the blog, Braun encouraged giving undergraduate mathematics student unsolved problems to does as class work or homework. To do this, he began by giving an unsolved problem to the class. After they worked on the problem for 15 minutes, he told the students that they had been working on an unsolved problem. He found students then shift their perspective of the problem as something that was simple to something that was nearly impossible. From presenting these problems, Braun reported that students were surprised at how simple the problems seem, they shift their perspective from only trying to get the correct answer to discussing the authentic nature of how mathematicians work, the students thought more about sense making and perseverance than finding a correct answer, and students learned that failure was completely acceptable and normal in the mathematics field. Braun had his students reflect in an essay on engaging with the unsolved problems. Some of the students stated they had relief and joy because they could work on the problems with no expectations of solving it. Other students said they felt defeated and frustrated because they knew they would not be able to solve the problem. However, both groups of students said that it was the best moment in the Braun's course.

Some people (e.g., Frenzel et al., 2012; Hamiton & Saarnio, n.d.; Schoenfeld, 1992; Patchter, 2015) have suggested that students should be presented with unsolved problems to experience mathematics similar to how mathematicians experience it, learn about the unknown, and ask questions in which the answer is not known, but none of them report on actually having done so. Only an anecdotal report with undergraduate students has reported on students' engagement with unsolved problems. With no research on students' engagement with unsolved problem, what do we know about elementary students engagement with problem solving?

Problem Solving

Problem-solving research primarily began and developed from Pólya's work (Kilpatrick, 1985; Lesh & Zawojewski, 2007). Pólya's (1957) reported on heuristic, which can be described as "studying the methods of solution" (p. vii). In other words, Pólya focused on the process and strategies students used to solve problems. He created the famous four steps to address a problem: (a) understand the problem, (b) devise a plan, (c) carry out the plan, and (d) look back. From Pólya's work in the 1950s to now the emphases in the research on problem solving has changed. Lester (1994) summarized the research focus shift from 1970 to 1994. From approximately 1970 to 1982 the problem-solving research emphasis was on problem difficulty and characteristics of people who were successful at solving problems. During the time of 1978 to 1985, the primary research focus was on comparing successful and unsuccessful problem solvers. Lester stated that the focus of problem solving research from 1982 to 1990 was focused on metacognition, training in metacognition, affects, and beliefs in relation to problem solving. However, other researchers, such as Lesh and Zawojewski (2007), argued that similar topics were the focus of research before the 1990s (e.g., types of problems students were asked to solve in schools, distinctions between good and poor problem solvers, and problem-solving instruction).

Since the 1990s, the suggested focus of research has been on mathematical problem solving outside of school settings (Lesh & Zawojewski, 2007). Mathematics educators have suggested that problem solving research should focus on modeling and theory development (Lesh & Zawojewski, 2007). Research in relation to problem solving has been completed on authentic problem solving or modeling (e.g., Izsák, 2003; Magiera & Zawojewski, 2011; Verschaffel & De Corte, 1997), critical mathematics or teaching mathematics for social justice (e.g., Gregson, 2013; Gutiérrez, 2013; Gutstein, 2016), contextualized problem solving (e.g., Lubienski, 2000), and problem posing (e.g., Cai, Hwang, Jiang, & Silber, 2015). However, little is known about how students learn in environments that are mathematically rich outside of a school setting (Lester & Kehle, 2003).

A dominant part of elementary mathematics is whole number concepts and operations (Verschaffel, Greer, & De Corte, 2007). Verschaffel, Greer, and De Corte (2007) summarized the research on whole number concepts and operations for elementary students. They concluded that when students beginning formal schooling (kindergarten), they can solve words problems that are additive and multiplicative; however, by the time students are learning multiplication and division, they have "some dissociations between calculations and situations, that is to say they have learned that, for the purposes of finding the answer, the calculation, once identified, can be done 'off line'" (p. 589). Vershaffel et al. furthered the argument by stating

> that the stereotyped and artificial nature of word problems typically represented in mathematics textbooks, and the discourse and activity around these problems in traditional mathematics lessons, have detrimental effects on students' disposition towards mindful and realistic mathematical modeling and problem solving. (p. 603)

Vershaffel et al. shared an example of a question students answered from to help explain how children just answer questions without making sense of what the question is asking, such as, "There are 26 sheep and 10 goats on a ship. How old is the Captain?" (Vershaffel et al., 2007, p. 587). Overall, many students answer questions but do not take "into account realistic considerations about the situations described in the text, or even whether the question and the answer make sense" (p. 586).

Researchers have documented that students can struggle with problem solving because of a dissociation and not making sense of the question and answer (Vershaffel et al., 2007). However, there have been studies on how to advance student thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Fraivillig, Murphy, & Fuson, 1999). After completing a study by observing 19 Grade 1 classrooms over a course of 5 years, Fraivillig, Murphy, and Fuson (1999) identified several strategies that could advance students' mathematical thinking. These strategies included having high expectations for all students as well as having students reflect and draw generalizations, find relations hips between concepts, share and reflection on different solution methods, create their own problems, and try different solution methods.

Carpenter, Fennema, Peterson, Chiang, and Loef (1989) found ways to help advance students' thinking through a project called cognitively guided instruction (CGI). This major research study was conducted with elementary students and their teachers on problem solving. The CGI teams worked with the elementary teachers to have the elementary teachers' students learn addition and subtraction through word problems. The CGI teachers focused on problem-solving and not number facts, they had students use multiple strategies to find solutions, and they built on students' existing knowledge during instruction. The results were that the students of the CGI teachers had better fact recall and problem solving skills than the control students, and the students self-reported they were more confident in participating in problem solving. Vershaffel et al. (2007) shared a major conclusion from all the research conducted by the CGI team, that "students' thinking and problem solving can profoundly affect teachers' cognitions and beliefs about arithmetic classroom learning and instruction, their classroom practices, and most important, their students' learning outcomes and beliefs" (p. 607).

Dispositions

Students' dispositions while they engage in problem solving are an important area of research in mathematics education (Goldin et al., 2016). A *disposition* is a tendency "to act in a certain manner under given circumstances" (disposition, 2017). So one's tendency to act a certain way when solving mathematics problems would be influenced by three factors: affective (tendency to believe or exhibit an emotion about what math is), conative (tendency to display diligence, effort, or persistence), and cognitive (tendency to use, or not, mathematical practices, such as justifying). The following three sections will discuss affective, conative and cognitive.

Affective

We know that children use a variety of problem solving strategies when they engage with problems in mathematics, however, "we know little about the affective factors that may contribute to this variation" (Ramirez, Change, Maloney, Levine, & Beilock, 2016). Goldin et al.

(2016) stated research on the area of affect in mathematics education “deserves greater attention” (p. 2) because it is such an important process when children engage in nonroutine problems. Affective dispositions towards mathematics include students’ beliefs about mathematics and emotions they display when doing mathematics (Beyer, 2011). In this section I examine research on affective dispositions in respect to the students’ beliefs about mathematics and the emotions displayed while problem solving.

Beliefs. It has been documented that mathematicians have very positive dispositions towards what mathematics is (e.g., Boaler, 2016; Burton, 1999; Thurston, 1994) and that many students have vastly different views of what mathematics is (e.g., Allen, 2004; Beilock, Gunderson, Ramirez, & Levine, 2010; Su, 2010). Not only do students have negative views of mathematics, their mathematical dispositions tend to decline during middle school (Eccles et al., 1993).

Students’ view of mathematics. Many people have been documented to have fear or anxiety towards mathematics, including teachers and children as young as first grade (Beilock et al., 2010). Based on research, all students tend to find mathematics to be procedural and something unenjoyable (Allen, 2004). In 1992, Schoenfeld listed beliefs students have about mathematics:

- Mathematics problems have one and only one right answer.
- There is only one correct way to solve any mathematics problem—usually the rule the teacher has most recently demonstrated to the class.
- Ordinary students cannot expect to understand mathematics; they expect simply to memorize it and apply what they have learned mechanically without understanding.

- Mathematics is a solitary activity, done by individuals in isolation.
- Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
- The mathematics learning in school has little or nothing to do with the real world. (Schoenfeld, 1992, p. 359).

In 2007, Lesh and Zawojewski stated the students' beliefs about the nature of mathematics identified by Schoenfeld in 1992 have not changed. Masingila (2002) studied how middle school students viewed mathematics. She found that many of them have a very narrow view of what mathematics is. They described mathematics as something to learn in school, a set of rules, or numbers. Only three of the 20 students she studied had a broader definition of mathematics and viewed it as a way of thinking about something.

Research was found on how people at the undergraduate level view or described mathematics (e.g., Crawford, Gordon, Nicholas, & Prosser, 1994; Petocz et al., 2007). Crawford, Gordon, Nicholas, and Prosser (1994) conducted a study with 300-university freshman on their perspectives of mathematics. They found that students viewed mathematics as a procedure to learn through memorization or a set of rules to follow. Petocz et al. (2007) evaluated 1,200 undergraduate students' dispositions towards mathematics from five different countries through an open-ended survey containing three questions. They found the students' dispositions ranged from ideas that mathematics is calculations to be done with numbers, a way to manipulate numbers, and a "collection of isolated techniques" (Petocz et al., 2007, p. 446). Students' views of the nature of mathematics are different when they are compared to how mathematicians describe what mathematics is.

Nature of school mathematics. There are three kinds of mathematics students can experience in a school setting: school mathematics, mathematicians' mathematics, and everyday mathematics (Civil, 2002). Civil (2002) described school mathematics as traditional mathematics in which students do computations, follow procedures, and memorize algorithms; there is a teacher, a textbook, and an answer key; and the teacher always knows the answer. Mathematicians' mathematics in school is when students engage in discussions about mathematics, collaborate on challenging tasks, have to give justifications, persist in a mathematics task, and develop their own strategies for solving problems. Everyday mathematics is described as the mathematics that is learned or that occurs outside of a school setting. This is the type of mathematics when a person might not be aware he or she is doing mathematics or the mathematics may be hidden. An example of this could be how a child might learn about giving change back to someone when they conducting a lemonade sale.

It has been documented that students who experience school mathematics tend to find mathematics as a subject that is procedural, structured, and rule-bound (Boaler, 2016; Lampert, 1990). There have also been studies on the topic of everyday mathematics that have documented how people complete mathematics in different situations with little or no error in their everyday lives, but when given a similar task on paper they might not do as well (e.g., Masingila, 1994; Nunes, Schliemann, & Carraher, 1993; Saxe, 1988). Studies have been completed when students explore mathematics like mathematicians through the use of robotics. They view the activity not as learning mathematics and vastly different from school mathematics (e.g., Nickels & Cullen, 2017; Norton, 2006; Sklar, Eguchi, & Johnson, 2003). These studies document how mathematics is viewed differently when it takes place outside of a school setting or through nontraditional learning.

Traditionally mathematics in classrooms gives students a limited view of what mathematics is, and students view mathematics differently when it is outside of a school setting (Masingila, 2002). It is important that we examine the different practices in place in a classroom. Classroom practices are not only linked to what students learn (Boaler, 2016; Boaler & Greeno, 2000), but also "the nature of these practices has been shown to affect the ways that students come to think about the domain" (Gresalfi & Cobb, 2006, p. 50).

Mathematics outside of school. Saxe (1988) and Nunes, Schliemann, and Carraher (1993) explored the difference between street mathematics and school mathematics with Brazilian children. Saxe's study compared street vendors and school children that were 10-12 years old. Saxe found that the street vendors had developed strategies and were able to solve arithmetic and ratio problems that contained large numerical values whereas the school children were not able to. Nunes et al. found that nonschool children (street vendors) were better at solving a problem in a real situation than during a formal testing situation. It was conjectured that the children could solve the problem in context because they were able to have context and understand the meaning of the problem while in the formal setting the context was dropped.

Similar to the studies done with Brazilian children using everyday mathematics, researchers have found that when students are learning mathematics through robotics, they do not make the connection that they are learning mathematics, do not view it as learning mathematics, or view it as different from school mathematics (Nickels & Cullen, 2017; Norton, 2006; Sklar, Eguchi, & Johnson, 2003). Nickels and Cullen (2017) researched critically ill students' learning of mathematics through robotics. Nickels and Cullen focused on one student, Amelia, who had Acute Lymphoblastic Leukemia. Prior to engaging in robotics, Amelia stated that she "hated math." After working with robots, which she did not view as school mathematics, she described her experience as, “I was learning math that was important to me. You never do something only once with a robot, so the math I learned I used over and over again. I’ll never forget it now” (p. 69). She viewed learning mathematics through robotics differently from how she viewed school mathematics.

Norton (2006) found that when students engaged with robots, their mathematical content improved, but they did not connect the mathematical concepts they learned to the mathematics they learned in school. Norton explained that the connections needed to be discussed with the students. Sklar, Eguchi, and Johnson (2003) researched students’ perceptions of their mathematical improvement through competing in robotic competitions. They found only 30% of students thought their mathematics knowledge improved through competing in robotic competitions. However, 80% of the children’s leaders thought their mathematics skills improved. In all three studies on students’ use of robotics, the students were learning mathematics through the robotics, but they did not view it as learning mathematics or thought of it differently than school mathematics.

Collaboration. Studies have been completed that document students’ thoughts or opinions on collaboration (e.g., Florez & McCaslin, 2008; Gillies, 2003; Mulryan, 1994). Florez and McCaslin (2008) documented that the Grades 3–5 students in their study enjoyed collaboration. Gillies (2003) reported that the Grade 8 students in their study thought group work allowed them the opportunity to complete quality work and found the group work enjoyable. Mulryan (1994) described that the Grades 5 and 6 students in their study thought collaboration allowed them chance to learn how to work with others and receive different ideas of how to solve the mathematics. All three of these studies were self-reported data from the students in the study.

Two studies were found that documented collaboration as a way to develop students' productive dispositions towards mathematics (e.g., Gresalfi, 2009; Jansen, 2012). Gresalfi (2009) researched the ways dispositions were exhibited during moments of interaction in mathematics classrooms and how classroom dynamics contributed to the dispositions. The data were collected in two classrooms of eighth-grade students who were taught the same content but had different teachers. One teacher taught using a collaborative approach, showed students strategies, and communicated expectations for working together. The other teacher would have students work in groups but focused on the mathematical work instead of emphasizing productive collaborative practices. Based on the analysis of collected data and comparison of classrooms, Gresalfi concluded that students' dispositions towards productive beliefs about how mathematics was learned were enhanced with opportunities for interaction with other students and those interactions allowed students more opportunities to engage with mathematics content. She claimed that one aspect for developing productive dispositions towards mathematics is through successful collaboration and that teachers play an important role in the development of students' dispositions.

Jansen (2012) observed two Grade 6 classrooms that contained a total of 54 students. The classroom teachers held the belief that a person's mathematical competence could be improved, understanding a topic is more important than completing a task, and collaboration is valuable. Jansen found that through teachers' facilitation of group work, students could develop these productive dispositions towards mathematics. It has been documented that collaboration could be a way for students to develop positive dispositions towards mathematics (Yackel, Cobb, & Wood, 1991). However, Jansen (2012) concluded, "more research is needed to identity conditions that lead to the development of students' productive dispositions toward mathematics in the context of group work" (Jansen, 2012, p. 38).

Emotions. Researchers have claimed that students experience positive and negative emotions when they engaged in problem solving (Goldin, 2000a; Hannula, 2015). However, most of the research conducted on emotions and affect has been completed through surveys and not during students' engagement with problem solving (Hannula, 2015). Several research studies were found that were conducted on children's emotions during problem solving situations in the classroom (e.g., Daher, 2015; Evans, Morgan, & Tsatsaroni, 2006; Hannula, 2015; Williams, 2002) and at home (e.g. Else-Quest, Hyde, & Hejmadi, 2008). Those researchers have conjectured that students experience positive and negative emotions while engaged with mathematical tasks (Hannula, 2015); students' emotions are linked to their positionings (Daher, 2015; Evans et al., 2006); and when students displayed positive emotions, they were able to develop mathematical understanding (Williams, 2002; Else-Quest et al., 2008).

Positive and negative emotions. Hannula (2015) researched the emotions of a 10-year-old student while solving a geometric solids problem that was open-ended. The researchers analyzed the student's interactions with his teacher and two of his classmates who sat next to him. The researchers main findings included that emotions are a crucial aspect of problem solving. Different emotions are beneficial to student learning, and teachers need to be aware of students' emotions while problem solving. The emotions that had negative aspects on students' problem solving skills included boredom and being emotionally flat. The emotions that help have a positive aspect on students' problem solving included the teacher being enthusiastic and the students and teacher having emotion regulation. The teacher can demonstrate emotion regulation through modeling as a way to increase emotion regulation in students. Students talking and joking with peers is a way for the students to deal with the different emotions they encounter while problem solving.

Link between emotions and positionings. Two research studies on emotions during problem solving in the classroom linked students' emotions with their positionings (Daher, 2015; Evans et al., 2006). Daher (2015) researched the positions and emotions of four Grade 7 students while they engaged with a modeling activity relating to ratios. The students worked on the task for 3 consecutive hours. The researchers found that three of the four students acted as insiders (accepted by peers) on the task and one student acted as an outsider (rejected by peers) until the completion of the task when he switched his positioning to an insider. Positive emotions were experienced and expressed by the insiders while negative emotions were experienced and expressed by the outsider. The researcher concluded that the positionings and emotions the students experienced were based on the students' previous learning experiences, being familiar with group work, and the students' characteristics such as authoritative and demanding. The one student who was considered an outsider, by himself and his group members, experienced negative emotions.

Evans, Morgan, and Tsatsaroni (2006) researched the emotions of three boys in Grade 8 while they worked together on a mathematical task that involved finding the edge lengths of a rectangular trapezium to measure the distance a sprinkler could throw water. Two of the boys took positions of insider and the other boy was considered an outsider. The researchers anticipated that their positioning was already formed prior to the study. The insider students displayed emotions of excitement while the outsider student exhibited emotions of anxiety while working on the mathematical task. The researchers concluded the emotions presented by the students were linked to the students' positionings. Both research studies by Daher (2015) and

Evan et al. found students who felt like insiders experienced positive emotions, and students who were considered outsiders displayed negative emotions.

Positive emotions influence on mathematical understanding. Williams (2002) and Else-Quest, Hyde, and Hejmadi (2008) found there was an association between students displaying positive emotions during problem solving and being able to develop mathematical understanding. Williams (2002) researched students' emotions and their mathematical behavior to see if there was an association. However, this study has limitations because Williams only examined three high school students for 13 minutes. The three students were seniors in a calculus class as they engaged in a challenging problem they were not familiar with. One student was considered an outsider in the problem-solving situation because of engagement in other activities during the problem solving. The other two students displayed numerous positive emotions during the exploration and developed a new cognitive structure and mathematical insight. Overall, the researcher concluded there was an association between positive affect (emotions) and the students' ability to develop a new cognitive abstraction.

Else-Quest et al. (2008) researched the emotions mothers and their children displayed while they solved mathematical tasks at home. The children were 11 years old, and the tasks were pre-algebra activities. The researchers analyzed 160 mother-child dyads. Both mothers and children displayed positive and negative emotions during the study. The most displayed negative emotions consisted of distress, frustrations, and tension. The most displayed positive emotions were affection, joy, pride, and positive interest. The emotions displayed by the child were similar to the emotions of the mother. The children who displayed positive emotions while working on the tasks with their mothers had significantly higher performance on the posttest with controlling for the children's baseline performance than the children who displayed mostly negative emotions. These studies indicate that students who display positive emotions while engaged in mathematics tasks can have a positive effect on mathematical outcome (Else-Quest et al., 2008; Williams, 2002).

Conative

NCTM (2014), the NGA & CCSSO (2010), and the National Research Council (2001) support students' development of the productive disposition of perseverance, and research has indicated that it is productive for students to develop perseverance (e.g., Hiebert & Grouws, 2007; Kapur, 2010; Warshauer, 2015). A person would display a positive conative disposition when they persevere while solving a challenging task. Conative dispositions are defined as "a tendency or inclination to purposively strive, exercise diligence, effort, or persistence in the face of mathematical activity" (Beyers, 2011, p. 23). Schoenfeld (1989) questioned secondary students' mathematical beliefs on perseverance through a survey. He asked them how long they would spend on a problem before they knew it was impossible. The average response was only 12 minutes. That amount of time on a task would not be considered to having displayed a positive conative disposition. Previous research in mathematics education has found that perseverance is related to mindset (Duckworth, Peterson, Matthews, Kelly, 2007) and self-efficacy (Pajares, 1996). In the next three sections, I will discuss mindset, self-efficacy, and productive struggle as it relates to research on perseverance in problem solving.

Mindset. Dweck (1999) has identified two different types of mindsets: growth and fixed mindsets. A growth mindset is a belief that difficult tasks and challenges can be mastered through one's willingness to try. A fixed mindset is the idea that competence is innate and not changeable. When a person has persistence or perseverance, they would be considered to have a growth mindset (Duckworth et al., 2007). Hong, Chiu, Dweck, Lin, and Wan (1999) found that people with different mindsets solve problems in different ways. When someone with a fixed mindset has trouble completing a mathematics task, her or she would take this as a lack of ability or intelligence. They might believe working hard on a task would only result in a waste of time or embarrassment. When a person with a growth mindset has trouble completing a mathematical task or has negative results, he or she would then take it as a sign to work harder. A person with a growth mindset has a tendency to show perseverance during a challenging task whereas someone with a fixed mindset may not show perseverance. Other studies(e.g., O'Shea, Cleary, & Breen, 2010; Shen, Miele, & Vasilyeva, 2016) have found similar results to Hong et al. (1999); however, these studies were conducted with undergraduate students. Liu, Chiu, Chen, and Lin, (2014) also found similar results to Hong et al. Liu et al. studied 264 high school seniors. They surveyed participants that had a high fear of being laughed at to explore their perceived ability, their perception of threat in relation to an unfamiliar challenge, how they confronted challenging situtations, and how they perceived completing a cognitive task they were not familiar with. Liu et al. concluded that students that viewed great value in confronting unfamiliar challenges and confronting challenging tendencies had a lower fear of being laughed at than students who had the opposite view. These results indicate that students who enjoy confronting challenges and are not embarrassed if they make a mistake display a growth mindset.

To develop persistent and flexible problems solvers who have a growth mindset, they need to experience mathematics instruction that enhances this idea. Suh, Graham, Ferrarone, Kopeing, and Bertholet (2011) proposed that classroom practices can develop this idea. These practices include: posing mathematical tasks that have different entry levels, engaging in mathematics communication and discussions during activities, and establishing a community that embraces challenges and mathematics inquiry.

While Suh et al. (2011) documented ways to develop a growth mindset, other researchers have found that a mindset can change from fixed to growth (e.g., Blackwell, Trzesniewski, & Dweck, 2007; Good, Aronson, & Inzlich, 2003). Blackwell, Trzesniewski, and Dweck (2007) conducted a comparison study with students in Grade 7. Students in both groups, the treatment and the control, had declining grades prior to the study. The treatment group of students ($N = 48$) participated in an eight sessions in which they learned about study skills and having a growth mindset. The control group ($N = 43$) had the same number of sessions but only learned about study skills. In the end, the treatment group had improved grades; however, the control group continued to have a decline in grades. The researchers concluded that the participants in the treatment group could change their mindset from fixed to a growth.

Self-efficacy. Self-efficacy is defined as, “beliefs in one’s capabilities to organize and execute the courses of action required to produce given attainments” (Bandura, 1997, p. 3). Bandura (1997) emphasized that if someone feels knowledgeable and proficient in something, he or she will be more likely to persevere when engaging in a challenge. Pajares (1996) further explained that self-efficacy determines “how much effort people will expend on an activity and how long they will persevere—the higher the sense of efficacy, the greater the effort expenditure and persistence” (p. 3)

There have been two major areas of focuses on research in self-efficacy (Goldin et al., 2016). The first area of focus was on an examination of self-efficacy and career choice or course selection. The second area of focus has been on the relationship between self-efficacy and motivation and achievement. Several research studies has documented that self-efficacy predicts persistence in solving problem, achievement, and interest in mathematics (Larson, Piersel, Imao, & Allen, 1990; Pietsch, Walker, & Chapman, 2003). Larson Piersel, Imao, and Allen (1990) found from research conducted with undergraduate students that self-efficacy was a significant predictor of successful problem solving and that the students with high self-efficacy had positive coping strategies for problem solving such as focusing on the task and cognitive restructuring of the task. Pietsch, Walker, and Chapman (2003) surveyed 416 high school students on their self-concept and self-efficacy. They also examined students' mathematics tests they took at the end of a term. Pietsch et al. found that students' "mathematics self-efficacy was more highly related to performance in mathematics when compared with the competency component of mathematics self-concept" (p. 598).

Schunk and Richardson (2011) described strategies for improving students' self-efficacy towards mathematics. They shared to use activities with students that would be of high-interest to them and where they can work in small groups because the students will be more motivated to learn and engaged which in turn increases their self-efficacy. Next, they suggested having the students set goals and evaluate their own progress. Lastly, they suggested that teachers teach self-regulatory skills. Schunk and Richardson concluded that most of the research conducted on self-efficacy has been in relation to computation and procedural skills, but a new trend and something to investigate is "motivation and self-efficacy with higher-order mathematical concepts and thinking" (p. 26).

Productive struggle. A student may exhibit a range of emotions when persevering with a mathematics problem. Struggle is when students use intellectual effort to make sense of a mathematics topic or solve a problem that does not make immediate sense and is challenging (Hiebert & Grouws, 2007). It is expected when students engage with unsolved problems, they will experience struggle. The research literature in mathematics education has reported that having students engage in productive struggle is beneficial for their mathematics learning

(Hiebert & Grouws, 2007; Kapur, 2010; Reinhart, 2000; Warshauer, 2015), but according to Warshauer (2015) and Zeybek (2016), there have been few studies completed on examples of students engaged in productive struggle and the previous studies have been limited to only that struggles occur. Because of this, Warshauer and Zeybek completed an exploratory study to examine what productive struggle looks like with middle school students and preservice teachers.

Warshauer (2015) studied 327 middle school students engaged but struggling through proportional reasoning tasks at three different middle schools. Warshauer not only found that productive struggle was an important tool in supporting students in doing mathematics and could enhance their understanding of mathematics, but she also found that students' persistence in productive struggle was related to the classroom sociomathematical norms. Classrooms that focused discussion and allowing students to explain their solutions had more success with students staying engaged and on task while struggling as compared to classrooms that did not have students explain their solutions. Lastly, Warshauer found that teacher responses or moves while students were engaged in struggle had an effect on their struggle and whether it remained productive or became unproductive. These responses or moves are based on individual students' beliefs and content knowledge. An example of a move or response might be to focus the student without lowering the cognitive demand by questioning, clarifying, probing, or confirming what a student is thinking.

Zeybek (2016) completed a study on the types of struggles preservice teachers experience while they are engaged in a high cognitive demand task that is nonroutine. The participants were 48 middle level preservice mathematics teachers. Zeybek concluded that the preservice teachers struggled to begin and carry out the task. Also, they made errors, exhibited misconceptions, and were unable to explain their ideas.

Kapur (2010) researched productive struggle with Grade 7 students; he compared two groups of students, one that had students learn through productive struggle with complex problems and the other that had students participate in a traditional lectures. The students who experienced productive struggle significantly outperformed the students who learned through lectures on a posttest. Upon completing his research, Kapur recommended that to keep students engaged in productive struggle, students should choose the problem they want to engage with, explain or discuss their work, and compare different types of solutions or strategies to solve problems.

Cognitive

Beyer (2011) proposed a conceptual framework for dispositions with respect to mathematics, however it is not well developed. This framework includes affective and conative dispositions, which is consistent with the mathematics education literature towards dispositions. He also included the idea that there are cognitive mental functions. He developed this idea from a 1958 psychology dictionary from English and English, which defined cognition, affection, and conation as "the three categories under which all mental process are classified" (pp. 92–93). He also used Snow, Corno, and Jackson (1996) to further agree why cognitive should be a disposition. They argued that all of the mental processes organisms enact are distinguished through cognition, affect, and conation (Snow, Corno, & Jackson, 1996). That is different from how the National Research Council (2001) defined a productive disposition. They said it is "the tendency to see sense in mathematics, to perceive it as both useful and worthwhile, to believe that steady effort in learning mathematics pays off, and to see oneself as an effective learner and doer of mathematics" (p. 131). The National Research Council's definition includes the affective and conative aspect but does not completely include the cognitive aspect. However, I believe it is important to include a cognitive aspect so people can learn if they students can make sense of the unsolved problems and have successful, meaningful engagement. Therefore, I will include one cognitive aspect—attempting to justify—in this research study.

According to Beyer (2011) a cognitive mental function in mathematics is when a person engages in a cognitive mental process, such as giving a justification or proof. Mathematical proving is something that is not typically introduced to students until middle school (Lin & Tsai, 2016) or high school (Stylianides, 2007). At the elementary level, mathematics is typically focused on finding answers and correctness and does not include students justifying their answers (Kieran, 2004). Carpenter, Franke, and Levi (2003) have stated that there are three ways students tend to give a justification or argument for a mathematical idea: (a) appeal to authority, (b) justification by example, and (c) generalizable argument. Appeal to authority would be when a student shares a rule or procedure that a teacher or someone in authority has taught them. An example would be when a student makes a case about something being true through an example such as three plus three is six and six is an even number, therefore whenever you add an odd with an odd, you will get an even number. For a generalized argument, students would share a logical argument that would apply to all cases in the conjecture.

Elementary students do not typically give generalizable arguments, but as they advance in grade level, students should see how just giving examples limits their arguments and should be encouraged to develop more generalized arguments (Carpenter, Franke, & Levi, 2003). This idea has been documented at the middle school level. O'Dell et al. (2016) found Grade 8 students originally could only give an explanation of area of a circle through appealing to authority by stating a formula they had been given. However, after completing three tasks of guided discovery, they were able to justify the area formula for a circle through using a triangle that had the height of a radius and a base equal to the length of the circumference.

At the elementary level, most justification is completed through examples, but this can be improved with teacher support and students engaging in mathematical reasoning and discussion (Keith, 2006). Ball (1993), Keith (2006), and Lin and Tsai (2016) all explored mathematical justification of the sum of even and odd numbers with second or third grade students. Keith found second grade students' justifications were based on the definition of even and odd numbers and they used blocks to determine whether the sums were shared equally when divided in half or if one block was left over. Ball's third grade students gave similar justifications but also concluded that an even number is a multiple of two and an odd number is a multiple of two plus one. Lin and Tasi found students were able to advance from examples or a sequence of examples to give generalizable arguments that included the definition of odd and even similar to the way the students did it in Ball and Keith's studies. Similarly, Rumsey (2012) examined if students' justification at the elementary level could develop when instruction promoted mathematical argument. She focused on the development of arithmetic properties and found that argumentation was beneficial for teaching students arithmetic properties. However, due to the short span of the study, she did not find changes in the development of students' arguments.

It has been documented that elementary students can make conjectures and develop mathematical claims (Ball, 1993; Keith, 2006; Lin & Tsai, 2016). Carpenter et al. (2003) stated that to help students develop justification teachers could restate a conjecture, give examples, and build on conjectures that have been justified. Visual representation also helps students' development of justifications (Schifter, Monk, Russell, & Bastable, 2008). Many researchers have recommend that students engage in proof and argumentation in all grade levels and in all mathematics content areas (e.g., Ball & Bass, 2003; Carpenter et al., 2003; NCTM, 2014; Stylianides, 2007).

Identity

A person's actions based on their mathematical identities can be called a social position, and these social positions are demonstrated through dispositions (Holland et al., 1998). Two mathematics educators, Martin (2006) and Bishop (2012) both define identity in relation to activity, positioning, self-concept, and beliefs/disposition. Martin defined it as

> mathematics identity refers to the dispositions and deeply held beliefs that individuals develop, within their overall self-concept, about their ability to participate and perform effectively in mathematical contexts and to use mathematics to change the conditions of their lives. A mathematics identity encompasses a person's self-understanding of himself or herself in the context of doing mathematics." (p. 206)

Bishop defined identity as

> the set of beliefs that one has about who one is with respect to mathematics and its corresponding activities. An identity is dependent on what it means to do mathematics in a given context; as such, it is individually and collectively defined. Identities include ways of talking/acting/being as well as how other positions one with respect to mathematics. (p. 41)

Bishop further described that "identity is enacted through discourse, and at the same time, it influences one's discourse" (p. 45). I explored students' in-the-moment identities while they engaged in unsolved problems. Because of this, I describe previous research conducted on identity during problem solving situations.

There has been research completed on how identities can shift or develop through different mathematical experiences (Andersson, Valero, & Meaney, 2015; Bishop, 2012; Cobb, Gresalfi, & Hodge, 2009; Turner, Dominquez, Maldonado, & Empson, 2013; Wood, 2013; Yamakawa, Forman, & Ansell, 2009). Mathematical identities can shift during a period of a year (Andersson et al., 2015) or be influenced during one problem-solving situation (Wood, 2013). Discourse, interactions, and teachers' responses to students can influence students' mathematical identities (Bishop, 2012; Turner et al., 2013; Yamakawa et al., 2009).

Identities can be Shifted or Developed

Identities have been documented that they are able to shift or develop over time during a mathematics class, during problem-solving situations, and in-the-moment identities can shift in response to different contexts during one class period. Cobb, Gresalfi, and Hodge (2009) found that how students position themselves during problem-solving situations can influence their mathematical identities. Next, I describe how Andersson, Valero, and Meaney (2015) and Wood (2013) described and found shifting identities.

Andersson et al. (2015) researched secondary students' identities over a 1-year period to examine their shifts in participation. The students' identities were determined through interviews, surveys, and observations. The researchers focused on two students. One student thought mathematics was boring, and the other student said that she hated mathematics. Through observations, the researchers determined that these identities were not stable over the course of the year. The two students' identities shifted with context. When the mathematics class was focused on textbook work, the students reported that their mathematics identities were consistent, but when the mathematics activity shifted to open classroom discourses and group projects, the students' identities shifted through their use of terms like good, interesting, and useful to describe the mathematics. The researchers concluded that the identity narratives were determined over a longitudinal period but were "changeable when different aspects of context changed" (p. 157) and that identities are not stable as past research and literature indicated.

Wood (2013) researched fourth grade students' micro-identities (similar to how I am using the term in-the-moment identities) to examine how these identities can be shifted, negotiated, resisted, or taken-up during a mathematics lesson. Wood focused on one student, Jakeel, during a mathematics class to explore the micro-identities he displayed. Jakeel displayed different positions such as the mathematical student, the mathematical explainer, and the menial worker based on his interactions with the teacher, other students, and the mathematical tasks. These different positions gave him micro-identities as both mathematically capable and mathematically incapable. Similarly to Andersson et al. (2015), Wood concluded that mathematical identities could shift in response to different contexts.

Influence on Mathematical Identities

It has been documented that identities can change or shift (Andersson et al., 2015; Cobb et al., 2009; Wood, 2013). There are different reasons researchers have found for these shifts. I will describe three studies that demonstrate different ways that can influence the shifts in identity.

Bishop (2012) researched two seventh grade students' discourse of 13 days while they were engaged in a technology based mathematics unit to examine how the discourse patterns might affect students' mathematics identities. Through examining the students' discourse, Bishop concluded that the students positioned one student, Teri, as being the "smart one" and the other, Bonnie, as being the "dumb one." Teri took an authoritarian role during the problem solving and positioned herself as the mathematical expert while Bonnie positioned herself as dependent and mathematically helpless. Bishop concluded that with respect to mathematics how people interact and talk with each other can and does influence their identities.

Turner et al. (2013) researched students', who were English Learners, and their positioning during a mathematical discussion on problem-solving activities. The study took place in an after-school program for Grade 4 and 5 students. The researchers examined the different ways the students were positioned as agentive problem-solvers. They concluded that the teachers inviting students to share their thinking, giving students explicit statements to validate a solution or mathematical reasoning, and positioning students' ideas as important were ways to develop or contribute to positive identity development.

Yamakawa, Forman, and Ansell (2009) examined a third grade classroom twice a week for 4 months during their mathematics instruction. The researchers focused on two of the students, Ophrah and Pulak. Ophrah tended to take leadership roles during group work and liked to share her solutions during class discussions. Pulak was considered a "good math thinker" by other students and was highly proficient in mathematics. The classroom teacher revoiced Ophrah's and Pulak's contributions differently. She used Ophrah to communicate strategies and positioned her as mathematically proficient. The teacher positioned Pulak as advanced, however, very rarely would revoice his strategies to solve problems during class discussions. Throughout the study, Ophrah changed her positional identity from average to above average and Pulak's positional identity did not change. The authors concluded how teachers and others position a student can effect and influence the student's positional identity. For this dissertation study, I examined students' in-the-moment identities and described whether they shifted or developed based on students' positioning.

Chapter Two Summary

My goal for this study is to describe students' mathematical dispositions while they engage with unsolved problems. In this chapter I discussed the four main components of this research study: unsolved problems, problem solving, dispositions, and identity. In the next chapter I describe the design of the study, the problem-solving sessions, the analytic framework, and the phases of analysis. In the fourth chapter, I describe students' dispositions towards their beliefs about mathematics, the emotions they experience, and their perseverance with problem solving. I also describe several cognitive aspects students attempted to make while engaging with the unsolved problems. Lastly, I share detailed case studies of two participants in the study in reference to their in-the-moment identities and positioning.

CHAPTER III: METHODS

This study was conducted using a qualitative research methodology. The major purpose of qualitative research in education is to "discover and understand a phenomenon, a process, or the perspective and worldviews of people involved" (Merriam, 1998, p. 11). It helps us understand a phenomenon from the perspective of the participant and to "make sense of their work and the experience they have in the world" (p. 6). This methodology was appropriate for examining the research question that guided this study:

> What are the characteristics of students' dispositions toward mathematics when they engage in the exploration of unsolved problems?

I used a descriptive case study approach to examine students' dispositions as they engaged with unsolved mathematics problems. A descriptive case study "presents a detailed account of the phenomenon under study" (Merriam, 1998, p. 38) and gives a basic description of what happens. Descriptive case studies are useful in education research to examine an idea in which little research has previously been conducted (Merriam, 1998). This idea is consistent with this dissertation study because elementary students' engagement with unsolved problems has not been previously studied before. Miles, Huberman, and Saldaña (2014) define a case as "a phenomenon of some sort occurring in a bounded context" (p. 29). The case for this study is students' dispositions as they experience unsolved problems. This is the appropriate method for the study because it will allow a vivid and illuminating story to emerge.

Participants

The participants of the study included 10 Grades 4 and 5 students from an after-school program in the Midwestern region of the United States. I used purposeful sampling to identify the participants for the study. Purposeful sampling was an appropriate method because it "is based on the assumption that the investigator wants to discover, understand, and gain insight and therefore must select a sample from which the most can be learned" (Merriam, 1998, p. 61).

There were several reasons for choosing elementary students to be the focus of this study; these reasons were based on the research literature and grounded in my own experiences as a teacher. Research has shown that as students transition from elementary to middle school, their level of motivation in mathematics tends to decline (Haselhuhn, Al-Mabuk, Gabriele, Groen, & Galloway, 2007). It is important then, that students in the elementary grades develop strong, positive dispositions about mathematics. According to Middleton and Jansen (2011),

> if students have positive experiences in mathematics—experiences that present an appropriate level of challenge, coupled with a sense of control—they begin to anticipate their future engagement in mathematics optimistically, with a sense of enjoyment. If, however, students are not challenged, they may perform well in mathematics but their interest will wane over time. If they lack a sense of control, they can develop seriously negative motivations, including math anxiety and learned helplessness. (p. 29)

Frenzel et al. (2012) found that positive emotional experiences in mathematics were more beneficial for younger students (i.e., fifth graders) than older students (i.e., ninth graders). Moreover, they suggested that a way to create a positive emotional experience for students would be by having students work on exciting, real-life problems including unsolved or unexplored problems. This resonated with my own experience as an elementary school teacher. I witnessed Grade 4 students becoming excited with mathematics when it was presented to them in a problem-solving setting, and I suspected that working with unsolved problems would have the same effect.

I chose to conduct the study in a nonschool setting to avoid possible negative associations with students' school-mathematics experiences. Goldin (2000b), whose research has involved engaging students in task-based interviews, warned that

> children drawn from their regular classes to participate in interviews may see the interview as fundamentally a school activity, and respond both mathematically and emotionally as if the expectation is for them to produce "school mathematics" correct answers through previously learned algorithms although that is not the interviewer's intent. (p. 534)

He further explained:

> Consider again an interview taking place in an elementary school, whose purpose the child may believe is to test his or her understanding of the mathematics taught in school. The child may respond very differently to the mathematical content of the interview tasks both cognitively and affectively according to whether or not the mathematical topics were discussed in class recently, discussed long before, or never encountered previously, or according to whether or not they were tested in class. (pp. 534–535)

The nonschool setting I selected, the Midwestern After-School Program (pseudonym), is part of a community center. The program director, Leann (pseudonym), organizes a staff of volunteers that include community members and college students. Students work with program volunteers either one-on-one or in pairs. The goal is to help students reach their grade level in reading and to be promoted to the next grade level. During the after-school program, students are given a healthy snack, read for a minimum of 20 minutes, and complete all of the homework assigned by their classroom teacher. They also have some free time and participate in different enrichment activities.

The after-school program is funded almost entirely by donations that allow the students to attend for free or at a very low cost. To attend, students must be from low-income families or considered at risk (i.e., students who do not have a safe place to go after school because their parents or guardians are working or not at home). Students are recommended for participation by their school principal. At the time of this study, the after-school program had approximately 50 students in Kindergarten through Grade 5.

The Midwestern After-School Program was suggested to me by a professor from Illinois State University whose students have served as volunteers. I emailed Leann a description of my study and was granted permission to conduct research during the after-school program. I began volunteering to become a recognized figure for the students so they would be more comfortable working with me during the study.

Although the setting of the study was purposefully selected, the identification of students to participate in the study was based on whether they typically were assigned homework by their classroom teacher. Because completing homework was an important part of the after-school program, Leann identified potential participants from those students who usually had few homework assignments. I invited all of the students she identified to participate in the study. Eleven students agreed to participate, but one was present for only the first problem-solving session. Thus, there were 10 participants in the study: Alia, Amanda, Becca, Bernice, Edward, Hector, Iris, Joella, Karly, and Trevor (all names are pseudonyms). Five of the students spoke English as their second language. All of the students and their guardians signed informed consent (See Appendix A) and assent (see Appendix B) forms in order to be a part of the study.

Study Design

In order to study the students as they engaged with unsolved problems, I involved them in a sequence of seven problem-solving sessions. These problem-solving sessions followed Goldin's (2000b) methods for conducting semi-structured, task-based interviews, which he developed from his experiences researching mathematical problem solving. According to Goldin, task-based interviews involve an interviewer and the subjects (problem solvers) "interacting in relation to one or more tasks (questions, problems, or activities) introduced . . . in a preplanned way" (p. 519). He suggested that "subjects should engage in free problem solving during the interview to the maximum extent possible, in order to allow observation of their spontaneous behaviors and their reasons given for spontaneous choices before prompts or suggestions are offered" (p. 542). The problem-solving sessions in my study were not structured as interviews, per se. Rather, the students worked in table groups and I circulated among the groups. However, students were encouraged to engage in free problem solving as Goldin suggested. In this way, I was able to acquire detailed and illustrative records of students' interactions and focus specifically on the processes students used to tackle the tasks I posed.

Context of Problem-Solving Sessions

The seven problem-solving sessions took place over 3 weeks—three in Week 1 (Nov. 28, 29, and Dec. 1), two in Week 2 (Dec. 5 and Dec. 8), and two in Week 3 (Dec. 12 and Dec. 13). Each problem-solving session lasted between 35 minutes and 45 minutes. Prior to conducting the problem-solving sessions, I had an overview of the different ideas I wanted to include but due to not knowing how far or where the students would take each activity I modified or adapted each lesson plan in between every problem-solving session.

I was the person in charge of all the problem-solving session; however, at times, other people were present during the sessions. During Session 1, two volunteers from the after-school program were in the room, one was a fellow doctoral student and the other was an undergraduate student majoring in education; I refer to them as Volunteers A and B. During Sessions 3 and 5 a professor from Illinois State, Dr. Amanda Cullen attended the sessions and helped several students. During Session 5, another professor from Illinois State, Dr. Schupp (a pseudonym) joined the problem-solving session to observe and interact with several students. For Sessions 6 and 7, a different undergraduate student who volunteered at the after-school problem was present for the problem-solving sessions. I refer to her in the rest of the study as Volunteer C.

During each problem-solving session, students worked on an aspect of the Graceful Tree Conjecture; the Collatz Conjecture was introduced in the last session. A conjecture in mathematics is a statement that is thought to be true but has not been proven to be true. Mathematicians think these two conjectures are most likely true. Infinite classes of cases have been settled for both conjectures, but no formal proof has been reported for either; thus, they are referred to as unsolved problems. Although the students in this study explored these unsolved problems only through the infinite classes that have been settled, I refer to their work as engaging with unsolved problems.

Graceful Tree Conjecture. Graph theory is a study of mathematical structures, called graphs, involving points (nodes or vertices) and lines between pairs of points (edges). The order of a graph is the number of nodes or vertices and the size is the number of edges. A tree graph is a connected (one piece) graph with no cycles (see Figure 1). A cycle is a connected graph where each node is an endpoint of two edges (see Figure 2). An acyclic graph contains no cycles. An acyclic graph in layman terms means that if you follow a path from node to node along edges on the tree you will never cycle back to the same node without repeating an edge; that is, the nodes do not make a circuit. This means that the number of edges (the size) is always one less than the number of nodes (the order).

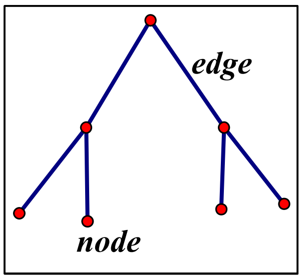


Figure 1. Tree graph

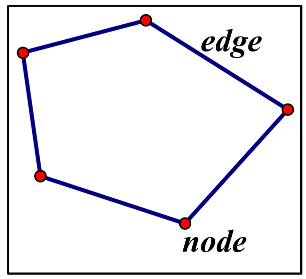


Figure 2. Cycle graph

There are many different types of tree graphs (e.g., paths, stars, and caterpillars). For example, a path is a tree graph where each node is an endpoint to at most two edges (see Figure 3). A star is a tree that has one central node (called the center) and each edge has the center as an endpoint (see Figure 4). A caterpillar tree starts like a path graph but then edges and nodes extend from the central path creating a tree that looks like a caterpillar (see Figure 5).

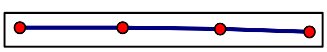
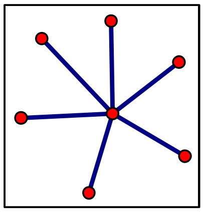


Figure 3. Path graph



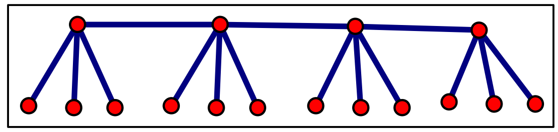
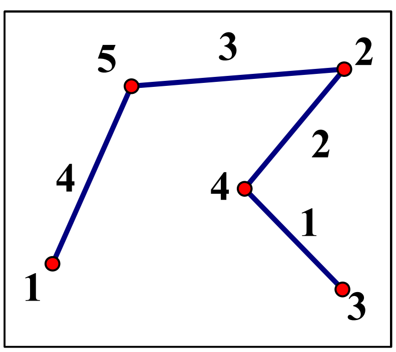


Figure 4. Star graph

Figure 5. Caterpillar graph

Graphs are labeled by assigning numbers to the nodes that induce labelings on the edges. In the late 1960s, Alexander Rosa first introduced the notion of graceful labelings. A tree graph of order m is labeled gracefully if every node is labeled distinctly from 1 through *m* and when the edges are labeled with the absolute value of the difference of the labels on their endpoints, the resulting edge labels are distinct (see Figure 6 for a tree labeled gracefully and Figure 7 for a tree that is not labeled gracefully).



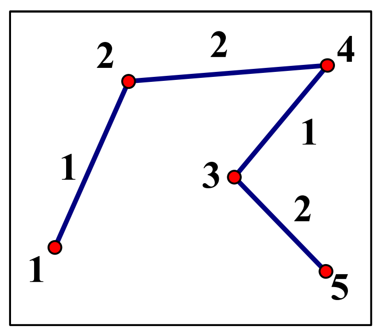


Figure 6. Graph labeled gracefully

Figure 7. Graph not labeled gracefully

Following Rosa's introduction of graceful labeling, Ringel and Kotzig advanced the Graceful Tree Conjecture (Superdock, 2013), which posits that all trees can be labeled gracefully. I selected the Graceful Tree Conjecture as the main unsolved problem for this study because it is accessible to fourth- and fifth-grade students—students need only the ability to subtract and identify and analyze patterns. In 2003, there was a conference at the Banff International Research Station for mathematicians and educators called Unsolved K–12 (Hamiton & Saarnio, n.d.). At the conference, one unsolved mathematics problem was selected for students in each grade of Kindergarten through Grade 12. The criteria for selecting each problem was that it matched the curriculum, was fun for students, would not confuse students, and would be easy for a teacher to implement. The problem selected for Grade 3 was the Graceful Tree Conjecture.

Collatz Conjecture. This conjecture was first proposed by Lothar Collatz in 1937 (Hamilton & Saarnio, n.d.). The conjecture is: Take any positive integer n. If n is even, divide it by 2. If n is odd, multiply it by 3 and add one. Repeat this process. The conjecture states that no matter the positive integer you begin with, you will always reach the number one. I chose to have the students explore this unsolved problem so they were able to see that there were other types of unsolved problems and to see what happened to their engagement, dispositions, and in-the-moment identity with a different problem. This problem was the chosen problem for Grade 4 students at the Unsolved K–12 conference conducted at Banff International Research Station (Hamiton & Saarnio, n.d.). Because this was the focus problem chosen for fourth grade, I decided to implement it during the last problem-solving session so students would be able to experience a second unsolved problem.

Overview of Problem-solving Sessions

The first six problem-solving sessions involved the Graceful Tree Conjecture. I structured the activities from the easiest types of graphs to make a generalized statement about to the types of graphs to were more difficult to make a generalized statement about: star graphs, path graphs, caterpillar graphs, and comet graphs. To begin, I first had to introduce students to what a tree graph is and have them develop a definition for themselves before they could really explore the Graceful Tree Conjecture. Because of this, at the beginning of the Sessions 2–4, in addition to gracefully labeling graphs, I also had the students explore in more depth what a tree graph is compared to what it is not and that tree graphs can be oriented differently but still be the same class of graph. During the last session, I introduced a new unsolved problem because I thought it was important for students to see a different unsolved problem.

Problem-solving session 1. My goal for Session 1 was to generate excitement about unsolved problems and the work of mathematicians and to introduce the Graceful Tree Conjecture. To begin the session, I asked students if they knew who mathematicians are. One student responded that it was a person who was good at math. I told them it is a person who studies patterns and numbers. I then told the students that we would be working on an unsolved problem called the Graceful Tree Conjecture. I explained that a mathematician at Illinois State University was working on the same problem. I also explained that because the problem was unsolved, if they were able to figure it out, they would become famous and could get one million dollars. The students were very excited about the possibility of getting one million dollars. Although they were not sure what a mathematician was, they were very excited about the prospect of getting one million dollars.

Next, I asked students if they had ever heard of a graph. Many students came up with examples of graphs they had learned about in school such as a line graph, bar graph, picture graph and several other graphs I have never heard of like a book graph and full graph. I introduced the types of graphs we would be working on, tree graphs, by drawing a tree graph that had six nodes and five edges in the shape of a path graph. I used a drawing to define and illustrate what an edge and node are. I had the students count the edges and nodes on the graph (see Figure 8 for image on the poster paper).

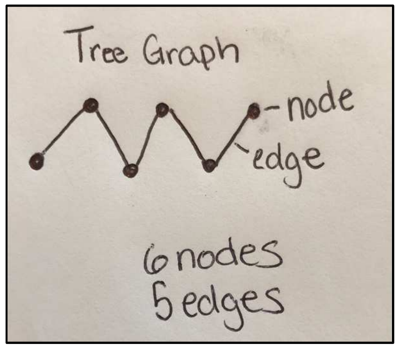


Figure 8. Drawing on poster paper when students were introduced to tree graphs, edges, and nodes

Then I asked them to draw a tree graph with five nodes and four edges. Almost all of the students were able to do this by drawing a path graph similar to the example I had showed them. One student noticed that she was not able to make the edges connected and still have a graph with five nodes and four edges. We came back together and the students shared their graphs with the whole group. I explained that the graphs we were exploring, tree graphs, were neither cyclic nor connected. Knowing that I would have the students further explore this idea the next day, I was not concerned if they did not understand that idea.

To introduce the idea of graceful labeling, I drew two star graphs, one with graceful labels and one without graceful labels (see Figure 9). I asked the students to talk at their tables about what was different between the graphs. They saw that one had all different numbers and no repeats. I explained that was a graceful label. I then told them that the problem of the Graceful

Tree Conjecture was to discover if all tree graphs could be labeled gracefully. Many students thought they could find a graph without a graceful labeling.

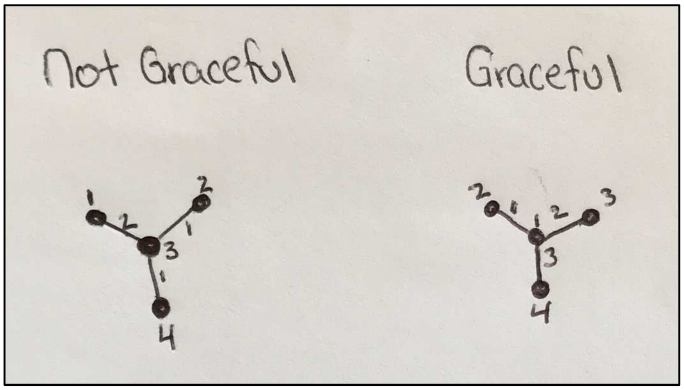


Figure 9. Image of poster paper with two star graphs. One has a graceful labeling and one has an ungraceful labeling.

The last thing we did during the first problem-solving session was attempt to label a graph gracefully (see Figure 10). Students were given circle chips with the numbers one, two, three, four, and five for the nodes and square chips numbered one, two, three, and four for the edges (see Figure 10). They where also given a page that had multiple representations of the graph so they could record their answers.

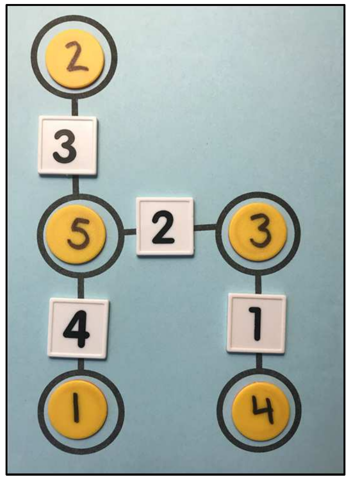


Figure 10. First graph for students to label gracefully. The graph is gracefully labeled with the circle and square numbered chips the students were given.

When the problem-solving session was about to end, three students had yet to complete a graceful labeling. I partnered those students with a student who had a solution so they could achieve a graceful labeling with help from a peer. At the end, all students were able to achieve at least one graceful labeling of the graph. Several students were able to find multiple labelings for the graph.

Problem-solving session 2. During this session, students examined tree graphs, labeled star graphs, and began to develop patterns for how to label any star graph gracefully. First, students worked collaboratively in table groups of two and three to sort examples and non-examples of tree graphs. They were given a set of 10 graph cards (Figure 11) to sort and tape to posters labeled *Tree Graphs* and *Not Tree Graphs.* After the students finished the sort, we had a whole-group discussion during which each table shared what they found through the sort. The goal of this activity was to prompt the students to reflect on the previous problem-solving session, notice that tree graphs have one less edge than node, and develop their understanding of what distinguishes a tree graph from other graphs.

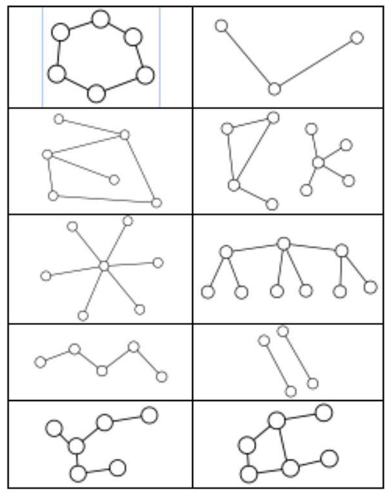


Figure 11. Graphs students had to sort as either a tree graph or not a tree graph.

Next, students were given a page with the first four graphs in the star class (see Figure 12), enlarged copies of each star graph, and the numbered chips. The enlarged graphs and numbered chips were given to the students so they could explore the graphs without having to erase, which I thought might ease their frustration. Due to time, the problem-solving session ended before students were able to share their labelings with the whole group or explain why their strategy would create a graceful labeling for any star graph.

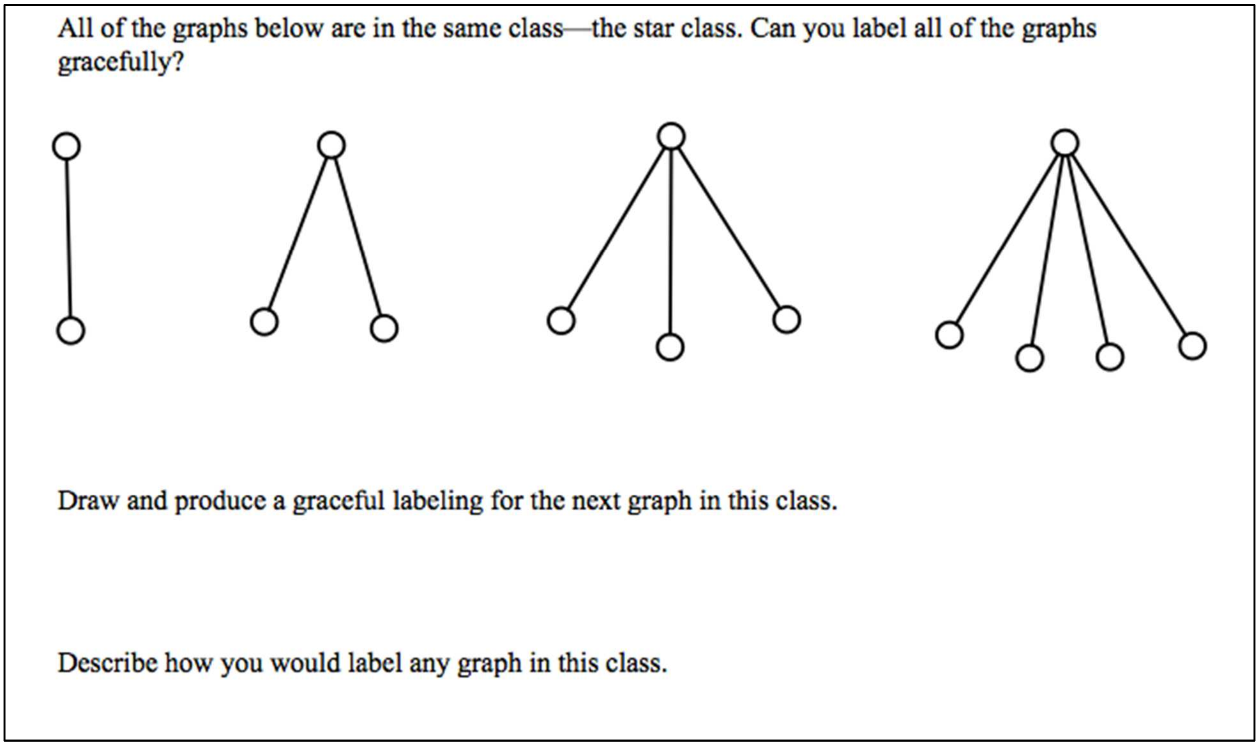


Figure 12. Star class page for students to label each graph, produce the next graph in the sequence, and describe how they would label any graph in the star class.

Problem-solving session 3. This session began with a discussion of students' labelings for star graphs. Students used what they had learned the previous day about star graphs and tried to develop an informal justification for labeling any star graph gracefully. Next, the students were given AngLegs (see Figure 13) and a task sheet (Figure 14) that contained five tree graphs, all arranged differently. I posed the question of whether or not they were different or represented the same tree graph. The students used the AngLegs to explore each graph.

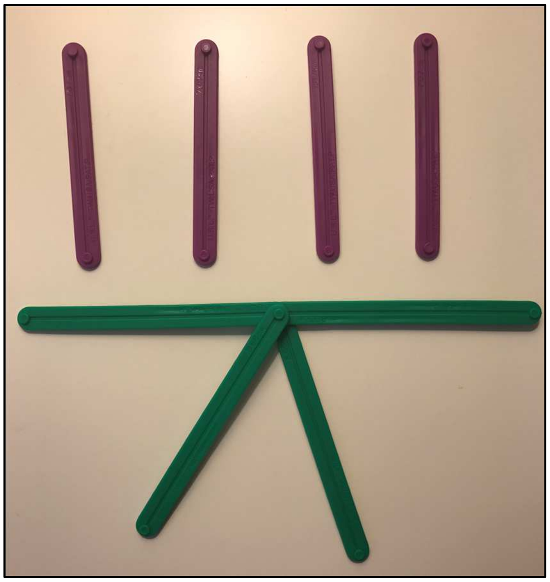


Figure 13. AngLegs. They have colored legs (edges) that are similar to a popsicle stick and they have notches (nodes) on each end so legs can be attached together. They are made for creating polygons and angles. The AngLegs allowed students to create a tree graph and move the edges and nodes around while still maintain the same tree graph).

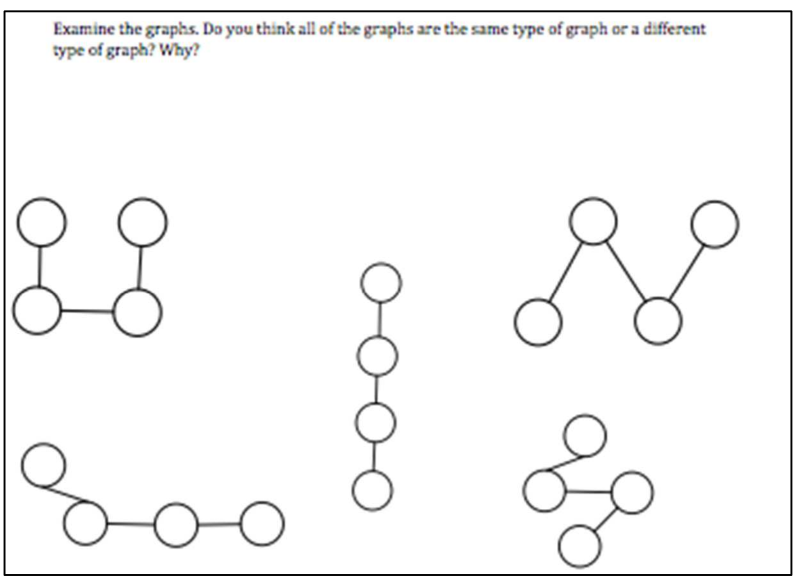


Figure 14. Same graph task sheet #1.

Lastly, students began to explore path graphs with the numbered chips. They were given a page to record their answers (see Figure 15) and a packet that had all of the graphs enlarged. However, most of the students became overwhelmed and confused with the way I had structured the activity. They were not sure which enlarged graph came next in the packet. Almost all of the students were able to work through the first three graphs but became frustrated after that. After 35 minutes of working, two students asked if they could quit for the day but come back the next time, and another three students were engaged in unproductive struggle, so I decided to end the session a few minutes early and told the students we would explore path graphs in the next session but in a different way.

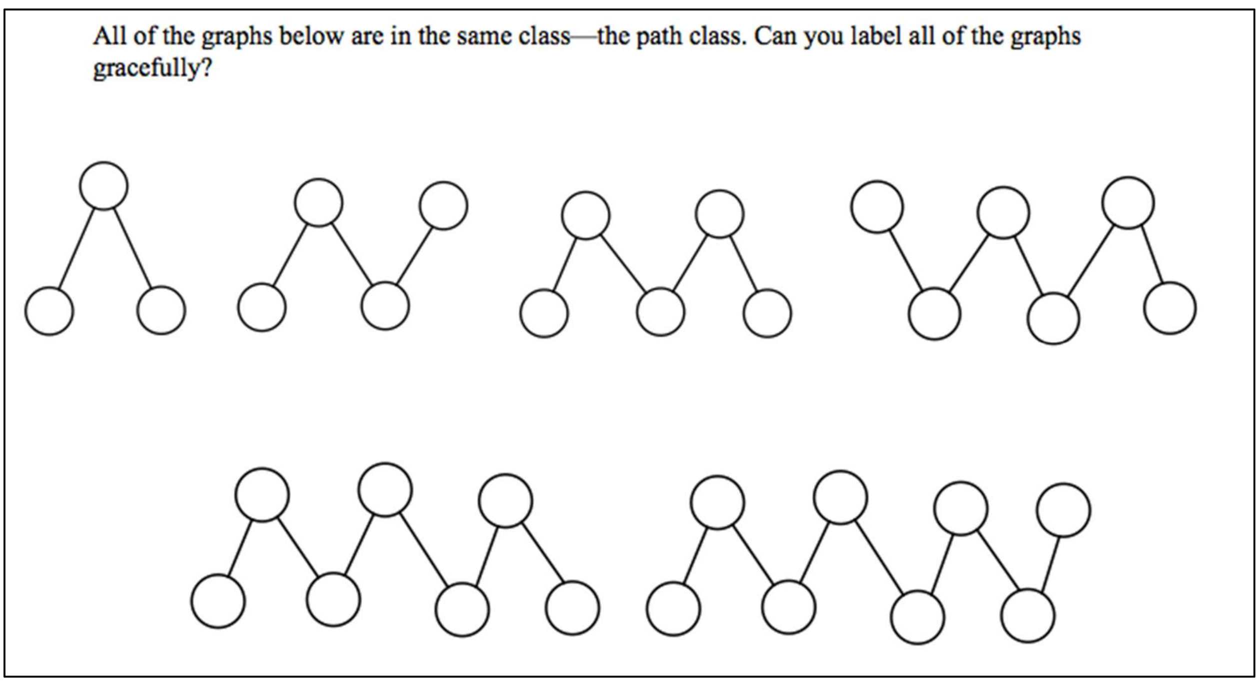


Figure 15. Task sheet of path graphs for students to label gracefully.

Problem-solving session 4. I wanted to encourage the students who struggled in Session 3 and to continue to explore path graphs but with a different approach. In the previous session, the students had explored five different tree graphs and conjectured that they were all the same graph because they had four nodes and three edges. Therefore, I wanted to have the students complete a similar activity but instead of all the graphs being the same, I created four graphs, two the same and two different (see Figure 16). The students used AngLegs to explore the graphs on the task sheets. This activity allowed the students to understand that the same graph can be positioned differently but just because it has the same number of nodes and edges does not mean it is the same graph.

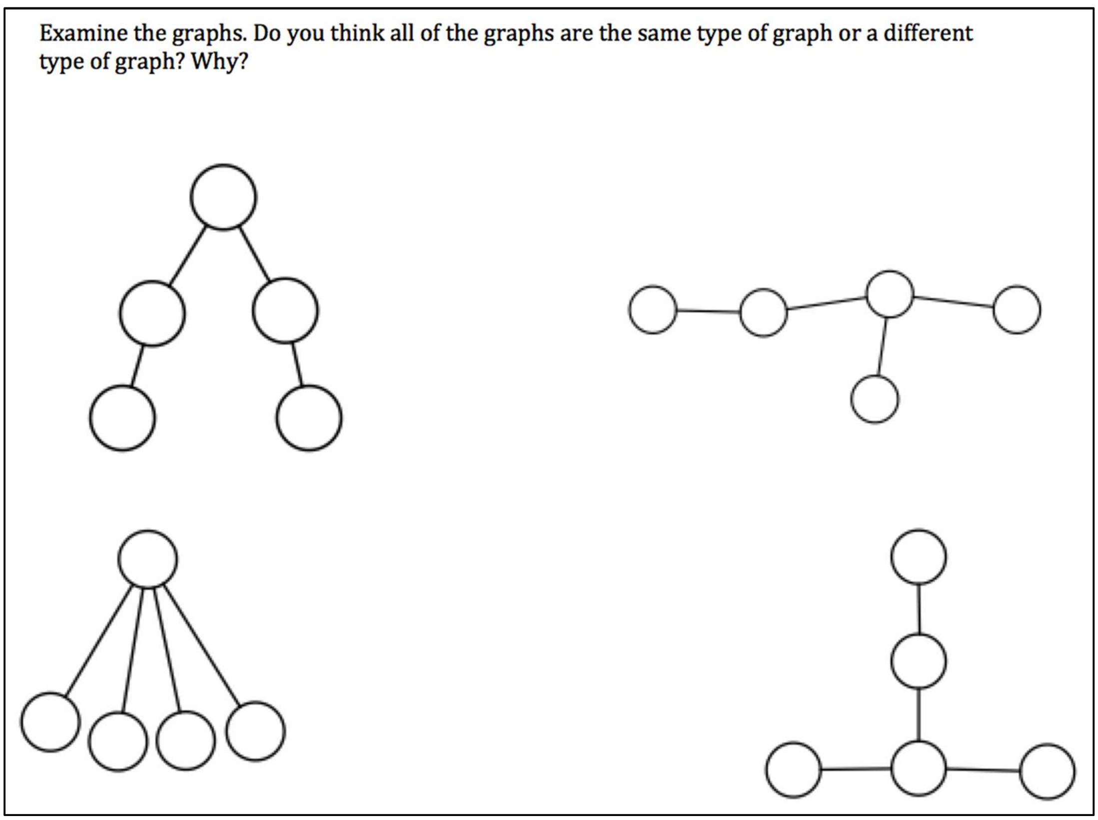


Figure 16. Same graph task sheet #2.

Next, I had the students explore path graphs. We began this activity as a whole group to encourage the students to work together and to look for patterns and relationship with graphs in the same class. Also, the students had mentioned many times previously that they wanted to spend more time writing on the poster paper. Therefore, I created the first three graphs in the path class on poster paper. Each table got one poster with a path graph on it. The first two graphs were completed quickly by the students at the tables. The third graph was not completed by the time the other tables were done, so all of the students worked on it together. The students arranged their posters in a row and as a whole group the students tried to solve the fourth graph in the class. This graph contained four nodes and three edges. I encouraged the students to look for patterns. They were able to complete a graceful labeling for that graph using the three previous graphs. Several students said they saw a pattern relating to the edges.

Next, students were given the next graph in the class, a graph with five nodes and four edges (see Figure 17). The problem solving session ended before most of the students were able to complete a graceful labeling for this path graph.

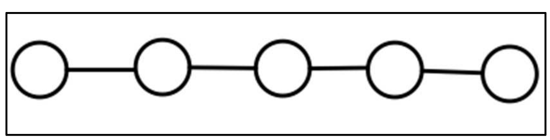


Figure 17. Path graph with five nodes and four edges.

Problem-solving session 5. Students continued exploring path graphs and attempted to find patterns or create an informal justification for labeling any graph in the path class. Also, students began to explore double star graphs and caterpillar graphs.

To begin the students were given a page that contained the first five graphs in the path class (see Figure 18). The first four graphs were the ones the students had labeled gracefully on the poster paper the previous problem-solving session. Those posters were hung in the front of the room. The students were also given an enlarged copy of the graph with six nodes and five edges and a set of circle and square numbered chips. The students were tasked to try to label the graphs gracefully, find a pattern, draw and gracefully label the next graph in the class, and generalize a way to gracefully label all path graphs.

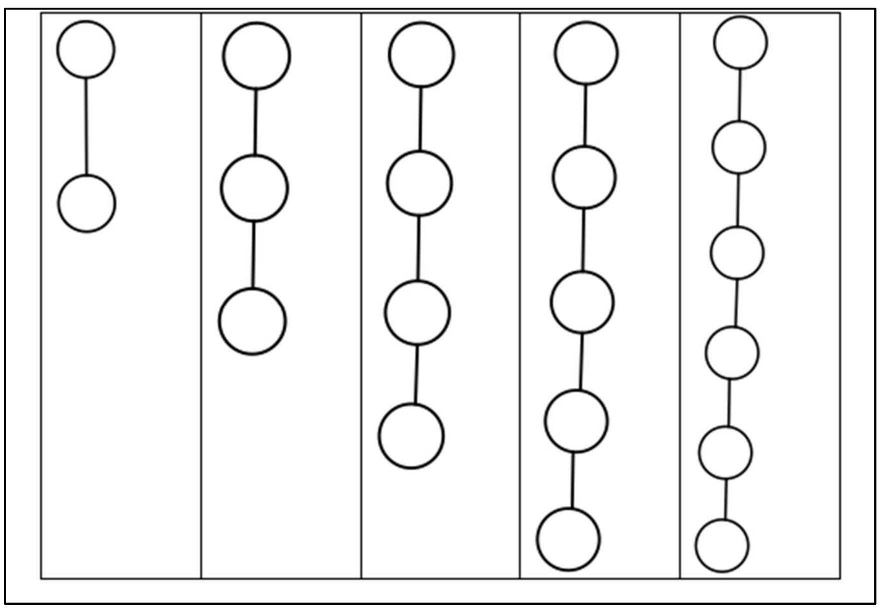
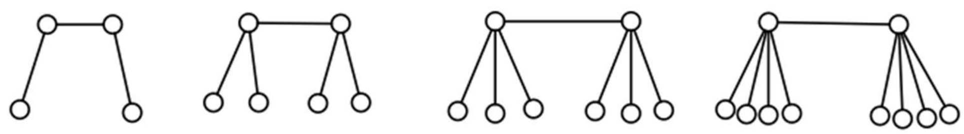


Figure 18. Path graph task sheet the students were given.

After the students had shared their patterns for path graphs and how to gracefully label any path graph, they moved on to caterpillar graphs (see Figure 19). They first worked on double stars, which are a part of the caterpillar class. They were given enlarged copies of each graph and the numbered chips. Right away, the students said they could label the first graph easily because it was a curved path graph. The problem-solving session ended with the students working on the double star graphs.

All of the graphs below are in the same class—the double star class. Can you label all of the graphs gracefully?

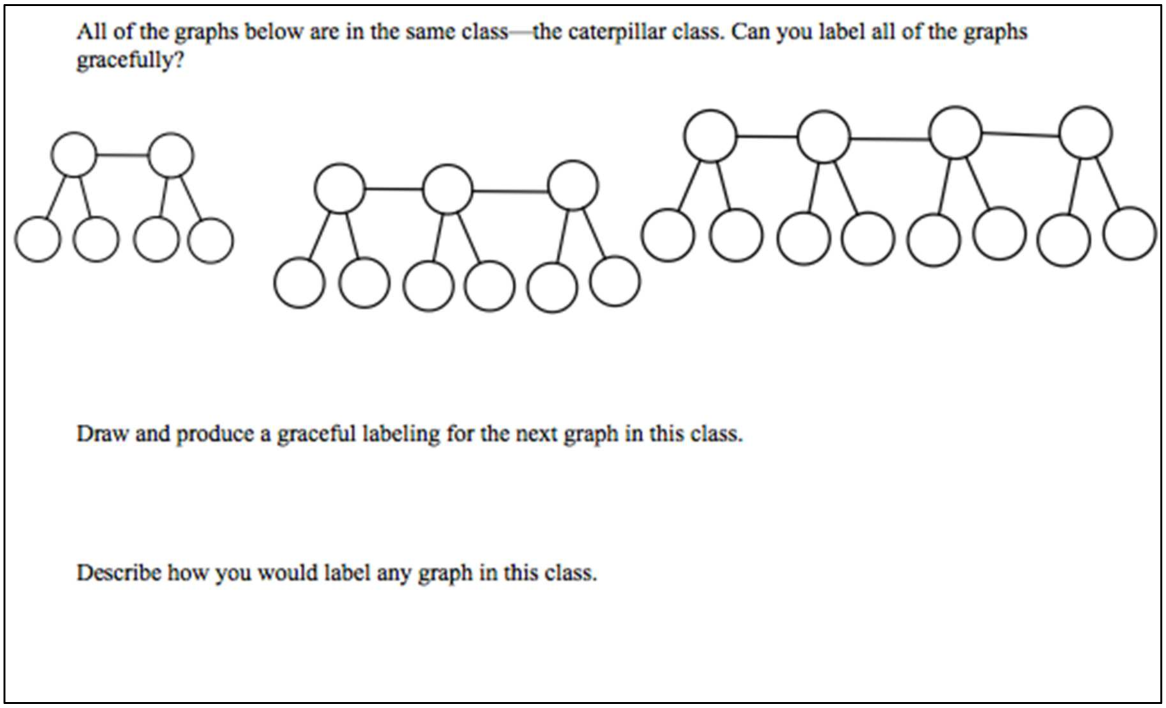


Draw and produce a graceful labeling for the next graph in this class.

Describe how you would label any graph in this class.

Figure 19. Double stars task sheet students worked on.

Problem-solving session 6. In this session, I allowed the students more time to explore double star graphs (see Figure 19), which they had started the previous day. When they had completed labeling these graphs, I gave them a page containing caterpillar graphs (see Figure 20), enlarged copies of each graph, and numbered chips to facilitate their labeling. I encouraged them to develop patterns or informal justifications for labeling any graph in this class. At the end of the session, the students shared their findings, patterns, and descriptions for how to gracefully label double star graphs and caterpillar graphs.


All of the graphs below are in the same class—the caterpillar class. Can you label all of the graphs gracefully?

Draw and produce a graceful labeling for the next graph in this class.

Describe how you would label any graph in this class.

Figure 20. Caterpillar graphs task sheet the students were given.

Problem-solving session 7. For the last session, I wanted the students to explore a different unsolved problem, the Collatz Conjecture. I began by telling the students the story of Icarus and Daedalus:

> King Minos has just imprisoned Daedalus and his son Icarus on a high tower. Daedalus was charged in the murder of King Minos son, the Minotaur. Daedalus is an inventor so he decided to try to find a way to escape. He comes along the idea of gathering bird feathers and he fashioned wings for his son and him to fly off the tower.
>
> On the day before their fateful flight Icarus and Daedalus both have dreams. In Icarus' dream, he writes a number on a rock and hurls it off the tower where they have been imprisoned. If then number is even, it is halved. If it is odd, the number is tripled and one

> added to the results. This is continued until 1 is reached (falls into the sea and is killed) or you know that you will not reach 1 (can fly away and is free). For example, if Icarus writes a 3 on the rock … you can follow the sequence of numbers until CRASH – he falls into the sea and is killed. This dream had turned into a nightmare. But Icarus knows that if he can just find a number to write on that rock so that he doesn't end up crashing into the sea … so that it doesn't end up at 1… then he will be all right. Daedalus has a similar dream. Your job is to help save the lives of Daedalus and Icarus. (Hamilton & Saarnio, n.d.)

The unsolved portion of the problem is the Icarus portion. The Daedalus side of the problem is solvable, and there are multiple solutions.

Next, I gave each student a page that included the patterns for Icarus and Daedalus and an example of what to do for the number 3 (see Figure 21) and calculators. As a whole group, we worked through the number 3 as an example for both Icarus and Daedalus so the students were able to see how the pattern worked and how examples both ended in one. I then tasked them to find a number that did not end in one. I told them if they tried a number and reached one, to write it on the poster paper so the other students would know the number had already been tested.

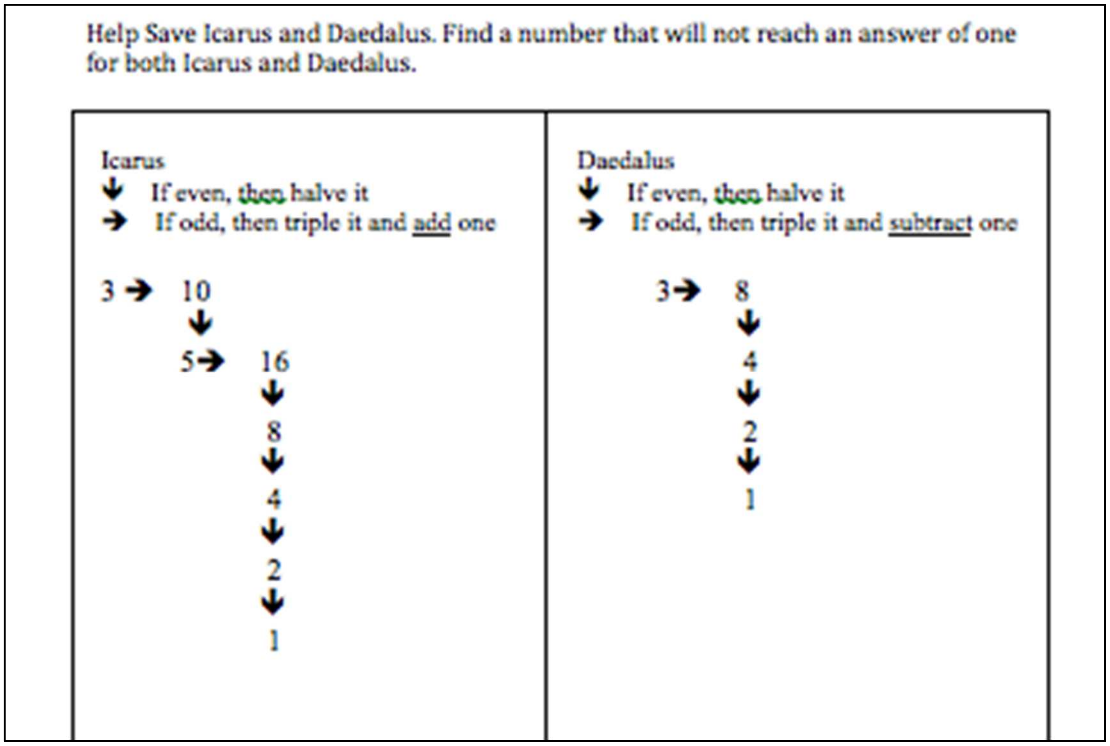


Figure 21. Task sheet for Icarus and Daedalus given to students.

Many students were confused on what they were supposed to do and were not sure what it meant to triple something. Those students worked with me through a few more examples to understand the process. After 30 minutes, the students were asking to work on “sticks and nodes.” I handed out a page with the first four comet graphs in the comet class (see Figure 22). At that time almost all of the students had to leave because they were either getting picked up by a parent or had to go to a music special. The music teacher showed up on the wrong day, so the students were told after they finished working with me on mathematics, they could then go and work with the music teacher. Two students stayed to work on the unsolved problems, but I let them go when the rest of the students at the after-school program had begun free time. Unfortunately, there was no time to wrap up the problem-solving sessions because most of the students had already left.

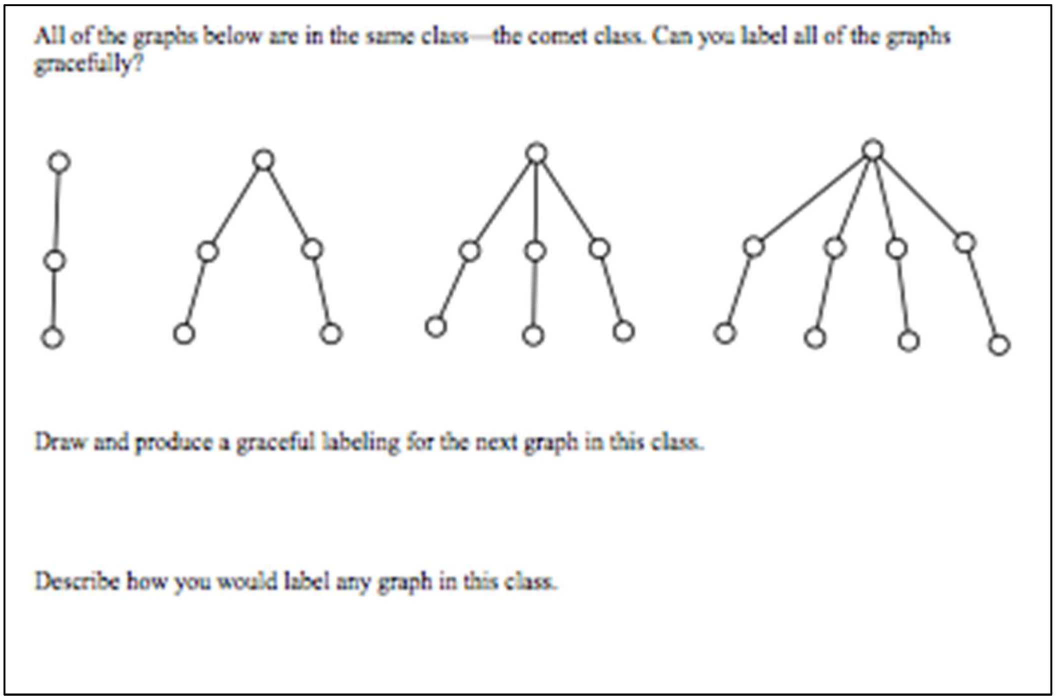


Figure 22. Task sheet with the comet graphs that were given to students.

Data Sources

Data for the study were collected from a variety of sources, including each problem-solving session, two individual interviews with each student, and an interview with the director of the after school program, Leann.

Problem-Solving Sessions

Each session was video recorded to document the interactions and conversations between students as well as non-verbal actions, such as facial expressions conveying excitement or frustration. The students worked in groups and each group sat at a table. The tables were positioned in a “u” shape around the room. This was so they were able to see each other and engage with each other and in group discussions. Each table group had a camera focused on them. The camera was positioned at end of the table to avoid distracting the students but to document their reactions, work, and conversations. A fourth camera was positioned in the back of the room to document whole-group discussions.

Individual Interviews

Prior to and upon completion of the problem-solving sessions, my dissertation chair, Dr. Cynthia Langrall interviewed each student to gain insight into his or her views and dispositions of mathematics before and after the study. The interviews were conducted by Dr. Langrall instead of myself to eliminate potential bias; that is, so I would have no knowledge of the students' views and opinions of mathematics, their abilities, or their identities prior to working with them during the problem-solving sessions. Both interviews were video recorded.

There were seven interview questions for the pre- and post-interview (see Figure 23). The post-interview also included reflection questions for students about the problem-solving sessions. All interview questions were open-ended to encourage students to describe their thoughts and opinions on each topic. The interview questions were created as a way to describe students' mathematical identities before and after the problem-solving sessions. The questions also gave insight into students' dispositions towards mathematics prior to the problem-solving sessions and after the problem solving sessions. The additional questions on the post interview were to have students reflect on the problem-solving sessions and have the students identify if their view of mathematics changed or if the problem-solving sessions made them think differently about mathematics, learning mathematics, or what mathematics is. All interviews were video recorded.

Pre- and Post Interview

1. Tell me about yourself (e.g., tell me about your school? How old are you? What do you like to do? What is your favorite subject in school? What is your favorite thing to do for fun?)
2. Talk to me about mathematics
    a. What is mathematics?
    b. Who does mathematics?
    c. What does a mathematician do?
    d. What is math like in school?
3. Are you interested in mathematics?
    a. What interests you about it?
    b. What doesn't interest you about it?
    c. Do you like to do puzzles and solve brain teasers related to mathematics?
4. Do you think math might be important for a job or career you might want to pursue?
5. What do you think a mathematics lesson should look like?
    a. How does someone learn mathematics?
6. How long should it take to solve a typical homework problem?
    a. What is a reasonable amount of time to work on a problem before you know it's impossible? (Schoenfeld, 1989)
7. If math were a food, it would be …
    a. Because (Cai & Merlino, 2011)

Additional Questions Asked on Post Interview

8. What does mathematics look like in your classroom?
    a. How is this similar or different to the math you did with Jenna?
    b. Would you like to spend time in your school mathematics working on unsolved problems?
9. What do you think you have learned about math by working on unsolved problems?
10. What is something you enjoyed about working on unsolved problems?
11. What was something challenging about working on the unsolved problems?
12. How has your work with Jenna made you think differently about…
    a. mathematics?
    b. learning mathematics?
    c. what mathematicians do?
13. If math were a animal, it would be …
    a. Because (Cai & Merlino, 2011)

Figure 23. Interview questions for pre- and post interview with students.

Interview with Program Director

My dissertation chair also conducted an interview with the program director, Leann, after the problem-solving sessions had finished. The purpose of this interview was to gain insight into the students' backgrounds, their experience with school mathematics, and things the students might have said in between problem-solving sessions (e.g., if they shared their feelings with her). There were two interview questions: What are your perceptions of the interest of the students in Jenna's study? Do you have background knowledge or information that you could share in general for each student such as their academic standing in school? The interview was video recorded.

Data Analysis

While I analyzed my data, I stayed focused on the research question through each phase of my analysis. I conducted four phases of analysis. First, I transcribed all data sources including motions or actions that could show nonverbal dispositions. Second, I broke the transcript up into conversations. Third, I constructed analytic frameworks for analyzing my research question: What are the characteristics of students' dispositions toward mathematics when they engage in the exploration of unsolved problems? I then analyzed the data based on the analytic frameworks. Fourth, I created another set of analytic frameworks to narrow my focus to explore students' in-the-moment identities that were displayed through their dispositions. I then analyzed the data again using those frameworks.

Phase 1

For Phase 1 of analysis, I transcribed all of the video recordings for the interviews and the problem-solving sessions using Transana (Woods & Fassnacht, 2016). Transana is qualitative software created to allow researchers to import video, transcribe, segment, categorize, and code media data. Using Transana, I transcribed all the words, actions, and emotions the students displayed in the video. I first transcribed the pre- and post interviews. Next, I transcribed the problem-solving sessions. To do this, I first transcribed the recordings from the camera that was set up in the back of the room, which captured the group discussion for each of the seven sessions. Next, I transcribed each of the recordings from the table cameras, which documented all of the students' individual and collaborative work. Last, I time-stamped each transcript to coordinate with the video so I could click on the transcript and watch the video that matched. While I time-stamped the video, I checked for errors or missed actions.

While I transcribed the video, I documented my thoughts and ideas in a word document about the students' dispositions. For example, when a student displayed an idea about what mathematics was or gave a justification for a graph I made a note for myself. After writing a draft of Chapter 4, I referred back to the notes and confirmed that I had not missed an important idea.

Phase 2

For Phase 2 of the analysis, I broke the transcript up according to conversations (see Figure 24), which consisted of a segment of dialog about a particular topic or idea and the surrounding activities. A conversation becomes a new conversation when there is a change of focus. In the example in Figure 24, the conversation changes when the students shift to labeling a new graph. Breaking the transcript up into conversations helped me eliminate transcript that held off-task behavior. Also, I used the conversations when I was writing Chapter 4 so the reader would be able to make sense of what was happening during the problem-solving session.

Conversation 1

Jenna:	How do you do it?
Becca:	We count it and the difference.
Jenna:	Okay, how did you count? What did you do?
Karly:	Minus
Becca:	I um,
Jenna:	Okay, you minused.
Becca:	First, I did one right here (pointing to the first node at bottom left). Then I did one, two, three, four five, six seven (moving her pen along the edges from right to left).
Jenna:	Okay, so you labeled the edge.
Becca:	And then, um, the first one I went
Iris:	One
Becca:	Then added it to there (pointing to bottom one and the edge)
Jenna:	Oh so you added it?
Becca:	Yeah and then I minused it and I kept on doing it.

Conversation 2

Jenna:	Okay, so do you think you could solve this one (giving them a caterpillar with three legs off each point)?
Becca:	Yes, that is so easy.
Jenna:	[Handed out the next caterpillar graph to all three girls]
Becca:	[puts a one in bottom right corner of graph]
Iris:	[labels edges using previous pattern]
Becca:	12 [writes 12 in top node on left]
Iris:	[Puts 12 in top left corner]
Becca:	Wait, this is the same one.
Iris:	Different
Jenna:	Yeah that one is different
Becca:	How?
Iris:	[Points to the new page and touches the first three nodes on the page] One, two, three. Look [pointing to the previous graph page]. Different, different.
Karly:	That has one more.
Iris:	I thought it was naturally the same. I was going to say I am copying it wrong.
Becca:	Oh, those are easy.

Figure 24. An example of two conversations that were identified in the transcript. This example comes from the sixth problem-solving session. In the first conversation, the three girls are explaining their pattern for labeling caterpillar graphs. The second conversation is the students beginning to label the next graph in the sequence and discussing that it is a different graph with another node. A conversation could have multiple codes (e.g., the student could display frustration, frustration again, and they joy when they have a graceful labeling).

Phase 3

For Phase 3, I first needed a way to code the data for the dispositions students exhibited while they engaged with the unsolved mathematics problems. To do this, I constructed analytic frameworks based on the ideas of Beyer (2011) and Else-Quest et al., (2008). I then used those frameworks to code my data.

Analytic frameworks for dispositions. To analyze dispositions, I first drew on an analytic framework developed by Beyer (2011), which focuses on three modes of dispositional functioning—cognitive, affective, and conative. Beyer described these modes according to definitions from English and English (1958) who considered cognition, affection, and conation as the "three categories under which all mental processes are classified" (pp. 92–93).

Cognition is "any process whereby an organism becomes aware or obtains knowledge of an object….It includes perceiving, recognizing, conceiving, judging, [and] reasoning" (English & English, 1958, p. 92). Beyer considered a cognitive function as dispositional "if a person has a tendency or inclination to engage (or not) in a particular cognitive mental process associated with perceiving, recognizing, conceiving, judging, reasoning, and the like in mathematics" (p. 23). He included two subcategories for this dispositional function: connection, which is "a tendency to try and connect ideas with or across mathematical topics" (p. 30) and argumentation, which is "a tendency to evaluate the mathematical correctness of statements, make mathematical arguments, justify mathematical statements, etc." (p. 30).

Affect is a "class name for feeling, emotion, mood, [and] temperament" (English & English, 1958). For Beyer (2011), an affective disposition is "a tendency or inclination to have or experience particular attitudes, beliefs, feeling, emotions, moods, or temperaments with respect to mathematics" (p. 23). He described seven subcategories that address beliefs about mathematics—it's nature, usefulness, worthwhileness, and sensibleness; mathematics self-concept and emotions; and experiences of anxiety in mathematics.

English and English (1958) defined *conation* as "a conscious striving…[an] impulse, desire, volition, purposive striving" (p. 104). Beyer (2011) further explained that an action is a dispositional conative function "if a person has a tendency or inclination to purposively strive, exercise diligence, effort, or persistence in the face of mathematical activity" (p. 23). His framework includes one subcategory for this dispositional function, "a tendency to persist or exert effort if necessary" (p. 30).

Beyer (2011) developed this framework as a way to assess preservice teachers' dispositions towards mathematics. Although it has not been formally validated, I have adopted it (with modifications) for my study of elementary students because it includes the component of cognitive dispositional functioning. Much of the literature on mathematical disposition does not include cognition as a disposition (National Research Council, 2001). However, I felt that as a foundation for examining students' dispositions, it was important to show that the Grades 4 and 5 students in this study were able to engage in challenging problems, to gracefully label tree graphs, and to convince people that their labelings would work. Including cognition as a component in my analytic frameworks enabled me to do this, but the variety of cognitive functions that exist required me to narrow the parameters of this component of my analysis. Thus, I included only one code with regard to cognitive functioning—*labeling graphs*—which I define as the tendency to find patterns, make mathematical arguments, and justify mathematical claims. Beyer included self-concept as a component of affective dispositions but following the conceptual perspective of communities of practice (Wenger, 1998), positional identities (Holland et al., 1998), and positional theory (van Langenhove, & Harré, 1999), I expanded the construct of self-concept to include students' positioning and identity to conduct a separate phase of analysis (see Phase 4 below).

I made other adjustments to Beyer's framework as well. I adapted the code effort/persistence to be perseverance because I felt the term was better suited for the definition or description of a conative mental function. Lastly, I changed Beyer's wording of attitude to emotion. Emotion is a better fit for the definition. The term attitude typically includes beliefs about mathematics as well (e.g., Goldin et al., 2016), which would contradict with the other codes of affective mental functions. My version of the Disposition Framework is presented in Table 1.

Table 1

Disposition Framework

Key Word	Code	Definition	Example
Cognitive	Labeling graphs	A tendency to find patterns, make mathematical arguments, justify mathematical statements.	I can solve all star graphs by putting the smallest number or largest in the middle because when I subtract all the edges will have a different number.
Affective	Nature of Mathematics	A belief in how a student interprets the ways mathematics should be learned or practice.	Mathematics is only addition, subtraction, multiplication, and division.
	Usefulness	A belief that mathematics will be helpful in the future for things such as a career.	Math is important because you need to know how much to pay for things.
	Worthwhileness	That mathematics is valuable and worth putting effort into learning it.	All the work I put forth in this activity has be worth it to me.
	Sensibleness	A idea that mathematics is made up of ideas that make sense.	Math is a connected system that can be made sense of.
	Emotion	The student's emotional reactions to mathematics such as joy or frustration.	Math is my favorite subject. I like doing mathematics.
	Mathematics self-concept	Students' beliefs about him or herself as a person engaged in mathematics.	These graphs are easy. I can solve any graph.
	Math Anxiety	The student demonstrating excessive stress towards mathematics.	These graphs are too hard. This is stressful.
Conative	Perseverance	An inclination to persist or not give up.	I want to figure this out. I don't want to leave.

Note. The definitions and examples have all been changed or modified from Beyer's framework to better fit or define this dissertation study. Many of the key words have been changed or adjusted as well.

As I used the Disposition Frameworks to code the data, I found that an additional modification was needed. While I was coding the first problem-solving session, I noticed that the code *emotions* was assigned much more frequently than the other codes. Students' emotions varied in many different ways, from students showing joy and excitement to the students being frustrated. I then realized that the code of emotions was not specific enough, and I needed a way to distinguish the different types of emotional reactions exhibited by the students. Through a review of the literature on students' emotions, I learned that a number of researchers have used a framework developed by Else-Quest et al. (2008) to characterize the emotions students' display while engaged with mathematics.

To explore the relationship between success in mathematics and the emotions students display with regard to mathematics, Else-Quest et al. (2008) examined a mother's and child's emotions while they were engaged in mathematics homework. They reported that students exhibited the following 13 types of emotions: tension, distress/dismay, frustration, sadness, boredom/apathy, anger/disgust, contempt, positive interest, affection/caring, joy/pleasure, humor, pride, and off-task. I adopted these emotions as an expansion of *emotions* in my analytic frameworks. After a first round of coding for these emotions, I modified the list by combining distress/dismay with frustration and joy/pleasure with pride. I found these emotions to be too similar to distinguish based on just verbal or facial statements without interviewing the students on how they were feeling at that time. I eliminated the off-task code because I did not consider it to be an emotion. I also eliminated positive interest because I found that to be the same as perseverance, which was already a component of my Dispositions Framework. See Table 2 for a list of my codes with a description and example for each.

Table 2

Emotion Framework

Category	Description	Example
Tension	Tautness, conflict, uncertainty, worry,	Just because your older doesn't mean you are smarter
Distress/Dismay/Frustration	Impatience, upset, complaining, disappointment, irritation, criticism	This is so hard I don't want to do this I messed up
Sadness	Grief, removal	I don't know, I am really sad
Boredom/Apathy	Tedium, indifference, lethargic	This is way to easy
Anger/Disgust	Annoyance, irritation, fury, rage, revulsion	She didn't help nobody I don't want to switch tables
Contempt	Disdain, scornful, jeering, ridiculing, mimicking, insult, "brattiness"	Bad job for you Stop being scary
Affection/Caring	Encouragement, helpfulness, friendliness, support, respect	Good job Let me help you
Joy/Pleasure/Pride	Happiness, self-respect, gratification, enjoyment, excitement, pleased, delight, having fun	I did it Boom, we did it Graceful!
Humor	Comedy, wit, funniness	Google knows the answer

After creating my analytic frameworks, I examined all of the conversations from the interviews and problem-solving sessions and used the codes that are defined in Tables 1 and 2 to code all of my transcripts in Transana (Woods & Fassnacht, 2016). To do this, I first created key words in Transana for all of the codes defined in Tables 1 and 2. In Transana, key words can be selected to code different portions of a conversation to describe what is happening and are later useful when you are creating reports. I then created a collection for each student and each problem-solving session. The collections are a place to sort and organize all of the coded conversations. Collections are useful in Transana because you are able to turn them into reports, which would document all of the different things that were coded, and you are able to search through them using the key words to create files for each time something was coded. Any time a student displayed any disposition or emotion, I used Transana to document that moment and labeled that moment with a key word and placed it in the appropriate student's collection. For example, if Alia shouted that she had labeled a graph, I would highlight that conversation, give it the key word of joy/please/pride, and place it in her collection for the appropriate problem-solving session (see Figure 25 for a screen shot of Transana). I did this process twice for all of the interviews and problem-solving sessions as way to check myself for consistency and make sure I did not miss something or interpret something in different ways.

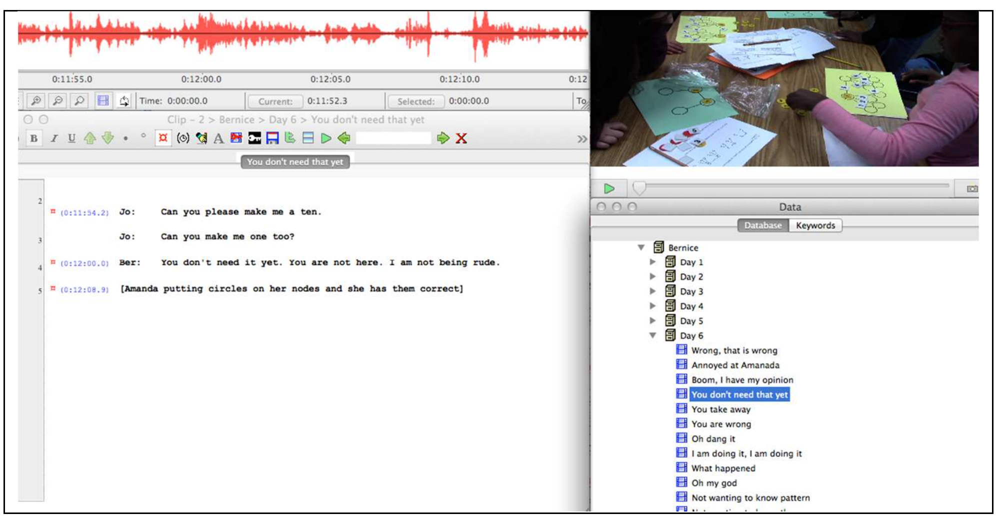


Figure 25. Example of a highlighted conversation coded as contempt for Bernice during Session 6. This is a screen shot of Transana.

Next, I created reports in Transana using the key words (codes) search for each of the different dispositions and students. In Transana a report is a document that reports every time you coded something using a key word. It includes the conversation, the key word, the transcript, and the video for each of the times an item was coded with a key word. For example, if I did a search for justification for Bernice, the report would include every conversation that I coded as a justification and placed in Bernice's collection file.

Phase 4

In this phase of analysis, I examined students' in-the-moment identities, which were displayed through their dispositions. In order to examine and make sense of the students' in-the-moment identities, another analytic framework was needed. Positioning theory allows me to examine the shifting patterns during the interactions between people from a personal point of view and the view of others (van Langenhove, & Harré, 1999).

Positioning theory. Position theory contains three main types of positioning including a person being positioned by someone else, a person positioned by themselves, and when a person positions someone else, they are at the same time positioning themselves. To expand and explain these three categories, there are different subcategories of positioning that can influence micro-identities: (a) performative positioning is when someone challenges an order, such as saying they do not want to do something; (b) personal positioning is when student refers to just himself or herself, such as saying I am good at math; (c) self and other positioning is when a student positions themselves and at the same time other person, such as when a student tells students to copy their work; (d) tacit positioning is when someone positioning others or themselves unintentionally or unconsciously, such as saying do the work; (e) deliberate self-positioning "occurs in every conversation where one wants to express his/her personal identity" (van

Langenhove, & Harré, 1999, p. 24), such as telling everyone how smart they are; (f) forced self-positioning is the same as deliberate self-positioning but is initiated by someone else, such as someone asking what they think of their mathematics ability; (g) deliberate positioning of others is when someone else expresses a personal identity of another person, such as telling another person how smart they are; and (h) forced positioning of others which is the same as forced self-positioning but not about one's own personal identity, such as asking someone how they have been.

I used these modes of positioning to create codes to analyze my collected data. Then from the analyzed data, I was able to make claims about the students' in-the-moment identities from the way they positioned themselves and how others positioned them.

To code for positioning, I focused on two of the three main types of positioning, person that is positioned by someone else and a person positioned by himself or herself (see Table 3). I did not include the aspect of when a person positions someone else they are at the same time positioning themselves because I just referred to that as self-positioning to keep the idea of who was positioning who clear and not confused the in-the-moment identity that was being displayed. Also, I split the category of positioned by someone else into two categories, student and teacher, so when analyzing the data I was able to see if it was a teacher or another student that positioned the student. By focusing on this idea, it allowed me to see who was positioning who.

Table 3

Positioning by Who

Code	Definition	Example
Self-positioning	Student makes a statement to position self.	I am the one that knows how to do this. I need help on this.
Student positions student	Student positions student through a statement.	You are not on that part yet. Iris, you figured it out. Can you help me?
Teacher positions student	Teacher positions student through a statement.	Just ask Bernice. She knows what to do.

Next, I used the created and adapted codes from Bishop (2012) to code the types of in-the-moment identities the students were exhibiting. The codes that I used from Bishop were authoritarian, controlling problem-solving situations, superiority, providing encouragement, and face-saving moves. However, I felt more documentation was needed. I created codes that included inferior or self-effacing, expert, unique idea, and collaboration. These were ideas I noticed during Phase 3 of analysis that seemed to be relevant. I noticed that many times myself or other students would make statements that caused a students' in-the-moment identity to be inferior, and I knew this needed to be documented because it was or could affect the student's in-the-moment identity; thus I created a code called inferior/self-effacing. Students made statements throughout the sessions that showed they knew what they were doing or were an expert. I knew this was needed because they were not just being superior or an authoritarian but making a statement indicating that they knew what they were doing. I also found they would make a statement that was different from anything that had been previously said and that would position them in a different way; hence, the code, unique idea, was created. The last code, collaboration, was created because many times the students would be working together and treating each other as equal so again the codes authoritarian, controlling, or superior would not work and something new was needed (see Table 4 for codes and definitions).

Table 4

How Students were Positioned

Code	Definition	Example
Expert	Knowing/thinking they know how to do something or solve a problem	I got it, I got it! My pattern would work for that.
Controlling problem-solving situation	Directing the problem-solving activities	Write it down
Authoritarian	"adopting a critical and evaluative stance to what another is doing; constant monitoring and correcting of behavior and mathematical solutions; … criticism; and laying blame" (Bishop, 2012, p. 52)	Five minus one equals four. Everybody knows that. You put three minus one and that does not equal four. You were supposed to write that down on the purple page.
Superiority	Statements of their underlying beliefs about who they were as mathematics students.	I finished before you. I am the only one who understands this.
Providing Encouragement	Statements that show support while working on problem solving activities.	I will help you
Face-saving move	Desire for approval; lack of understanding	Oh, I did that too.
Inferior/self-effacing	Statement that show disapproval or lowers someone's belief in their mathematics ability	You did that wrong. I don't know how to do this.
Collaboration	Statements or actions that show students working together on problem solving instead of one student trying to control the situation.	I wrote backwards for the first part. What did you write for the second part?
Unique Idea	Stating something new or giving a new idea.	You solved the graph with a one in the middle. You could also probably solve it with the largest number in the middle.

I began the analysis of Phase 4 by coding the problem-solving sessions using Transana (Woods & Fassnacht, 2016) the same way I had done for Phase 3. I created collections for each student and each problem-solving conversation and key words for each of the codes in Tables 3 and 4. Next, I documented in the transcript who positioned who (see Table 3) and how were they positioned (see Table 4). Table 5 shows an example of this for one conversation during the sixth problem-solving session (See Table 5).

Table 5

Example of Positioning

Codes	Person	Transcript
Self/Expert	Becca	Done.
	Jenna	You figured it out?
	Becca	Yeah
	Jenna	Let's see. So you did one, two, three, four, five, six (checking her labeling of the edges). Are you seeing a pattern?
	Becca	Yeah
	Jenna	What is the pattern?
	Becca	Um…
Student/Expert	Trevor	Becca was at our group. [Becca had worked with Trevor in the previous problem-solving sessions and wanted to continue to work with her. He was claiming her for his group.]
	Jenna	That is okay. She can be over here.
Self/Unique idea	Becca	Look it is one, two, three, four, five (pointing to the edges). You go up backwards (edges) and I added here and subtracted here (pointing to the nodes).
Teacher/Expert	Jenna	Okay. Write it down on this sheet. You can help your whole table.
	Iris	Yeah Becca
	Karly	Yes
	Jenna	Yeah, you guys work together
Self/Controlling	Becca	Okay [moving her sheet so Iris and Karly could see], so write it down. [Karly moves it so Iris can see it better but much farther away from Becca.]
Self/Controlling	Becca	Hey, I can't even see my own sheet [moves the sheet back to the middle of the table]. Every time I just do it and you all copy down. Okay?
	Karly	So six…
	Becca	Shh…

I first coded all the problem-solving session for all students and discussed my results with my dissertation chair to clarify my codes and their definitions. After the discussion, I coded all seven problem-solving sessions, including the first again, for all of the students based on who positioned who and how they were positioned (see Table 5 for example). Next, my dissertation chair coded four problem-solving sessions using the same frameworks and Transana (Woods & Fassnacht, 2016). After she finished, we met and check our codes for similarities and differences. The situations during the problem-solving sessions that we both coded matched at least 80% of the time. Every conversation that we both coded matched 100%. There were a few conversations that one of us had coded and the other person did not code. Once these conversation were discussed, there was 100% agreement on the coding.. At the meeting, we both agreed that Becca and Bernice were good selections for the students to be used in the descriptive case study. The next step I did was to repeat the positioning coding for both Becca and Bernice for each problem solving session based on who positioned who and how they were positioned. This was done as a way to check for consistency and make sure that all situations were documented.

Lastly for the Phase 4 of analysis, I created collection reports for Becca and Bernice using Transana (Woods & Fassnacht, 2016) from both the first time and the second time I analyzed their seven problem-solving sessions. These collection reports included all of the situations I coded for both Becca and Bernice along with the assigned codes of who positioned who and how they were positioned. I used these reports to write the descriptive case study. To do this, I first wrote a detailed description of what happened for both Becca and Bernice during the interviews and problem-solving sessions. At the end of each problem-solving session, I documented their positioning and who positioned them in a table from the reports collected from Transana. Next, I explained how they were positioned and explained the different dispositions they exhibited during the sessions.

Chapter Three Summary

For this chapter, I first described the methods used for conducting this qualitative research study. The study was conducted at an after school program with 10 Grades 4 and 5 students. They participated in seven semi-structured, task-based interviews (problem-solving sessions) that focused on unsolved problems and two individual interviews. In this chapter I also explained the analytic frameworks that were constructed to analyze the students' cognitive, conative, and affective dispositions while they engaged with unsolved problems and their self-concept and in-the-moment identities that were displayed through their dispositions. In the next chapter, I share the results for the study, and in Chapter 5 I share the conclusions, limitations, and recommendations for future research.

CHAPTER IV: RESULTS

In this chapter, I share the findings of my analysis of data to address my research question: What are the characteristics of students' dispositions towards mathematics when they engage in the exploration of unsolved problems? First, I give an overview of each student based on their pre-interview and the interview conducted with the program director, Leann. Next, I share an overview of students' dispositions while they engaged in the exploration of unsolved problems in an attempt to answer research question broadly. Third, I share an overview of each student's self-concept during the problem-solving sessions. Next, I focus my analysis to a detailed overview of two students—Becca and Bernice—explaining what happened during for them during the interviews and problem-solving sessions and share their in-the-moment identities and the dispositions they displayed. Last, I share details from the post-interview for each student.

Pre-Interview: Background of the Students

Given that the focus of my study is to describe the students' dispositions while they engage in unsolved mathematics problems, it is helpful to have a sense of their dispositions and identities towards mathematics prior to their work with unsolved problems. This was the purpose of the pre-interview. The students' responses to the interview questions provided insights into their beliefs about the nature of mathematics, the importance or usefulness of mathematics, and their interest toward mathematics. Also, the interview with Leann, the director of the after-school program, provided additional insight into the students' backgrounds.

Overall, it seemed that for many of the students, mathematics was about doing computations. They described mathematics as adding, subtracting, multiplying, and dividing. When asked about mathematicians, almost all of the students said that they did not know what that was. Several students did not even know what the word mathematics meant. When the students described mathematics in school, they described memorizing facts and following the rules or steps the teacher told them to do. Many students did state that mathematics would be important in the future for a job like a teacher or working at store and needing to count money. Almost all of the students said that they enjoyed mathematics. A description of each student is presented in the following paragraphs.

Alia was a Grade 4 student. She lived with her grandmother and is a cousin to Trevor (a boy also in the study). She identified reading and mathematics as her favorite things and mathematics as her favorite school subject. Alia self-identified as a student who was good and fast at doing mathematics. She said that she liked “being smart at math.” Alia described mathematics as, “It’s where you can add, subtract, multiply, divide, or do fractions.” She said that she had never heard of a mathematician. When asked what mathematics is like in school, she responded, “Trying to know all of your facts, like by heart.” According to the program director, she has raw talent, a positive outlook on life in spite of a tough family environment, is a natural leader, is incredibly bright, and when she is given a little bit of encouragement, she will run with the project.

Amanda was a Grade 4 student. She enjoyed playing volleyball, soccer, and being able to go to her mom’s house. She spoke both English and Spanish. Iris is her sister. Amanda said that mathematics is her favorite subject in school. She described herself as a learner of mathematics by saying, “I don’t really need that much help because I sometimes, I get it right.” When Amanda was asked who does mathematics, she said that children at school, the teachers, and the after school program does mathematics. When she was asked if anyone else does, she shook her head no. She said that she had never heard of a mathematician. When she was asked if mathematics was a food, what would it be, she said, “Pizza because it is my favorite.” The program director explained that Amanda struggles with reading and mathematics and that she has missed a lot of school in the past, which may account for the difficulties she encounters. She also said that Amanda has a hard time with memorizing things. The program director said that Amanda is more willing to participate in things than her sister Iris is.

Becca was a Grade 4 student. She liked to play, do gymnastics, read, and do mathematics. She was the youngest child from a family of 13 children. When asked what mathematics is, Becca said, “Math is like, you are basically trying to find an answer. Answer or product or something.” She said that she did not know what a mathematician was. When explaining how someone learns mathematics, she said that her teacher tells the steps to follow. She thought mathematics would be helpful in the future for a job such as at a store because “I would need to count the money and add it up.” She was described by the program director as a “bright little girl who was a thinker.” She did not participate in the first problem-solving session but attended the rest of the sessions.

Bernice was a Grade 4 student. She enjoyed doing art and her favorite subject in school was science. Bernice spoke French and English. At the time of the study, her family was going through a transition. Her father had passed away from cancer, and her aunt had moved from Africa to live with them. The aunt’s children were also moving from Africa and into their house later in the year. Bernice described herself as being “really good at math.” During an interview, she said, “Math, like in school, is really easy for me. So me and some other people are always the ones that finish first.” She explained that “everybody in the world” does mathematics. She also explained that someone learns to do mathematics “step by step to do addition, then subtraction, and when you are done with that it goes to multiplication. When you finish all the multiplication steps, you can do division." The program director described Bernice as being a leader, bright, confident, and someone who brings people together.

Edward was a Grade 4 student. He enjoyed playing sports such as basketball, soccer, golf, and baseball. His favorite subjects in school were science and social studies because his teacher let them do hands-on things. Edward described mathematics as "Numbers, um… addition, subtraction." He said that everyone does mathematics. When asked if mathematics were a food, what would it be, he responded, "it would be kind of complicated." When Edward described what a mathematics lesson should look like, he explained that it should get harder when you get older and you memorize how to do it. He thought mathematics would be important for a job because "it is so you can know like what you are doing with, how much you are buying." The program director said that Edward was bright but struggled with comprehension and multi-step problems. She also said that he was funny, loving, and sweet.

Hector was a Grade 5 student. His favorite things to do were soccer and basketball. His favorite subjects in school were physical education, art, and technology. He spoke both Spanish and English. Hector's sister Karly also took part in the study. When asked to talk about mathematics, Hector said, "I am very good at long division." This showed a high-self concept. When he was asked what is involved in doing mathematics, he responded, "Learning facts and getting them right." He said he had never heard of a mathematician. When describing how someone learns mathematics, he said that you start with adding then go on to multiplication. He described that a mathematics lesson should be "sort of hard but not too hard for you…not too easy and not too hard." The program director stated that Hector really struggled in school but was very willing to work with anyone, worked hard, and was very open to learning. Hector was absent for three of the seven problem-solving sessions.

Iris was a Grade 5 student. She enjoyed spending time with her mom, and her favorite subject in school was mathematics. Iris spoke English and Spanish. Her sister Amanda also took part in the study. Iris said that she liked mathematics and that it was her favorite subject in school. She said that she had never heard of a mathematician. She said, “I like doing math because to me it is fun.” When describing why it was fun for her, she said, “Because you need to times all of them and sometimes you need to plus them and then we do, subtract them.” She described mathematics in school as listening to her teacher. The program director said that Iris had potential but really struggled in school and had been through a very emotional family situation. The program director explained that Iris had to spend time being the parent for her three younger siblings, which was a burden on her learning, and that Iris has a hard time comprehending what she reads.

Joella was a Grade 4 student. She said that she liked singing, dancing, karate, and jumping rope. She said that she did not really like mathematics. When she described what mathematics was she said it is, “where you add and subtract.” She said that she did not know what a mathematician was. When asked how someone learns mathematics or what is it that helps her learn mathematics, Joella responded, “Um, my fingers, my hands help me the best.” Joella described mathematics in school as “it is like, we do chapters, like chapter one, chapter, two. We get a packet.” She said that she did not like mathematics when she got a thick packet to do. The program director described Joella as someone who has a lot of potential and is bright but struggled with paying attention, staying focused on a task, and had little self-confidence.

Karly was a Grade 4 student. She liked to ride her bike, jump on her trampoline, and do crafts. She spoke English and Spanish. Hector is her brother and she also missed three of the seven problem-solving sessions. When Karly was describing mathematics, she said, “It is this thing that you learn in school and like you do times, plus, minus, division. Um, that is all we have learned in school from adding like stuff. And we do, we do fractions." Karly said that mathematics is really hard but that she likes mathematics and it is important for a career in the future, such as someone who builds houses. She explained what she liked about mathematics, "I like that it is challenging so I can learn even more. Instead of one plus one and then I keep on doing. I like doing math, I like to do different stuff." The program director stated that Karly got along with everyone and was a good problem-solver, but she struggled with reading and she was able to do mathematics as long as there was not a lot of reading involved.

Trevor was in Grade 4. His favorite thing to do was to play football and at school he enjoyed recess and mathematics. He is a cousin with Alia. Trevor did not seem very interested in answering the question during the interview, most of his responses were about getting all A's so he could play football. He did say that doing mathematics in school was following directions and that he liked mathematics. When asked what a mathematician was, he said, "They're the people that, that have secrets of math and that is how they know math." He explained that to learn mathematics you, "follow directions and listen." According to the program director, Trevor thinks totally differently than most children and has a strong memory for detail. The program director said that he struggles in school because of his behavior, however, when he is challenged and engaged, he is willing to work really hard.

Findings of Phase 3 Analysis: Dispositions

In the following sections, I describe the different aspects of disposition—cognitive, affective, and conative—that students displayed while they engaged in unsolved mathematics problems.

Cognitive

Much of the literature in mathematics education does not include cognition as a disposition (e.g., National Research Council, 2001). However, I included cognition as a component of students' dispositions to document whether these Grades 4 and 5 students were able to engage with the unsolved mathematics problems I posed. My analyses indicated that, for some classes of tree graphs, students could produce graceful labelings for specific graphs through trial-and-error, describe a pattern for labeling any graph in a class, and use a known pattern to gracefully label a new class of graphs (see Figure 26). I found that it was difficult for students to justify a generalized pattern or to explain why a pattern produced a graceful labeling. These findings' are illustrated in the paragraphs below.

	Star	Path	Caterpillar	Comet
Alia	③	③	⑩	④
Amanda	③	④	④	-
Becca	③	③	②	④
Bernice	③	③	⑩	④
Edward	③	④	④	-
Hector	③	④	⑩	-
Iris	③	④	②	-
Joella	③	④	④	-
Karly	③	④	②	-
Trevor	③	④	④	-

-	Student was absent, did not have a chance to work on that type of graph, or did not find a graceful label.
④	Found graceful labels for graphs of specific cases through trial-and-error.
⑩	Describe a pattern for labeling edges for any graph in a class and labeling the nodes through trial-and-error
③	Describe a pattern for labeling the edges and nodes for any graph and producing graceful labels for any graph in the class.
②	Use a known pattern to gracefully label a new class of graphs.

Figure 26. Cognition chart for tree graph patterns.

Trial-and-error. Many times students used trial-and-error to label the graphs. The students were given enlarged copies and circle and square numbered chips so they could label the graphs without having to erase their mistakes. Every student used trial-and-error to label the graph first graph they labeled during Session 1. For star graphs, all of the students were able to describe how to gracefully label star graphs.

For path graphs, I had the students work on the first three graphs in the sequence together and then encouraged the students to look for patterns as we labeled the fourth graph in the sequence together, reducing the opportunity to use trial-and-error. However, during that same session, the students were tasked to label the fifth graph in the path sequence gracefully. Bernice was the only student who was able to find a graceful labeling before the end of the session (Alia, Karly, Trevor, and Hector were absent for that session). The next day only Alia, Becca, and Bernice described how to gracefully label path graphs. The rest of the students attempted to label the path graphs but were not able to describe a pattern when verbally asked, therefore I concluded that they used trial-and-error to label the fifth graph in the sequence.

During the fifth and sixth sessions, students worked on labeling caterpillar graphs gracefully on their own or in small groups at their tables. Several students (Amanda, Joella, Trevor, and Edward) used trial-and-error to find graceful labels. When they were doing this, they moved their chips around until the found a graceful labeling. This meant that they did not have a pattern, they gracefully labeled each graph in the sequence differently, and it took them longer to label the graphs gracefully, therefore labeling fewer graphs (see Figure 27 for Joella's work on caterpillar graph). Joella was only able to label two gracefully. Edward was able to label three graphs, and Amanda and Trevor were able to label four graphs gracefully. The rest of the students labeled about eight graphs gracefully.

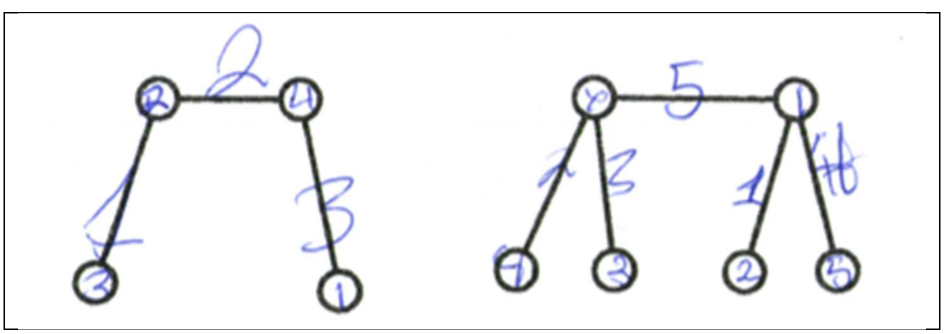


Figure 27. Joella's labeling of caterpillar graphs through trial-and-error.

Describing patterns. During the problem-solving sessions, students attempted to describe the different patterns that they found. At times the students would create graceful labeling of graphs through trial-and-error and these are not included in this section. All of the students were asked to either give a verbal or written description for all of the graphs they worked on, but most were unable to explain and so were not coded as describing a pattern. At other times the students would gracefully label the graphs, and I thought they might be using a pattern, but then they stated that they did not find a pattern or did not share their pattern.

The students attempted to gracefully label star graphs, path graphs, caterpillar graphs, and comets. All of the students describe a pattern to label star graphs. Alia, Becca, and Bernice described a pattern for path graphs. Alia, Becca, Bernice, Hector, Iris, and Karly described a pattern for caterpillar graphs. No student was able to find a pattern for comet graphs, but Becca was able to explain that her previous pattern did not work. In the following section, I share different patterns the students described for each class of tree graph we explored.

Star graphs. For the class of graphs referred to as stars, the students found two patterns that enabled them to produce graceful labels. One pattern involved labeling the center node of the star as a 1 and then labeling the rest of the nodes 2 through the number of total nodes (see Figure 28 for student's pattern and general case of this pattern). All of the students except Trevor used this pattern. However, most students described their pattern only by what they placed in the center node (many of the students referred to the center node as either the middle or top node) Alia described this pattern by saying that the one goes "in the middle, always in the middle." Bernice wrote on her task page, "In the top there would always be one in the top node." Neither Alia nor Bernice described how they labeled the remaining nodes and the edges until they were attempting to describe why they pattern would work for all star graphs. Becca, who used the same pattern as Alia and Bernice, explained that her nodes went in order. She pointed to an example and said, “I went one (pointing to center node), two (pointing to right node at the bottom left), three (pointing to the next node on the bottom left), four (pointing to the third node from the left on the bottom), and that is a five (last node on bottom).” She then explained she labeled the edges from left to right in order. She said that pattern would work for any star graph. When she was asked how she would do a graph with 20 nodes, she said, “Um, one at the top (center node) and then two, three four, five, six all the way to twenty (referring to the other nodes).

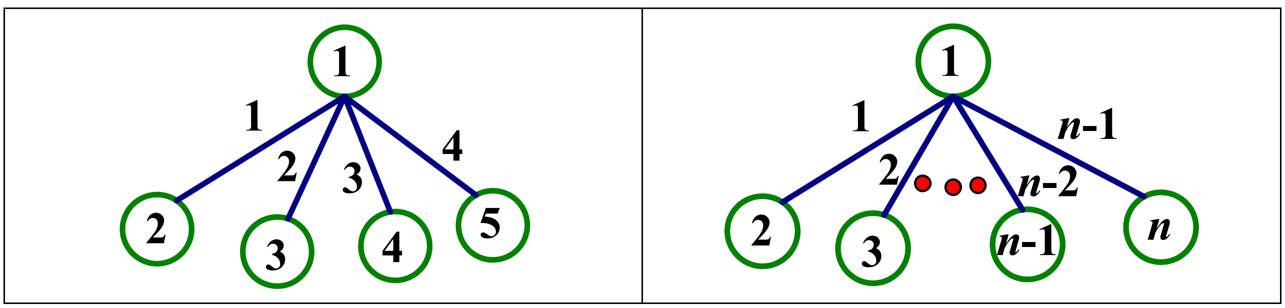


Figure 28. Pattern 1 for star graphs. The image on the left is an example of the pattern described by all the students besides Trevor. The image on the right is the general case for star graphs when one is in the center node.

Trevor discovered a different pattern. His approach was to label the center node of the star with the largest number (n) and the rest of the nodes one through the largest number minus one (see Figure 29 for pattern). However, Trevor had a hard time describing his pattern. First, he said that he had different numbers in the middle (this is how he referred to the center node). Becca helped him explain his pattern by saying; “He put the bigger number in the top [middle] and the littler numbers in the bottom." When he was asked to write his solution on the poster paper in the front of the room, he wrote, "The biggest number is in the middle." Trevor explained his pattern for labeling the edges, but he was not able to explain why it worked.

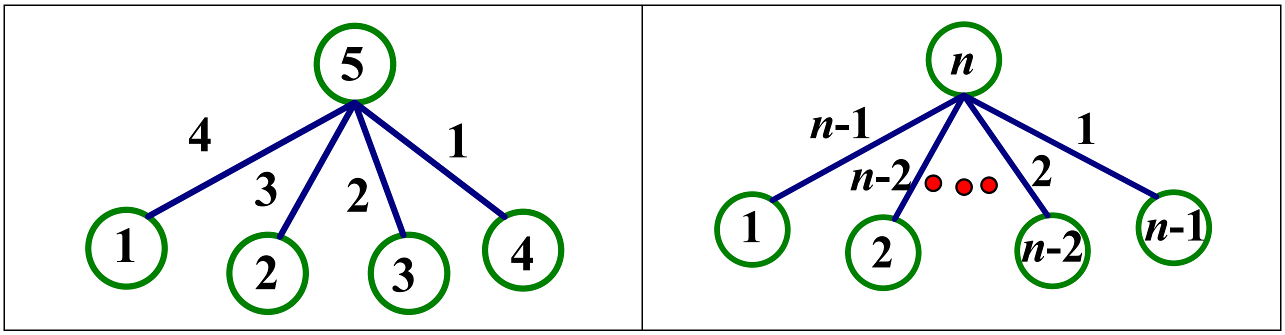


Figure 29. Pattern 2 for star graphs. The image on the left is an example of the pattern described by Trevor. The image on the right is the general case for star graphs when the largest number is in the center node.

Path graphs. For the class of graphs referred to as paths, Alia, Becca, and Bernice were able to describe a pattern for producing a graceful labeling for any path graph. Alia and Bernice found the same pattern, and Becca found a different pattern. However, all the patterns labeled the edges the same way, and all of the girls described how to label their pattern first through the edges (see Figure 30). Bernice wrote about her pattern for labeling path graphs: "The edge all the way in the bottem [sic] is one then go up (had drawn an arrow pointing upward)." Alia explained the way to label the edges as, "Before you put all of these right here (pointing to nodes), do five, four, three, two, one (pointing to the edges) or whatever number that you get. But first you count the nodes and there are six of those so you would start, you would subtract six away. Okay you would do six minus one so it would be five so then you start off with five and you go five, four, three, two, one and that is how you do nodes." Becca used the same strategy for labeling the edges.

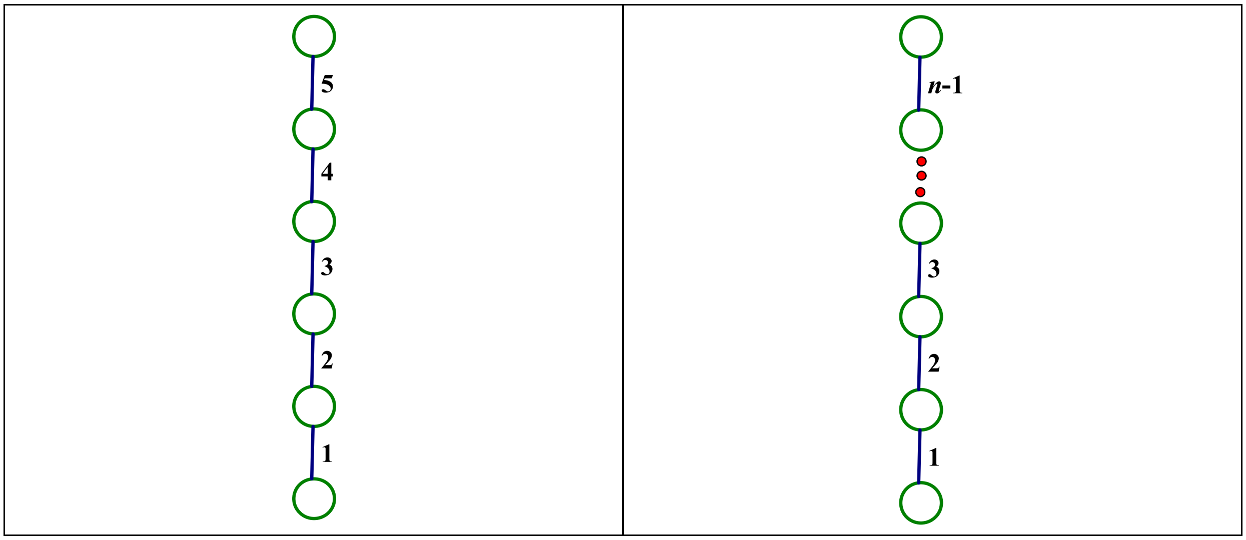


Figure 30. Example of edge labeling for a path graph. Image on the left is the students' pattern and the image on the right is the general case for label a path graph with the same pattern.

Alia and Bernice verbally described how to label the edges, but, all of their graphs in the path sequence had the edges labeled the same way. They did not describe how they determined the nodes (see Figure 31 for the pattern for labeling the edges).

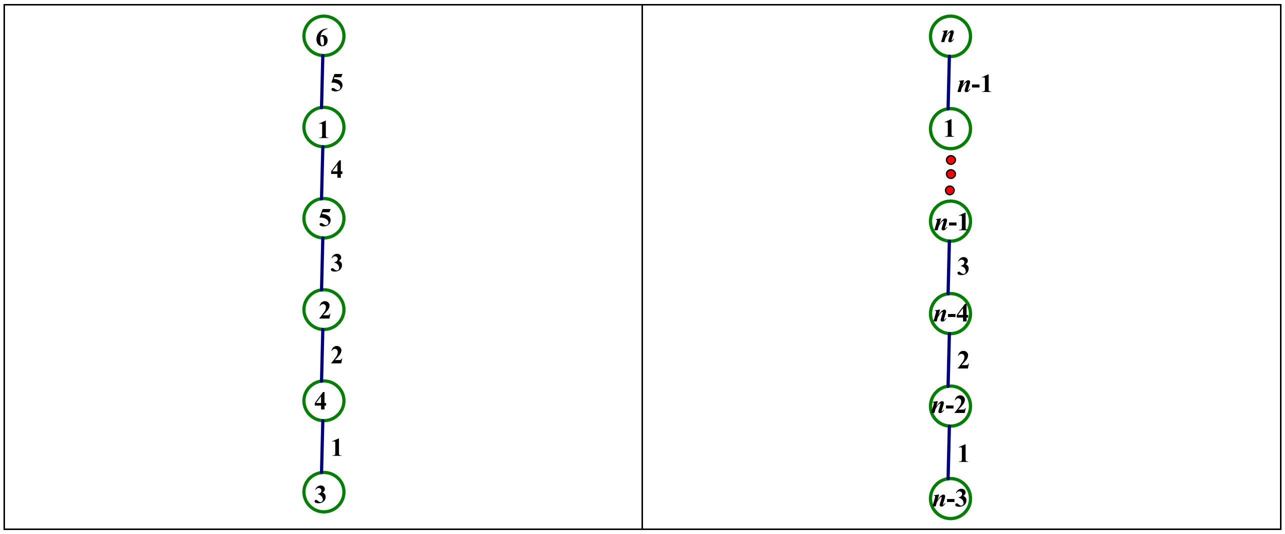


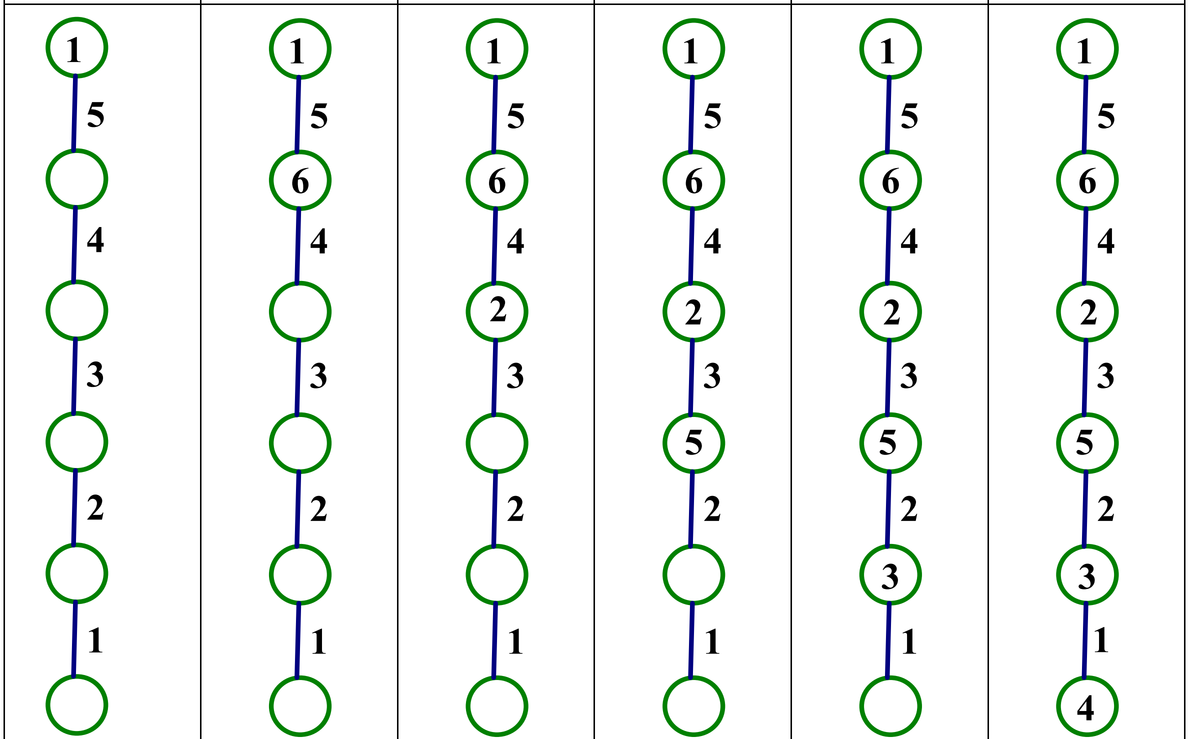
Figure 31. Example of Alia and Bernice's pattern for labeling path graphs. Image on the left is an example of how Alia and Bernice labeled the path graphs. Image on the right is the general case for path graphs using that pattern.

Becca explained how to gracefully label any path graph. She said that she could label any graph in the sequence with her pattern and used a graph with six nodes to describe her pattern. First, she labeled the edges using previously described pattern. She placed a one in the top node (see Figure 32). Then she explained how to label the rest of the nodes:

Becca: Yeah. I noticed that these go up (referring to the second node on the graph) and that um, this is one goes up and this one goes up (repeating herself and referring to the second node on the graph) and then this one goes down (referring to the third node) and then this one add (referring to the relationship between the node and edge), then this divides (misspoken, she actually subtracted), then this adds.

Becca added the first node and first edge together to get the labeling of the second node (this one goes up), then she subtracted the edge from the second node to get the third node (this one goes down), she repeated this pattern until her graph was labeled (see Figure 32 for example).

First node	Second node	Third node	Fourth node	Fifth node	Sixth node



“Top” node is 1.	Second node “goes up." Node number (1) plus edge number (5) equals number of next node (6).	Third node “goes down.” Node number (6) minus edge number (4) equals number of next node (2).	This node “goes up.” Node number (2) plus edge number (3) equals number of next node (5).	This node “goes down.” Node number (5) minus edge number (2) equals number of next node (3).	This node “goes up.” Node number (3) plus edge number (1) equals number of next node (4).

Figure 32. How Becca explained labeling the nodes for path graphs.

Caterpillar graphs. For the class of graphs referred to as caterpillar, Alia, Becca, Bernice, Hector, Iris, and Karly were able to describe patterns. All of the students who described patterns labeled their edges the same way, explaining that they labeled the edges from right to left going in order (see Figure 33). Alia said the pattern for labeling these types of graphs was, "One, two, three, four [moving her finger to each edge she had previously labeled]."

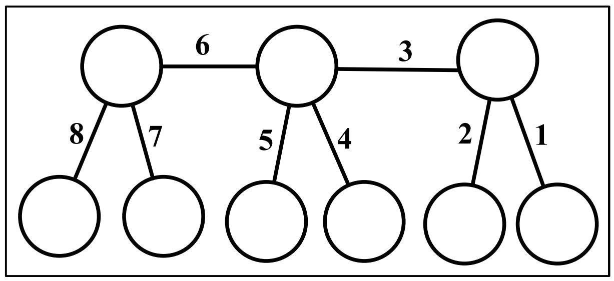


Figure 33. How the students labeled the edges of caterpillar graphs.

Becca, Iris, and Karly also described how to label the nodes. These girls worked together; however, it was Becca who originally found the pattern and shared it with Iris and Karly. I asked them to explain their pattern:

Jenna: How do you do it?

Becca: We count it and the difference.

Jenna: Okay, how did you count? What did you do?

Karly: Minus

Becca: I um,

Jenna: Okay, you minused.

Becca: First, I did one right here [pointing to the first node at bottom left] (see Figure 34a). Then I did one, two, three, four five, six, seven [moving her pen along the edges from right to left] (see Figure 34b).

Jenna: Okay, so you labeled the edges.

Becca: And then, um, the first one I went

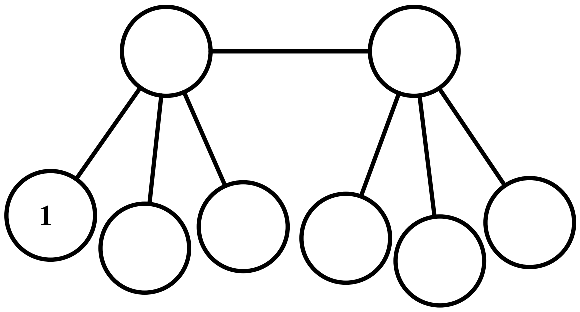
Iris: One

Becca: Then added it to there [pointing to the node labeled 1 and the edge labeled 7] (see Figure 34c).

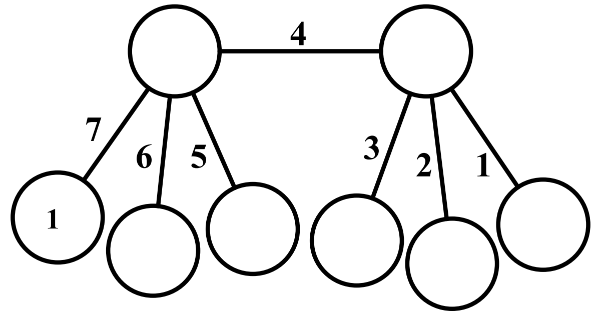
Jenna: Oh so you added it?

Becca: Yeah and then I minused it (see Figure 34d) and I kept on doing it ("it" seems to mean subtracting or adding the nodes and edges; see Figure 34e).

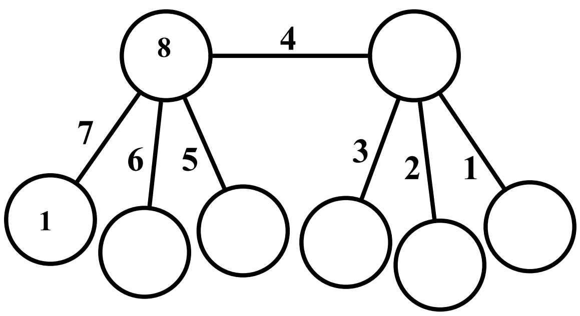
Their process was to first label the edges in order from right to left. They then placed a one in the bottom left corner node. They described their pattern as an addition subtraction pattern, similar to the pattern they used for path graphs. However, although they described the pattern for caterpillar graphs the same way they described the pattern for path graphs, the pattern is actually different. The pattern for caterpillar graphs was addition then subtraction for the left-most cluster of nodes and edges and subtraction then addition for the right-most cluster of nodes and edges.



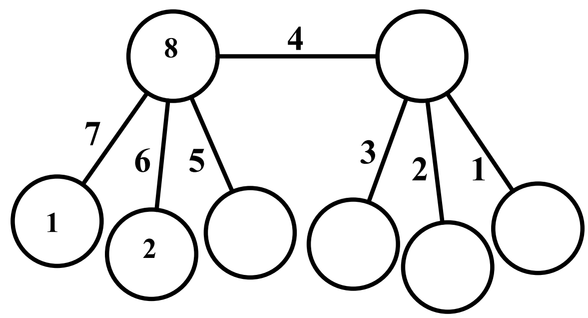
A. Step 1: Place a one in the bottom left node.



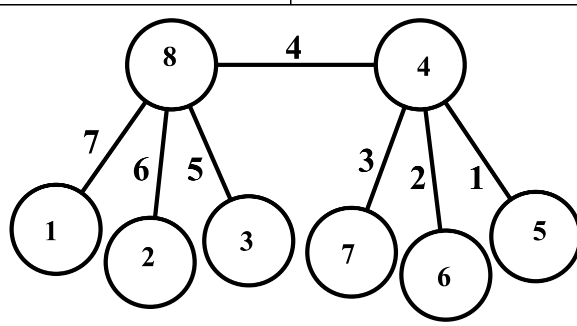
B. Step 2: Label the edges from right to left in order



C. Step 3: Add the bottom left node (1) with the edge (7) to get the node label (8).



D. Step 4: Subtract the edge label (6) from the known node label (8) to get the unknown node label (2).



E. Completed labeling for caterpillar graph. Continue with Becca's addition and subtraction pattern to complete graceful labeling. Subtract the edge (5) from the center left node (8) to get node (3). Next, subtract the edge (4) from the center left node (8) to get center right node (4). To find the rest of the nodes add center right node (4) to edges to get the last three node labels.

Figure 34. Becca's steps for labeling caterpillar graphs.

Alia, Bernice, and Hector only described how to label the edges of this sequence of graphs. All three of them labeled their nodes differently for every graph, so I do not believe they found a pattern for labeling the nodes and were only able to gracefully label the nodes through trial-and-error.

Comets. None of the students found a pattern for producing graceful labels for comets. This class of graphs was introduced in Session 7 when most of the students had already left and only Iris, Bernice, Alia, Becca, and Joella remained to work on these graphs. It is noteworthy that although Becca did not recognize the first graph as a path graph that she had already labeled, she did attempt to use a strategy that was similar to how she labeled the edge for caterpillar graphs. This is, she attempted to right-to-left numbering of the edges for the other graphs. Her work is presented in Figure 35 and shows that she was successful in gracefully labeling only the second graph (of order 4). She appears to have tried two different numbering strategies. Alia and Bernice both labeled the first two graphs gracefully using their pattern for path graphs (the first two graphs in the comet sequence are path graphs). Both students used trial-and-error to attempt a graceful label for the third graph in the sequence; however, neither girl was successful. Also, neither Joella nor Iris found any graceful labels for the comet graphs. Both girls left the problem-solving session after working on the graph for less than 1 minute.

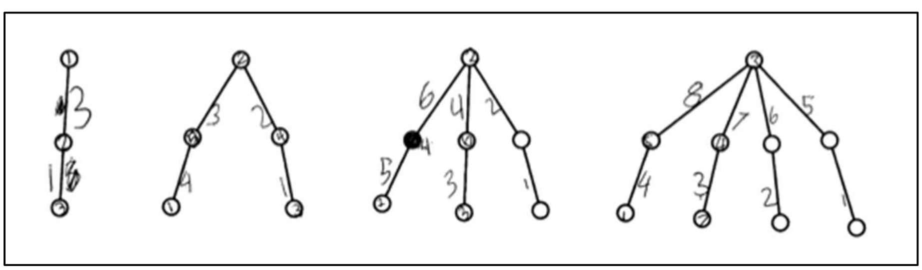


Figure 35. Becca's attempt at using her previous pattern to gracefully label comets.

Using a pattern. Becca was the only student to describe how she used a pattern she found from the previous graph to label the next sequence of graphs she attempted to gracefully label. During Session 5 Becca stated that she had a pattern for labeling path graphs. At the ending of that session she stated, “I have a strategy to do it and it works” referring to her pattern for labeling the path graphs (see Figure 29). She walked into Session 6 and used her idea of how to label path graph to quickly label all of the caterpillar graphs gracefully. During the last problem-solving session, Session 7, Becca asked to work on “sticks and nodes.” I gave a page with comet graphs on it. She sat down and quickly filled in the first two graphs using her pattern for solving path and caterpillar graphs; however, her first graph is not labeled correctly (she placed a three on the edge; see Figure 36). After 5 minutes of working on the third and fourth graphs in that sequence attempting to use the same pattern, Becca stated that she “needed a new pattern.” Her solution for gracefully labeling the other graphs did not work on comet graphs.

Trying to justify. There were only a few times in which students attempted to justify a generalized pattern or to explain why a pattern produced a graceful labeling. These attempts all pertained to star graphs and were made by Alia, Alia’s table group members (i.e., Karly and Iris), and Bernice.

During the third problem-solving session, I asked Alia to explain why her for graceful labeling for star graphs would work for all star graphs. I wanted her to develop this idea and explain why that worked:

Alia: Because one always goes in the middle… You subtract. You put the one in the middle and then you subtract each number in the circles to get a number of one, two, three, four, and keep going.

Jenna: But how do I know if I put a one here (pointing to middle node), cause that is what you said, and I put a two here and a three. How do I know this (the different edges) is going to be a different number?

Alia: You know because one minus two does not equal two. One minus two equals one. This is how you know.

A few minutes later, I questioned Alia again to have her continue to try to explain this idea. Karly joined in the conversation. She was absent on the previous day when they had begun to explore the graphs. The following is the transcript of this interaction:

Alia: This is one, this is two, this is three, this is four (referring to the edge numbers; see Figure 36)

Jenna: But how do you know they are always going to be different numbers?

Karly: Because you have to minus it.

Alia: Yeah, every number, say six, is in the circle then there is… and then you put seven, eight, nine, ten (in the nodes; see Figure 36 for drawing of Alia's work).

Karly: One, two, three (referring to the edges in Alia's drawing; see Figure 36)

Jenna: Are they (the edges) going to be different?

Alia: Yes, these are going to be different, see. One, you subtract that and it is one, two, three, and that is four (edges; see Figure 36). So that is how you know they are all going to be different because you are subtracting. You are subtracting that is in the middle to the number outside and the number outside are different numbers.

Alia attempted to explain why her pattern would work for labeling star graphs but struggled with the language needed to make a general argument for why her pattern worked. She attempted to describe why the edges labels would be different through an example of putting six in the center node.

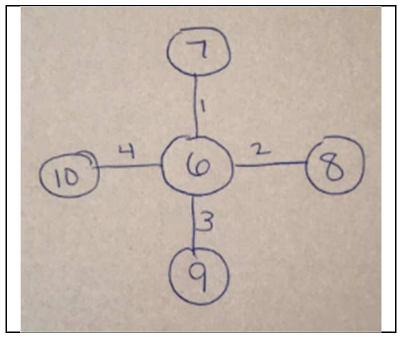


Figure 36. A drawing of Alia's work when she attempted to give a generalized argument for star graphs.

Bernice also tried to give an argument for why her pattern for labeling star graphs worked. After Bernice labeled the first six graphs in the star class gracefully, I asked her if she could label any star graph. She said that she could and that it would be a one in the middle node of the star graph. To explain why it worked, she said, "It would be because the first number and the second number equals the first number" (pointing to two different nodes that were not the center node). I do not know what she meant by that statement. Several minutes later I asked her again why her pattern worked and she only gave the argument that "there will always be a one [in the middle node]." Again, like Alia, she attempted to explain why her pattern worked, but she struggled with the language to really explain why it worked.

Summary for cognitive. The students were able to describe patterns for star, path, and caterpillar graphs but not comets. Some students were able to use or modify a known pattern for one graph to gracefully label a new type of graph. Only two students attempted to explain why their pattern for star graphs could allow all star graphs to be labeled gracefully, but they had difficulty articulating their thoughts and did not produce complete or coherent explanations. Because of the struggles the students had when trying to justify their pattern for star graphs and due to time, I did not ask or push the students to give justifications for the other graphs. This idea was challenging for the students to do and they had a hard time articulating their explanation.

Affective

My analysis for affective dispositions focused on the beliefs students held about the nature of mathematics and the emotions they experienced. My analyses indicated that students hold different beliefs about what mathematics is and how it should be done and that they experience multiple emotions while they are engaged in problem solving. These findings are described in the detail in the paragraphs below.

Nature of mathematics. Throughout the problem-solving sessions the students demonstrated dispositions about the nature of mathematics that related to five themes. Students exhibited beliefs that (a) mathematics should be done alone, (b) students could work together and learn from each other, (c) you are good at mathematics if you are fast at it, (d) mathematics problems always have solutions, and (e) mathematics is about computations. Two additional comments were coded as referring to the nature of mathematics, but they did not fit any of these five themes and are categorized as *other*. Table 6 displays the number of times in each session that students made comments about the nature of mathematics. Each of the themes are discussed in the following paragraphs.

Table 6

Comments about the Nature of Mathematics by Session

Beliefs	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Mathematics should be done alone.	2		1			1	
Students can work together and learn from each other.						2	3
Being good at mathematics is being fast at it.	1	2		2		2	
Mathematics problems always have a solution.		1	2	3	4	1	1
Mathematics is about computations.	1				3		
Other					1		1

Mathematics should be done alone. The idea that mathematics should be worked on individually was more prevalent than in the beginning of the problem-solving sessions even though I encouraged the students to work together and talk with their neighbors throughout all of the problem-solving sessions. During the first problem-solving session, when a student asked Iris for help, she responded, "You have to do it on your own." On the same day, after Bernice had found a graceful labeling, she told Amanda and Joella, "No peeking." During the third problem solving-session, I asked Bernice if she wanted to share her solution with everyone, and she replied, "No, they can figure it out themselves." Although several students began to work together as the study progressed, there was still evidence in the later sessions that some students believed that mathematics should be completed on one's own. For example, during Session 6, Edward claimed that another student had copied his work and called that student a "cheater."

Working and learning together. By Sessions 6 and 7, several students acknowledged or seemed to accept that doing mathematics can be a collaborative activity. During Session 6, Iris, Becca, and Karly worked together for the whole sessions. They were able to gracefully label all of the caterpillar graphs I made available to them and found patterns for how to gracefully label the caterpillar graphs. During Session 7, Joella switched tables to work with Iris and Becca (Karly was absent). At the end of Session 6, Joella stated that she needed help and said, "Nobody is helping me." Thus, I believe Joella switched tables for Session 7 because she saw Becca, Karly, and Iris working together during the previous sessions and making progress. She made suggestions that she wanted them to work together. First she said, "Aren't we suppose to work together?" She then followed that statement up by stating that she was trying to work with Becca. The following is the transcript from this interaction:

Joella: Don't go so fast. I am trying to keep up

Becca: You are doing math with me?

Joella: We are working together. What is that (pointing to something on Becca's page)?

Becca: Five

Iris: Are we working together?

Joella: Yes

Becca: (Nods head yes)

Fast at mathematics. Throughout the problem-solving sessions, students made comments about being the first one done or being ahead of others. During Session 1, Alia made a point of telling Bernice that she beat her when she found a graceful labeling before Bernice. Then in Session 2, Alia made a statement about being the first one done:

Alia: I am done. I got stars and hearts.

Jenna: Okay

Alia: I am done. I was the first one done.

Amanda also made a comment about finishing first by claiming, “I won.” During Session 4, twice Joella made comments about being the first done: “I am done. I am the first one done.” And a few minutes later she said, “Done first.” During the Session 6, Becca and Karly made a comment about being faster than others:

Becca: I need the next one.

Jenna: You need the next one? Yellow?

Becca: Yeah, yellow. I am so ahead of everyone.

Jenna: No, you guys are right… Those girls back there are where you are.

Karly: We have to hurry up.

These comments imply that many of the students perceived the work they were doing as a competition and that being fast at mathematics is important or something to strive for.

Mathematics problems always have a solution. During the problem-solving sessions, I explained to the students that the Graceful Tree Conjecture had never been solved. To solve this problem they would have to either prove that all tree graphs can be labeled gracefully or find a tree graph that could not be labeled gracefully. They seemed to understand if they figured the problem out, they would win a prize, but they seemed unsure of the idea that the problem had never been solved and how we would know when it was solved. They made several comments to make this idea clear. However, they seemed to think they would find a tree graph that could not be labeled gracefully, and they also seemed to think that someone knew the answer or were trying to process what it meant for a problem to be unsolved. They also made several statements about how many solutions a problem can have.

On Sessions 2 and 3 of the problem-solving sessions, Trevor found a pattern for solving star graphs that was different from the other students. This created a discussion among the students and myself about having different answers. On Session 2, Amanda and Alia said that the one goes in the middle. I questioned them if there was only one correct answer. The following is the conversation we had:

Amanda: The one in the middle.

Jenna: Is there only one right answer?

Alia: Yep

Amanda: The one goes in the middle

The following problem-solving session, I wanted to focus on the idea again of there the students solving it different ways. I asked Trevor to explain his answer because it was different from the rest:

Jenna: What were you thinking (to Trevor)?

Iris: We already know.

Jenna: No, because he was thinking differently, right?

Bernice: That would be wrong.

After Trevor explained his answer. I asked again. The following is what was said:

Jenna: What do you guys that about that? Do you think the star graphs could have two different ways to make them graceful?

Joella: No

At the end of the discussion, the rest of the students seemed to accept Trevor's answer and the idea that there could be more than one way to create a graceful labeling for star graphs. Unfortunately, we did not have more conversation about different ways to label graphs in later sessions, and this was the only example of students exhibiting the belief that mathematics problems have only one right answer.

During the fourth problem-solving session, I wanted to clarify with the students that nobody had ever solved this problem. The following is the transcript from this discussion:

Jenna: Does anybody know the answer?

Bernice: No

Others: Yes

Jenna: No, nobody does.

Iris: Yes

Trevor: Yes, I know a person that knows.

Jenna: Who knows?

Trevor: Google

Jenna: If you typed this into Google, it wouldn't know because this problem has never been solved.

Iris: So how do you know?

Jenna: How do I know that it has never been solved?

Iris: You don't know too?

Jenna: Well I know it has never been solved.

Later in that session, Joella wanted to clarify that if I did not know the answer, could I figure the problem out. She asked me, "Can you figure it out?" I responded that I was not sure if I could.

On the same day, Trevor also asked me, “Can you figure it out?” I gave the same response. Both, Trevor and Joella seemed either confused, concerned, or shocked that I did not know the answer to the problem I had asked them to work on or that I should figure the solution out; however, I did know the solutions to the classes of tree graphs I posed.

During the fifth problem-solving session, we had a group discussion about the problem again. The following is a snapshots of the idea that I did not know the answer and how would I know if they were correct. The following are three different short transcriptions from that discussion:

Jenna: Edward, we would really have to convince somebody that our solution is correct since nobody has ever figured out the answer.

Alia: So who is the person that made this, cause they know the answer because they are the ones that made the equation.

Trevor: It wasn’t made up.

Joella: All questions are made up.

A few minutes later:

Edward: Wait, do you know the answer?

Jenna: No, nobody has…

(some side discussion)

Edward: How do we know if it is correct?

Jenna: Well we don’t know so far. We have just figured out star graphs.

Edward: But if you don’t know it is correct then….

During that same discussion, Alia decided that we just needed to find a mathematician to give us the answer. The following is transcription of what was said:

Alia: So we need to find a mathematician to give us an answer.

Jenna: But do you think a mathematician knows the answer if nobody has ever figured this problem out? Do you think they know the answer?

Alia: Yep.

While working on the gracefully labeling the graphs on the same day, Joella asked Dr. Cullen, a professor from Illinois State who had volunteered to help, if she knew how to do it. Here is the transcription:

Joella: Do you know how to do it?

Dr. Cullen: Um hum, this one I do.

During the sixth problem-solving session, the idea that the problem had not yet been solved came up again and Hector wondered how I would know if they got the correct answer:

Hector: How do you know if we are right?

Jenna: How do I know if you are right?

Hector: You know how you said nobody knows the answer or nobody knew?

Jenna: Yeah

Hector: How do you know if someone gets the right answer?

Jenna: How do I know? Because you guys have to justify your answer.

The students did not seem to understand how I would know if they were right because no one knew the answer to the problem and they did not question what I meant by justifying they answer.

During Session 7, the students wanted to know when they would find out the answer:

Bernice: Today is our last day with math.

Trevor: No

Bernice: Yes, Ms. [Leann] told me.

Trevor: Wait, when are we going to find out?

Jenna: Find out what?

Trevor: Find out about the question (standing and shouting).

Jenna: Well, we haven't figured it out yet have we?

Trevor: No, but we aren't going to get it finished.

Throughout the problem-solving sessions, the students seemed to think that mathematics problems always have a solution. First, they had to grasp the idea that the problem had never been solved and no one knows the answer, next they struggled with the idea that of how we would know if the solution is correct or not. Several students, like Alia, thought that they just needed to find a mathematician to give them the answer. Trevor though that he could find the answer on the Internet and that on the last day they were eventually going to find out (be told) the answer (implying that there must be a proof for the conjecture). Also, in Session 4, Joella and Trevor could have been implying that although I did not know the answer, I should be able to figure it out (because of course there is a solution). This might have been the same with Joella asking Dr. Cullen if she knew "how to do it."

Mathematics is computations. In the pre-interview, all of the students described mathematics as addition, subtraction, multiplication, and division. Several times throughout the study, I asked students what a mathematician does. Many times I was told that they did not know. During the fifth problem-solving session, as a whole group, we discussed what a mathematician does. Alia said, "It is a person who knows almost all of the math equations and is good at math." Joella said, "It is a person who is good at math" and a "mathematician is when they know like every single number, every number to multiply, divide, plus, and subtract." For many of the students in this study, mathematics was viewed as computing with numbers.

Emotions. All of the students displayed a range of emotions throughout the study. These include: boredom/apathy, contempt, affection/caring, and humor, as well as, distress/frustration and joy/pleasure/pride. However, three of the emotions described in my analytic frameworks—sadness, tension, and anger—were never exhibited by the students. Joy and frustration were exhibited most frequently, and they were evident in every session and exhibited by every student at some point in the study. The number of times I coded each student as displaying the different emotions are presented in Table 7. In the following sections I describe examples of each emotion that was exhibited.

Table 7

Emotions Displayed During Problem-Solving Sessions

Emotion	Alia	Amanda	Becca	Bernice	Edward	Hector	Iris	Joella	Karly	Trevor	Total
Anger/Disgust											0
Tension											0
Sadness											0
Boredom/Apathy		2	1				1				4
Affection/Caring	4		3	6		1		1		3	18
Humor	1	2	2	2	5	1	1	1		6	21
Contempt	1	5	5	18			6	6	1	7	49
Joy/Pleasure/Pride	41	14	24	28	17	5	13	13	5	17	177
Distress/Frustration	19	9	26	58	9	13	5	35	7	42	223

Distress/frustration. The most common emotion that was exhibited during the problem-solving sessions was distress or frustration, and it was documented 223 times; Bernice, Trevor, and Joelle displayed this disposition the most. Overall, frustration was most commonly displayed when a student was using trial-and-error to find a graceful labeling for a graph. Statements such as “This is so hard,” “Oh my gosh,” “Dang it,” “I can’t do this,” “I messed up,” and “That doesn’t work” were the most common statements made to indicate frustration that were exhibited. At times students would bang their arms down on the table in frustration, and at one point Becca slammed a chip down on the table while stating, “How do you do this?”

Boredom/apathy. Throughout the problem-solving sessions, boredom/apathy was documented four times. Although the students never explicitly stated that they were bored, they made comments that they were tired. Three of these statements were in reference to the student being "I am tired" and "I am really tired. The only time apathy was documented was during Session 3 when Becca said, "blah, blah, blah" in a tone that indicated apathy.

Contempt. Contempt was exhibited 49 times during the seven problem-solving sessions. Bernice was the person that displayed the most contempt. She often critiqued other students with contempt in her voice as she commented: "Three minus four is not three." "One minus five equals four, everybody knows that." "Wrong, that is the wrong one." "You don't need it (a circle chip with the number nine on it) yet. You are not here (the graph that has nine nodes). I am not being rude." At one point, she described Amanda as "disgusting" and another time told Alia, "You must be a kindergartner or preschooler" when Alia said she did not know how to do something. Other students made contemptuous statements such as: "stop being scary," "this one doesn't even talk right," or calling a child a copycat or saying that someone cheated.

Affection/caring. The emotion of affection or caring was documented 18 times during the problem-solving sessions. Most of the time these were statements of a student wanting to help another student such as, "Can I help her," "I am going to give her a hint," "Ask if you need help," "You forgot the tippy top," and "Do you need help." At other times, students made statements of encouragement such as "you did it" and "good job."

Joy/pleasure/pride. The emotion of joy, pleasure, and pride was exhibited 177 times during the seven problem-solving sessions. Students frequently exhibited this emotion when they found a graceful label for a graph. They would exclaim, "I am done!" "I did it!" "Got it!" "I finished!” and often raise their arms above their heads in celebration. Every student exhibited joy, pleasure, or pride multiple times throughout the study.

Humor. Humor was documented 21 during the problem-solving sessions. Most statements of humor came from Trevor and Edward. Trevor made statements such as, “Your name will be cupcake” and told various jokes. Edward made statements such as “it is peanut butter jelly time” and made Bernice and Joella laugh by using the word “twee” instead of three. Bernice made the other students laugh when I asked if they saw a relationship between two things and she responded, “So they are dating; they are dating.” Becca made humorous statements about the numbered chips used for labeling graphs. For example, when I asked someone if they wanted chips, she said, “Yeah. Could we eat them?” This made the students in her table group laugh.

Summary for affective. Overall, the students displayed many different beliefs about the nature of mathematics. Most of the themes are very common in mathematics education, including to be good at mathematics you have to be fast, mathematics is about finding solutions, and mathematics is typically done alone. The students also displayed a range of emotions, most notably joy and frustration, while they engaged with unsolved mathematics problems.

Conative

A positive conative disposition would be when a student showed perseverance and continued to engage with the problem, even if they were challenged. Overall, all students displayed positive conative dispositions during the study. If they were present at the after-school program, they attended the problem-solving sessions and, for the most part, participated actively. Although the sessions were voluntary, not one of the students stopped attending. I identified three themes in my analysis of the data pertaining to students’ conative dispositions: (a) all students engaged in productive struggle, (b) some students considered quitting but persisted instead, and (c) some students exhibited increased engagement as the study progressed.

Productive struggle. While analyzing the data for emotion, I found interplay between the emotions of frustration and joy, which were the two more common emotions displayed by the students. This interplay gave an example of the productive struggle students displayed through their emotions. All of the students experienced positive and negative emotions while they engaged with the unsolved problems. They had frustration and they had joy in their work which was an indication of productive struggle.

I broke down the two emotions of frustration and joy by student and problem solving session as a way to see the interplay (see Table 8). Next, I describe two students' (Alia and Trevor) interplay between frustration and joy by sharing two students engagement during problem-solving Session 6. These students were chosen because they had an equal balance of joy and frustration during that session. The two examples of the emotions the students displayed were similar to the emotions the other students went through. In the given examples, the students experiencing productive struggle is evident. They would be frustrated with a task, but when they found a graceful labeling, they would show excitement. Because their struggle was productive, they demonstrated perseverance.

Table 8

The Interplay of Frustration and Joy Broken Down by Session and Student

Session #	1		2		3		4		5		6		7	
Emotion	F	J	F	J	F	J	F	J	F	J	F	J	F	J
Alia	2	4	0	12	8	5	-	-	3	9	4	5	2	6
Amanda	3	1	1	2	1	2	1	1	0	1	2	7	1	0
Becca	-	-	3	3	5	2	7	1	0	5	2	10	9	3
Bernice	6	6	8	0	15	2	9	5	5	4	6	6	9	5
Edward	2	5	2	2	1	2	1	4	1	1	1	3	1	0
Hector	0	1	-	-	4	0	-	-	2	0	7	4	-	-
Iris	0	0	0	3	1	3	1	1	2	2	1	2	0	2
Joella	3	3	3	2	10	3	2	1	5	2	11	1	1	1
Karly	1	0	-	-	5	2	-	-	0	1	1	2	-	-
Trevor	5	3	9	8	6	0	0	1	5	0	3	4	14	1
Total	22	23	26	32	56	21	21	14	23	25	38	44	37	18

Note. F stands for frustration and J stands for Joy. – means a student was absent for that session.

Alia. At the beginning of the sixth problem-solving session, Alia displayed a positive interest in the activity. Prior to beginning she told me she needed the pink page. The following is the conversation we had:

Alia: I want the pink one.

Jenna: Will you just move your stuff and put it in the center (speaking of their reading and homework they had done previously). Make sure you put your name on this stuff. What color did you need?

Alia: Found it.

Alia went and found the sheet she was working on because she wanted to begin the activity. She then began working. Several minutes later she stated, “I am lost” while erasing her page. This showed a disposition of frustration. Thirty seconds later, she stated, “Okay, I got it!” which indicated that her frustration shifted into pride. Alia continued to work. Several minutes later she shouted, “No! I need an eight.” This was referring to her noticing she still needed to label an eight on one of her nodes and showing frustration. Twenty-seven seconds later, Alia had the following conversation with Bernice:

Alia: I did it (clapping her hands and then throwing her arms up in air). Woo!

Bernice: You did it?

Alia: Yeah. Ha-ha, I did it!

In the 27 seconds, Alia changed her frustration into pride and joy. Alia continued to work on the next graph. Two minutes later, Alia slammed her hand down on the table and said, “Oh, I messed up” demonstrating frustration. Twenty-four second later, Alia stated while clapping her hands, “I was right. I was right.” Once again, Alia changed her frustration into pleasure. Alia continued to work. Six minutes later, Alia put her arms in the air and shouted, “Oh, I am good at this!” displaying pleasure and her mathematics self-concept. About 6 minutes later, Alia again put her arms in the air celebrating and showing joy and said, “Yes! I finished.” After filling in her graph she said with her arms in the air, “Okay, I am ready for the next one.”

While participating in the problem-solving sessions, Alia displayed a range of emotions that alternated between joy and frustration. Her most common emotion was joy/pride/pleasure. She also demonstrated distress/frustration and positive interest. This short description of her activity during problem-solving Session 6 showed her perseverance and productive struggle that was demonstrated through the range of emotions she experienced. This oscillation between joy and frustration was common for her in the other sessions as well.

Trevor. Trevor displayed productive struggle that was similar to all of the students through joy and frustration. He would go through a period of frustration or struggle and then experience joy when he would figure out a graceful labeling.

While other students that participated in the problem-solving sessions were still completing their homework, Trevor asked me while pointing to the math supplies, “Can you give me one.” This disposition displays a positive interest in the activity. After working on a label for a graph for several minutes, Trevor said, “Found it, found it” while smiling, showing joy in his accomplishment. Once he wrote the numbers for his labeling down he said to me, “I need a new one.” Once again showing positive interest in the activity. Six minutes later, Trevor displayed frustration over trying to label a graph. The following is the conversation we had:

Trevor: We don’t have enough things.

Jenna: Nope, put the one here. You had the one here.

Trevor: You can’t do that.

Jenna: Oh, you can’t? Hum?

Trevor: That would mess up the problem.

During this discussion, Trevor is encountering some struggle with the problem. Trevor continued to work on the graph and 13 minutes later, while shouting and clapping, Trevor said, “Got it! Got it! Look it, look it.” He turned his struggle into success and displayed joy and pleasure over labeling the graph. Trevor was then given the next graph to label. He stated, “This is hard.” This showed slight frustration. After working for 5 minutes, Trevor said frustrated, “I can’t figure this out because I don’t have enough chips.” He was given the chips that he needed to complete the graph and continued to work. Just 1 minute later Trevor said, “Got it.” He wrote down the solution on his page and said, “Got them all!”

Similar to Alia, Trevor also displayed a range of emotions and alternated between being frustrated and then exhibiting joy or pride when he produced a graceful labeling. I interpreted this interplay between joy and frustration as a sign of productive struggle. This was similar to the productive struggle he and the other students demonstrated during the other sessions.

Persisting rather than quitting. There were several instances in which students made statements about wanting to quit the study. The third problem-solving session was challenging for many of the students because the task sheets I created did not present the graphs in an order that was easy to follow. Students were not noticing patterns. One group in particular, Trevor, Becca, and Edward, had not made much progress. Trevor was frustrated and Dr. Cullen, who was helping with the session on that day, spoke with him about working hard in football (which was Trevor's favorite activity). Becca and Edward were listening. The following transcript presents the conversation that ensued after Dr. Cullen and Trevor finished the football discussion:

Dr. Cullen: You don't have to stay here.

Becca: You don't?

Dr. Cullen: (Nods head.)

Becca: (laughs)

Trevor: (laughs)

Trevor: I really want to leave.

Dr. Cullen: Alright.

Becca: Can we go?

Trevor: Can we come back here?

Dr. Cullen: You mean another day or another time?

Trevor: Yeah, another day?

Dr. Cullen: Yeah

Trevor: Is other people going to go here? Other people?

Dr. Cullen: Nope, you might just miss some stuff. You know how we keep building on.

Becca: (Nods head yes.)

Dr. Cullen: Yeah.

During this conversation, I noticed that other students were getting frustrated and decided to end the session. Thus, Becca and Trevor did not have an opportunity to follow through with their thoughts about leaving. However, the next day and every day that followed, both students attended the problem-solving sessions and fully participated. They never asked to leave early again. On the seventh problem-solving session, Trevor even showed concern because he knew his mom was going to go pick him up early and he did not want to miss the session.

During the sixth problem-solving session, Joella made several statements indicating that she wanted to quit working. After she had expressed her frustration several times—"I don't know this crap," "this is hard," "I don't know how to do this stuff," and "everybody knows this stuff but me." I asked if she would like to be finished for the day. She replied, "I want to be done," and I told her she could go into the main room with the rest of the students not in the study or she could stay in the room but not participate. However, Joella did not leave our room and continued to find a graceful labeling for the caterpillar graph. The next day, she came right into the room and when I told the students that it was the last day I would be working on unsolved problems with them, Joella said, "I don't want it to be the last day." Although she had made statements of wanting to quit during the previous session, she continued to persist and worked on all the tasks I gave her.

Increased engagement. All of the students demonstrated persistence throughout the study. Iris's participation increased over the course of the study. Alia was very persistent during each problem-solving session but had an increase in engagement during the fifth problem-solving session. At times other students did not want to stop with an activity, asked to return to the Graceful Tree Conjecture when I changed the problem, and one student even asked for task sheets to bring home.

Iris. During the first 3 sessions of the study, Iris did not want to work or show perseverance; however, she changed her mind and spent the following four sessions working throughout and demonstrated positive conative dispositions towards mathematics.

During the first problem-solving session, Iris stated, "I don't want to do this." She was then given the option to leave and was told that participation in the study was voluntary. Iris continued to sit at the table not working. I went to talk with her. The following is our conversation:

Jenna: Is everything okay?

Iris: I don't know. I am really tired.

Jenna: Well do you just want to watch for today and you can always do it tomorrow?

Iris: (Nods head yes.)

Jenna: Okay.

Iris: I am so tired (leaning on Karly).

Iris stayed in the room for the rest of the session but she did not work. At the beginning of the second session, I noticed Iris was working on a mathematics worksheet. I asked what she was doing. The following is the transcript of what was said:

Iris: Oh, I am doing this (pointing to the mathematics worksheet).

Jenna: Okay, okay. But are you going to pay attention today?

Iris: I don't know.

Jenna: Okay. If not we can have you go back out (referring to the main room where the rest of the students were working).

Iris: I want to stay.

Although Iris had not worked during the first session and began by not working on the second, she made the choice to stay in the room both days. Towards the end of the second session, Iris began to attempt to gracefully label several star graphs.

At the beginning of Session 3, I reminded Iris that this was a choice and she did not have to stay. She said, "I will stay." Then during the session Iris began to shift in her work ethic. She attempted to gracefully label path graphs. She labeled one path graph gracefully during the session.

During the fourth problem-solving sessions, Iris participated and joined in all of the group discussions. During the fifth problem-solving session, Iris worked throughout the whole session with no negative comments about not wanting to work. She labeled six path graphs gracefully. On the sixth problem-solving session, Iris was enthusiastic and completely engaged throughout the whole session. Before I had even started the students on the problem (several had homework to do and others were reading), Iris took out her mathematics work from the previous sessions and began to work. During that session and through working with her tablemates, she gracefully labeled all of the graphs I had assigned and wrote explanations for how to label all the types on graphs in each class she worked with. Iris continued to work throughout the seventh problem-solving session expressing joy and excitement as she found graceful labels for the graphs presented.

Alia. Throughout all of the problem-solving sessions Alia worked hard and never quit. She would repeatedly ask for the next graph or make statements such as, “Shush, let me figure it out.” She never wanted to be told how to label a graph, always wanted to figure it out by herself, and would ask to keep exploring a certain type of graph to find the pattern before moving on to the next type. Her engagement increased the more she engaged with the problem to the point that she did not want to stop working at the end of the session. During the fifth session, parents were beginning to arrive to pick up their students so I had ended the problem-solving session earlier than I intended. Alia then stated, “I need the purple one.” She was referring to the next graph in the class she was labeling. I handed her the page. One minute later her guardian came through the door to pick her up. Alia held up one finger to her, I infer that was her single to wait. Alia’s guardian then stated, “You can’t do it because we got to go.” I told Alia that I would be back the following Monday (it was a Thursday) and that she could continue to work then. However, Alia continued to work for another 3 minutes even with the reassurance I would be back for more sessions and with pressure from her guardian to leave.

Other instances. During Session 6, Becca, Iris, and Karly had gracefully labeled all of the graphs I had brought with me for that session. I asked if they would be willing to write their solutions for the graceful labeling on the poster paper. All three students appeared excited at the opportunity to do this. While doing this, the girls would shout things like, “I call the edges,” “I want to do it next,” and “I call the circles.” After drawing and labeling four caterpillar graphs each on a different sheet of poster paper, I said, “That is good girls. You do not have to do anymore.” Iris was quick to say, “Ah but she just…” and Karly interpreted with, “Please.” I agreed they could do one more. This was followed with looks of excitement and again comments like, “I want to do this one.” Although all three of these girls had been engaged in finding ways to label the caterpillar graphs during the session, the level of their engagement increased when they recorded their graphs on poster paper to be displayed in the classroom.

During Session 7, I switched problems so the students could engage with a different unsolved problem. However, throughout this different problem Becca asked repeatedly to work on, “sticks and nodes” again, meaning the Graceful Tree Conjecture. Iris, Alia, and Bernice also said they wanted to work on the Graceful Tree Conjecture again instead of working on the new problem. Then, after her post interview, she asked me for copies of task pages so she could work on them at home, show her mother, and see if her mother could find graceful labels.

Summary for conative. Overall, the students demonstrated productive conative dispositions towards mathematics. They worked on the same unsolved problem for six sessions and a different problem for the last session. The students made sense of the problem and persevered in the problem solving. They engaged in struggle throughout all of the sessions, but they continued to come back each day and work on the problems.

Findings of Phase 4 Analysis: In-the-Moment Identities

The students were analyzed based on their positioning and who positioned them. Next, a brief overview is presented on a summary of all the students’ in-the-moment identities, followed by a detailed overview of two students.

Overall, all of the students, with the exception of Joella, had a positive mathematics identity. They enjoyed mathematics and thought they were good at it. Alia positioned herself more frequently than she was positioned by other people. She was very positive, typically positioning herself as an expert and rarely positioning herself as inferior. At times she was very controlling and authoritative. Amanda seldom positioned herself. She was equally positioned as inferior as she was an expert, and at times she was encouraging to the other students. Becca was positioned by other students and self-positioned. Her positioning shifted throughout the sessions from inferior to expert. Bernice self-positioned herself as an expert throughout most of the sessions there were also times when she positioned herself as inferior. Edward was a very positive self-positioner, similar to Alia and Bernice. He positioned himself several times but was not positioned by other students or teachers very often; he mostly self-positioned as an expert. Hector and Karly were absent often and were both very quiet. The only exception was during Session 6; Karly positioned herself as an expert many times during that session. She had success that day, which in part may have been because she worked with Iris and Becca. Iris self-positioned often with an equal balance of inferior and expert. Trevor self-positioned many times but how he positioned was different depending on the session. During Session 2 he positioned himself as an expert nine times, but during Session 7 he positioned himself as inferior seven times. Joella self-positioned and was positioned by others throughout all of the sessions. However, her positioning was different from the other students who had more positive positioning. She was positioned as inferior many times throughout the problem-solving sessions. She was also the only student who said that she did not enjoy mathematics (see Appendix C for positioning tables of all 10 students).

Detailed Overview of Two Students

I conducted an in-depth analysis of the in-the-moment identities of two students, Becca and Bernice. They present two different perspectives from which we can learn how children engage with unsolved problems. The body of data for these students was more robust than most of the other students who were absent for some of the sessions or who did not say much throughout the study. Becca is an interesting case because her in-the-moment identity shifted the most throughout the sessions. When she first began, she was quiet and timid; however, by the ending of the sessions, she was loud and telling other students what to do. Bernice presents an interesting case because she provides some insight into how a student experiences productive struggle.

Becca. In this narrative about Becca, I illustrate how her in-the-moment identity shifted from a person who was negatively positioned by other students to someone who positioned herself as an expert. She was chosen to be included in the detailed overview of a two students because her positioning and in-the-moment identity shifted the most throughout the sessions. I begin by describing her views of mathematics and perceptions of herself as a learner of mathematics as they were conveyed in her responses to the first interview.

Pre-interview. Becca did not participate in the first problem-solving session because she had not returned her consent form. When I returned for the second session, Becca had her consent form and said that she would like to join the study. She then participated in the second problem solving session and was interviewed after that session had concluded. Becca stated that she liked mathematics and reading. When she was asked what she liked about mathematics, Becca stated, "Um… well she said that…and I want to learn because nobody else has learned it yet." I believe she was referring to the study and why she wanted to join it. When she was asked what mathematics was, Becca said, "oh, like, like, um… Math is like, you are basically trying to find an answer. Answer or product or something." Her view of mathematics was based on finding an answer. A few questions later, Becca gave some insight into her likes and dislikes of mathematics:

Dr. Cullen: What interests you about it?

Becca: Um, that I can like, like. I can like. Cause people from here, I was good, I don't know about it. So I probably will a little

Dr. Cullen: So what do you not like about math?

Becca: Um, I don't know.

Dr. Cullen: You like everything about math?

Becca: (Laughs) Well not like everything. I don't know. Not division or something. That is different. I don't like that because I can't do it.

Dr. Cullen: You don't like division because you don't think, you don't feel like you can do it?

Becca: No because I don't.

Dr. Cullen: Why not?

Becca: Because basically it is backwards multiplication but I don't get it at all.

Becca's answer about what interested her in mathematics might have been influenced by her participation in the problem-solving session she attended before the interview. She did not state whether or not she thought she was good at mathematics, but she did indicate that she did not like division because she could not do it or did not understand it. Based on her responses to the interview questions, it is difficult to make inferences about her mathematical identity but based on her quiet, hard to hear answers, she seemed quiet and shy. However, she appeared to enjoy mathematics as long as it was not too difficult.

Problem-solving Session 2. During the second problem-solving session, Becca worked at a table with Trevor and Iris. As soon as she joined the group, Iris said, "You don't go here." I told Iris that Becca did belong with her table group. Iris responded by saying, "Does she have to go to everything?"

Following this interaction, we had a whole-group discussion of what we had worked on during the previous session. Throughout that discussion, Becca remained quiet but appeared to be listening. Next, I asked the students to talk at their tables about the Graceful Tree Conjecture. Trevor made a quick comment about some trees being graceful and some being not graceful and shouted that they had a solution. I told him to take time and really discuss the problem with the Iris and Becca. Trevor then turned to Becca and said, "Stop being scary."

While the students were still discussing the previous session, Trevor turned to Iris and asked her what she thought. Iris only responded with a "hum":

Trevor: Okay. (Looking at Iris) You got a solution, fifth grader?

Iris: Hum

Trevor: Okay then say it. Exactly.

Trevor: (Trevor turns and speaks to Jenna about Iris and Becca) These [girls] weren't even doing solutions (reflecting on the previous day). They weren't even looking at me.

Jenna: So did you tell them what you thought?

Trevor: Yes! And they are over here, something, they are over here staring at me.

Jenna: So Becca, do you think all trees are graceful?

Becca: Umm…

Trevor: I said some. I said some.

Jenna: You said some. Well we figured out one was yesterday, right?

Iris: They are all not graceful.

After finishing the group discussion, I handed out cards of different graphs for students to sort in two categories—trees and not trees. Trevor and Iris began working on the sort. They did not hand Becca any of the graphs, and she did not ask for them. Becca just sat there and watched. After several minutes, Iris quit sorting and Trevor sorted the rest of the graphs himself. I came over to their table and encouraged Becca and Iris to help Trevor by saying, “Help Trevor. Don’t make him do it by himself.” However, Trevor finished sorting the rest of the graphs by himself, and Becca and Iris just watched. I came back to their table and asked if they were talking about it. Trevor said, “No, no one was helping me, so I just did it.” Trevor then said, “I am the only one doing this.” At this point Becca began picking up graphs, looking at them and placing them right back where she found them.

Nine minutes into the problem-solving session, Becca pointed to a graph Trevor had placed on the “is a tree graph page” and said, “That is not.” Trevor continued to look at other graphs he had already sorted and did not respond to Becca. Thirty seconds later, Becca picked up a graph that was sorted as “not a tree graph” and said, “That is graceful.” She moved it to the other page. Right away, Iris said, “No” and moved it back. Becca said, “It is the bottom of a tree.” Iris said, “No it isn’t.” Becca was actually correct—it was a tree graph.

After completing the sort, I encouraged the students to count the edges and nodes of the various graphs they had sorted as either a “tree graph” or “not a tree graph.” Trevor took the “tree graph” page and Iris took the “not a tree graph page” and Iris took a page. About a minute later, Becca attempted to write on the page Trevor had but he pushed her pen away:

Trevor: What are you doing (while pushing her pen)?

Becca: What?

Trevor: Would you (referring to me) tell her how to do this? Would you tell her how to do this part?

I explained to Becca how to count the edges and nodes. After another minute, Becca attempted to write on the page Iris had. Iris pulled the paper away and said, “No, girl, I had it first.” Trevor then told Becca she could do the nodes on his page, and she began to count and write. After Iris finished her page, she began doing something else and Becca took the page. Iris grabbed the page back and said, “You need to stop taking stuff away.”

After a whole-group discussion about the sort, the students were tasked to gracefully label star graphs. I helped Becca gracefully label her first star graph because she was not at the first session when students learned how to label their graphs. Becca began on the second graph in the sequence using an enlarged copy of the graph and the numbered chips. She placed the numbers 3, 4, and 6 on her nodes, not using 1 and 2. She then subtracted and labeled her edges 2 and 3. Several minutes later, Becca said, “I am done I think. I like it. I did that.”

I came over and told Becca she had the first graph correct and to record her answer. After she recorded her answer, I cleared her enlarged graph of the numbered chips including the second graph that she had done incorrectly and told her to use 1, 2, and 3 on the nodes. She confirmed with me that she was supposed to subtract the nodes. At that moment, Trevor asked if he could have the next graph, and I told him to make sure his whole table was together. Trevor took his pen and wrote one and two on Becca’s edge lengths. Becca did not say anything to Trevor but she did have an upset look on her face.

I handed the next enlarged graph (order 4) to Becca, Trevor, and Iris. Quickly, Trevor placed his chips on his page gracefully labeling the graph and shouted, “Look it! Boom!” Becca continued to place circle chips on her nodes and square chips on her edges, gracefully labeling her graph (see Figure 37 for Becca’s task page where she recorded her answer). She recorded her answer and labeled the next graph correctly using the same pattern (placing a 1 in the center node) and said, "Got it."

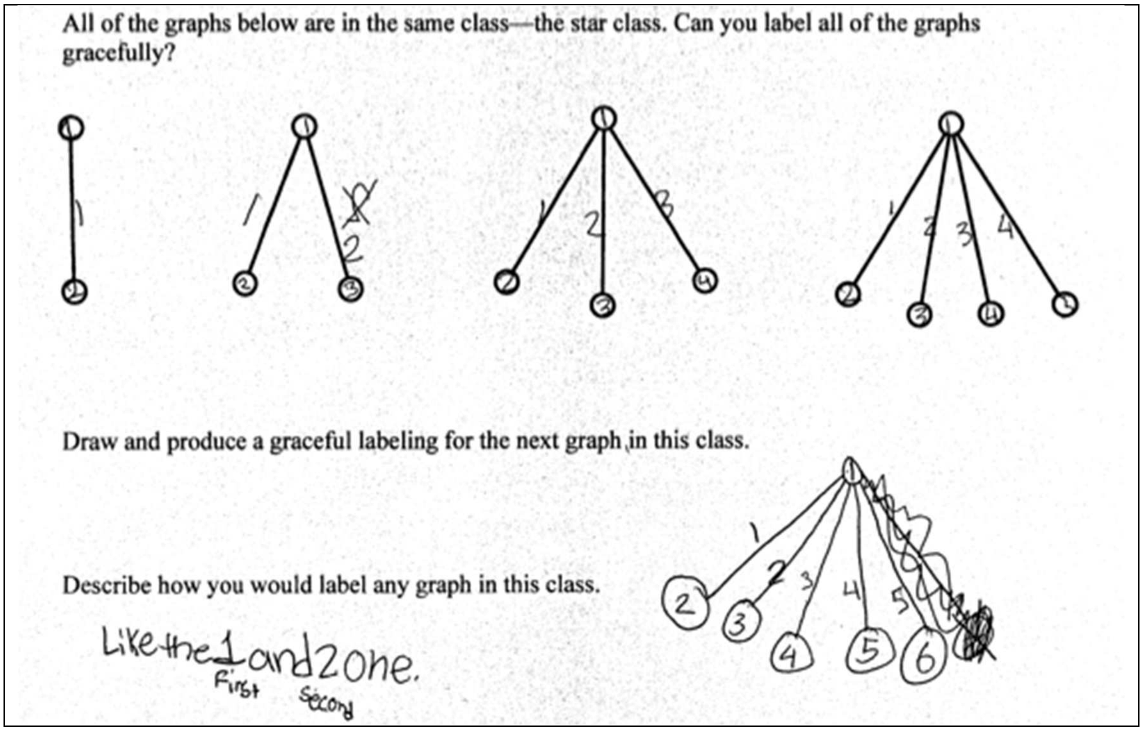


Figure 37. Becca's task page for star graphs.

After all the students at the table had gracefully labeled their graphs, I asked all three of them if they thought they could label any star graph gracefully. Trevor said that he could. When asked how, Trevor simply said that he would "go in order." He did not explain that he always had put the largest number in the middle. He said, "I have different numbers." Becca then said, "He put the bigger number in the top and the littler numbers in the bottom." I confirmed Becca was correct and said, "Becca, tell Trevor what he did." Becca explained to Trevor his solution. After several more minutes of discussion on star graphs, the session ended.

During this session, Becca was frequently positioned by other students as being inferior (see Table 9). Trevor and Iris made rude comments to her and did not let her interact with the materials and discourage her from using them by accusing her of taking stuff. However, towards the ending of the session, Becca had made progress in understanding the problem and positioned herself as knowing what to do by labeling the graphs herself and explaining Trevor's solution to him.

Table 9

Becca's Positioning Timeline for Session 2

	0:15	3:01	3:30	8:19	9:35	12:37	13:40	18:17	29:22	31:08	34:16	37:39
Inferior	O	O	O	O	O	O	O	O		O		
Expert									S		S	T
Control												
Authority												
Superior												
Encourage												
Collaboration												
Unique Idea												
Face-saving												

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 3. During the third session, Becca worked at a table with Trevor. Edward joined their table mid-way through the session when he finished his homework in the other room.

At the beginning of the session, I asked students to reflect on the previous problem-solving session:

Jenna: So I want you to spend a couple of minutes talking at your group about what you figured out about star graphs.

Trevor: We can't talk. We only have one person here.

Jenna: You can talk back and forth, okay.

Trevor: This one [referring to Becca] doesn't even talk right.

Jenna: Well…

Trevor: I can't even hear her when she talks.

Dr. Cullen: I can.

Becca: (Laughs)

Trevor: She is like…

Jenna: Now you are going to work with my friend Ms. Amanda (Dr. Cullen).

Dr. Cullen joined their group and encouraged a discussion between the three of them. Midway through their discussion, Becca explain a strategy for labeling the star graphs the following exchange took place:

Dr. Cullen: Do you see what she is saying? What is she saying? One is going forwards (the edges on Trevor's star graphs were in order from left to right) and one is going backwards (the edges on Becca's star graphs were in order from right to left). What is that saying?

Trevor: She is not.

Dr. Cullen: Where is it going backwards versus forwards?

Trevor: (Pointing to the edges from right to left) One, two, three.

Becca: One, two, three, four. One, two, three. One, two. One. And then mine is going…

Trevor: Oh, right here. Two, three, four, five, six (pointing to the middle node). That is not backwards, oh my gosh.

Becca: One, two, three, four. If they were going backwards that would be here. It goes forwards up here. This is actually.

Dr. Cullen: What do you think about that?

Becca: These go forward. These go backwards.

As a whole group, students discussed how they could label any star graph gracefully. For the next task, the students explored different graphs to see if they were all the same graph just positioned on the page differently or all different graphs. Becca and Trevor agreed that all of the graphs were the same.

Then students were given path graphs to determine whether they could be labeled gracefully. Becca labeled the first graph in the sequence (order 3) using her pattern for solving star graphs. She began working on the second graph. After working on the activity for about 5 minutes Becca said, “I need help. I don’t know.” After 2 more minutes trying to figure out a solution, Becca said, “I got it. I copied off his (pointing to Trevor’s).” Becca smiled and turned her page over to begin working on the next graph. After working on the third graph for 2 minutes, Becca said, “This is hard.”

Around this same time, Trevor became frustrated and stated that he wanted to go lay down. Dr. Cullen initiated a discussion with Trevor about working hard and then the discussion addressed the voluntary nature of the study and that the students did not have to stay and work on the problem. In hearing this, Becca said, “You don’t?” She then began cleaning up and asked if she could leave. At the same time, I ended this session because of the general frustration students were exhibiting with the task.

At the beginning of the session, Trevor positioned Becca as inferior by making claims about her speech. However, Dr. Cullen challenged Trevor by saying that she could hear what Becca was saying and then by highlighting Becca's strategy to Trevor. In this way, Becca was identified as being an expert. Then, as Becca worked to label the path graphs, she repositioned herself, identifying as someone needing help and not knowing what to do (see Table 10).

Table 10

Becca's Positioning Timeline for Session 3

	1:03	3:24	25:38	26:53	30:23
Inferior	O		S	S	S
Expert		T			
Control					
Authority					
Superior					
Encourage.					
Collaboration					
Unique Idea					
Face-saving					

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 4. Becca had homework that she had to finish at the beginning of the fourth problem-solving session and did not join until after the students had explored different tree graphs to see if they were similar or different. However, she did join the study again even after she had decided to quit at the end of the previous session. Becca worked at a table with Edward during that session.

After the unproductive struggle that many of the students encountered during Session 3, I changed the activity for labeling path graphs and had the students work as a whole group writing on the poster paper to find a pattern. Edward and Becca labeled the graph that had two nodes and one edge. Edward labeled the two nodes and Becca took the marker to label the edge. Edward said, “One right there (pointing to the edge).” Becca told him, “I know.” The rest of the group labeled their graphs.

I hung all of the posters in the front of the room and led a group discussion about the patterns that students saw and how to find a graceful labeling for the next graph in the sequence (five nodes). In prior class discussions, Becca was quiet and just listened. In this discussion, she was engaged and spoke throughout the conversation. She shared the different patterns she saw in the path graphs. Midway into the discussion, Joella made a comment about what she saw. Becca said, “That is what I said. I just told you all that.”

After several more minutes of discussion, the students as a group had gracefully labeled the graph with five nodes. I handed out a graph that had six nodes to each student. The students worked independently. While working Becca slammed the chip down on the table and said, “How do you do this?” Edward responded, “It is just so easy” and he covered his page with his arms.

By the end of the session, only one student, Bernice, had gracefully labeled the graph. I let Bernice write her graceful labeling on poster paper that was posted in the front of the room and asked her to share the pattern she found with the whole group. During her explanation to the whole group, Becca called out, “I don’t get it” and “Wait, I don’t get it, why?” A few minutes later after Bernice explained her pattern a second time, Becca said, “Oh, I get it now.”

Becca's in-the-moment identity shifted throughout this session. First, she positioned herself as being an expert by explaining the different patterns she saw during the whole-group discussion. She then exhibited an authoritarian stance by claiming that Joella was simply saying what she had already explained. As she worked to find a graceful label for the six-node path graph, Becca positioned herself as not understanding and Edward reinforced this inferior in-the-moment identity by claiming the task was easy and hiding his paper (as though she might copy his work). In fact, Edward had not found a graceful label for the graph either. After listening to Bernice describe how she had labeled the graph, Becca repositioned herself as a person interested in understanding and then as having figured out someone else's strategy—an in-the-moment identity of expert (see Table 11).

Table 11

Becca's Positioning Timeline for Session 4

	14:02	21:02	26:03	27:14	27:47	32:50	32:55	35:21
Inferior			S	S	S	S	O	
Expert	S							S
Control								
Authority		S						
Superior								
Encourage								
Collaboration								
Unique Idea								
Face-saving								

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 5. Becca came late to the fifth problem-solving session because she had to finish her homework before she could join the group. She worked with Karly and Iris during this session. Dr. Schupp, (a pseudonym) a professor from Illinois State University who volunteered to help at the problem-solving session, sat at their table. Becca came in while everyone was working on graceful labelings of path graphs, a continuation of the previous session. She spent a few minutes looking at the poster papers in the front of the room. Then she said, “I think I have a pattern.” She began writing on her page and said, “I think I know how to do that one. It is easy.”

Becca continued working through the labeling of path graph what was order five from the previous day on a copy of an enlarged graph. She did not use chips but just wrote with pen right on the paper. After working on the problem for 2 minutes and only being in the room for 4 minutes, Becca stood up and said, “I got it.” I drew the next path graph (order six) on Becca’s page. She announced, “She challenged me.” I responded, “I did challenge you.”

Becca sat back down and gracefully labeled that graph using the same pattern as the previous graph (see Figure 38). Less than 1 minute later, the following conversation took place:

Dr. Schupp: Wow, you are cruising!

Becca: Done

Dr. Schupp: She is pretty fast.

Iris: Yes.

Dr. Schupp: Kind of cool.

Jenna: So I want you to tell me how you would solve… How would you solve any of these?

Dr. Schupp: Maybe she needs a harder challenge.

Becca: No I don't. I am just good at it! I just came in here and sat…

Dr. Schupp: Pretend I don't know anything about it and I am trying to do the next one, what is the rule? How could you help me, yeah so how could you tell someone.

(Iris and Karly began copying down Becca's solution on their paper.)

Becca: You guys! Can you stop (taking her paper away)?

Karly: Did you get it all down?

Iris: (Showed Karly her paper with the solution written down.)

Becca: The edges. They go up (labeled the edges from the bottom to the top in order).

At that moment I called the group of students to have a whole-group discussion about path graphs.

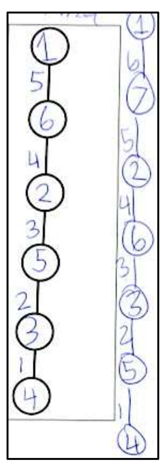


Figure 38. Becca's task page for path graphs.

As soon as the group had gathered, I said, “Everyone listen to Becca. She is going to explain what she figured out.” Becca explained her solution to the whole group (Becca’s description of her was presented above; see Figure 32). Alia explained her solution as well. After Alia was finished explaining, I told the students to use both Alia and Becca’s explanation to label path graphs.

Next, I handed out first set of caterpillar graphs. Becca stated, “This is going to be easy for me.” Becca began working, again just filling in the graph with her pen, not using the chips for labeling (Becca’s pattern is presented above; see Figure 34). Within 30 seconds, Becca filled in the first graph and shouted, “Done! I am done. I am done. I am done.” At that time the problem-solving session ended. However, Becca made a few more statements. She said, “Everything is easy for me.” Bernice responded with a “Yeah, right.” Becca said back, “That is easy. What are you talking about? I have a strategy to do it and it works.”

As I was cleaning everything up and Becca was waiting for a ride home, she came and spoke with me. She told me how she thought I had challenged her and she was so good at it and fast. You could hear the pride and excitement she had in herself.

In previous sessions, Becca’s in-the-moment identity was lowered multiple times. However, in the fifth problem-solving session Becca walked in and knew or quickly figured out how to solve the problem. She moved to an in-the-moment identity of the expert, and she may have had some thoughts about superiority as well by her statements of how good and fast she was at labeling the graphs. In the previous sessions, Becca was mostly positioned as inferior, but during this session, Becca shifted her positioning and in-the-moment identity to expert (see Table 12).

Table 12

Becca's Positioning Timeline for Session 5

	20:38	24:57	25:45	26:17	28:30	31:30	32:56	33:51	36:52
Inferior									
Expert	S	S	T	STO	T	T	S	S	
Control									
Authority									
Superior									S
Encourage									
Collaboration									
Unique Idea									
Face-saving									

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 6. Becca worked with Iris and Karly again during problem-solving Session 6. Again, Becca had homework to complete but this time she brought it into the room and was working on it as I handed her the first caterpillar task page that she had started the previous day. She said, "This is easy. This is too easy." Several minutes later when Becca had finished her homework I checked in with her table to let them know they could begin and reminded them to look for patterns when labeling the graphs. Becca responded, "I even know one." She started labeling the graph, writing directly on the page rather than using the chips and was finished in about 1 minute.

Becca: Done.

Jenna: You figured it out?

Becca: Yeah

Jenna: Let's see. So you did one, two, three, four, five, six (checking her labeling of the edges). Are you seeing a pattern?

Becca: Yeah

Jenna: What is the pattern?

Becca: Um,…

Trevor: Becca was at our group.

Jenna: That is okay. She can be over here.

Becca: Look it is one, two, three, four, five (pointing to the edges). You go up backwards (edges) and I added here and subtracted here (pointing to the nodes).

Jenna: Okay. Write it down on this sheet. You can help your whole table.

Iris: Yeah Becca

Karly: Yes

Jenna: Yeah, you guys work together.

Becca: Okay, (moving her sheet so Iris and Karly could see), so write it down. (Karly moves it so Iris can see it better but much farther away from Becca.)

Becca: Hey, I can't even see my own sheet (moves the sheet back to the middle of the table). Every time I just do it and you all copy down. Okay?

Karly: So six…

Becca: Shhh

I gave the students the next enlarged graph in the sequence to the students (order eight). Becca began labeling the next graph without using the numbered chips and just writing her labeling directly on the page. Karly filled in her graph, copying Becca, and Iris watched Becca label the graph and used the numbered chips to label the nodes on her graph the same way that Becca had done. After 1 minute Karly tried to interact with Becca:

Karly: Two. Wait, that can't be.

Becca: Wait that is five. No cause we don't do these bubbles (nodes) again.

Karly: Cause we have to add.

Becca: No, I added and then subtract.

Becca finished labeling the graph with eight nodes (see Figure 39). Karly and Iris copied down the solution on the page:

Becca: Done. We're done.

Karly: We are done. I found a pattern. On a turn, I found a pattern.

(Becca gave Karly a questioning look.)

Karly: I found a pattern. It goes one, two, three (pointing to the edges).

Becca: I already said that.

Jenna: Okay, hang on.

Karly: You didn't say that.

Becca: Yes I did.

Jenna: Okay. Put your names on everything. Make sure…

Becca: I did. I need the next one.

Jenna: You need the next one? Yellow (enlarged copy of the graph with 10 nodes)?

Becca: Yeah, yellow. I am so ahead of everybody.

Jenna: No, you guys are right. Those girls back there are where you are.

Karly: We have to hurry up.

The girls began to work on gracefully labeling the next graph, a double star graph that had 10 nodes (see Figure 39). Iris sat and copied down what Becca wrote. Karly kept pointing to spots on Becca's page and would say a number. Becca finally said, "Will you stop doing that." Again Karly tried to join the solving and said, "One, that is easy." Becca responded, "That is not actually because I have one here (pointing to a node on her page)." Becca finished labeling the graph and said, "Done. Copy it down."

Iris and Karly listened to Becca and copied her solution on their sheets. Iris asked me for the next page. I asked them to write about the pattern they found instead of giving them the next graph (see Figure 39). The following is the transcript from their conversation about writing their pattern:

Karly: Can I, can I do a pattern about this one (pointing to the graph with eight nodes)?

Jenna: Yeah.

Becca: Oh, we all, I am going to write about this (pointing to the same graph).

Karly: Okay, I am going to write this. Do this one (looking at Iris and pointing to the graph with eight nodes).

Iris: Okay.

Karly: I am going to write it says, so seven backwards.

Becca: It has, yeah.

Karly: It has one, two, three, four, five, six, seven backwards.

Iris: What did you do seven on?

Karly: One, two, three, four, five, six, seven, backwards.

(All three girls wrote that down and I walked over to the table).

Jenna: So it counts one, two, three, four five, six, seven?

Becca: Backwards.

Jenna: But on the edges, right? That is what you were telling me?

Becca: On the edges. Okay. And the next one will be the edge.

Karly: Okay, okay. We are suppose to draw it?

Becca: Yeah

Karly: Okay

Becca: Okay, so this would be one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve (drawing the next graph in the sequence).

Karly: I have a song stuck in my head. I am confused actually.

Becca: Well you shouldn't be.

Jenna: So you guys think you can solve anyone that is like that?

Becca: Yes

Jenna: What about…

Becca: I don't know about them.

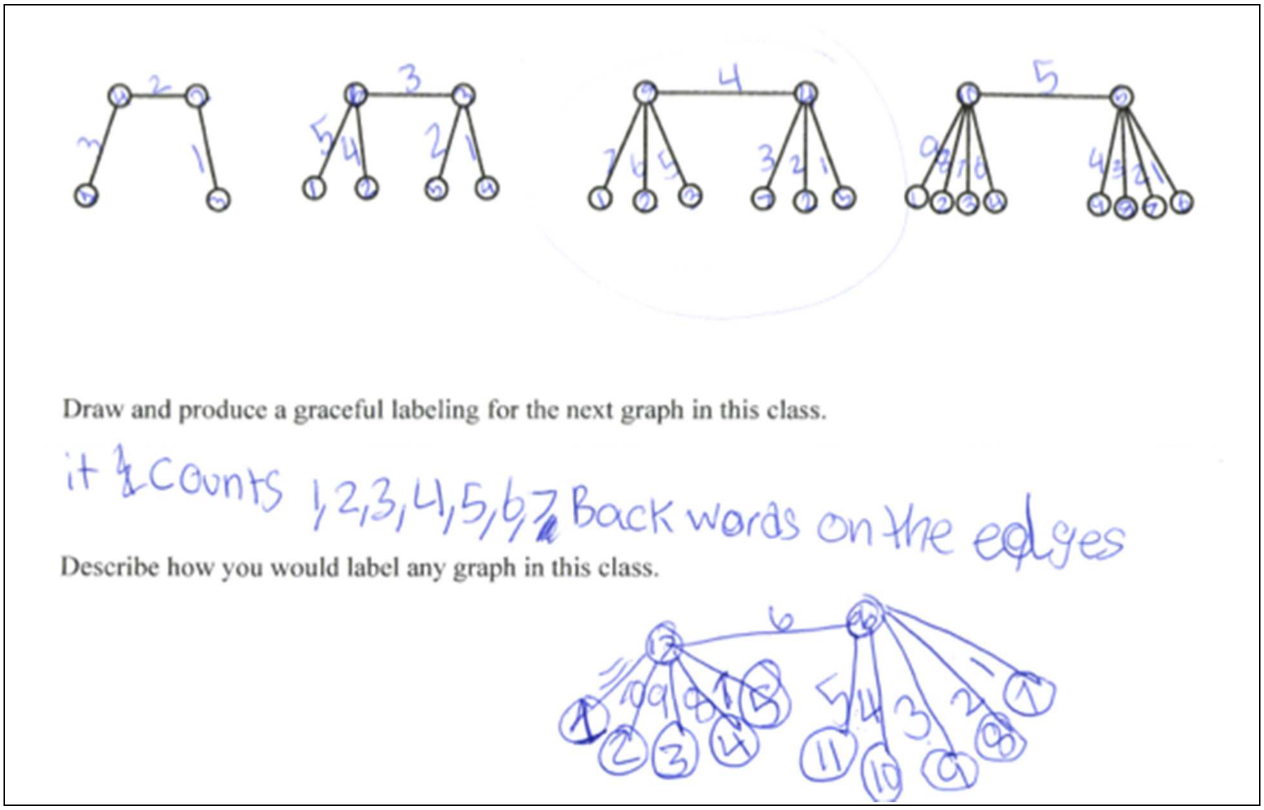


Figure 39. Becca's task page 1 for caterpillar graphs.

I then gave the girls the next sequence of graphs, the second set of caterpillar graphs (see Figure 40). Throughout this interaction, Becca began to collaborate with Karly and Iris.

After Becca looked at the page with caterpillar graphs on it the following conversation happened:

Becca: These look easy to me.

Iris: I need the next one too.

Becca: These look easy to me.

Jenna: So do you need this one (referring to the enlarged graphs)?

Karly: I don't care.

Becca: That is because of my help.

Iris: Hey!

Jenna: I think it is because you guys are working as a team.

Karly: Not really.

Jenna: Not really?

Becca: Actually, I am doing all the work and they're just copying me.

Jenna: Well does it make sense?

Iris: Yes

Karly: Yeah

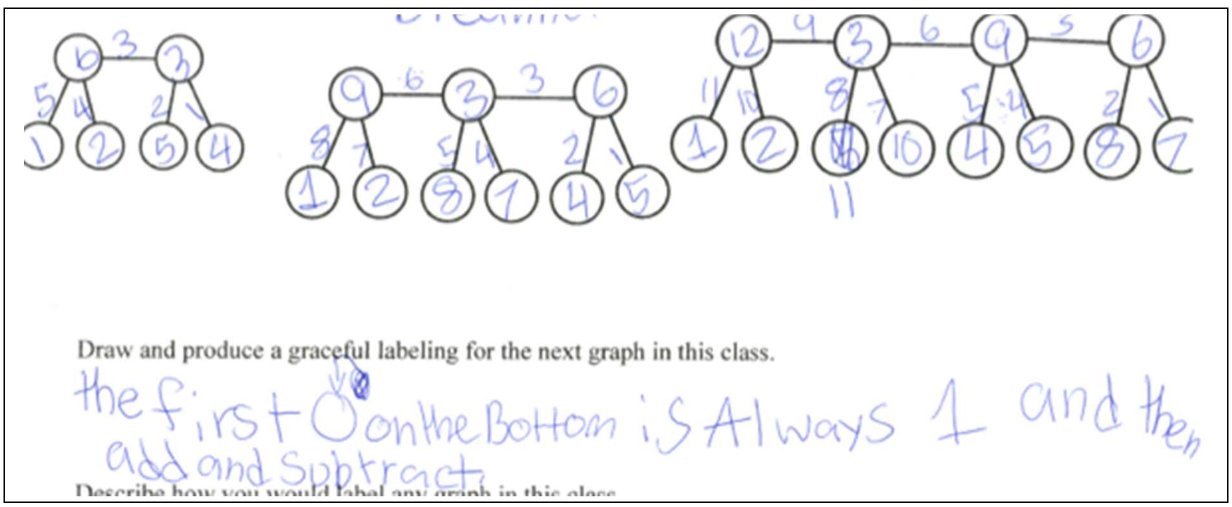


Figure 40. Becca's task page 2 for caterpillar graphs.

The girls began labeling the graphs. While working, the following conversation took place:

Karly: Next one. Next one. I am smart, too. You are not the only smart one. We are both smart. We are all smart.

Becca: Like you don't know my pattern.

Karly: I don't need your pattern.

Becca: Okay, you can do that. I don't care.

The girls continued to label the caterpillar graphs. Students at another the other tables had not found a pattern and made statements of struggle. Bernice said that she did not think there was a pattern. I told Bernice's table that Becca's table had found a pattern and they could share the pattern. Bernice's table said they did not want to know and wanted to keep working on finding a pattern. You could see Becca's joy throughout this conversation through her smile. At the end of my conversation with Bernice, Becca said, "I found the pattern first." A few second later, Becca had the following conversation with Karly:

Becca: I will tell them.

Karly: Um, um. You will tell them. I will copy off of you.

Becca: That is not. You are copying off of me.

The table of girls continued to work on finding graceful labels for caterpillar graphs. Throughout that time, Becca continued to make statements such as, "That is so easy," and "Oh, those are easy," and when Joella stated frustration by saying, "I don't know this crap," Becca responded with, "Well, figure it out then."

After Becca's group had gracefully labeled all the graphs I had planned on them doing for that session, I told them, "You guys just flew through these" and asked them if they wanted to write their solutions on the poster paper. They seemed very excited about doing that. The three girls went to the front of the room and began drawing graphs and gracefully labeling them on the poster paper working together and taking turns. After filling in several graphs, Becca provided encouragement to Iris:

Iris: Now that is messed up. That is just messed up. That is just messed up.

Becca: (Laughs)

Iris: You guys this is the final. I messed up. I was doing it too fast. I was doing it too fast.

Becca: I will help. You just do it later.

While the girls were filling in the graphs on the poster paper, Trevor, Hector, and Edward were not sure of the pattern and I knew the problem-solving session would end soon, so I asked if they wanted Becca to show them the pattern. They said, “No.”

To wrap up the problem-solving session for the day we had a group discussion. At the beginning of the discussion, Becca tried to share her pattern while Hector was describing what he found. I told her she had to wait until Hector shared his pattern. After he described his pattern, Becca responded, “We all know that.” Next, Becca explained how you always start labeling the edges. Next, she pointed to the bottom left node and said, “So you always put the one right there.” She then shared her adding and subtracting pattern that was the same as her pattern for path graphs.

After the session ended, I had a discussion with Becca and several of the students left in the room. I asked if they wanted to try a different problem the next day. Becca said, “I can do it.” Several minutes later while I was cleaning up and talking with a volunteer about the problem, Becca and Bernice came back into the room and had the following conversation with me:

Jenna: (To the volunteer) It is a famous unsolved problem that has never been solved.

Becca: Yes it has.

Jenna: No it hasn't. We only figured out three graphs. Are you going to keep working on this problem?

Becca: It is easy.

Jenna: Do you think all trees are graceful?

Becca: Yes.

Jenna: Do you?

Becca: Yes because we got all tree figured out.

Jenna: Well we have figured out three of them but have we figured out all of them?

Bernice: We have been doing this for fifty-four minutes.

Becca: How much are there? Ten? Are there all ten?

Jenna: I don't know. How many do you think here are?

Becca: A lot more than three.

Jenna: Probably a lot more than three. Do you think you could figure this problem out someday?

Becca: Wait, do you have the hard, do you have the hardest problem ever?

Jenna: Do I have the hardest problem ever? I don't know if I could figure out the hardest problem in math.

Becca: I can.

Jenna: You think you could?

Becca: Yeah. Do you have it?

Jenna: No, but maybe I can ask somebody what it is because I don't know.

Bernice then stopped the camera.

The sixth problem-solving session allowed Becca to change her in-the-moment identity from hesitant and shy to confident and demanding (see Table 13 and 14). She continued to position herself as the expert and even positioned herself as in charge of the problem solving. Her in-the-moment identity increased in positivity throughout the last two sessions with her success and the pattern that she had found. She did not display the same in-the-moment identity she had during the second and third sessions when she was told she did not talk correctly and was scary. She gained confidence to not only participate in the activities, but she also acted in charge of the activities and identified herself as someone who could solve the hardest problem in mathematics.

Table 13

Becca's Positioning Timeline for First Half of Session 6

	00:49	3:37	5:38	5:53	6:12	6:22	6:32	8:07	9:03	9:15	9:32	9:36	10:52	12:01	14:58	15:49
Inferior												T				
Expert	S	S	S	O	T				S						S	S
Control						S	S									
Authority										S			S			
Superior											S					
Encourage																
Collaboration								O						SO		
Unique Idea																
Face-saving																

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Table 14

Becca's Positioning Timeline for Second Half of Session 6

	16:07	16:39	18:25	18:37	18:41	19:00	23:23	24:36	26:27	27:04	29:32	35:43	38:11	38:33	38:41	39:12
Inferior																
Expert		OS		S		S	S	S	S	S	T		S		S	S
Control																
Authority														S		
Superior	O		T		S											
Encourage Collaboration												S				
Unique Idea																
Face-saving																

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 7. Becca worked at a table with Iris and Joella during this session. Karly, who Becca had worked with during the previous two sessions, was absent. At the beginning of the session, I told the students it was the last day we would be working together on unsolved problems. I then spent the next 15 minutes explaining the unsolved problem for that day, the Collatz Conjecture. Throughout this time Becca participated in the whole-group discussion by share if numbers were odd or even and completing the mathematics equations that were given for examples.

The students began to work on the problem at their tables. Alia, who was sitting at another table, made a statement to her table about four being an odd number. Becca responded, "Alia, four is an even number." Thirty seconds later, Becca stated, "This is hard. This is hard." She then sat and was off task for the next 3 minutes. She then stated to me, "You need to help me." I told her to hang on because I was working with another student. She followed that by saying, "You aren't helping me." After I finished working with the other student, I asked her what she needed help with. She responded, "Everything." I helped Becca, the girls at her table, and Trevor and Edward for the next several minutes make progress on the problem. Next, I encouraged them to try a number on their own and left their table. Becca attempted to try a number on her own. After she started to work, the following interaction took place between Joella, Iris, and Becca:

Joella: Don't go so fast. I am trying to keep up (looking at Becca's paper).

Becca: You are doing math with me?

Joella: We are working together. What is that (pointing to a spot on Becca's page)?

Becca: Five

Iris: Are we working together?

Joella: Yes

Becca: (Nods head yes.)

Joella: Becca, you already got it.

Becca: No, not all of it.

After Becca tried one number, she said, "Can I get the um, sticks and nodes? Can you, um, give me the hardest one?" I agreed and handed her the comet page, a sheet with four different comet graphs (see Figure 41). I said, "This is the hardest one I have." Iris asked to work on it as well. Becca then stated to the other participants in the room, "I am doing the hardest one" and again a minute later saying, "I am doing the hardest problem ever." After working on the problem for 5 minutes, the following conversation took place:

Becca: Yeah, I don't want to do this no more. I only got one done.

Alia: Yes! I need the next paper (also working on comets).

Jenna: You got this one done, or are you working on it? Did your pattern work though?

Becca: A little. I need a new pattern.

Jenna: You need a new pattern? Why don't you use the chips then and stuff?

Becca: Because

Jenna: You told me you wanted to do this. You said I really want to!

Becca: Because I did! I didn't know it was going to be that hard!

Jenna: Well you told me, give me the hardest one you have!

Becca: Yeah, because I thought I was good at it!

Jenna: You are good at it!

Becca: No, I am not!

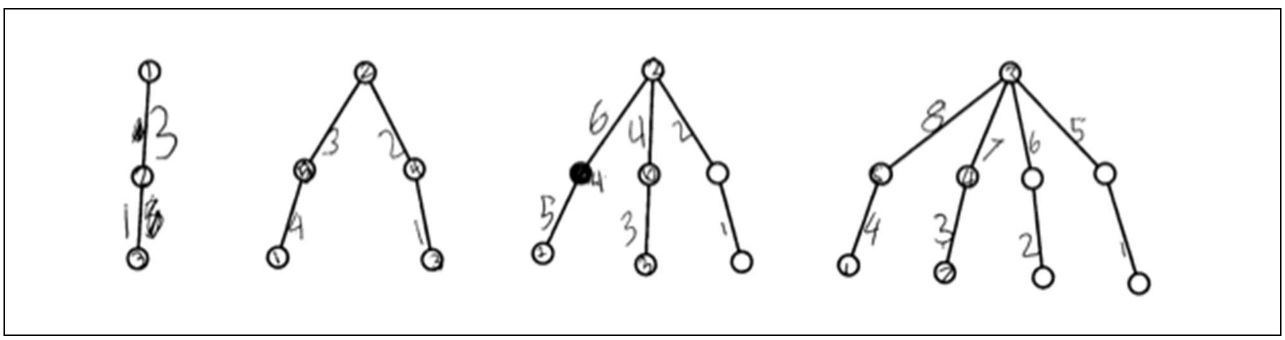


Figure 41. Becca's attempt at labeling comets.

The pattern Becca had used on the previous graphs did not work and because of the difficulty she encountered she lowered her in-the-moment identity (see Table 15). During this session, Becca did not have the success she had the previous two sessions. She showed resistance against working on a new problem and gave up after only 5 minutes because her pattern did not work.

Table 15

Becca's Positioning Timeline for Session 7

	16:52	17:16	19:34	20:26	29:38	31:34	32:02	34:02	38:06	38:15
Inferior		S	S	S					S	
Expert					O	S	S	S		T
Control										
Authority	S									
Superior										
Encourage										
Collaboration										
Unique Idea										
Face-saving										

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Post interview. During the post interview, Becca said that she liked mathematics and figuring out the products. She said that she did not like mathematics when it was hard. Later, Becca spoke about the similarity between school mathematics and unsolved problems. She said:

Becca: Because you get um, well I added and subtracted and that is close then. You add and subtract in school and you add and subtract in math.

She was then asked how the unsolved problems were different from school mathematics. She responded:

Becca: Because it is unsolved and people are in on it do that kind of math. Like if the teacher is showing you how to do it, they already know how to do it so it is not unsolved. It is already solved.

She then talked about how she was not able to label the comets gracefully but she wanted her mom to try. She was asked if she wanted to keep trying. She said:

Becca: No. That is too hard because I tried to do it how I do every single problem and it did not work out.

Becca was asked if she enjoyed working on the unsolved problems. The following is the transcript from that conversation:

Dr. Langrall: Is there something you enjoyed about working on unsolved problems?

Becca: When, cause, one day I was coming in the, coming in the art room late and they were already started and she gave me a piece of paper and… wait what was the question again?

Dr. Langrall: Well what did you enjoy?

Becca: And um, she gave me this piece of paper and I was like how do you do this and she showed me and I was like and then actually, first, I didn't see it. Actually first she gave me the piece of paper and I started to work on it and then I was done in like thirty, like five minutes. And then I said I am done. And she said you are done that quick? And then she gave me another paper and I was done quicker and yeah.

Dr. Langrall: So you enjoyed that?

Becca: Yeah, I was just quick on it.

Becca seemed to have confidence in her mathematics ability, which may have been because she thought that being quick at solving problems indicated that she was successful and good at mathematics.

Summary of Becca's in-the-moment identity. Becca's in-the-moment identity shifted throughout the problem-solving sessions. During the first session Becca attended, she was positioned by other students the most; however, by Sessions 4, 5, and 6 she began positioning herself more than allowing others to position her (see Table 16). Her in-the-moment identities for the first three sessions that she attended, she experienced an inferior identity, was reserved, and had negative comments said about her because of how quiet she was. However, through her success at labeling the graphs during the fifth problem-solving session, Becca's in-the-moment identity shifted. She positioned herself and others positioned her as an expert. She took an authoritarian and in charge role telling others how to solve the problems and correcting their mistakes. Even during the last session, when Becca struggled with the problem, she still took a role as a leader and participated loudly (see Table 17). By the end of the problem-solving session, Becca was not the same quiet girl she was during the first three sessions. This was evident even in the interviews. In the first interview, she was so quiet that the interviewer had to repeat everything she said to make sure it was correct and could be deciphered on the recording. In the post interview, with a new interviewer she had never met, she spoke loudly and was outgoing and willing to talk.

Table 16

Who Positioned Becca

	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	2	3	7	6	25	8
Student	9	1	1	1	5	1
Teacher	1	1		4	4	1

Table 17

Becca's Positioning

	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	9	4	5		1	4
Expert	3	1	2	10	20	5
Superior				1	4	
Authoritarian			1		3	1
Controlling					2	
Collaborator					3	
Encouraging					1	

Bernice. In this narrative about Bernice, I illustrate how her in-the-moment identity was displayed and that she was an outgoing student who was very verbal during the problem-solving sessions. She was selected as a case study because she was verbal about her feelings and demonstrated an example of productive struggles. Also, her experience was different from Becca's experience. She was a person who self-positioned often as an expert, but when she was engaged in productive struggle, she would position herself as inferior or as someone who could not do the given task. Bernice was a good person to be chosen for the case study because of how vocal she was and how she shared her feelings, which in turn creates a descriptive case study analysis. She positioned herself throughout the problem-solving sessions more than students positioned her or myself or other teachers positioned her. I begin by describing her views of mathematics and perceptions of herself as a learner and doer of mathematics as displayed through her responses during the first interview.

Pre-interview. During her pre-interview, Bernice made several statements about her feelings towards mathematics. When she was asked what mathematics was like in school, it opened up a conversation about her feelings towards mathematics and her views of herself and others:

Cindy: What is math like in school?

Bernice: Math like in school is really easy for me and a few other people but lots of other people, well lots of people really struggle, so me and some other people are always the first one to finish.

Cindy: Yeah, so why do think so many people struggle?

Bernice: It is because they don't concentrate that much, they don't see what the words are saying. So like when it says estimate the product, they don't estimate at all. They don't see what they want you to do.

A few questions later, Bernice was asked if she ever did mathematics puzzles or brainteasers. Her answer gave some more insight into her beliefs about her mathematics ability. The following was what she said:

Bernice: I love to do puzzles. I love puzzles. I haven't tried a brainteaser yet but my friends have been trying to get me to try some, since I am really good at math to just try some.

However, she did state during the interview that she did not like division:

Bernice: The thing that I don't get is division sometimes. It is really hard, like eight divided by zero. I sometimes say it is like eight or something because eight plus zero is eight and I thought division was kind of like addition but then division is like subtraction. Yeah multiplication is addition.

Overall, Bernice seemed to have a high feeling of her mathematics identity prior to beginning the study. She made it clear that she was fast at mathematics and really good at doing mathematics.

Problem-solving Session 1. During the first session, Bernice worked at a table with Joella, Amanda, and Alia. At the beginning of the session, I explained to the students what graphs, nodes, and trees were.

I showed the students a graph and asked how many nodes were on the graph. Bernice was quick to give the correct answer before anyone else did. I then asked how many edges there were on the graph. Once again, Bernice was the first student to answer correctly.

The students were then tasked to draw a graph that had five nodes and four edges (see Figure 42 for Bernice's drawing). After the paper was handed out, the students started to work. Before working Bernice said, "This is so hard."

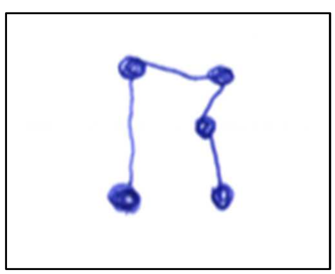


Figure 42. Bernice's drawing of a tree graph with five nodes and four edges.

Bernice began to draw a graph. She asked if it had to be a real shape, and I told her it did not. After several minutes, Bernice said, “I am done, I am done and a check.” She continued to work and even stated, “This is fun.”

After students had shared the graphs they had drawn, the students were told about graceful labeling. Throughout the conversation in which graceful labeling was explained, Bernice continued to participate, almost always sharing the correct answer before others did.

Once the students had a grasp on graceful labeling, they were tasked to gracefully label a graph with five nodes and four edges (see Figure 43 for Bernice’s labelings). Bernice worked on finding a graceful labeling stating the mathematics she was doing aloud, for example, “three minus five equals two.” After several minutes she said, “Two, oh my gosh. There almost.” Bernice continued to show productive struggle with another statement only 11 second later saying, “I need to switch these around. That is two. Two there, three. Oh my gosh.” She continued to try to find a labeling. She stated, “doing my fives first” and “goodness gracious.” After working on the problem for only 3 minutes Bernice stated, “Okay, I did it. I did it. I did it. I did it. I finished.” I told her to write down her solution. She wrote down her solution to record the labeling. I encouraged her to try to find another solution. She agreed and cleared the chips off her page.

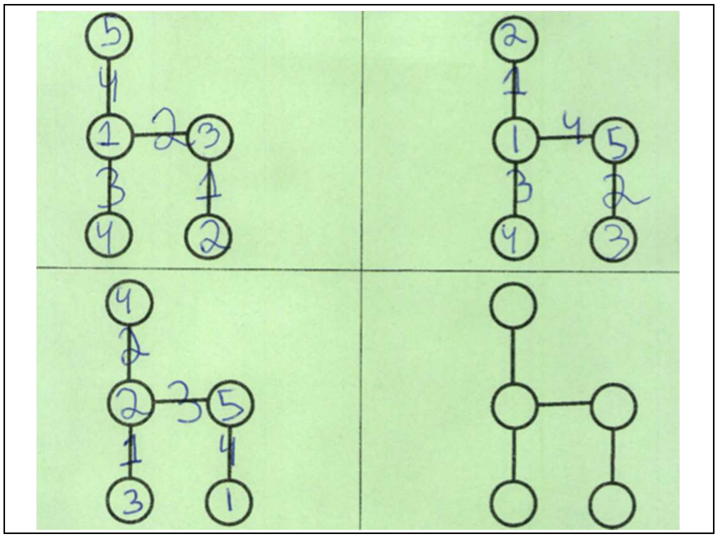


Figure 43. Bernice's labelings for tree graphs during Session 1.

After Bernice cleared her page, she told her tablemates, "No peeking." Bernice continued to work on labeling the graph a second way. While doing this, she made comments of productive struggle, such as, "Dang it, switch it around. So that would be four" and "oh my god, switch these two and switch these two." After 3 minutes of work Bernice had the following conversation with Alia and me:

Bernice: I did it. I got it again. I got a different one. See I did it.

Jenna: And is it completely different from your last one?

Bernice: Yes. See I did a five up there and a one and a four.

Alia: I did the exact same one.

Jenna: Will you guys help your table, your shoulder partners? Help them.

Quickly after the conversation ended, Alia state that she had messed up:

Alia: Wait, messed up. Four

Bernice: Three minus four is not three.

Alia: I know. Two or no, one, five shoot. I messed up on this one.

Amanda then joined the conversation:

Amanda: She messed up, oh my goodness.

Bernice: I got two times and I am going to try again.

Amanda: Let me see your paper.

Bernice: Try it yourself first and if you need help then ask.

Amanda: I need help.

Bernice: I got two.

Rather than help Amanda, Bernice informed her that she had found two solutions. She cleared her page and began looking for a third solution. She made statements such as, "So I can not put a four right there" and "I need to switch." After working for two minutes, she said, "I did it again, again." I then asked Bernice if she had helped Joella. She said that she had not. Amanda confirmed by stating, "she didn't help nobody." I asked Bernice and Alia to help the other people at their table. Bernice responded by saying, "She did it wrong," referring to Alia's paper. Alia confirmed that she did it wrong. Next, I had the following conversation with Bernice and her tablemates:

Jenna: Well, Bernice is going to show you guys a solution that she had.

Bernice: I had three, pick one.

Jenna: Well show them, move the pieces around.

Bernice: Okay, so what do you have people?

Bernice began to put chips on her page but did not speak to the students while she was doing it and did not have her paper so other students could see it. Alia continued to work and found a solution during this time period. Alia then said, "Bernice, I beat you." Bernice responded that she had not. Also during that time Joella and Amanda found a graceful labeling with my help. Bernice never did share a solution with her table. After a short group discussion about tree graphs and the problem, the session ended.

During the first problem-solving session, Bernice mostly displayed in-the-moment identities of an expert and authoritarian through her dispositions and positioning (see Table 18). At times, I encouraged these identities by asking her to share her solutions with her classmates.

Table 18

Bernice's Positioning Timeline for Session 1

	2:28	4:29	5:36	19:07	19:47	19:59	21:02	21:21	21:42	22:01	22:05	23:07	23:25	24:09	24:12	24:15	26:26
Inferior		S															O
Expert	S		S	S	T		S	T			O	S	T		T		
Control																S	
Authority						S			S					S			
Superior										S							
Encourage																	
Collaboration																	
Unique Idea																	
Face-saving																	

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 2. At the second problem-solving session, Bernice worked at a table with Joella and Edward. Edward joined the session late. Bernice and Joella quickly sorted their 10 different tree graphs into either "tree graphs" or "not tree graphs" categories. They taped their graphs onto each of their selected pages and with encouragement they started to count the number of edges and nodes on each graph. While doing this, Bernice notice that the tree graphs had one less edge than node. After they finished, Bernice waited for other groups to complete the sort and Joella was off task. While waiting, Edward came in and joined their table. I asked Bernice to help Edward catch up and to tell him what she had figured out. After all of the groups finished the sort, I asked Bernice to share with the group the relationship she had found between edges and nodes. At that time Joella said that she had figured it out and Bernice had not, but Bernice confirmed that she had figured it out.

After the sort, the students started to work on star graphs (see Figure 44). After working on the first graph for several minutes, Bernice said, "Yay! I did it." After Bernice and Edward had finished the second graph in the sequence (order three) Bernice said, "We are all done. We are all done except for Joella. Joella, that is wrong. It is supposed to be the circles." Joella then told Bernice that she needed her help. Joella was working on the graph with three nodes. Bernice looked at Joella's page and said, "Oh my gosh. Joella thinks this is a six. A six! One, two, six. Oh my gosh." Bernice was referring to Joella having labeled the nodes on the graph she was labeling a one, two, and six, not a one, two, and three. Joella changed the six to a three and Bernice responded, "There, you did it, you did it."

I gave Bernice and her table an enlarged copy of the next two graphs in the sequence. Bernice quickly labeled the first one (four nodes) gracefully, and said, "There. I am done. I am done." She then labeled the next graph. After she finished, I asked if she had found a pattern and could do the next graph. She said, "sure." She then left the room to get a drink of water. When she came back into the room she was off task for the next 2 minutes while the rest of her group finished labeling the graphs gracefully.

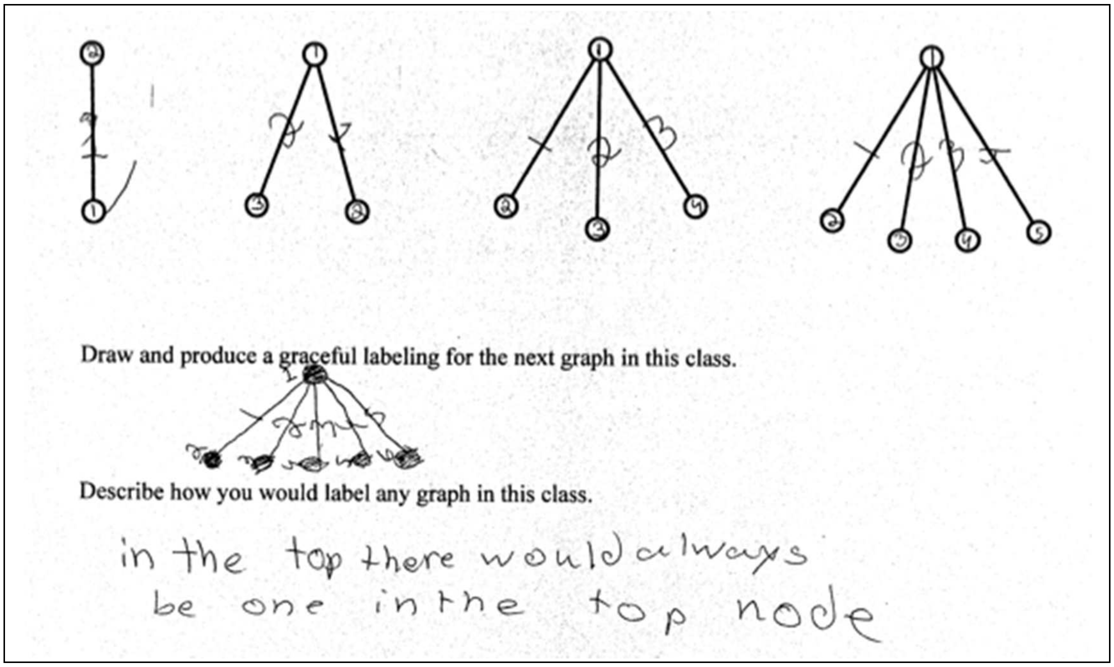


Figure 44. Bernice's task page for star graphs

After Bernice's table finished labeling the first four graphs in the sequence, I asked them to answer the next question on the page (see Figure 44). It was draw or produce a graceful labeling for the next graph in the sequence. Bernice said, "I don't get this. I don't get this question." I explained she needed to draw what the next graph would look like. She said, "Oh that is easy." Bernice drew the next graph and following her same pattern she labeled the graph gracefully. I asked Bernice what her pattern was and what number she had always placed in the middle. She told me one. She then wrote on her page to describe how she would label any graph in this class, "In the top there would always be one in the top node." She then said, "There I did it." After she finished writing, the session ended.

Bernice positioned herself and was positioned by others as an expert for most of the second problem-solving session (see Table 19). Her in-the-moment identity during this session was similar to the previous session. At times she positioned herself not only as an expert, but also as an authoritarian over the rest of her table by critiquing other students' work, such as teasing Joella for labeling a graph with a six instead of a three.

Table 19

Bernice's Positioning Timeline for Session 2

	12:00	12:33	12:49	12:52	22:16	24:29	24:33	25:05	25:18	26:30	28:12	30:11	33:50
Inferior			O									S	
Expert	T	T		S	S	S		O		S	S		S
Control													
Authority							S		S				
Superior													
Encourage													
Collaboration													
Unique Idea													
Face-saving													

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 3. On the third session, Bernice worked at a table with Iris, Karly, and Alia. Dr. Cullen, a professor from Illinois State University was present to help with the session.

The session began with a table discussion on graceful labeling of star graphs. At Bernice's table, Alia led the discussion. I came over to the table to hear what they were discussing:

Jenna: Bernice what do you think?

Alia: Let me. You subtract. You put the one in the middle and then you subtract each number in the circles to get a number of one, two, three, four, and keep going.

Jenna: But how do I know if I put a one here, cause that is what you said, and I put a two here and a three. How do I know this is going to be a different number?

Alia: You know because one minus two does not equal two. One minus tow equals one. That is how you know.

Jenna: Can you tell me more about that?

Alia: (Laughs)

Jenna: Bernice, what do you think about it?

Bernice: Um, um…

Alia: I am going to show you how to solve any.

Jenna: What do you think Bernice?

Bernice: I don't know.

Jenna: What did you say?

Alia: Here, this is how you know…

Bernice: All we had to do…

Jenna: Well how do you know every time these are going to be different numbers if you put one in the top?

Alia: Look like this.

Jenna: I want to hear Bernice right now.

Bernice: Because what?

Alia: Because like…

Next, Alia continued to try to explain why her pattern worked and Bernice said that she did not know why her pattern worked. Alia and Trevor shared their solution with the whole group and led the group discussion. When I said that Trevor labeled the graph differently, Bernice said, “That would be wrong.” Bernice did not say much else besides when asked if they thought there could be more than two solutions:

Jenna: You think there is more than one way, or more than two ways (to label star graphs gracefully)?

Bernice: No, I think there is only two.

Jenna: Only two, why?

Bernice: Yes, because that means less work for us.

Bernice made it seem as though she did not want to work hard that day.

Next, the students began to explore graphs that were connect the same way but in different positions to see if they were the same or different. Bernice seemed to change her attitude during this activity and she began to participate and join in her table discussion and exploration. After the students had a chance to make the different graphs with AngLegs, I started a whole group discussion on the graphs they had just explored. Bernice stated that she thought they were all the same because, “They all have four nodes and they all have three edges.” This statement furthered everyone’s knowledge that tree graphs have one less edge than node, which was what I wanted students to notice.

Path graphs were then handed out to the students. They were tasked to gracefully label the graphs. When I came to give the students at her table the chips for labeling the edges and nodes, Bernice told me that she did not want them. I still set them down on the table in case she changed her mind. She quickly changed her mind and started to use them.

Bernice quickly filled the first path graph using her pattern from star graph (it was a path graph and a star graph with only three nodes). Within a minute of working on the second path graph Bernice began to make statements displaying struggle such as, “I can’t do this. I can not do this,” and “Three. Dang it. So close.” After few more seconds of working Bernice said, “I did it! I did it.” She had found a graceful labeling and persevered through her productive struggle.

However, the productive struggle quickly changed to struggle that may not have been productive. The following statements show the struggle that followed (the time is in the parentheses):

(22:32) Two, three, four, five, six, and seven (playing circle chips on the nodes). Okay one, I can’t do that.

(23:02): One. I can’t do this. What if I switch it around? This would be a two, no wait, this would be a two. Oh my gosh!

(23:19): That would be…

(23:28): That would be a three.

(23:40): I can’t get it.

(24:04): This is so hard.

(24:34): (crying sound)

(24:39): Yes, I hate it. I hate it.

I then notice she has skipped a few graphs and was working on the fifth graph in the path sequence (seven nodes). I helped her find the next graph in the sequence and encouraged her to try to find patterns. I hoped this would help her change from unproductive struggle back to productive struggle. Bernice made one more statement of struggle, "this is hard" and then was off task. After several minutes off task, I spoke with Bernice again. She told me, "It was horrible" and said, "It is giving me a headache." After several more minutes of Bernice briefly working and being off task, Bernice had the following conversation me:

Bernice: This doesn't make sense.

Jenna: I think you can figure it out.

Bernice: No, I can't. It is horrible, really horrible. Are we the only people that go here?

Dr. Cullen: Yep.

Bernice: That is not special at all. At least it is better than reading.

At this time, almost all of the students had moved from productive struggle to unproductive struggle and I ended the session.

During the first two sessions, Bernice positioned herself as the expert and the authoritarian. However, in this session, Bernice lowered her in-the-moment identity to inferior and as someone who could not figure the problem out, did not want to participate, and was not enjoying herself (see Table 20). She experienced some productive struggle at the beginning of the exploration of path graphs but this soon changed to unproductive struggle and much of her time-spent off-task.

Table 20

Bernice's Positioning Timeline for Session 3

	7:29	10:09	19:26	20:42	21:26	21:36
Inferior	S			S	S	
Expert			S			S
Control						
Authority		S				
Superior						
Encourage						
Collaboration						
Unique Idea						
Face-saving						

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 4. Bernice worked with Iris during the fourth problem-solving session. To begin the session, I asked the students if they ever had times during their mathematics in school where they might not know the answer. This was give them a brief review of the unsolved problem and to help them understand that it is okay to not know an answer in mathematics. Bernice responded, "No, I always get the answer correct."

Next, I handed out the sort for the students to do. Bernice predicted they were all the same graph. Bernice quickly changed her answer after exploring the graphs with AngLegs and said they were all different. A class discussion followed. Joella said that she thought they were the same and tried to show how they were the same but she was taking apart the AngLegs, essentially changing the graph and reconstructing new graphs. Bernice told her so and explained why she thought they were all different. After the class discussion, Bernice called me over to her table. She showed me that two of the graphs were the same graph, something I had not realized while creating the page. I gave her verbal praise and told her she was correct.

Next, I turned the group of students' attention back to path graphs. Each table got one path graph on poster paper to gracefully label and a marker. Bernice and Iris received a path graph with three nodes. The first thing Bernice said was, "It is hard. I don't want to do it. Please." She followed that statement with "That one is hard. We can't do this." With encouragement from me, she attempted to label the graph and got both edges to be one. I told her to change one node. She did and she got a graceful labeling. Iris and she recorded it on their poster paper.

Joella and Amanda were given the path graph with four nodes, and by the time the other group had finished, they were still working. All of the students joined their table to help label the graph. Bernice began, in a bossy manner, telling the other students how to label the graph gracefully. However, on the first attempt she got it wrong. She tried again and created a graceful labeling.

The graphs were hung in the front of the room. A group discussion followed to figure out how to label the next graph in the sequence, a path graph with five nodes. The first node on every graph that was hanging in the front of the room was labeled with a one. I asked what they thought we should label the first node as on the graph with five nodes. Bernice was quick to say, "One." I confirmed that correct and placed a one in the node. From there, I asked if they saw patterns. Many students had ideas they wanted to share. However, Bernice made statements like, "No, I don't," "No," "I see nothing," and "No, it just looks like a bunch of numbers to me." She repeatedly said that she did not see a pattern, but other students were very willing and anxious to share. Even Iris who had not wanted to participate during the first sessions was very willing to state what she saw.

After several minutes of discussion, the edges were labeled with a four on the top edge, a three on the second edge, a two on the third edge, and a one on the bottom edge. The first node was also still labeled with a one. Bernice then chimed in the discussion. She said, “Five, put five.” She was referring to the second node. I asked why. She responded, “One minus five equals four. Everybody knows that. Except for kindergarteners” I wrote the five where she told me.

Several minutes later while still trying to label the graph, Joella pointed to a node and said it should be a four. Bernice was quick to tell Joella she was wrong because, “five minus four is one not three.” Joella tried to change her answer to one but Bernice told her, “We already have one.” Only 10 seconds later the following conversation took place:

Bernice:	I know. I know. I know.
Jenna:	What do you know? What should be right here (pointing to the third node)?
Bernice:	Two
Becca:	Two
Jenna:	Why?
Bernice:	Because five minus two equals three.
Jenna:	Okay. What should go right here then (pointing to the fourth node)?
Bernice:	Three.
Amanda:	Yeah. Three.
Bernice:	Not three. Four, four, four.
Becca:	I don’t get it.

Bernice: Two minus four is two.

Jenna: So what should go right here (pointing to the last node needing to be labeled)?

Amanda: Three.

Bernice: Boom, we did it!

With the path graph containing five nodes labeled, I handed out a page that had the next graph in the sequence, a path graph with six nodes. I challenged the students to complete a graceful labeling by using the patterns created from the posters in the front of the room. I asked Bernice if she knew the pattern. She said, “No.” Several other students said they knew the pattern. Bernice began making statements of struggle similar to the previous day before even beginning to work. She said, “I don’t get it” and “I don’t get this.” While working on the label, the statements continued. She said, “I don’t get this. This is so hard” and “It is going to take forever” but she continued to work. After only four minutes of working, Bernice said, “I am done. Yes! I finished.” I then told her she could write her solution on the poster paper, which she responded with, “Sure I can do it.”

Bernice was the only student to create a graceful labeling of the path graph with six nodes by the end of the problem-solving session. When other students were cleaning up they made comments of frustration, for example, Iris said, “I messed up.” Bernice responded to the comment with, “ Well I got it. Finished.” At the every end of the session, I asked Bernice to share her solution with the all of the students. She explained her strategy, “Okay, the bottom edge always has to be one and then you keep going up till the last edge at the tippy top.”

Bernice self-positioned 21 many times throughout the session (see Table 21 and 22 for Bernice’s positioning). She went from positioning herself as someone who could not find a graceful labeling to being superior because she could do it and no one else could. She was authoritative at times but she also spent some of the session off-task. As the teacher, I positioned her several times as an expert based on her graceful labelings.

Table 21

Bernice's Positioning Timeline for First Half of Session 4

	2:32	5:29	6:37	10:17	13:26	13:49	14:41	16:00	16:30	18:13	21:07	22:44	24:58
Inferior						S					S		
Expert	S	S	S	T					S				S
Control							S	S		S			
Authority					S							S	
Superior													
Encourage													
Collaboration													
Unique Idea													
Face-saving													

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Table 22

Bernice's Positioning Timeline for Second Half of Session 4

	25:57	26:13	26:25	27:18	27:49	29:08	29:23	31:01	31:05	33:45	34:06	34:27	35:25
Inferior				S			S						
Expert		S						S	T		T	T	O
Control	S				S								
Authority			S			S							
Superior										S			
Encourage													
Collaboration													
Unique Idea													
Face-saving													

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 5. Throughout the fifth problem-solving session, Bernice worked at a table with Alia, Joella, and Amanda. Drs. Cullen and Schupp were present to help with the problem-solving session.

After I had the rest of the students started on path graphs, I went and spoke with Bernice. I told her, "What I want you to work on is explain how you would figure out the pattern because you said there was a pattern. Right?" Bernice told me that yes she had found a pattern and I told her to write about it. I assigned her different work than the rest of the students.

Bernice wrote about her pattern (see Figure 45). She wrote, "The edge all the way in the bottem [sic] is one then go up. One is always in the last edge." After Bernice had finished writing, she went and filled in the path graphs on her page using her edge pattern. After Bernice had filled in all the graphs, I told her I was going to challenge her and drew the next path graph in the sequence on her page. Bernice began to work. After several minutes Bernice crossed the graph out (there was no error in her graph) and drew it again. I came back and noticed Bernice was engaged in productive struggle. I asked her if she had solved it or if she was still working. She told me that she had messed up. Bernice continued to label the new graph with seven nodes gracefully. After she had it labeled, I asked Bernice if her pattern would work for all path graphs. She told me it would. I then gave her the double star graph sequence to begin working on while all the other students finished path graphs.

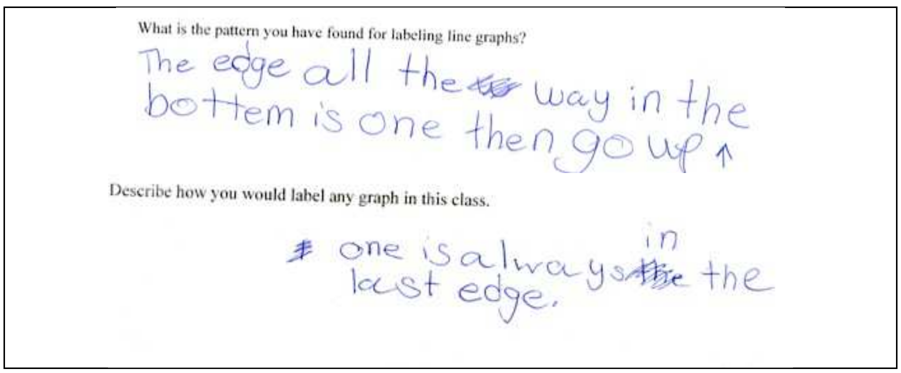


Figure 45. Bernice's solution for labeling path graphs.

Bernice began to gracefully label the first double star graph (also a path graph with four nodes; see Figure 46). After one minute, Bernice stated, "Oh there it is. I did it (put arms up in air). I did the first one!" Bernice then began working on the second one. She worked on the graph for three minutes without saying anything and over the next two minutes she said, "Ah, so close" and "no." Then Bernice said that it was impossible and had the following conversation with Dr. Cullen:

Bernice: Impossible

Dr. Cullen: It's not. Let me see what you got left. Talk me through it. What have you tried?

Bernice:	I have tried one, two, three, four, five (pointing to each edge in order) and that does not work.
Dr. Cullen:	Okay.

Bernice moved several chips around on her page and said:

Bernice:	Dang it, still can't do it.

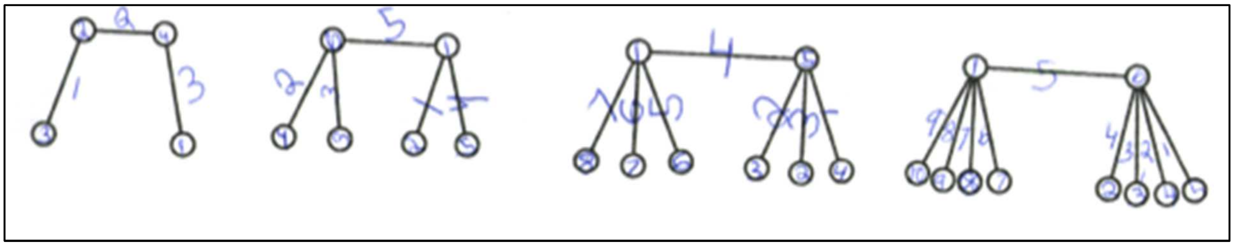


Figure 46. Bernice's labeling of caterpillar graphs on task page 1.

Right after she said she still couldn't do it, I asked the group of students if they were ready to move on to the next type of graphs (referring to moving on from path graphs to double star graphs). Bernice was quick to shout, "No," not realizing she was already on that graph.

During the group discussion on path graphs, Bernice continued to work on a graceful labeling of the double star graph with six nodes. At one point, I asked if they could gracefully label any path graph, Bernice responded, "No." I then reinforced the idea that she was on a different graph in front of the rest of the students. Dr. Schupp said, "Oh, wow!" As the rest of the group finished the discussion on path graphs, Bernice made several remarks of struggle trying to label the double star graph, such as, "No!" While I was handing out the double star graphs to the rest of the students (about 10 minutes after Bernice began working on the double star graph with six nodes), Bernice shouted, “I did it!” As soon as the students were working and the discussion had ended, Bernice called me over:

Bernice: Ms. Jenna.

Jenna: Yes?

Bernice: I did the second one.

Jenna: You figured it out?

Bernice: Yes.

Jenna: Are you trying to find a pattern?

Bernice: No, not anymore.

Then Bernice began to work on the third double star graph that contained eight nodes.

Alia, who had just started working on the double star graphs, Dr. Cullen, myself, and Bernice had the following conversation:

Alia: I finished the green one! (Graph with four nodes and first one in the sequence.)

Dr. Cullen: Okay, write it down.

Bernice: You just finished?

Alia: Yeah, all done.

Jenna: So write it on your white sheet because we are going to start here.

Several side discussions took place, then their discussion continued:

Alia: I am on the blue one. I think it is hard.

Bernice: It is super (emphasizing super) hard. It took me like thirty minutes or something. How long have we been work? (Gets up to look at camera.)

Becca: It is not that hard.

Bernice: Yes it is. The blue one, not the green one. It has been 34 minutes.

Bernice continued to work on the third graph in the sequence. She first placed her edges and then did the nodes. She made several comments to herself while working, including, "this goes right here," and "So hard. This is impossible. Impossible." Only 45 seconds after saying it was impossible, Bernice put her arms in the air and shouted, "I did it! I did it." I responded with, "You did it?" Dr. Cullen told Bernice to tell Dr. Schupp what she had figured out. Bernice told Dr. Schupp, "I did a double star graph. The first one was easy, the second one was super hard, and this one was easy." Although she stated what graphs she had labeled and shared how easy or hard they were, she never explained her process. At this time the problem-solving session ended as parents arrived to pick up their children.

Overall, Bernice positioned herself and was positioned by others as an expert during this problem-solving session (see Table 23). The teachers that helped with the session reinforced this positioning because of her success and even positioned Bernice with an in-the-moment identity of superior by sharing with the group that she was on a class of graphs ahead of the rest of the students. Bernice's in-the-moment identity was very positive at the end of the session.

Table 23

Bernice's Positioning Timeline for Session 5

	9:21	13:51	15:04	19:06	20:54	28:02	31:56	32:52	36:53	38:56
Inferior										
Expert	T	S	T	T	S		S	S	S	
Control										
Authority										S
Superior						T				
Encourage										
Collaboration										
Unique Idea										
Face-saving										

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 6. Bernice worked at a table with Joella, Alia, and Amanda during the sixth problem-solving session. An undergraduate student who volunteered at the after school program sat at their table during the session and briefly interacted with them.

The beginning of the session, Bernice still had the last caterpillar graph to do for the first group of caterpillar graphs on the task page (see Figure 47). She had completed the three previous graphs in the sequence during the prior session. Alia was on the same graph as Bernice. Alia began by placing the circles on the edges. Bernice was quick to say, "Wrong, that is the wrong one. The square ones are supposed to go in the middle, the circle." Alia seemed to ignore Bernice and kept working. Bernice said a few seconds later, "No, you put these first." This time Alia responded. She said, "They won't fit." Bernice called me over and asked, "Don't you put the squares things first and then the nodes?" I told her to do what had worked in the past. Bernice said, "Boom, I have my opinion." Alia appeared to be ignoring Bernice again.

Bernice began labeling her graph again. She noticed she did not have eight or nine circle and square chips because this was the first time they had labeled graphs with that high of a number of nodes and edges. Bernice asked the volunteer to make her an eight and nine but she said, “I need it first for squares.” This reinforced the idea that Alia was wrong for starting with the circles. Bernice started to work on her labeling again. About 1 minute later, the following conversation took place:

Bernice: Can you make me a ten too?

Joella: Can you please make me a ten? Can you make me one too?

Bernice: You don’t need it yet. You are not here. I am not being rude.

The girls at the table continued to work on labeling their double star graphs. Amanda then asked, “Wait, do we take it away or add?” Bernice was quick to jump in and say, “You take it away, you subtract it.”

Bernice continued to work on a graceful labeling for the double star graph with 10 nodes. She made a comment of struggle, “Oh dang it. I need a nine.” Quickly her dispositions changed to, “I am doing it. I am doing it. I am doing it.” However, at the same time Amanda had figured out the second graph in the sequence and said, “Do I get another paper?” Bernice was quick to jump in and say, “What is the next color? I think you are ready for the next one, purple I think. I don’t know.” At the same time as this interaction, Alia said, “Okay, I got it.” Bernice responded, “No, you don’t.”

While working on her labeling of the fourth graph, Bernice continued to help Amanda with comments like, “Do you need a circle ten?” While Bernice’s helped Amanda, she made comments on her own work and showed signs of struggle. She said, “One, two. What happened?

Oh no, I missed one" and "Seven minus three equals four. Oh my god. Ms. Jenna, look how close I was but this last one doesn't make sense." When I came over to speak with her, Bernice told me, "There is no pattern." I informed her that another table had found a pattern and they could share it with her group. She told me, "No, no, no." She continued to work. Her following statements included:

(19:44): I put ten up there.

(19:46): Ten minus five.

(19:50): Ten minus one is

(20:01): Ten minus

(20:05): Ten minus

(20:08): Three

(20:12): Ten minus minus two equals eight.

(20:22): (To Joella) You don't have the ten thing.

(20:37): The next one is not.

(20:51): The circle kind

(21:09): Let's see. Four. Ten.

(21:21): That makes. Oh my god this doesn't make sense.

(21:30): It does not make sense.

At (21:45), Alia, who was working on the same graph, shouted, "I did it! Whooo!" Bernice responded, "You did it?" Alia confirmed, "Yeah! Haha, I did it." I came over to the table and Alia showed me the pattern she had found for the edges. Bernice quickly changed her edges to match Alia's pattern. Two minutes after Alia had labeled her graph, Bernice shouted with her arms in the air, "I did it. I am done. I need a pen. Ms. Jenna I did it." I asked Bernice if she had found a pattern and she said, “No, all I see is it is backwards I guess (moving her hand across the edges from right to left).”

While Bernice was recording her answer on her worksheet, Alia was talking aloud while she worked. She said, “Six minus five is one. Yes it is.” Bernice responded with, “Yes, six minus five is one. Do your math.”

I handed out the next page, caterpillar graphs task page 2 (see Figure 48 for Bernice’s work), to Alia and Bernice. Alia looked at the page and said, “Oh, those are easy.” Bernice looked at the page and said, “You call this easy? This is all easy. This one looks easy (pointing to the first graph). This one looks hard (pointing to the second graph). The one looks mega hard (pointing to the third graph).” Alia replied, “The last one was easy. The last one was easy.” Bernice started on the first graph and noticed it was the same as the second graph on the previous caterpillar task page. She said that she would just copy it and she did. Then she announced, “I am done with the first one.” Bernice was given the second graph in the sequence.

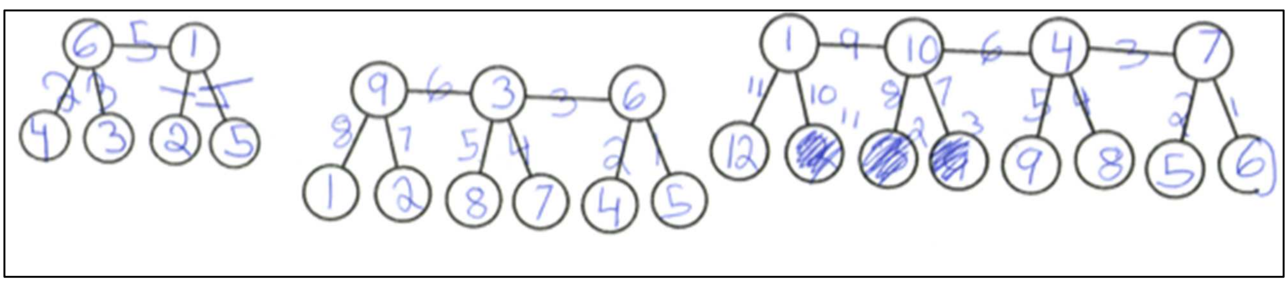


Figure 47. Bernice’s labeling of caterpillar graphs on task page 2.

Amanda was working on the previous double star page. She announced that she had finished and was going to onto the next graph. Bernice looked at her graph and told her, “You forgot the tippy top.”

Bernice began to place her square chips on the edges on her graph. She used her pattern she had found from her work on the double star graphs for her labeling. She then placed the circles on her nodes, stating the numbers aloud as she placed them. She announced, “Finished” when she completed a graceful labeling. This time she did not make statements of struggle while she worked, but she did position herself as an expert by announcing her completion of the labeling of the second caterpillar graph. Bernice wrote her solution down on her worksheet and said, “Yep, I figured out a pattern for this.” She used her pattern and began placing chips on the third graph in the sequence with twelve nodes. After only working on the labeling for 2 minutes, Bernice through her arms in the air and announced, “And I finished!”

I asked Bernice if she had found a pattern and could label any caterpillar graph. She told me, “Yes.” We began a whole group discussion on graceful labelings and patterns for double star and caterpillar graphs. Hector and Becca explained their patterns. I asked Bernice to explain her pattern, and she said that she did it the same way Hector did by labeling the edges on the caterpillar from right to left in order. When I asked her how she did the nodes, she said that she did not know. Becca then stepped in and explained how to label the nodes.

Bernice continued to position herself as an expert during this session but also positioned herself as an authoritarian by critiquing and evaluating Amanda throughout the session (see Table 24 for Bernice’s positionings). She repeatedly told Amanda what she was had done correctly or what she was forgetting to do. At other times she also critiqued Alia, such as telling her that she was putting the square and circle number chips in the wrong spots.

Table 24

Bernice's Positioning Timeline for Session 6

	8:48	10:05	12:00	14:00	15:23	16:50	18:30	20:22	23:45	24:35	24:57	26:14	28:45	32:27	33:18	34:51	39:23
Inferior							T				S				O		
Expert					S				S			S		S		T	T
Control																	
Authority	S	S	S	S		S		S		S			S				
Superior																	
Encourage																	
Collaboration																	
Unique Idea																	
Face-saving																	

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Problem-solving Session 7. At the seventh problem-solving session, Bernice worked at a table with Amanda and Alia. The same volunteer from problem-solving Session 6 sat at their table and briefly interacted with them.

The session began with a discussion about it being our last problem solving session. I asked the students if this math was similar or different to the mathematics they did in school. Bernice was quick to respond with, "Different." She did not elaborate. I then told the students the story of Icarus and Daedalus and led them through a discussion of the unsolved problem and what they were tasked to do. Bernice participated in the discussion and then began to work on the problem by herself.

Bernice tried several numbers for the Icarus side. On the Daedalus side Bernice started with the number 5 (the Daedalus side had many different solutions and was solvable). After she went through the pattern a number of times she said, “I did it wrong. I think I need to restart it.” I asked her if she thought it would end in one. She responded, “five, fourteen, no.” She thought it would not work. I told her if she found a number that did not end in one, they would survive the fall. She said, “So I won?” I told her that she had won for the Daedalus side, the part of the problem that was not unsolved, but not the Icarus side. She continued to try numbers on the Icarus side.

Amanda, who was sitting at Bernice’s table, told me that she was confused. I told her, “Just ask Bernice, she understands; she knows what to do. Okay.” Bernice helped her for only several seconds and then told her, “Put it on the three. I am not listening.”

A short while later, Bernice was testing seven to see if it would end in one. After following the pattern for eight attempts, Bernice said, “Ms. Jenna, I did it.” I asked her if she had found one. She said, “Yes. It goes 7, 22, 21, 34, 17, 52, 26, 13, 38.” I explained that it had not repeated yet so she needed to keep going. She continued working on the number. Next she tried 20 and got it to end in one, saying, “Dang it, it doesn’t work.” After 20, Bernice tried nine and when it ended in one she said, “Ah, nine doesn’t work.” Next, Bernice tried 12 and said that one did not work either. During this time Alia became frustrated because she had yet to find a pattern that did not end in one for the Icarus side. Bernice told her that it was okay and that she had not found one either.

Several minutes later, Becca asked to work on the Graceful Tree Conjecture instead of Collatz Conjecture. I asked Bernice if she wanted to work on it as well and she said yes. Bernice gracefully labeled the first two graphs (see Figure 48) and then asked if she could leave and go to music. I encouraged her to work for a few more minutes but then I told her she could go to music and she left.

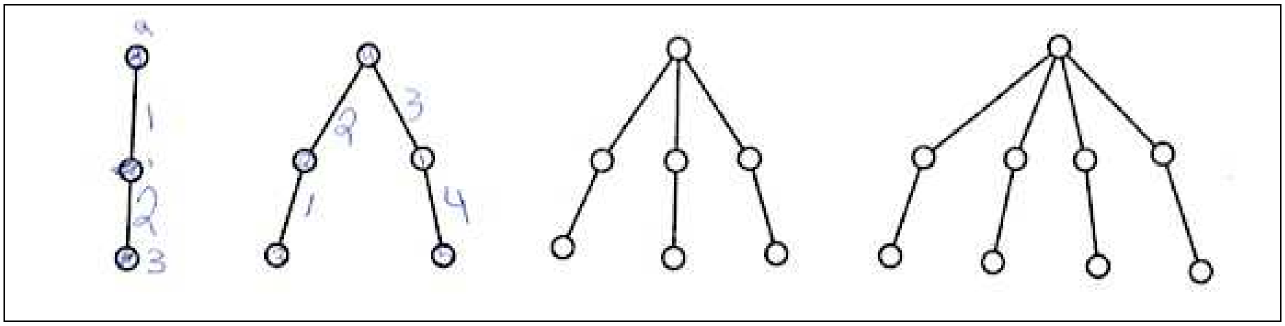


Figure 48. Bernice's attempt at labeling comet graphs

In the seventh problem-solving session, Bernice continued to position herself as an expert, but she did not act as authoritative as she had in the previous days (see Table 25). I positioned Bernice as an expert when I told Amanda to ask Bernice if she had questions because she knew how to go about solving the problem. I also positioned her as inferior by telling her that she had not found a number that worked for Icarus and she needed to continue to work. Bernice self-positioned as an encourager by sharing with Alia that she had yet to find a number that would not end in one for Icarus.

Table 25

Bernice's Positioning Timeline for Session 7

	18:06	18:30	20:41	21:53	22:10	23:03		34:01
Inferior	S				T		S	
Expert		T	T	S				
Control								
Authority								
Superior								S
Encourage						S		
Collaboration								
Unique Idea								
Face-saving								

Note. O represents being positioned by *other* (i.e., another student); S represents being positioned by *self*; and T represents being positioned by *teacher*.

Post interview. During the post interview, Bernice was asked about mathematics. She described the study, her feelings about it, and her skill level:

Cindy: So what does it mean to do mathematics? So when you do math, what is it?

Bernice: Math for me now, last time it was super hard but now that I have been working on it, it has been getting easier and easier and easier.

Cindy: So what have you been working on exactly?

Bernice: We have been working on tree graphs, path graphs, and all that stuff. All the different graphs we are trying to figure out the question that nobody has since answered, if all trees are graceful.

Cindy: Ah ha, and what does it mean to be graceful?

Bernice: It means that the nodes will not continue. Like, when you, that means when it is graceful they don't connect so we are checking all the graphs to see if every single one of them is not connect.

Cindy: Okay, I see. The nodes. What do the nodes look like?

Bernice: Nodes are the circle parts and edges are the line parts that put them together.

Cindy: Ah ha, okay. And what kinds of graphs did you work with? You said something about trees.

Bernice: Tree graphs, we have done path graphs, star graphs, um I don't know the rest of the graphs but those were the two ones that we were mostly focusing about.

Cindy: Okay, interesting stuff, huh? So what, well you told me about the unsolved problems that you were working on. What is math like in school?

Bernice: Math is school is like multiplication, division, we are not working on the stuff we are doing at the after school program at school. So it is kind of harder on me.

Cindy: Oh at school it is harder or harder here because you are not doing the same?

Bernice: Here because yeah.

Cindy: Okay, well are you interested in the math you do here?

Bernice: Well yeah, I am interested because personally eight weeks ago when we started it, I was like really bad at it and now I am really good.

Cindy: Why do you think you are better?

Bernice: Because I have been practicing and we have gone over it. We have solved some short cuts that we can use to do it. We made patterns.

Later in the interview, Bernice reflected back on the study again when she compared the unsolved mathematics problem to school mathematics:

Bernice: Well the math that we do here is super hard. Everybody has to team up and do it as a group but at school we don't team up and do it as a group because it is really easy. Because it is hard and you have to figure it out as a group.

Later in the interview Bernice said that she was "happy that I was learning something new" and that she liked challenges and if you liked challenges these would be good problems for fourth grade students. Her final statement about the unsolved problems was a reflection on how working on the unsolved problems made her change her view of mathematics:

Bernice: I thought the only problems in life were multiplication, division, addition, subtraction but when she taught me that, I was like there is more math stuff. I was like blown away. I was like what I thought there was only those but I guess not.

However, at the end of the interview, Bernice was asked if she would call herself a mathematician and she said, "No."

Summary of Bernice's in-the-moment identity. In the pre- and post-interviews, Bernice displayed an overall identity as someone who is good at mathematics. In the post interview she said that she found the activity challenging at first but much easier at the end of the sessions. Bernice positioned herself more than other positioned her (see Table 26). Overall, Bernice's in-the-moment identity shifted depending on the success she found. The more success she had the more she identified herself as the controller of the problem solving and as authoritative by critiquing other students work. When she was not successful, she would identify as inferior (see Table 27).

Table 26

Who Positioned Bernice

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	11	9	6	21	6	13	5
Student	2	2		1		1	
Teacher	4	2		4	4	3	3

Table 27

Bernice's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	2	2	3	4		3	3
Expert	10	9	2	12	8	6	3
Superior	1			1	1		1
Author	3	2	1	4	1	8	
Controlling	1			5			
Encourage Collaboration							1
Face-saving							

Post Interview: Students' Views on Doing Mathematics

In the post-interview the students were asked the same questions as the pre-interview as well as several additional questions to have them reflect on their work with unsolved problems. The students shared insights into their views about the nature of mathematics, what mathematicians do, and their perceptions about unsolved problems.

During the pre-interview students did not know what a mathematician was. Overall, they still did not know what a mathematician was after engaging with unsolved problems. It was a topic we talked about briefly several times but not something that was focused on. In the post interview, Alia described a mathematician as someone who would "try to solve problems." Joella said, "They study real hard to find out what is the hardest question." Karly stated, "They figure out the problems, they try to figure out difficult problems." Several students, such as Iris and Amanda, said that they did not know what a mathematician was.

Common views of mathematics is that the purpose of mathematics is to find an answer and that if you are fast at mathematics, you are good at it. There were two instances these dispositions were presented during the post-interview. The students were asked if mathematics were animal, what animal would it be. Alia responded, "A tiger because tigers are fast and I am fast at math." She persisted with the idea that to be good at mathematics you must be fast. When Hector was asked what he had learned from doing unsolved problems, he said, "That they do have answers but people just haven't figured it out." He seemed to persist with the idea that mathematics is about finding an answer.

During the post interview students were asked if their work with unsolved problems made them think differently about mathematics. Iris said that the unsolved problems made her think differently about mathematics but she did not know how. Hector had an interesting conversation with Cindy about school mathematics and how the work with unsolved problems made him think differently about mathematics:

Cindy: What about math in school? What is that like?

Hector: It is just like easy as pie.

Cindy: What makes it so easy?

Hector: Cause let's think. Okay, it is just like a pie, you cut it up into four pieces. One slice is adding and the other slice is subtraction and the other one is multiplication and the other one is division. It is just like that.

A minute later Hector and Cindy had the following conversation:

Cindy: So has your work with Ms. Jenna made you think differently about what mathematics is?

Hector: Um hum

Cindy: How?

Hector: It is, mathematics isn't just four things, it is like five pieces of pie.

Cindy: Oh, there is another piece? Tell me about that piece.

Hector: The other piece is called, what is it called? I don't know what it is called, let's just call it mystery and you don't know what you are ever going to get because you could get some hard stuff because in it, when you do like the, what's called, caterpillar, you do like the dots and you do that line and then you subtract. It, so, it's basically sort of subtracting and getting the right answer. Yeah.

When Amanda was asked to explained if the work with unsolved problems made her think about mathematics differently, she said that it did not make her think of mathematics differently.

The students were asked to compare school mathematics to unsolved problems. Most of the students said unsolved problems were different from school mathematics. Alia said, "It is not [similar]." She further said, "Cause we doing multiplication and stuff in a classroom and with Ms. Jenna we are doing, we are doing, like are all trees graceful and we are tying to find out the answer to the problem is." When Amanda said it was different, "Because sometimes she makes it harder but at least you will figure it out and we got the answers." Trevor said that the unsolved problems were not similar to school mathematics besides they were sort of similar to "touch points" that they learned in kindergarten. Karly described the school mathematics as different from unsolved problems because "they don't use the circle things." Iris and Joella both explained that it was different and similar. Iris said it was different, "because we do times over there [school] and plus and dividing, but they do the dots and the straights but it is kind of the same to be because we get to plus and take-away and minus them." Joella said the unsolved problems were similar to school mathematics because they wrote on paper and subtracted. She said it was different because it was hard and there were edges and nodes. She said that she learned math is hard when you are doing unsolved problems.

The answers were mixed on if they students would want to do the unsolved problems in school. Alia said that she does not want to do these problems at school "because it gets confusing sometimes." When Amanda was asked if she would like to do unsolved problems at school she said, "Yes because it was kind of fun." Hector said that he would like to do unsolved problems in school and he said that his teacher told him they would be doing some similar problems in a few weeks. I wonder if they are unsolved problems or graphs on coordinate planes?

Several students shared their feelings about working on unsolved problems. When Alia was asked, she said, "Well luckily with Ms. Jenna doing it, it is fun because we get to write on the board and sometimes not fun because I can't figure out the problem, so it kind of frustrates me." She claimed that she enjoyed working on the problems and figuring out the answers was challenging. Trevor said that he enjoyed doing the unsolved problems. Karly described the unsolved problems as challenging, enjoyable, and fun. Iris said that she would like to do unsolved problems at school and that she enjoyed doing the unsolved problems.

Chapter Four Summary

For this chapter, I first described the cognitive, affective, and conative dispositions the students displayed while they engaged with unsolved problems. For the cognitive dispositions, I described the different ways students went about gracefully labeled graphs. With regard to the affective dispositions, I described the five themes that characterized the students' beliefs about the nature of mathematics and the mixture of emotions the students displayed. For conative dispositions, I described the perseverance the students demonstrated during the study and the interplay between the emotions of frustration and joy. Next, I shared a detailed overview of two students' in-the-moment identities that were evident from their interview responses and participation in the problem-solving sessions. In the next chapter, I share a summary of the findings, a discussion of the findings, the limitations, implications for teaching, and recommendations for future research.

CHAPTER V: CONCLUSIONS

As an elementary school teacher and future college professor, I want students to view mathematics as a study of relationships and patterns and develop a love for the subject. However, we know many students believe mathematics is a set of rules to follow and something to memorize, not explore (Allen, 2004; Boaler, 2016; Delvin, 2000). My purpose in conducting this study was to show the participants a different perspective of mathematics by allowing them to experience mathematics like a mathematician through exploring unsolved problems and then to tell the story of their experiences and the mathematical dispositions the exhibited.

In the previous chapters, I weaved together a picture of events that took place during the study. I described the dispositions of students as they engaged in the exploration of unsolved problems, something they had not experienced previously as mathematics students. I also shared a detailed look at two students' journeys though the seven problem-solving sessions and described their in-the-moment identities. In this chapter, I will address my research question by summarizing and discussing the findings of my analyses. Following that, I share the limitations of this study and present implications for teaching, recommendations for future research, and share my closing thoughts.

Summary of Findings

During the pre-interview, students were asked questions about mathematics, such as what is mathematics, who does it, what is a mathematician, how does someone learn mathematics, and if they are interested in mathematics. Overall, students described mathematics as something that is procedural and about doing operations. They said it was just adding, subtracting, multiplying, and dividing. During the pre-interview, none of the students were able to describe what a mathematician was.

I focused on the cognitive, affective, and conative aspects of students' mathematical dispositions as they were engaged with unsolved problems. For the cognitive dispositions I focused on attempted justifications and patterns. All of the students were able to find a pattern for star graphs. For path graphs, three of the students were able to describe patterns, but all of the students were able to find graceful labels through trial-and-error. In respect to caterpillar graphs, Bernice, Alia, and Hector described a pattern for labeling the edges, but it appeared they used trial-and-error for labeling the nodes. Becca, Karly, and Iris used their pattern from path graphs to find a way to label any type of graph in the caterpillar sequence. The rest of the students were only able to find graceful labels through trial-and-error. Several students (Becca, Alia, Iris, and Bernice) had the opportunity to work on comet graphs during the last session. They were only able to find a few labels through trial-and-error. Becca attempted to use her pattern for path and caterpillars to label the comets; however, she stated that her pattern did not work and she needed a new pattern.

With regard to affective aspects of students' dispositions, I found that students made statements throughout the problem-solving sessions about their beliefs on the nature of mathematics. Students' statements about the nature of mathematics revolved around five themes. First, students made multiple statements about there always being a solution in mathematics. They thought that either the unsolved problems should have an answer or someone knew it. The students questioned the adults in the room about whether they could figure out the answer or they seemed to think that the answer could be found on the internet. Next, there were several statements made about how they were fast at mathematics or they were the first one finished, displaying a disposition that it matters how fast you are at mathematics. Third, they made statements about mathematics being all about operations or equations and not about an exploration of numbers and patterns. Fourth, they exhibited the idea that mathematics should be done alone, which was displayed through statements such as "no peeking." Lastly, after four problem-solving sessions, several students displayed dispositions that mathematics problems like unsolved problems should be worked on in groups.

Students exhibited a variety of emotions throughout the study. The two emotions exhibited most frequently were frustration and joy. However, students also showed emotions of humor, contempt, sadness, boredom, and displayed acts of caring.

I also examined a third type of disposition, the conative construct of perseverance. This was more challenging to identify than the other types of dispositions because it had to be displayed over multiple sessions and was not something someone could just make a statement about. Although three times students said that they wanted to quit the study, they continued to come and work on the problems, showing perseverance after unproductive struggle. For example, Iris wanted to stay in the room but not work for the first two sessions. By the fourth and fifth sessions, Iris was taking materials out before being prompted, working hard, and enjoying herself. The third thing I found with respect to perseverance was the students' emotions experiencing the interplay between frustration and joy, which could also be described as productive struggle. The students would repeatedly demonstrate frustration while they worked through trial-and-error to find a way to gracefully label the graphs. As soon as they found a graceful label, they would shout out with joy that they had done it, many times with their arms raised above their head in excitement.

Lastly, I explored two students in-depth to tell the story of their experience with unsolved problems. Becca was a quiet student when she began the study. She was repeatedly positioned as inferior during the first two problem-solving sessions that she attended. However, with increased time to think about the problem and work on a solution, she shifted her in-the-moment identity to being expert and even superior by finding a pattern that enabled her to label path graphs and caterpillar graphs. She shared her knowledge with the people that sat around her and began to tell other students what to do and how to label the graphs. In the first interview, it was almost impossible to hear what Becca was saying, but in the post-interview, she was loud and appeared proud of what she had accomplished. Bernice, the other focus student, began the problem-solving sessions acting as an expert. She continued to have this same in-the-moment identity throughout most of the problem-solving session but throughout all of the sessions she experienced productive struggle, again with the interplay between joy and frustration.

In the post-interview the students' responses were very similar to the pre-interview. A few students made broad statements about mathematicians, but none really seemed to understand that they were acting as mathematicians while they explored the unsolved problems. Students still had to the same view that mathematics was about computations; however, a few students stated that they learned there was something else to mathematics besides adding, subtracting, multiplying, and dividing. They learned there were also tree graphs. Almost all of the students said that they enjoyed working on the unsolved problems. Their answers varied, however, regarding whether or not they would like to do these types of problems in school.

Discussion of Findings

I begin the discussion of findings with a reflection on the theoretical perspective that informed this study—communities of practice (Wenger, 1999). Next, I discuss the findings from Phase 3 of analysis where I focused broadly on students' dispositions. Then, I address the main findings from my examination of the two students' in-the-moment identities. Lastly, I reflect on unsolved mathematics problems.

Communities of Practice

I believe learning occurs and is developed when students are active participants in a learning community. The theoretical perspective, communities of practice (Wenger, 1999), was the overarching framework for the development and enactment of this study. The participants had an opportunity to engage with unsolved mathematics problems for seven problem-solving sessions during which they were expected to be active participants in a learning community. However, I encountered several difficulties when I attempted create a community of practice with the students for the study. First, the students already had created a community within the after-school program. Many had been attending the after-school program for multiple years or at least for the first 4 months of school, and the relationships they had established presented challenges, such as students who wanted to work with certain students but were disruptive and other students who did not want to work together. Second, six students in the study had a family member also in the study. Amanda and Iris were sisters, Karly and Hector were siblings, and Alia and Trevor were cousins. These family dynamics created challenges for collaborative group work. Karly did not want to listen to her brother, and Amanda and Iris did not interact with each other. The most challenging pair was Trevor and Alia. Just before the study began, they were living in the same house and did not get along. They fought with each other constantly at the after-school program and during the problem-solving sessions, which meant they had to stay separated. Last, Alia, Amanda, and Joella were all in the same class at school. Although they got along and wanted to work together, they were very competitive with each other.

There were aspects of a learning community, however, that did develop over the course of the study. The students were able to negotiate meaning by participation in the social aspect of the problem-solving situations, working with all of the students and in small groups, and by making sense of graph theory, a topic normally reserved for undergraduate and graduate mathematics students. They were able to make sense of tree graphs, develop an understanding of edges and nodes, and explore graceful labelings. The students were able to develop mutual engagement and make sense of the problem. They experienced joint enterprise by holding each other accountable for the goals of gracefully labeling graphs. The students also developed a shared repertoire. For example, Edward began calling the Graceful Tree Conjecture, "Sticks and nodes." The students took up this idea and many of them began calling the activity "sticks and nodes." During the seventh session, Becca repeatedly asked me to work on "sticks and nodes," and I even began calling the problem that. The students were quick to adopt the shared repertoire or actions of creating graceful labels for their graphs and were able to talk about them. Communities of practice (Wenger, 1999) was helpful in the development and enactment of the problem-solving sessions even though a learning community did not have time to fully develop.

Overall Dispositions

The research question that guided my study was: What are the characteristics of students' dispositions toward mathematics when they engage in the exploration of unsolved problems? In the sections that follow, I describe the cognitive, affective, and conative dispositions the students displayed.

Cognitive. Beyer's (2011) conceptual framework for dispositions includes cognitive mental processes as a disposition but in the mathematics education literature, a student's cognition is usually considered separately from his or her disposition. For example, the National Research Council's (2001) definition of dispositions does not include a cognitive aspect. I included cognition as a disposition for this dissertation study because I thought it was important to document that the students were able to make sense of the unsolved problems, label graphs gracefully, and find and use patterns. Although the students were engaged in these mathematical practices, they did not necessarily exhibit a tendency to do so. Rather, it was an expectation of our work together. However, *disposition* is defined as a tendency to act in a certain way (disposition, 2017). Thus, I do not believe that cognition is a dispositional function although it played an important role in my analysis of data.

The students who took part in this study would not have been considered the "top" or high-achieving students in their regular classrooms. In fact, a few of the students had to use their fingers to model the basic subtraction computation to label the graph edges. However, even though these were not the top students, they were able to make sense of the problem and make progress on the problem. They were able to explore what a tree graph was, learn about graph theory and graceful labeling, and find graceful labels for four different types of tree graphs. Many of the students were even able to give a generalized pattern for different types of tree graphs, albeit in an informal way.

According to Carpenter et al. (2003) there are three ways students present an argument or justification. They appeal to authority, use an example, or give a generalized argument. The interesting thing about having the students experience graph theory is that they were not able to appeal to authority because they did not have any knowledge of how to do this and with unsolved problems there was no authority with a solution. They did try to explain their patterns by giving an example. On their task pages they were asked to explain how they would get a graceful labeling. Many times the students would circle a graph they had labeled and try to explain their pattern using the specific graph they had circled, giving a justification by example. Another time Alia said that she could show me how to label a graph through an example with 20 nodes. At times the students would attempt a generalizable argument, but they struggled with the language and knowledge of how to do that.

Affective. The two major aspects of affective dispositions I reported on in Chapter 4 were students' beliefs about mathematics and the emotions they experienced. Overall, the children in my study had positive feelings about mathematics. They enjoyed it, said that they liked school mathematics, and chose to participate this study, which was extra mathematics at their after-school program. However, their views of what mathematic were very different from those of a mathematician. Mathematicians believe that mathematics is a subject that is something to explore, collaborate on, and think deeply about (Boaler, 2016; Burton, 1999). The students in this study found mathematics to be procedural and about equations. They made many statements that mathematics should be done alone and that it was cheating to look on your neighbor's paper.

Schoenfeld (1992) created a list of students' beliefs about mathematics and these same views were very evident in this study. Students made statements that there is only one right answer. They also made statements about the teacher knowing the answer, and in their interviews they explained that you learn mathematics by following the rules the teacher gives you. Also, in the interviews, the students explained that mathematics was about memorizing their facts and getting faster at doing the facts pages, similar to what Schoenfeld reported. Schoenfeld reported that students thought they should be able to finish a mathematics problem in 5 minutes or less, and the participants in this study had similar views; several gave even shorter time periods. The major one difference in beliefs stated by the students in this study as compared to Schoenfeld's participants was that these students thought school mathematics would help them for future jobs if they were going to be a teacher or work in a store and needed to count money. Schoenfeld reported that students thought school mathematics had nothing to do with the real world.

Masingila (2002) reported that only three out the 20 students in her study had a broad definition of what mathematics is. In my study, none of the children had a broad definition of what mathematics is—they all reported that it was adding, subtracting, dividing, and multiplying. Only two children took their definition of mathematics farther to mention it also contained fractions.

I conducted this study outside of a school setting because I wanted the children to view the activity as something different from school mathematics. I tried to position myself as a facilitator rather than a traditional teacher; yet, through the dispositions they exhibited, I believe the students still saw me as a teacher and viewed the problem-solving sessions similar to school mathematics. Although this experience was not about performing computations, throughout the study the students would get excited when they "got it"—an answer, indicating about answers and not seeing the big picture. For example, when I told them it was the last session, Trevor said, "But when will we find out" indicating that he thought we should not end until the answer was given. Another reason I believe the students viewed this as school mathematics is that Hector and Bernice both spoke about learning graphs in the future. Hector said that his teacher told him he would be learning about graphs in a few weeks and Bernice thought she would learn about graphs in Grade 5. The students thought they would be learning the same thing in school, but it is likely that their instruction will involve graphs on a coordinate plane rather than tree graphs, which are not part of the elementary school curriculum.

Many studies have documented that children enjoy collaborating in mathematics (e.g., Florez & McCaslin, 2008; Gillies, 2003; Mulryan, 1994). The students in this study did not seem to understand the idea of collaboration at the beginning of the study. They made statements to each other such as "No peeking" and "You have to do it yourself." However, after five sessions during which I repeatedly told the students they could work together, three students began to collaborate. The next day another student switched tables to be with them and commented about working on the mathematics together. This student had been frustrated at her own lack of progress, and I believe she decided to switch tables because she saw the success those students were having by working together.

The second major affective disposition I identified pertained to the students' emotions. The findings of this study match previous research, which has reported that students experience both positive and negative emotions while they are engaged with problem solving (Goldin, 2000a; Hannula, 2015). In the past, the research has focused on students' feeling as expressed through surveys. In this study, I have documented the range of emotions students experienced as they were engaged in solving problems. I have described how the students were feeling and why they were feeling the way they were. Daher (2015) and Evan et al. (2006) stated that there was a link between the emotions students experience and their positioning, such as being authoritative or controlling of the situation when they experienced positive emotions. The findings of this study were similar: Students positioned themselves based on their emotions. When they were frustrated, they would position themselves as inferior and when they were excited or joyful, they would position themselves as an expert or superior.

Conative. The students in this study were very willing to work on the Graceful Tree Conjecture. They persisted on the same problem for six full problem-solving sessions, and after I introduced a new problem, they asked to return to the Graceful Tree Conjecture instead. Overall, the Graceful Tree Conjecture and how the problem-solving sessions were set up, allowed the students to show perseverance throughout the whole study.

I did not ask questions or determine whether the students in this study had a growth mindset or a fixed mindset. But according to Hong et al. (1999), a person with a fixed mindset would view working on a challenging task as a waste of time and a person with a growth mindset would have a tendency to persevere throughout a challenging task. The students in my study did persevere through seven problem-solving sessions to continue to work on an unsolved problem. I interpret this as an indication that children in Grades 4 and 5 can persevere through unsolved problems no matter what their mindset is. We know that students' enjoyment of mathematics tends to decline during middle school and to keep that sense of enjoyment they need to have positive experiences in mathematics (Middleton & Jansen, 2011). This study was an overall positive experience for the students. Also, these types of problems could help students maintain a positive view about mathematics, teach the students about perseverance, and help the students develop a growth mindset. We know that to develop students who are persistent and flexible problem solvers they need to be challenged (Suh et al., 2011) and the Graceful Tree Conjecture appeared to do that for the students in this study.

Warshauer (2015) and Zeybek (2016) claimed that research has not documented what productive struggle looks like. A major finding of this study is that students demonstrated productive struggle through the interplay of frustration and joy. Students experienced repeated frustration as they struggled through a problem, and when they found a graceful labeling, they demonstrated joy and pride in their work. At times the students moved from productive struggle to unproductive struggle (e.g., in Session 3); however, when I changed the activity slightly and allowed the students to explain their solutions and strategies and work together, they became more productive in their struggle. As Kapur (2010) suggested, students should be allowed to discuss, explain, and compare their solutions to other students.

In-The-Moment Identities

A positional identity develops over the long-term based on students' social positions (Holland et al., 1998), and it has been documented that identities can shift or change (e.g., Andersson, 2015; Cobb et al., 2009; Wood, 2013). These shifts or changes can be seen in one's in-the-moment identity, which are the dispositions students have towards themselves based on what they believe they can achieve, the emotions they experience, and how they react to different situations.

Becca experienced many shifts in her in-the-moment identities. In the first two sessions she attended, the other students continually put down Becca. She was quiet and reserved. She did not engage in many conversations and decided to quit the study for the day during the third session. Her social position in the study tenuous and she was not very confident or outgoing. In the fourth problem-solving session, Becca remained quiet but began to participate in the discussions and even showed signs of being an authoritarian. However, in the fifth, sixth, and seventh problem-solving sessions this all changed for Becca. She displayed social positions as an expert, authoritarian, controller of the problem-solving, and at times acted superior. She told the other students what to do, saying, "Copy it down." She seemed to believe that she was good at solving the problems; exhibited positive, happy emotions; and took a leadership role. Becca's in-the-moment identity developed throughout the problem-solving sessions from being timid and shy to being a leader.

Bernice's in-the-moment identities did not shift the same way Becca's did. She was confident when she entered the first session and left the last session still being confident. Bernice did enact many different social positions; she positioned herself as an expert, authoritative, and superior throughout most of the sessions. However, at times she also lowered her position by being vocal when she did not know an answer or solution. She experienced many different emotions while engaging with the problems and displayed signs of productive struggle throughout all of the sessions.

The findings of this study resonate with Wood's (2013) claim that in-the-moment identities can shift and change when students are placed in different contexts. Both Becca and Bernice displayed different in-the-moment identities throughout the sessions based on the situation they were placed in. When Becca found success with her pattern, her in-the-moment identity shifted to become authoritative and she began controlling the other students at her table. Becca and Bernice both displayed multiple in-the-moment identities as exhibited through their dispositions, actions, and emotions during their engagement with unsolved mathematics problems.

Unsolved Problems

The topic of having elementary students engage in unsolved problems has been recommended by multiple sources (e.g., Frenzel, Pekrun, Dicke, & Goetz, 2012; Hamiton & Saarnio, n.d.; Schoenfeld, 1992; Patchter, 2015), but it had not been researched. However, the findings of this study were similar to ideas from a blog written by Braun (2015). He assigned university students different unsolved problems and noticed that his students understood the problems and experienced different feelings of joy and frustration while they worked on the problems. The students that participated in this study were also able to understand the problem I posed and exhibited multiple emotions. However, there was one major difference between these elementary students and Braun's college students. The elementary students either believed they could find the answer or did not completely understand that no one knew the answer. Braun reported that the college students had no expectation that they would be able to solve the problems they were assigned.

Limitations

Every study has limitations and this study had several. The first major limitation was that the exploration of unsolved problems only lasted through seven problem-solving sessions over a period of three weeks. If students had greater exposure to these types of problems, they may have developed stronger ideas of about the nature of mathematics and their dispositions towards mathematics may have changed. Seven, 40-minute sessions across three weeks was not a lot of time for students to develop a different perspective of mathematics; however, this was not a study about changing dispositions but rather to describe what dispositions students displayed while they engaged with the unsolved problems. Even with the short time the students explored the problems, they were able to make sense of the tasks posed, make progress on producing graceful labels for tree graphs, and develop generalizable arguments about tree graphs and the Graceful Tree Conjecture. However, because of often chaotic setting of the after-school program, there was not enough time to discuss patterns and there was a lack of summary discussions. Having the students discuss their findings would have benefited the study.

A second limitation of the study was that it only involved 10 students who attended the same after-school program. Although the students attended different schools, including laboratory, private, and public schools, all of the students were from the same town and spent 3 hours together everyday after school. Although these students chose to join the study, attend the sessions, and engaged in the exploration of the problems, it cannot be concluded that all students would make this choice.

Although the next three ideas may not be considered limitations, they are things that I believe would have made this a better study. The first was that I did not collect any data to measure mathematical gains or progress the students may have made through their participation in the study. This was not the purpose of the study, but it would have been beneficial to learn about mathematical progress that the students may have experienced. The second idea that would have made this better study would have been to speak with the students' classroom teachers during and after the problem-solving sessions. It would have been helpful to know if any student went to school and told their teacher about tree graphs or the experience they had with unsolved problems. One student said his teacher told them they would be learning about graphs soon (probably the typical graphs students learn about in the elementary grades), and it would be interesting to hear what type of conversation that student might have brought to a classroom after exploring graph theory and their ability to persevere in other contexts before verses after the study. The last thing that I believe would have made this stronger study would have been a retention interview after several months. It would be informative to see if the ideas the students learned about mathematics transferred several months later or with time to reflect on the sessions, their views or understanding of mathematics were affected several weeks or months later.

Looking back at the study, I believe there are several limitations from the problem-solving sessions that I would have done different in hindsight. First, I would have encouraged more discussion and sharing of patterns between the students as a whole group. Second, I would have not switched the unsolved problem during the last session. I would have instead spent the time having the students reflect on what they learned and wrapped up all of the sessions. Last, I would have sent students home with graphs to work on, such as lobster graphs. Lobster graphs are a class of graphs that have not been generalized. All of the cases of graphs that the students worked on gracefully labeling have already been generalized.

Implications for Teaching

The findings of this study contribute to the current research literature on mathematics problem solving (e.g., Cai, 2003; Lester 1994; Schoenfeld, 1992), but through a different context, outside of a school setting, and goes beyond students solving problems in which the answers are already known. The results of this study also have implications for anyone who is interested in helping students develop positive and productive dispositions towards mathematics.

This study could potentially influence teachers and curriculum developers to use unsolved problems for upper elementary and middle school students to help them understand what mathematics truly is. These types of problems can teach students that mathematics is not only addition, subtraction, multiplication, and division but about the exploration of patterns and relationships. Unsolved problems might not be part of the curriculum standards but they match many of the recommendations NCTM (2014) has made for effective teaching and learning, including, reasoning and problem solving, purposeful questions, productive struggle, and meaningful mathematical discourse. Also, this study allowed students to create generalizable arguments and not just appeal to authority for labeling tree graphs, which creating generalizable arguments is a missing part of many elementary school classrooms (Carpenter et al., 2003). This is critical for students. In this study, the students had a hard time even articulating their patterns and trying to explain how their patterns worked. These types of problems could help children develop the language needed to articulate their thinking and this needs to be a bigger portion of mathematics classes.

According to Leann, the students in the study were not the top students in their classrooms. Many of the fourth and fifth grade students did not even have fact recall on subtraction problems such as four minus three and had to use their fingers to figure out the solution. However, these students were able to engage in these problems and make progress on the problems. This implies that these types of unsolved problems are accessible to all fourth and fifth grade students, not only the “top” students in a classroom. With the students working on these problems for only seven short sessions, imagine the potential these types of tasks could have in a more structured instructional setting and over a longer time period.

A significant finding of this study pertains to the interplay between frustration and joy that the students experienced. I have interpreted this as an illustration of the construct of productive struggle. The study documented that students need time to work through a problem. Many times teachers step in to help a child because they are struggling but as demonstrated with this study, all of the students persisted through the struggle and found joy and pride in their work. Another important aspect I learned about productive struggle is that when the student has entered unproductive struggle, to change that struggle back to productive, all they need is the problem to be adjusted or have them work in groups as demonstrated between Sessions 3 and 4.

Teachers and curriculum developers could use unsolved problems as a way for students to develop positive and productive dispositions towards mathematics. The unsolved problems could be a task that students work on when they finish their work, problems that are explored throughout the whole year, or something the students work on once a week. The exploration of unsolved problems could also be used in after-school programs for students.

Recommendations for Future Research

There are many questions raised by this exploratory study that indicate a need for further studies focused on unsolved problems. First, it would be beneficial to repeat this study with older children, maybe fifth-, sixth-, and seventh-grade to see if they can create more generalizable arguments about labeling tree graphs and develop formal proofs about whether a graph can be labeled gracefully or not. Second, analyzing students' discourse and changes that may occur while they engage in unsolved problems would be advantageous to document because I believe through exploring unsolved problems, students are able to advance their written and spoken language surrounding mathematics, such as developing an idea of how to form a generalized argument. A third recommendation would be to use unsolved problems in a classroom setting—because this study was done outside a school setting—to examine students' dispositions while they engage in these types of problems in a more structured learning environment. Fourth, there is a need for assessments to document the mathematical progress students experience when they work on unsolved mathematics problems. The last recommendation would be to repeat a similar study but for a longer period of time to document what might happen with students' dispositions, discourse, proofs, and mathematical achievement.

Closing Thoughts

Students' views of mathematics have been documented to be limited and narrow. I wanted to explore a way to expand those views and allow children to experience mathematics similar to how a mathematician does. I wanted to explore elementary students' dispositions toward mathematics and unsolved problems seemed like an interesting context to do this. However, it was an idea that made me nervous because I have specialized in elementary school mathematics and did not know a lot about different unsolved problems. When Dr. El-Zanati showed me the Graceful Tree Conjecture, I knew it was the perfect problem and one that could allow elementary children to experience mathematics as more than simple calculations.

Although each problem-solving session was exciting and challenging, the students continued to work and struggle through each graph I gave them. And, thankfully, they agreed to come back to the next session. The progress these students made was outstanding, and for them to engage in graph theory and attempt to make generalizable arguments all while having productive dispositions towards mathematics was more than I could ask for. After completing this study, I believe all students would benefit from the opportunity to explore unsolved problems.

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APPENDIX A: CONSENT FORM

Dear Parents or Guardians of Elementary Students at Midwestern After School Program:

My name is Jenna O'Dell and I am a doctoral student at Illinois State University studying mathematics education. Having taught elementary school for six years, I returned to graduate school to learn about how to prepare upper elementary school students for success in their future mathematics courses. I would like permission for your child to participate in my dissertation research project, which has been approved by Midwestern After School Program. The project, Beyond Problem Solving, will study how children view and learn mathematics while they engage with mathematics problems that have never been solved. This letter contains a description of the project followed by permission forms for you and your child to sign.

Problem solving in mathematics is important for children. We want children to learn how to make sense of mathematical problems, solve mathematics problems without giving up, and learn to be flexible and resourceful problem solvers. Mathematicians have very positive views of mathematics and we want student to develop views towards mathematics that are similar. The aim of my research is to examine students' views of mathematics as they engage with exciting and challenging mathematics problems.

Cynthia Langrall, a mathematics education professor in the Department of Mathematics at Illinois State University, will also help with this project. Dr. Langrall is directing my dissertation study. She has a wide range of research and teaching experience, including teaching at the elementary school level.

We are seeking your permission to (a) work with your child during 10 video-recorded problem-solving sessions and (b) conduct two video-recorded interviews with your child. For the problem-solving sessions, students will work collaboratively in small groups on mathematics problems or tasks. Each problem-solving session will be about 45 minutes and the individual interviews will be about 15 minutes. The purpose of the interviews is to ask students questions about mathematics, if they are interested in mathematics, and what they think mathematics should look like in a classroom. There are no right or wrong answers to any questions; our interest is only in how students think and feel about mathematics. The interviews and problem-solving sessions will occur at a time approved by the program leaders so your child will not miss their reading time or any special activity. We anticipate the study to occur in late October through early December.

Additionally, if you agree to allow your child to be part of the problem-solving sessions, please consider giving us permission to use selected video clips in conference presentations of the research. Note that this could be a potential loss of confidentiality because your child could be identified by those viewing the video; however, these video clips would be available for your preview should you wish to see them prior to giving your approval. If you prefer that you child not be included in video recording that might be included in conference presentations, please use the attached consent form to indicate that. In any publications associated with this research, pseudonyms will be used in place of your child's name and the name of the after school program.

The risks of participating in this research project are no greater than the risks associated with everyday life or what your child experiences in school. The most likely risks your child would be exposed to involve the challenge of answering mathematics questions where the answers are not known and your child may feel some discomfort being video recorded. If being video recorded makes your child feel uncomfortable, they may withdraw from the study at anytime without any penalty. However, our experiences are that children soon ignore the video recorder and enjoy engaging in these unsolved problems. Research has shown that such experiences can be empowering and increase positive views of mathematics.

If you choose not to have your child participate in any aspects of this study or if you choose to withdraw your permission at any time, there will be no penalty. Participation in the project will not affect your child's participation in the after-school program. Likewise, if your child chooses not to give permission or to withdraw his/her permission at any time, there will be no penalty.

If you have any questions concerning the research study or your child's participation in this study, please call Jenna O'Dell at xxx-xxx-xxxx or contact Cynthia Langrall at xxx-xxx-xxxx. If you have any questions about you or your child's rights as a participant in this research, or if you feel you or your child have been placed at risk, you can contact the Research Ethics & Compliance Office at Illinois State University at xxx-xxx-xxxx. To give consent for your child to participate in this study, please complete that attached form and have your child return it to their program leader in the enclosed envelope. Also, please talk with your child about the study and have him or her complete the student assent form and return it in the same envelope.

Sincerely,

Cynthia Langrall
Professor of Mathematics Education
Department of Mathematics
Illinois State University
Normal, IL 61761
(xxx) xxx-xxxx

Jenna O'Dell
Doctoral student at Illinois State University
Department of Mathematics
Illinois State University
Normal, IL 61761
(xxx) xxx-xxxx

Please complete these forms and return to Midwestern After School Program in the enclosed envelope.

Parent/Guardian Consent Form

I have read the information presented above and have had an opportunity to ask questions and receive answers pertaining to this research project. I am aware that my permission is voluntary and that I am free to withdraw my permission at any time without any penalties to my child or me.

I **give permission** for Jenna O'Dell and Cynthia Langrall to: (please check all that apply)

______ Conduct two video-recorded interviews with my child

______ Conduct no more than ten problem-solving sessions with my child

______ Use short video clips that include my child during conference presentations

______ I **do not give permission** for my child to participate in any aspect of the data collection for this study.

____
Child's Name

____
Signature of Parent/Guardian

APPENDIX B: ASSENT FORM

Dear Student,

I would like you to consider participating in the study your parent talked to you about (and which is described in the letter included with this form). Your participation will involve talking with my research partner and me about math and doing some math problem solving with other students. We plan to talk with you about your feelings about math in two 15-minute interviews during the study. There will be about 10 problem-solving sessions that will be 45-minutes long during the homework time of your after-school program. In the problem-solving sessions you will be working in small groups on interesting math problems. We will be video recording the problem-solving sessions and interviews. You might feel uncomfortable being video recorded and if it bothers you too much you can withdraw from the study. Also, you might feel challenged by some of the math problems, but we are not concerned about right or wrong answers and other children we have worked with have enjoyed these challenging problems. In fact, you might be surprised how much fun you have working like a mathematician!

Your participation is always voluntary and if you choose not to participate in the study, there will be no negative consequences and it will not affect your participation in the after-school program. Please know that if you give us permission to use any of your work or share any videos we will change your name (rather than using your real name we will use a pseudonym) or make sure your name cannot be heard on the video.

Please complete the checklist below, place it in the envelope and return.

Sincerely,
Jenna O'Dell
Doctoral Student

I **give permission** for Jenna O'Dell and Cynthia Langrall to: (please check all that apply)

______ Conduct video-recorded interviews with me

______ Include me in videoed problem-solving sessions

______ Use short video clips that include me during conference presentations

______ I **do not agree** to participate in any aspect of the data collection for this study.

____

Student's Name

____

Signature of Student

APPENDIX C: POSITIONING TABLES

Who Positioned Alia

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	5	16	13	-	13	8	2
Student	1	2	1	-	3		
Teacher	1	2	2	-	4	2	1

Note. - means student was absent

Alia's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	1	1	3	-	4	2	
Expert	4	10	7	-	12	5	3
Superior	1	2	1	-			
Authoritarian	1		1	-			
Controlling		7	4	-	4		
Collaborator				-			
Encouraging				-		1	

Note. - means student was absent

Who Positioned Amanda

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	2	1	1	1	1	3	2
Student	1	3				3	
Teacher	1		3	1			

Amanda's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	2	2		1		2	2
Expert	2	1	3			2	
Superior							
Authoritarian			1			1	
Controlling				1			
Collaborator							
Encouraging		1			1	1	

Who positioned Becca

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	-	2	3	7	6	25	8
Student	-	9	1	1	1	5	1
Teacher	-	1	1		4	4	1

Note. - means student was absent

Becca's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	-	9	4	5		1	2
Expert	-	3	1	2	10	20	5
Superior	-				1	4	
Authoritarian	-			1		3	1
Controlling	-					2	
Collaborator	-					3	
Encouraging	-					1	

Note. - means student was absent

Who Positioned Bernice

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	11	9	6	21	6	13	5
Student	2	2		1		1	
Teacher	4	2		4	4	3	3

Bernice's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	2	2	3	4		3	3
Expert	10	9	2	12	8	6	3
Superior	1			1	1		1
Authoritarian	3	2	1	4	1	8	
Controlling	1			5			
Collaborator							
Encouraging							1

Who Positioned Edward

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	3	2	2	8	2	2	3
Student	2			1			
Teacher		1	1			3	

Edward's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	1	1	1	1	1	1	1
Expert	4	2	1	2	1	2	1
Superior				4			
Authoritarian							1
Controlling			1	2			
Collaborator							
Encouraging						2	

Who Positioned Hector

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self		-	1	-	3	6	-
Student		-	1	-	1		-
Teacher	1	-	1	-		2	-

Note. - means student was absent

Hector's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior		-	2	-	2	3	-
Expert	1	-	1	-	1	3	-
Superior		-		-	1		-
Authoritarian		-		-		2	-
Controlling		-		-			-
Collaborator		-		-			-
Encouraging		-		-			-

Note. - means student was absent

Who Positioned Iris

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	1	4	4	7	5	1	7
Student		2		2		1	
Teacher	1	1				1	

Iris's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	2	3	1	4	2	1	1
Expert		1	3	4	3	2	3
Superior		1					1
Authoritarian							2
Controlling		2					
Collaborator							
Encouraging				1			

Who Positioned Joella

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	5	10		5	3	5	4
Student		2		2		2	
Teacher		1	4	4	1	2	

Joella's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	2	4	2	8	2	8	3
Expert	2	3	2	1	2	1	1
Superior				1			
Authoritarian		1					
Controlling	1	3		1			
Collaborator							
Face-saving		2					
Encouraging							

Who Positioned Karly

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	1	-	2	-	2	7	-
Student		-		-		2	-
Teacher	1	-		-	2	1	-

Note. - means student was absent

Karly's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior		-		-	1	3	-
Expert	1	-	1	-	3	5	-
Superior		-		-			-
Authoritarian		-	1	-		1	-
Controlling	1	-		-			-
Collaborator		-		-			-
Encouraging		-		-		1	-

Note. - means student was absent

Who Positioned Trevor

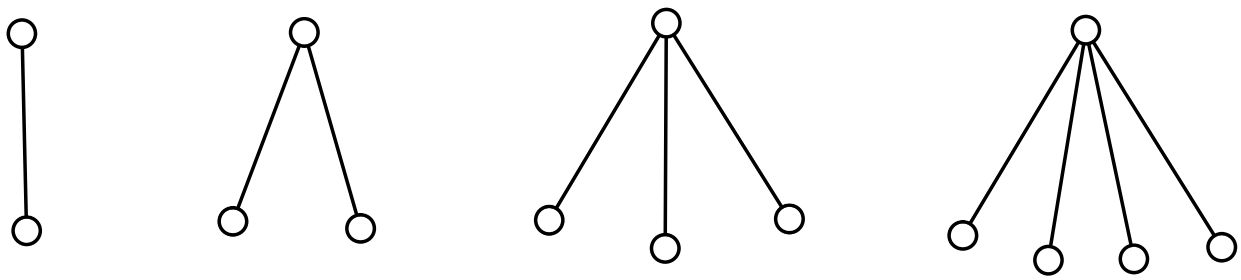
	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Self	5	19	3	2	2	3	10
Student		1	2				
Teacher	2	3				1	

Trevor's Positioning

	Session 1	Session 2	Session 3	Session 4	Session 5	Session 6	Session 7
Inferior	1	2	2			1	7
Expert	5	9	2	2	1	3	2
Superior		1					
Authoritarian	1	2			1		1
Controlling		9	1				
Collaborator							
Encouraging							

APPENDIX D: TASK PAGES

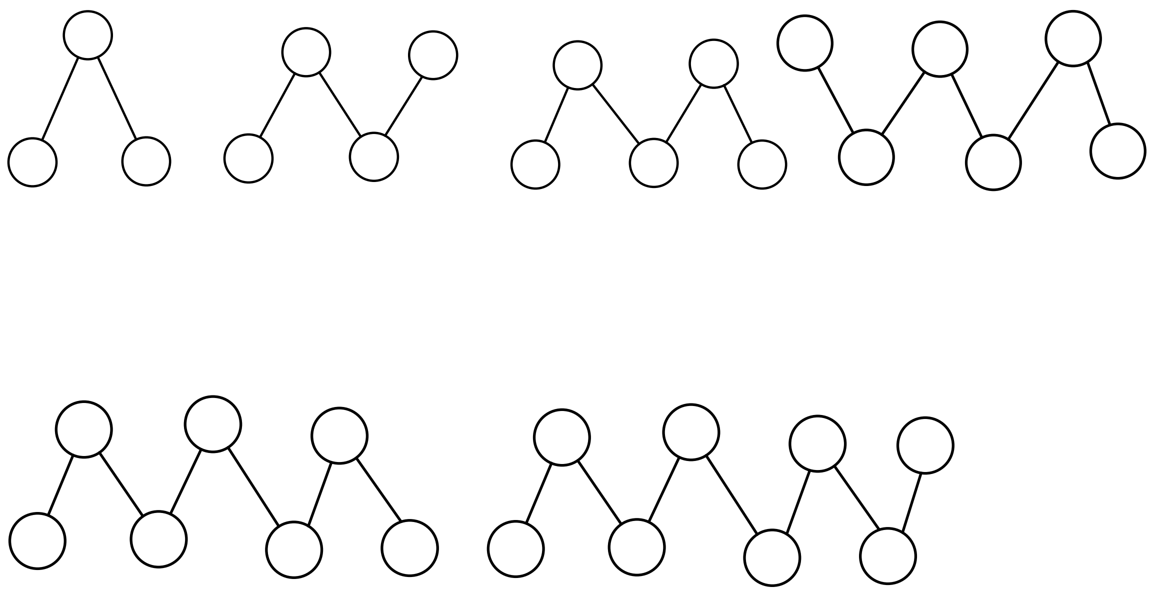
All of the graphs below are in the same class—the star class. Can you label all of the graphs gracefully?



Draw and produce a graceful labeling for the next graph in this class.

Describe how you would label any graph in this class.

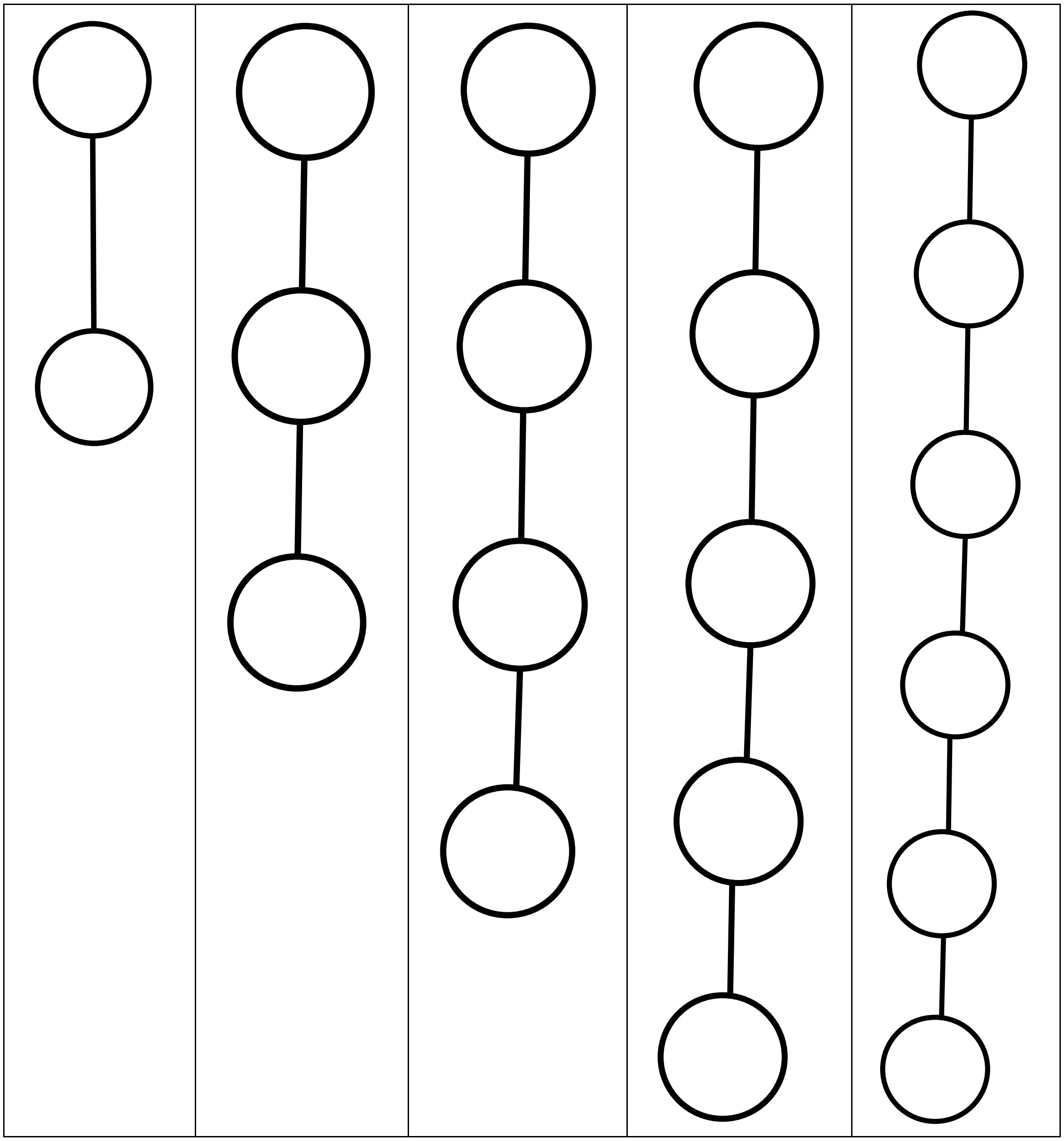
All of the graphs below are in the same class—the path class. Can you label all of the graphs gracefully?



Draw and produce a graceful labeling for the next graph in this class.

Describe how you would label any graph in this class.

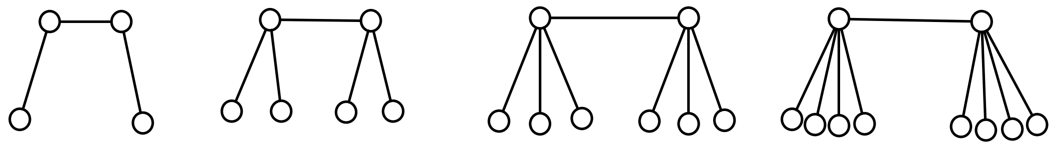
Path Graphs



What is the pattern you have found for labeling line graphs?

Describe how you would label any graph in this class.

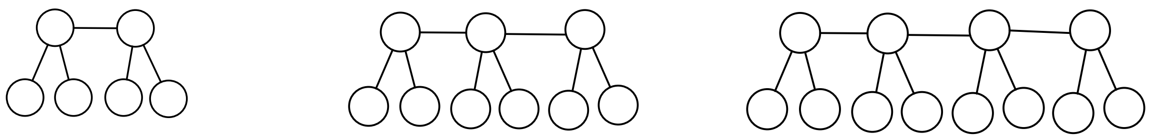
All of the graphs below are in the same class—the caterpillar class. Can you label all of the graphs gracefully?



Draw and produce a graceful labeling for the next graph in this class.

Describe how you would label any graph in this class.

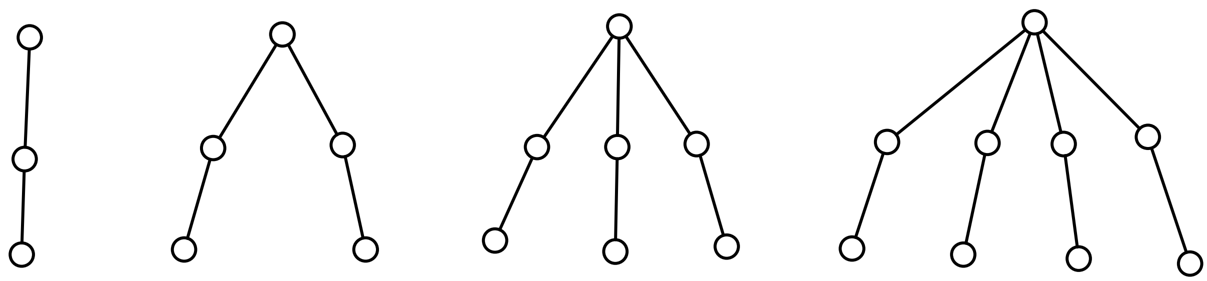
All of the graphs below are in the same class—the caterpillar class. Can you label all of the graphs gracefully?



Draw and produce a graceful labeling for the next graph in this class.

Describe how you would label any graph in this class.

All of the graphs below are in the same class—the comet class. Can you label all of the graphs gracefully?



Draw and produce a graceful labeling for the next graph in this class.

Describe how you would label any graph in this class.