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# MATHEMATICAL MODELLING IN MEDICAL SCIENCES

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**Abstract**

*This article mainly concerns with mathematical modelling in medicine. There is one area in medicine namely pharmaco- kinetics which have already been mathematicized. There are large areas in medical science which are not amenable to mathematical treatment, and modelling constantly endeavour to widen the areas to which mathematical techniques can be applied for gaining a better insight, and help deepen our understanding of those areas which have already been mathematicized. The skills needed to be successful in applying mathematics are quite different from those needed to understand concepts, to prove theorems or to solve equations. The difficulty is not in learning and understanding the mathematics involved but in seeing where and how to apply it. In this paper an attempt is made to demonstrate the essentials of mathematical modelling without going deep in to the details on specialised topics.*

## Introduction

Mathematics is a powerful, flex& Language, models are representations and mathematical models are representations framed in mathematical terms. A good metaphor for mathematical model is a map. Models are representations for a purpose and some knowledge is required to build and use them. Medical theories entail a concern with among other things, generality, precision and parsimony. The knowledge required for mathematical model building includes good intuition and ability to abstract, some knowledge of a range of models, some facility with the manipulation of the mathematical representations, an ability to assess critically the models, some flair and modesty. In this article, an attempt is made to give a brief introduction to mathematical modelling to get an insight into the problems of medical sciences.

## Early Expositions of the Methodlogy of Modelling

Most of the early developments in methodologies are due to physical sciences. Klamin (1971) proposed a five stage model of the problem solving process. The stages being recognition, formulation, solution, computation and explanation. Klamin described problem solving as a linear process starting at stage one and proceeding to stage five at which point the problem is solved. Lin (1976) added the evaluation of modelling process as sixth stage. In a subsequent paper Lin (1978) addressed to another important concept of a range of models instead of a single best model.

Wood (1969) described modelling process as an iterative process with evaluation of the results in the light of obsenable reality leading to modification of the model and repetition of the stages - a form of parameter identification problems. Hall (1972) further developed the iterative methodology and Bajpai *et al* (1975) used a flow block diagram to illustrate their concepts of methodologics of mathematical modelling. A further improvement in this direction is due to d'Irveno and McLone (1977) where approach is to start by constructing the simplest model. If this is inappropriate then go back and make the model more sophisticated usually by dropping or altering one or more of the assumptions. A novel treatment of modelling process by Penrose (1978) consists of six stages in a circular progression His representation is reproduced in Fig. 1.

The boxes are numbered and joined by arrows to indicate the normal direction of travel from one to another, but it is usual to

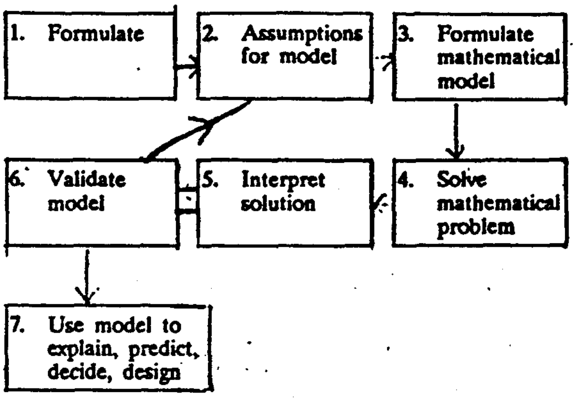


Fig. 1

return to some of the boxes several-times as one's ideas develop. A good account of complex linkage methodology of mathematical modelling was given by Clements (1989).

In this article model building is considered within the broad methodologies having epistemological, substantial, technical and purposive components. Within the epistemological component, criteria for the construction and evaluation of the models are set forth. Within the substantial components the nature of theories, its theoretical terms and the nature of relevant data are formulated. The technical component deals with the nuts and bolts of modelling - the mathematical systems used, their properties, measurement, and estimation. Within the purposive component, we discuss the goals of the modelling effort. These include explanation, description and explication. They also include a consideration of the uses of a model. These components are highly interrelated.

**Classification of models** : Mathematical models are frequently co-ordinated by three distinctions: The first differentiates, process models from structural models, the second deterministic models from probabilistic models, and the third, using discrete from continuous. In principle, an eight-cell table can be constructed and each cell fitted with mathematical models sharing the criteria defining the cell.

Process models explicitly, attempt to model the changes and provide an understanding of the mechanisms of change. Among the frequently used tools are differential equations and difference equations. Structural models attempt to represent and understand the structure of systems relations. The tools used include graph theory, matrix algebra, groups, semi groups., Boolean algebra, algebraic topology. Statistical models are used to model processes whose outcomes are governed by a probabilistic mechanism(s). Deterministic models eschew stochastic mechanisms in favour of deterministic mechanisms and relations. Process models may be deterministic or stochastic, but structural models tend to be deterministic.

The distinction between mathematical models and statistical analysis is a very bluned one. Formal approaches can and do incorporate error specification which other approaches generally do not. This informs estimation. Second, the properties of the statistical tools are stated and established mathematically. Finally, new mathematical models and their uses generate estimation problem and statistical questions.

The Theory, Model and Data Mangle : This triangle provides a method for selecting models best suited for a substantive problem where skill shows alternative model candidate6 and selection of most. fruitful models. There are three pairs of mappings - between theory and model, between model and data and between theory and data and all are important.

The theory - model linkage is concerned with expressing a congruence between a theory and its representation in a mathematical model The theory has to map into the model with little distortion or loss. Deductively, there is a mathematical formalization of the theory while inductively this can be a formal generalizaton of the theory. The mathematical model then has to be useful. These results can be mapped to theory and data. Deductively model maps to data by specifying or predicting empirical outcomes. Also deductively mathematical results map to theory by specifying theoretical implications of the derivations through mappings linking theory and model. The theory, the model and the data have to make sense and be consistent with one another - which is the nub of evaluating models (Doreian, 1987).

**Metaphor, Analogy and Model:** There are no clear cut simple definitions for the terms like metaphor, analogy and model, but although they are by no means synonyms, they do share certain features. Broadly speaking it can be said that they all attempt to enhance our understanding of new information in terms of what is already familiar. According to Sutton (1978), "Analogies are extended similes in which an attempt is made to trace multiple points of comparisons, Metaphors arc ... less explicit and much more mentally teasing ... forcing the hearer to search among his associated ideas for possible connections and, a model can be thought of as an extended metaphor".

## An Applications of Modelling with Differential Equations

**The model represented by the linear differential equation.**

$$\frac{dy}{dx} = ky \qquad 3.1$$

represents a vast variety of situations in medical sciences. Problems in population growth, drug absorption and elimination, alcohol absorption, water cooling carbon dating etc can be modelled by equations of type (3.1). Once this differential equation has been solved we have effectively solved numerous problems. The solutions to the above problem can be expressed in the form

$$y = yo \exp(kt) \qquad (3.2)$$

where yo is a initial value. The behaviour of the solution depends on the sign of the constant k. If k is positive we have exponential growth, if k is zero, y remains equal to its initial value, and if k is negative we have exponential decay, $y \to 0$ as $x \to \infty$.

The study of the way in which a drug loses its concentration in the blood of a patient is fundamental to pharmacology, The 'dose-response' relationship plays a vital role in determining the required dosage level and the interval of time between doses for a particular drug.

Suppose $y = y(t)$ represents the quantity of drug in the blood stream at time t. The simplest way to model such behaviour is assume that the rate of change of the concentration is proportional to the concentration of the drug in the blood stream. In mathematical terms

$$\frac{dy}{dx} = -ky \qquad (3.3)$$

where k is a positive constant. Experiments have. shown that (33) is a good approximation to reality for many drugs, the most important being penicillin and streptomycin

Having determined the constant k for a particular drug we now use equation(3.3) as a model. Suppose the patient is given an initial dose yo, which is assumed to be instantaneously absorbed by blood at time t = 0 resulting in **a quantity y = yo at t = 0 in the blood. The actual time of absorption is usually in comparisons small with the time between doses. From our model, the solution is**

$$y = yo \exp(-kt) \qquad (3.4)$$

-k showing that the drugs' concentration decays exponentially. After a fixed time T, a second dose yo is administered. Just before the dose, the amount in the blood is given by

$$Y(T-) = yo \exp(-kT) \qquad (3.5)$$

Just after the second dose, at time
T = T+

$$y(T+) = yo + yo \exp(-kT)$$
$$= yo\,(1+\exp(-kT)) \qquad (3.6)$$

The new quantity decays according to law (33) with initial condition

$$y = yo\,(1 + \exp(-kT)) \text{ at } t = T$$

Thus for t > T

$$y(t) = yo\,(1 + \exp(-kT)) \exp(-k(t-T)) \qquad (3.7)$$

Again giving the patient a dose yo at t = 2T results in

$$y(2T+) = yo\,(1+\exp(-kT) + \qquad (3.8)$$

Again solving (33) with

$$y = yo\ (1+\exp(-kT)+\exp(-2kT)) \quad (3.9)$$

at t > 2T gives

$$y(t) = yo\ (1 + \exp(-kT) + \exp(-2kT)\ \exp(-k(t-2T)) \quad (3.10)$$

Continuing this way

$$y(nT+) = yo\ (1 + \exp(-kT) + .... + \exp(-nkT) \quad (3.11)$$

for n= 1, 2, . . . . .

$$y(nT+) = yo\ (1-\exp(-(n+1)kT))/(1-\exp(-kT)) \quad (3.12)$$

As n gets larger, exp(-(n+1)kT -> 0 so that

$$y(nT+) = yo / (1-\exp(-kT))$$

Since this is independent of n, the model predicts that the quantity of drug is tending to a saturation level say Ys, where

$$Ys = yo / (1-\exp(-kT))$$

This formula can be used for determining (i) the required time interval T, between doses for a given dose yo and prescribed final level Ys. (ii) the dose level yo required to obtain a final dose level Ys with a prescribed interval between doses, T.

## Discussion

Since the situations in medical sciences are quite complex, one should have some insight into the problem before he attempts to formulate a new mathematical model. A good review in this aspect has been given by Kapur (1985). Once a model is formulated its consequence can be deduced by using mathematical techniques and the results can be compared with observations. The discrepancies between theoretical consideration and observations suggest further improvement in the model as suggested by Clements (1989) and Penrose (1978). This process is repeated till a readily satisfactory model is obtained. It is important to realise that learning to apply mathematics is a very different activity from learning mathematics. There is no theory to learn mathematical modelling and there are only a few guiding principles. There are many examples of very simple mathematics giving useful solutions to very difficult problems in biomedicine, although generally speaking the complexity of the problems and the required mathematical treatment go hand in hand There is growing fraternity of biomedical scientists involved in mathematical modelling now. Many journals cater to a large extent for the needs of mathematical modelling and three of them need mention are : (i) Applied Mathematical modelling (ii) International Journal of Mathematical model&g and (ii) International Journal of Mathematics and Computer simulations. To conclude, we quote

"All knowledge is, in the final analysis, History,

All Sciences are, in the abstract, Mathematics'.

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