TREE-DEPTH AND VERTEX-MINORS

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ABSTRACT. In a recent paper [6], Kwon and Oum claim that every graph of bounded rank-width is a pivot-minor of a graph of bounded tree-width (while the converse has been known true already before). We study the analogous questions for "depth" parameters of graphs, namely for the tree-depth and related new shrub-depth. We show that shrub-depth is monotone under taking vertex-minors, and that every graph class of bounded shrub-depth can be obtained via vertex-minors of graphs of bounded tree-depth. We also consider the same questions for bipartite graphs and pivot-minors.

1. INTRODUCTION

Various notions of graph containment relations (e.g. graph minors) play an important part in structural graph theory. Recall that a graph H is a minor of a graph G if H can be obtained from G by a sequence of edge contractions, edge deletions and vertex deletions. In their seminal series of papers, Robertson and Seymour introduced the notion of tree-width and showed the following: The tree-width of a minor of G is at most the tree-width of G and, moreover, for each k there is a finite list of graphs such that a graph G has tree-width at most k if, and only if, no graph in the list is isomorphic to a minor of G. This, among other things, implies the existence of a polynomial-time algorithm to check that the tree-width of a graph is at most k.

There have been numerous attempts to extend this result to (or find a similar result for) "width" measures other than tree-width . The most natural candidate is clique-width, a measure generalising tree-width defined by Courcelle and Olariu [2]. However, the quest to prove a similar result for this measure has been so far unsuccessful. For one, taking the graph minor relation is clearly not sufficient as every graph on $n > 1$ vertices is a minor of the complete graph K_n, clique-width of which is 2.

However Oum [8] succeeded in finding the appropriate containment relation – called *vertex-minor* – for the notion of rank-width, which is closely related to clique-width. (More precisely, if the clique-width of a graph is k, then its rank-width is between $\log_2(k+1) - 1$ and k.) Vertex-minors are based on the operation of local complementation: taking a vertex v of a graph G we replace the subgraph induced on the neighbours of v by its edge-complement, and denote the resulting graph by $G * v$. We then say that a graph H is a vertex-minor of G if H can be obtained from G by a sequence of local complementations and vertex deletions. In [8] it was shown that if H is a vertex-minor of G, then its rank-width is at most the rank-width of G.

Another graph containment relation, the *pivot-minor*, also defined in [8], is closely related to vertex-minor. Pivot-minors are based on the operation of edge-pivoting: for an edge $e = \{u, v\}$ of a graph G we perform the operation $G * u * v * u$. Then a graph H is a pivot-minor of G if it can be obtained from G by a sequence of edge-pivotings and vertex deletions. It follows from the definition that every pivot-minor is also a vertex-minor.

This brings an interesting question: What is the exact relationship between various width measures with respect to these new graph containment relations? Recently, it was shown that every graph of rank-width k is a pivot-minor of a graph of tree-width at most $2k$ [6]. In this paper

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we investigate the existence of similar relationships for two "shallow" graph width measures: tree-depth and shrub-depth.

Tree-depth [7] is a graph invariant which intuitively measures how far is a graph from being a star. Graphs of bounded tree-depth are sparse and play a central role in the theory of graph classes of bounded expansion. *Shrub-depth* [4] is a very recent graph invariant, which was designed to fit into the gap between tree-depth and clique-width. (If we consider tree-depth to be the "shallow" counterpart of tree-width, then shrub-depth can be thought of as a "shallow" counterpart of clique-width.)

Our results can be summarised as follows. We start by showing that shrub-depth is monotone under taking vertex-minors (Corollary 3.6). Next we prove that every graph class of bounded shrub-depth can be obtained via vertex-minors of graphs of bounded tree-depth (Theorem 4.4). Note that, unlike for rank-width and tree-width, restricting ourselves to pivot-minors is not sufficient. Indeed, this is because, as we prove in Proposition 4.7, graphs of bounded tree-depth cannot contain arbitrarily large cliques as pivot-minors. Interestingly, we are however able to show the same result for pivot-minors if we restrict ourselves to bipartite graphs, which were, in a similar connection, investigated already in [6]. In particular, our main result of the last section is that for any class of bounded shrub-depth there exists an integer d such that any bipartite graph in the class is a pivot-minor of a graph of tree-depth d.

2. Preliminaries

In this paper, all graphs are finite, undirected and simple. A *tree* is a connected graph with no cycles, and it is *rooted* if some vertex is designated as the root. A leaf of a rooted tree is a vertex other than the root having just one neighbour. The height of a rooted tree is the maximum length of a path starting in the root (and hence ending in a leaf). Let G be a graph. We denote $V(G)$ as the vertex set of G and $E(G)$ as the edge set of G. For $v \in V(G)$, let $N_G(v)$ be the set of the neighbours of v in G.

We sometimes deal with *labelled graphs* G, which means that every vertex of G is assigned a subset (possibly empty) of a given finite label set. A graph is *m-coloured* if every vertex is assigned exactly one of given m labels (this notion has no relation to ordinary graph colouring).

We now briefly introduce the *monadic second order logic* (MSO) over graphs and the concept of FO (MSO) graph interpretation. MSO is the extension of first-order logic (FO) by quantification over sets, and comes in two flavours, MSO_1 and MSO_2, differing by the objects we are allowed to quantify over:

Definition 2.1 (MSO_1 logic of graphs)**.** *The language of* MSO_1 *consists of expressions built from the following elements:*

- *variables $x, y, \dots$ for vertices, and X, Y for sets of vertices,*
- *the predicates $x \in X$ and $\mathrm{edge}(x, y)$ with the standard meaning,*
- *equality for variables, quantifiers $\forall$ and $\exists$ ranging over vertices and vertex sets, and the standard Boolean connectives.*

MSO_1 logic can be used to express many interesting graph properties, such as 3-colourability. We also mention MSO_2 logic, which additionally includes quantification over edge sets and can express properties which are not MSO_1 definable (e.g. Hamiltonicity). The large expressive power of both MSO_1 and MSO_2 makes them a very popular choice when formulating algorithmic metatheorems (e.g., for graphs of bounded clique-width or tree-width, respectively).

The logic we will be mostly concerned with is an extension of MSO_1 called *Counting monadic second-order logic* (CMSO_1). In addition to the MSO_1 syntax CMSO_1 allows the use of predicates $\mathrm{mod}_{a,b}(X)$, where X is a set variable. The semantics of the predicate $\mathrm{mod}_{a,b}(X)$ is that the set X has a modulo b elements. We use $\mathrm{C_2MSO_1}$ to denote the parity counting fragment of CMSO_1, i.e. the fragment where the predicates $\mathrm{mod}_{a,b}(X)$ are restricted to $b = 2$.

A useful tool when solving the model checking problem on a class of structures is the ability to "efficiently translate" an instance of the problem to a different class of structures, for which we already have an efficient model checking algorithm. To this end we introduce simple FO/MSO_1

graph interpretation, which is an instance of the general concept of interpretability of logic theories [10] restricted to simple graphs with vertices represented by singletons.

Definition 2.2. *A* FO (MSO_1) graph interpretation *is a pair $I = (\nu, \mu)$ of* FO *(*MSO_1*) formulae (with 1 and 2 free variables respectively) in the language of graphs, where μ is symmetric (i.e. $G \models \mu(x,y) \leftrightarrow \mu(y,x)$ in every graph G). To each graph G it associates a graph G^I, which is defined as follows:*

- *The vertex set of G^I is the set of all vertices v of G such that $G \models \nu(v)$;*
- *The edge set of G^I is the set of all the pairs $\{u,v\}$ of vertices of G such that $G \models \nu(u) \wedge \nu(v) \wedge \mu(u,v)$.*

This definition naturally extends to the case of vertex-labelled graphs (using a finite set of labels, sometimes called colours) by introducing finitely many unary relations in the language to encode the labelling.

For example, a complete graph can be interpreted in any graph (with the same number of vertices) by letting $\nu \equiv \mu \equiv \mathit{true}$, and the complement of a graph has an interpretation using $\mu(x,y) \equiv \neg\,\mathrm{edge}(x,y)$.

Vertex-minors and Pivot-minors. For $v \in V(G)$, the *local complementation* at a vertex v of G is the operation which complements the adjacency between every pair of two vertices in $N_G(v)$. The resulting graph is denoted by $G*v$. We say that two graphs are *locally equivalent* if one can be obtained from the other by a sequence of local complementations. For an edge $uv \in E(G)$, *pivoting* an edge uv of G is defined as $G \wedge uv = G*u*v*u = G*v*u*v$. A graph H is a *vertex-minor* of G if H is obtained from G by applying a sequence of local complementations and deletions of vertices. A graph H is a pivot-minor of G if H is obtained from G by applying a sequence of pivoting edges and deletions of vertices. From the definition of pivoting every pivot-minor of a graph is also its vertex-minor.

Pivot-minors of graphs are closely related to a matrix operation called pivoting. To give the exact relationship (Proposition 2.5) we will need to introduce some matrix concepts.

Pivoting on a Matrix. For two sets A and B, we denote by $A\Delta B = (A \setminus B) \cup (B \setminus A)$ its symmetric difference. Let $\boldsymbol{M}$ be a $S \times T$ matrix. For $A \subseteq S$ and $B \subseteq T$, we denote the $A \times B$ submatrix of $\boldsymbol{M}$ as $\boldsymbol{M}[A,B] = (m_{i,j})_{i\in A, j\in B}$. If $A = B$, then $\boldsymbol{M}[A] = \boldsymbol{M}[A,A]$ and we call it a *principal submatrix* of $\boldsymbol{M}$. If $a \in S$ and $b \in T$, then we denote $\boldsymbol{M}_{a,b} = \boldsymbol{M}[\{a\},\{b\}]$. The *adjacency matrix* $\boldsymbol{A}(G)$ of G is the $V(G) \times V(G)$ matrix such that for $v, w \in V(G)$, $\boldsymbol{A}(G)_{v,w} = 1$ if v is adjacent to w in G, and $\boldsymbol{A}(G)_{v,w} = 0$ otherwise.

Let

$$\boldsymbol{M} = \begin{array}{c} \\ S \\ X\setminus S \end{array} \begin{array}{c} \begin{array}{cc} S & X\setminus S \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$$

be a $X \times X$ matrix over a field F.

If $\boldsymbol{A} = \boldsymbol{M}[S]$ is non-singular, then we define *pivoting* S on the matrix $\boldsymbol{M}$ as

$$\boldsymbol{M} * S = \begin{array}{c} \\ S \\ X\setminus S \end{array} \begin{array}{c} \begin{array}{cc} S & X\setminus S \end{array} \\ \begin{pmatrix} A^{-1} & A^{-1}B \\ -CA^{-1} & D - CA^{-1}B \end{pmatrix} \end{array}.$$

It is sometimes called a *principal pivot transformation* [11]. The following theorem is useful when dealing with matrix pivoting.

Theorem 2.3 (Tucker [12])**.** *Let $\boldsymbol{M}$ be a $X \times X$ matrix over a field. If $\boldsymbol{M}[S]$ is a non-singular principal submatrix of $\boldsymbol{M}$, then for every $T \subseteq X$, $(\boldsymbol{M} * S)[T]$ is non-singular if and only if $\boldsymbol{M}[S\Delta T]$ is non-singular.*

Proof. See Bouchet's proof in Geelen [5, Theorem 2.7]. □

Theorem 2.4. *Let $\boldsymbol{M}$ be a $X \times X$ matrix over a field. If $\boldsymbol{M}[S]$ and $(\boldsymbol{M} * S)[T]$ are non-singular, then $(\boldsymbol{M} * S) * T = \boldsymbol{M} * (S\Delta T)$.*

Proof. See Geelen [5, Theorem 2.8]. □

We are now ready to state the relationship between pivot-minors and matrix pivots. The proof of the following proposition uses Theorem 2.3 and Theorem 2.4, and we refer the reader to [6] for detailed explanation.

Proposition 2.5. *Graph H is a pivot-minor of G if and only if H is the graph whose adjacency matrix is $(\boldsymbol{A}(G) * X)[Y]$ where $X, Y \subseteq V(G)$ and $\boldsymbol{A}(G)[X]$ is non-singular.*

Tree-depth. For a forest T, the closure $\mathrm{Clos}(T)$ of T is the graph obtained from T by making every vertex adjacent to all of its ancestors. The *tree-depth* of a graph G, denoted by $\mathrm{td}(G)$, is one more than the minimum height of a rooted forest T such that $G \subseteq \mathrm{Clos}(T)$.

3. Shrub-depth and Vertex-minors

In this section we show the first of our results – that shrub-depth is monotone under taking vertex-minors. The shrub-depth of a graph class is defined by the following very special kind of a simple FO interpretation:

Definition 3.1 (Tree-model [4])**.** *We say that a graph G has a* tree-model of m colours and depth d *if there exists a rooted tree T (of height d) such that:*

i. the set of leaves of T is exactly $V(G)$,
ii. the length of each root-to-leaf path in T is exactly d,
iii. each leaf of T is assigned one of m colours (i.e. T is m-coloured),
iv. and the existence of an edge between $u, v \in V(G)$ depends solely on the colours of u, v and the distance between u, v in T.

The class of all graphs having such a tree-model is denoted by $\mathcal{TM}_m(d)$.

For example, $K_n \in \mathcal{TM}_1(1)$ or $K_{n,n} \in \mathcal{TM}_2(1)$. We thus consider:

Definition 3.2 (Shrub-depth [4])**.** *A class of graphs $\mathscr{S}$ has* shrub-depth d *if there exists m such that $\mathscr{S} \subseteq \mathcal{TM}_m(d)$, while for all natural m it is $\mathscr{S} \not\subseteq \mathcal{TM}_m(d-1)$.*

It is easy to see that each class $\mathcal{TM}_m(d)$ is closed under complements and induced subgraphs, but neither under disjoint unions, nor under subgraphs. However, the class $\mathcal{TM}_m(d)$ is not closed under local complementations. On the other hand, to prove that shrub-depth is closed under vertex-minors it is sufficient to show that for each m there exists m' such that all graphs locally equivalent to those in $\mathcal{TM}_m(d)$ belong to $\mathcal{TM}_{m'}(d)$. As shrub-depth does not depend on m, this will be our proof strategy. Note that Definition 3.2 is asymptotic as it makes sense only for infinite graph classes; the shrub-depth of a single finite graph is always at most one. For instance, the class of all cliques has shrub-depth 1. More interestingly, graph classes of certain shrub-depth are characterisable exactly as those having simple CMSO_1 interpretations in the classes of rooted labelled trees of fixed height:

Theorem 3.3 ([4, 3])**.** *A class $\mathscr{S}$ of graphs has a simple* CMSO_1 *interpretation in the class of all finite rooted labelled trees of height $\leq d$ if, and only if, $\mathscr{S}$ has shrub-depth at most d.*

Proof sketch. In [4] this statement occurs with a little shift—involving MSO_1 logic instead of CMSO_1. However, since the proof in [4] builds everything on one technical claim (kernelization of MSO on trees of bounded height) which has been subsequently extended to CMSO in [3, Section 3.2], the full statement follows as well. □

Note that the above theorem implies that any class of graphs of bounded shrubdepth is closed under simple CMSO_1 interpretations, i.e., the class of graphs obtained via a simple CMSO_1 interpretation on a class of graphs of bounded shrub-depth has itself bounded shrub-depth. This is one of the two essential ingredients we need to prove that shrub-depth is closed under vertex-minors. The other ingredient is the following technical claim:

Lemma 3.4 (Courcelle and Oum [1])**.** *For a graph G, let $\mathscr{L}(G)$ denote the set of graphs which are locally equivalent to G. Then there exists a simple* $\mathrm{C_2MSO_1}$ *interpretation such that each such $\mathscr{L}(G)$ is interpreted in vertex-labellings of G.*

Proof sketch. Again, [1, Corollary 6.4] states nearly the same what we claim here. The only trouble is that [1] speaks about more general so-called transductions. Here we briefly survey that the transduction constructed in [1, Corollary 6.4] is really a simple C_2MSO_1 interpretation (we have to stay on an informal level since a formal introduction to all necessary concepts would take up several pages):

i. In [1] local complementations of a graph G are treated via a so called isotropic system $S = S(G)$. It is, briefly, a set of $V(G)$-indexed three-valued vectors, and so S can be described on the ground set $V(G)$ by a collection of triples of disjoint sets. This representation is definable in C_2MSO_1 [1, Proposition 6.2].
ii. The set of graphs locally equivalent to G then corresponds to the set of isotropic systems strongly isomorphic to S. A strong isomorphism of isotropic systems on the ground set $V(G)$ is expressed in MSO_1 with respect to a suitable 6-partition of $V(G)$ by [1, Proposition 6.1].
iii. Finally, a graph H is locally equivalent to G if and only if H is the fundamental graph of some (not unique) $S' \simeq S$ with respect to a special vector of S', which again has a C_2MSO_1 expression with respect to a triple of subsets of $V(G)$ describing the vector (as in point i.) by [1, Proposition 6.3].

Note that all the aforementioned C_2MSO_1 expressions are on the same ground set $V(G)$. In the desired interpretation I we treat the nine parameter sets of (ii.) and (iii.) as a vertex-labelling of G, which consequently can interpret any H locally equivalent to G using C_2MSO_1. □

Theorem 3.5. *For a graph class $\mathscr{C}$, let $\mathscr{L}(\mathscr{C})$ denote the class of graphs which are locally equivalent to a member of $\mathscr{C}$. Then the shrub-depth of $\mathscr{L}(\mathscr{C})$ is equal to the shrub-depth of $\mathscr{C}$.*

Proof. Let d be the least integer such that, for some m as in Definition 3.2, it is $\mathscr{C} \subseteq \mathcal{TM}_m(d)$. Let I denote an FO interpretation of $\mathscr{C}$ in the class $\mathscr{T}_d$ of rooted labelled trees of height d which naturally follows from Definition 3.1, and let J be the simple C_2MSO_1 interpretation from Lemma 3.4.

For every $H \in \mathscr{L}(\mathscr{C})$ there is a suitably labelled graph $G \in \mathscr{C}$ such that $H \simeq G^J$, and a tree $T \in \mathscr{T}_d$ such that $G \simeq T^I$. As this T can additionally inherit any suitable labelling of G, we can claim $H \simeq (T^I)^J$. Therefore, the composition $J \circ I$ is a C_2MSO_1 interpretation of $\mathscr{L}(\mathscr{C})$ in $\mathscr{T}_d$. By Theorem 3.3, $\mathscr{L}(\mathscr{C})$ is of shrub-depth at most d and, at the same time, $\mathscr{C} \subseteq \mathscr{L}(\mathscr{C})$. □

Corollary 3.6. *The shrub-depth parameter is monotone under taking vertex-minors over graph classes.*

Proof. By the definition, a vertex-minor is obtained as an induced subgraph of a locally equivalent graph. Since taking induced subgraphs does not change a tree-model, the claim follows from Theorem 3.5. □

4. From small Tree-depth to small SC-depth

We have just seen that taking vertex-minors does not increase the shrub-depth of a graph class. It is thus interesting to ask whether, perhaps, every class of bounded shrub-depth could be constructed by taking vertex-minors of some special graph class. This indeed turns out to be true in a very natural way—the special classes in consideration are the graphs of bounded tree-depth.

Before proceeding we need to introduce another "depth" parameter asymptotically related to shrub-depth which, unlike the former, is defined for any single graph. Let G be a graph and let $X \subseteq V(G)$. We denote by $\overline{G}^X$ the graph G' with vertex set $V(G)$ where $x \neq y$ are adjacent in G' if either

(i) $\{x, y\} \in E(G)$ and $\{x, y\} \not\subseteq X$, or
(ii) $\{x, y\} \notin E(G)$ and $\{x, y\} \subseteq X$.

In other words, $\overline{G}^X$ is the graph obtained from G by complementing the edges on X.

Definition 4.1 (SC-depth [4])**.** *We define inductively the class $\mathcal{SC}(k)$ as follows:*

i. let $\mathcal{SC}(0) = \{K_1\}$;

ii. if $G_1, \dots, G_p \in \mathcal{SC}(k)$ and $H = G_1 \dot\cup \dots \dot\cup G_p$ denotes the disjoint union of the G_i, then for every subset X of vertices of H we have $\overline{H}^X \in \mathcal{SC}(k+1)$.

The SC-depth *of G is the minimum integer k such that $G \in \mathcal{SC}(k)$.*

Proposition 4.2 ([4]). *The following are equivalent for any class of graphs $\mathscr{G}$:*

- *there exist integers d, m such that $\mathscr{G} \subseteq \mathcal{TM}_m(d)$;*
- *there exists an integer k such that $\mathscr{G} \subseteq \mathcal{SC}(k)$.*

From Definition 4.1, one can obtain the following claim:

Lemma 4.3. *Let k be a positive integer. If a graph G has SC-depth at most k, then G is a vertex-minor of a graph of tree-depth at most $k + 1$.*

Proof. For a graph G of SC-depth k, we recursively construct a graph U and a rooted forest T such that

i. G can be obtained from U as a vertex-minor via applying local complementations only at the vertices in $V(U) \setminus V(G)$, and
ii. $U \subseteq \mathrm{Clos}(T)$ and T has depth k.

If $k = 0$, then it is clear by setting $G = U = T = K_1$. We assume that $k \geq 1$.

Since G has SC-depth k, there exist a graph H and $X \subseteq V(H)$ such that $G = \overline{H}^X$ and H is the disjoint union of $H_1, H_2, \dots, H_m$ such that each H_i has SC-depth $k-1$. By induction hypothesis, for each $1 \leq i \leq m$, H_i is a vertex-minor of a graph U_i and $U_i \in \mathrm{Clos}(T_i)$ where the height of T_i is at most k. For each $1 \leq i \leq m$, let r_i be the root of T_i, and let T be the rooted forest obtained from the disjoint union of all T_i by adding a root r which is adjacent to all r_i. Let U be the graph obtained from the disjoint union of all U_i and $\{r\}$ by adding all edges from r to X. Validity of (ii.) is clear from the construction.

Now we check the statement (i.). By our construction of U, any local complementation in U_i has no effect on U_j for $j \neq i$, and local complementations at vertices in $V(U_i) \setminus V(H_i)$ do not change edges incident with r. Hence, by induction, we can obtain H as a vertex-minor of U and still have r adjacent precisely to $X \subseteq V(H)$. We then apply the local complementation at $r \in V(U) \setminus V(H)$, and delete $V(U) \setminus V(G)$ to obtain G. □

This, with Proposition 4.2, now immediately gives the main conclusion:

Theorem 4.4. *For any class $\mathscr{S}$ of bounded shrub-depth, there exists an integer d such that every graph in $\mathscr{S}$ is a vertex-minor of a graph of tree-depth d.* □

Comparing Theorem 4.4 with [6] one may naturally ask whether, perhaps, weaker pivot-minors could be sufficient in Theorem 4.4. Unfortunately, that is very false from the beginning. Note that all complete graphs have SC-depth 1. On the other hand, we will prove (Proposition 4.7) that graphs of bounded tree-depth cannot contain arbitrarily large cliques as pivot-minors. We need the following technical lemmas.

Lemma 4.5. *Let G be a graph and $X \subseteq V(G)$ such that $\boldsymbol{A}(G)[X]$ is non-singular and $|X| \geq 3$. If $u \in X$, then there exist $v, w \in X \setminus \{u\}$ such that $vw \in E(G)$.*

Proof. Let $u \in X$. Suppose that for every pair of distinct vertices $v, w \in X \setminus \{u\}$, $vw \notin E(G)$. That means $G[X]$ is isomorphic to a star with the centre u. However, the matrix $\boldsymbol{A}(G)[X]$ is clearly singular, and it contradicts to the assumption. □

Lemma 4.6. *Let G be a graph and let $X \subseteq V(G)$ such that $X \neq \emptyset$ and $\boldsymbol{A}(G)[X]$ is non-singular. Let $s \in X$. Then G has a sequence of pairs of vertices $\{x_1, y_1\}, \{x_2, y_2\}, \dots, \{x_m, y_m\}$ such that*

a) $\boldsymbol{A}(G) * X = \boldsymbol{A}(G \wedge x_1y_1 \wedge x_2y_2 \cdots \wedge x_my_m)$,
b) $(\{x_i, y_i\} : 1 \leq i \leq m)$ is a partition of X (in particular, $|X|$ is even), and
c) $s \in \{x_m, y_m\}$.

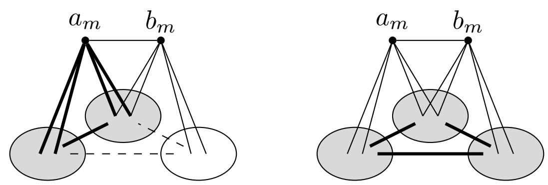

FIGURE 1. Two cases of a new clique obtained from G' by pivoting the edge $a_m b_m$ in Proposition 4.7 where $r \in \{a_m, b_m\}$. By induction hypothesis, the each coloured part can have a clique of size at most 3^{d-2} in $G' \setminus r$, and therefore the size of a new clique cannot exceed 3^{d-1}.

Proof. We prove the theorem by induction on $|X| \geq 1$. If $|X| = 1$, then $\boldsymbol{A}(G)[X]$ cannot be non-singular, as we have no loops in G. If $X = \{x_1, x_2\}$, then x_1, x_2 must form an edge of G since, again, $\boldsymbol{A}(G)[X]$ is non-singular. Since $\boldsymbol{A}(G) * \{x_1, x_2\} = \boldsymbol{A}(G \wedge x_1 x_2)$, and either $s = x_1$ or $s = x_2$, we conclude the claim.

For an inductive step, we assume that $|X| \geq 3$. Since $\boldsymbol{A}(G)[X]$ is non-singular, by Lemma 4.5, there exist two vertices $x_1, y_1 \in X \setminus \{s\}$ such that $x_1 y_1 \in E(G)$. Also, by Theorem 2.3, $\boldsymbol{A}(G \wedge x_1 y_1)[X \setminus \{x_1, y_1\}]$ is non-singular. By Theorem 2.4, we have

$$\begin{aligned}\boldsymbol{A}(G) * X &= \boldsymbol{A}(G) * (\{x_1, y_1\} \Delta (X \setminus \{x_1, y_1\}) \\ &= (\boldsymbol{A}(G) * \{x_1, y_1\}) * (X \setminus \{x_1, y_1\}) \\ &= \boldsymbol{A}(G \wedge x_1 y_1) * (X \setminus \{x_1, y_1\}).\end{aligned}$$

Since $s \in X \setminus \{x_1, y_1\} \neq \emptyset$, by the induction hypothesis, $G \wedge x_1 y_1$ has a sequence of pairs of vertices $\{x_2, y_2\}, \ldots, \{x_m, y_m\}$ such that

a) $\boldsymbol{A}(G \wedge x_1 y_1) * (X \setminus \{x_1, y_1\}) = \boldsymbol{A}((G \wedge x_1 y_1) \wedge x_2 y_2 \cdots \wedge x_m y_m)$,
b) $(\{x_i, y_i\} : 2 \leq i \leq m)$ is a partition of $X \setminus \{x_1, y_1\}$, and
c) $s \in \{x_m, y_m\}$.

Thus, $\boldsymbol{A}(G) * X = \boldsymbol{A}(G \wedge x_1 y_1 \wedge x_2 y_2 \cdots \wedge x_m y_m)$ and we can easily verify that $\{x_1, y_1\}, \{x_2, y_2\}, \ldots, \{x_m, y_m\}$ is the desired sequence. □

Now we are ready to prove the promised negative proposition.

Proposition 4.7. *Let d, t be positive integers such that $t > 3^{d-1}$. Then a graph of tree-depth at most d cannot contain a pivot-minor isomorphic to the clique K_t.*

Proof. Let $K(d) = \max\{q : \mathrm{td}(G) \leq d \text{ and } G \text{ has a pivot-minor isomorphic to } K_q\}$. The statement is equivalent to $K(d) \leq 3^{d-1}$. If $d = 1$, then each component of a graph of tree-depth 1 has one vertex and we have $K(1) = 1$. We assume $d \geq 2$.

We choose minimal d such that a graph G of tree-depth at most d has a pivot-minor isomorphic to K_t where $t > 3^{d-1}$. Let T be a tree-depth decomposition for G of height at most d. Since G is without loss of generality connected, T has a unique root r which is a vertex of G, too. Since G has a pivot-minor isomorphic to K_t, there exists $X \subseteq V(G)$ and $S \subseteq V(G)$ such that

a) $\boldsymbol{A}(G)[X]$ is non-singular, and
b) the graph whose adjacency matrix is $(\boldsymbol{A}(G) * X)[S]$ is isomorphic to K_t.

By Lemma 4.6, for $s = r$ if $r \in X$ or $s \in X$ chosen arbitrarily otherwise, there exists a sequence of pairs of vertices $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_m, b_m\}$ in G such that $\boldsymbol{A}(G) * X = \boldsymbol{A}(G \wedge a_1 b_1 \wedge a_2 b_2 \cdots \wedge a_m b_m)$ and $r \notin \{a_i, b_i\}$ for $1 \leq i \leq m - 1$.

Let $G' = G \wedge a_1 b_1 \wedge a_2 b_2 \cdots \wedge a_{m-1} b_{m-1}$. Then $(G' \wedge a_m b_m)[S]$ is isomorphic to K_t, and there are two cases:

i. $r \notin \{a_m, b_m\}$, which means that $G \setminus r$ has the pivot-minor $(G' \wedge a_m b_m) \setminus r$ containing a K_{t-1}-subgraph. Since the tree-depth of $G \setminus r$ is $t - 1$ as witnessed by the decomposition $T \setminus r$, and $t - 1 \geq 3^{d-1} > 3^{d-1-1}$, this contradicts our minimal choice of d.

ii. $r = a_m$, up to symmetry. After the pivot $a_m b_m$, a new clique K in G (which is not present in G') is created in two possible ways: K belongs to the closed neighbourhood of one of a_m, b_m, or K is formed in the union of the neighbourhoods of a_m, b_m (excluding a_m, b_m). See Figure 1. In either case, K is formed on two or three, respectively, cliques of $G' \setminus \{a_m, b_m\}$. Again, by minimality of d, the largest clique contained in $G' \setminus r$ can be of size 3^{d-1-1}. Therefore, $t \leq \max\left(1 + 2 \cdot 3^{d-2}, 3 \cdot 3^{d-2}\right) = 3^{d-1}$, a contradiction.

Indeed, $t = K(d) \leq 3^{d-1}$ as desired. □

5. Bipartite Graphs of small BSC-depth

In the previous section we have seen that every graph class of bounded shrub-depth can be obtained via vertex-minors of graphs of tree-depth d for some d. Moreover, we have also proved that this statement does not hold if we replace vertex-minors with pivot-minors. However this raises a question whether there is some simple condition on the graph class in question which would guarantee us the theorem to hold for pivot-minors. It turns out that one such simple restriction is to consider just bipartite graphs of bounded shrub-depth, as stated by Theorem 5.4.

To get our result, we introduce the following "depth" definition better suited to the pivot-minor operation, which builds upon the idea of SC-depth. Let G be a graph and let $X, Y \subseteq V(G)$, $X \cap Y = \emptyset$. We denote by $\overline{G}^{(X,Y)}$ the graph G' with vertex set $V(G)$ and edge set $E(G') = E(G) \Delta \{xy : x \in X, y \in Y\}$. In other words, $\overline{G}^{(X,Y)}$ is the graph obtained from G by complementing the edges between X and Y.

Definition 5.1 (BSC-depth)**.** *We define inductively the class $\mathcal{BSC}(k)$ as follows:*

i. let $\mathcal{BSC}(0) = \{K_1\}$;

ii. if $G_1, \ldots, G_p \in \mathcal{BSC}(k)$ and $H = G_1 \dot\cup \ldots \dot\cup G_p$, then for every pair of disjoint subsets $X, Y \subseteq V(H)$ we have $\overline{H}^{(X,Y)} \in \mathcal{BSC}(k+1)$.

The BSC-depth *of G is the minimum integer k such that $G \in \mathcal{BSC}(k)$.*

In general, graphs of bounded SC-depth may have arbitrarily large BSC-depth, but the two notions are anyway closely related, as in Lemma 5.2. Here $\chi(G)$ denotes the chromatic number of a graph.

Lemma 5.2. *a) The BSC-depth of any graph G is at least $\lceil \log_2 \chi(G) \rceil$.*

b) The SC-depth of G is not larger than three times its BSC-depth.

c) If G is bipartite, then the BSC-depth of G is not larger than its SC-depth.

Proof. a) If $H' = \overline{H}^{(X,Y)}$, then $\chi(H') \leq 2\chi(H)$ since one may use a fresh set of colours for the vertices in Y. Then the claim follows by induction from Definition 5.1.

b) We have

$$\overline{H}^{(X,Y)} = \overline{\left(\overline{\left(\overline{H}^{X}\right)}^{Y}\right)}^{X \cup Y}$$

and so the claim directly follows by comparing Definitions 5.1 and 4.1.

c) Let $G \in \mathcal{SC}(k)$. Let $V(G) = A \cup B$ be a bipartition of G, i.e., that A and B are disjoint independent sets. We use here for G the same "decomposition" as in Definition 4.1; just replacing at every step a single set X with the pair $(X \cap A, X \cap B)$ (point ii. of the definitions). The resulting graph $G' \in \mathcal{BSC}(k)$ then fulfils the following: both A, B are independent sets in G', and every $uv \in A \times B$ is an edge in G' if and only if uv is an edge of G. Therefore, $G = G' \in \mathcal{BSC}(k)$. □

In particular, following Lemma 5.2 a), the BSC-depth of the clique K_n equals $\lceil \log_2 n \rceil$, while $K_{m,n}$ always have BSC-depth 1.

Lemma 5.3. *Let k be a positive integer. If a graph G is of BSC-depth at most k, then G is a pivot-minor of a graph of tree-depth at most $2k+1$.*

Proof. The proof follows along the same line as the proof of Lemma 4.3. For a graph G of BSC-depth k, we recursively construct a graph U and a rooted forest T such that

i. G can be obtained from U as a pivot-minor via pivoting edges only between vertices in $V(U) \setminus V(G)$, and
ii. $U \subseteq \mathrm{Clos}(T)$ and T has depth at most $2k+1$.

If $k = 0$, then it is clear by setting $G = U = T = K_1$. We assume that $k \geq 1$.

Since G has BSC-depth k, there exist a graph H and disjoint subsets $X, Y \subseteq V(H)$ such that $G = \overline{H}^{(X,Y)}$ and H is the disjoint union of $H_1, H_2, \ldots, H_m$ such that each H_i has BSC-depth $k-1$. By induction hypothesis, for each $1 \leq i \leq m$, H_i is a pivot-minor of a graph U_i and $U_i \in \mathrm{Clos}(T_i)$ where the height of T_i is at most $2(k-1)+1$. For each $1 \leq i \leq m$, let r_i be the root of T_i, and let T be the rooted forest obtained from the disjoint union of all T_i by adding an edge between two new vertices r_x and r_y and by connecting r_Y to all r_i. Let U be the graph obtained from the disjoint union of all U_i and the vertices $\{r_x, r_y\}$ by adding an edge between r_x and r_y and all edges from r_x to X as well as all edges from r_y to Y. Validity of (ii.) is clear from the construction.

Now we check the statement (i.). By our construction of U, any pivoting on edges in U_i has no effect on U_j for $j \neq i$, and pivoting on edges in $V(U_i) \setminus V(H_i)$ does not change edges incident with r_x or r_y. Hence, by induction, we can obtain H as a pivot-minor of U and still have r_x adjacent precisely to r_y and $X \subseteq V(H)$ and r_y adjacent to r_x and $Y \subseteq V(H)$. We then pivot the edge $\{r_x, r_y\} \in V(U) \setminus V(H)$, and delete $V(U) \setminus V(G)$ to obtain G. □

The main result of this section now immediately follows from Lemmas 5.3, 5.2 c) and Proposition 4.2.

Theorem 5.4. *For any class $\mathscr{S}$ of bounded shrub-depth, there exists an integer d such that every* bipartite *graph in $\mathscr{S}$ is a pivot-minor of a graph of tree-depth d.*

6. Conclusions

We finish the paper with two questions that naturally arise from our investigations. While the first question has a short negative answer, the second one is left as an open problem.

A *cograph* is a graph obtained from singleton vertices by repeated operations of disjoint union and (full) complementation. This well-studied concept has been extended to so called "m-partite cographs" in [4] (we skip the technical definition here for simplicity); where cographs are obtained for $m = 1$. It has been shown in [4] that m-partite cographs present an intermediate step between classes of bounded shrub-depth and those of bounded clique-width.

The first question is whether some of our results can be extended from classes of bounded shrub-depth to those of m-partite cographs. We know that shrub-depth is monotone under taking vertex-minors (Corollary 3.6) and an analogous claim is asymptotically true also for clique-width [9]. However, the main obstacle to such an extension is the fact that m-partite cographs do not behave well with respect to local and pivot equivalence of graphs. To show this we will employ the following proposition:

Proposition 6.1 ([4])**.** *A path of length n is an m-partite cograph if, and only if, $n < 3(2^m - 1)$.*

By the proposition, to negatively answer our question it is enough to find a class of m-partite cographs containing long paths as pivot-minors:

Proposition 6.2. *Let H_n denote the graph on $2n$ vertices from Figure 2. Then H_n is a cograph for each $n \geq 1$, and H_n contains a path of length n as a pivot-minor.*

Proof. It is $V(H_n) = \{a_i, b_i : i = 1, 2, \ldots, n\}$ and $E(H_n) = \{b_i b_j : 1 \leq i < j \leq n\} \cup \{b_i a_j : 1 \leq i \leq j \leq n\}$. The graph H_n can be constructed iteratively as follows, for $j = n, n-1, \ldots, 1$: add a new vertex a_j, complement all the edges of the graph, add a new vertex b_j, complement again. Consequently, H_n is a cograph (and, in fact, a so called threshold graph).

For the second part, we let inductively $G_1 := H_n$ and $G_j := G_{j-1} \wedge a_j b_j$ for $j = 2, \ldots, n-1$. Then, by the definition, G_2 is obtained from H_n by removing all the edges incident with b_1 except $b_1 a_2, b_1 b_2$. In particular, $G_2 \setminus \{a_1, b_1\}$ is isomorphic to H_{n-1}, and a_3, b_3 are adjacent in G_2 only to vertices other than a_1, b_1. Consequently, by induction, G_j is obtained from G_{j-1}

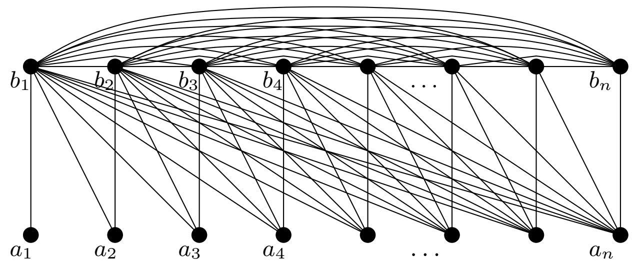

FIGURE 2. A graph on $2n$ vertices [4] which is a cograph and pivoting on $a_2b_2, a_3b_3, \dots, a_{n-1}b_{n-1}$ results in an induced path on $a_1, b_1, b_2, \dots, b_n$.

by removing all the edges incident with b_{j-1} except $b_{j-1}a_j, b_{j-1}b_j$, and G_{n-1} has the edge set $\{b_1b_2, b_2b_3, \dots, b_{n-1}b_n\} \cup \{a_1b_1, a_2b_1, a_2b_2, a_3b_2, \dots, a_nb_n\}$. Then $G_{n-1}[a_1, b_1, b_2, \dots, b_n]$ is a path. □

Building on this negative result, it is only natural to ask whether not having a long path as vertex-minor is the property exactly characterising shrub-depth.

Conjecture 6.3. *A class $\mathscr{C}$ of graphs is of bounded shrub-depth if, and only if, there exists an integer t such that no graph $G \in \mathscr{C}$ contains a path of length t as a vertex-minor.*

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