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Contingent claim pricing through a continuous time variational bargaining scheme

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Abstract We consider a variational problem modelling the evolution with time of two probability measures representing the subjective beliefs of a couple of agents engaged in a continuous-time bargaining pricing scheme with the goal of finding a unique price for a contingent claim in a continuous-time financial market. This optimization problem is coupled with two finite dimensional portfolio optimization problems, one for each agent involved in the bargaining scheme. Under mild conditions, we prove that the optimization problem under consideration here admits a unique solution, yielding a unique price for the contingent claim.

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1 Introduction

In this paper we extend the variational sequential bargaining pricing scheme studied in Azevedo et al. (2013) to the setup of continuous-time financial markets. The ultimate goal of such variational pricing scheme is to provide a novel behavioural explanation for the pricing of contingent claims and similar financial assets, traded in realistic setups leading to market incompleteness. Therefore, the approach developed here, extending previous work in Azevedo et al. (2013), Boukas et al. (2011), Pinheiro et al. (2013), Xanthopoulos and Yannacopoulos (2008), is an alternative point of view to the pricing of contingent claims in incomplete markets, a relevant problem in financial mathematics. Recall that incompleteness of the market can arise from all sorts of market imperfections and, in particular, it may be due to the non-existence of a large enough number of assets in the market so that all contingent claims can be hedged, to lack of liquidity in the financial markets, to taxation rules and transaction costs, among other reasons. Unlike the complete markets setup extensively studied in the mathematical finance literature (see, e.g. Karatzas and Shreve 1998; Pliska 1997), under which a unique pricing kernel exists and uniquely determines the price of every contingent claim, in the incomplete markets setup the existence of an infinity of pricing kernels is still compatible with the assumption of absence of arbitrage. Hence, for the setup under consideration here, absence of arbitrage and the risk neutral valuation principle are not sufficient to provide a pricing rule for any given contingent claim.

Let us provide some additional motivation for the model under consideration here. Specifically, we consider two interacting agents, one playing the role of "seller" of a given asset, while the other plays the role of "buyer". The asset to be traded is modeled as a contingent claim whose payoff at some future time T equals the value of a random variable defined on an appropriate probability space. We note that this general modeling point of view enables us to include in our analysis examples as distinct as, for instance, financial assets traded over the counter; company mergers and acquisitions; physical infrastructures; and real options, among others. We assume that both agents goal is to reach an agreement for the price of such contingent claim by an a priori fixed instant of time $T_0 < T$. Moreover, we assume that the two agents are allowed to have subjective beliefs, modeled by probability measures, reflecting their personal points of view concerning the likelihood of occurrence of the future states of the world. Finally, we suppose that the two agents actively engage in a beliefs update process under which both collect information about each other pricing for the contingent claim and the corresponding sensitivity to changes in such price. The agents then combine this collective information to change their beliefs concerning the future value of the asset to be traded and, therefore, the contingent claim price at time T_0. The resulting common price turns out to be such that the agents end up assuming more risk than they would optimally like to, by giving away some potential profit, so that the agreement on the price of the contingent claim to be traded becomes feasible.

The problem described above from an intuitive point of view can be modelled by a variational problem which may be decomposed into the following two types of coupled partial problems: a final wealth stochastic optimal control problem for each agent and a joint belief update problem for the two agents. In what concerns the stochastic optimal control problems for the agents final wealth, it is enough to mention that standard techniques for the control of Itô diffusions apply (Oksendal and Sulem 2005; Yong and Zhou 1999), guaranteeing existence and uniqueness of solutions. Our main focus is then the variational problem modeling the interaction between the two agents, whereby their beliefs, identified with appropriately picked probability measures, are updated and a common price is reached for the asset being traded. We remark that this problem turns out to be a continuous-time Calculus of Variations problem. However, a key difficulty lies on the fact that the phase space on which the dynamics take place is some space of probability measures, an object of difficult mathematical treatment. To address such issue, we resort to an assumption based on the "bounded rationality" of the two agents trading the contingent claim, enabling us to specify a space of probability measures suitable for analytical treatment. More precisely, we assume that each agent assigns some positive probability to a finite number of events forming a partition of the probability space where the underlying financial market is defined. One possible construction yielding the desired outcome is to assume that each agent assigns some positive probability to the following special class of events: the price of a given asset traded in the underlying financial market is in some element of a finite partition of the positive half-line on a given element of a finite list of instants of time. It is our opinion that such assumption is not only realistic, but also that it accurately mimics the qualitative behaviour of many financial market agents. Finally, we remark that the finiteness assumption detailed above enables us to identify the phase space of our Calculus of Variations problem with a simplex in a high dimensional Euclidean space. Given such identification, the dynamics of this Calculus of Variations problem turn out to be given by absolutely continuous paths (on some simplexes) that minimize an "action functional" modelling the agents reluctance in moving to fast towards a final agreement, as well as the increased utility gained from moving closer to a unique price for the contingent claim, subject to appropriate boundary conditions.

This paper is organized as follows. In Sect. 2 we provide the mathematical formulation of the model under consideration leading to a continuous time variational problem. In Sect. 3 we study the existence of solutions to this variational problem under general assumptions on the bidding functions used by the agents in the bargaining scheme. In Sect. 4 we provide an example of bidding rule realizing the assumptions used for the proof of our main theorem in Sect. 3. We conclude in Sect. 5.

2 Mathematical formulation of the model

In this section we will fix notation and introduce the setup under consideration throughout the rest of the paper.

2.1 The financial market

Let $(\Omega, \mathcal{F}, P)$ be a complete filtered probability space. We consider a financial market with finite horizon $T > 0$ determined by the following processes:

(i) a M-dimensional Brownian motion $\{W(t), \mathcal{F}_t : 0 \leq t \leq T\}$ defined on $(\Omega, \mathcal{F}, P)$, where $\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the augmentation of the filtration $\{\mathcal{F}_t^W\}_{0\leq t\leq T}$ generated by $W(\cdot)$;
(ii) a continuous deterministic risk-free rate process $r(\cdot)$;
(iii) a continuous deterministic N-dimensional mean rate of return process $\mu(\cdot)$;
(iv) a continuous deterministic $(N \times M)$-matrix-valued volatility process $\sigma(\cdot)$;
(v) a vector of positive, initial stock prices

$$S(0) = (S_1(0), \dots, S_N(0))'.$$

We refer to this financial market as $\mathcal{M} = (r(\cdot), \mu(\cdot), \sigma(\cdot), S(0))$. Thus, the financial market under consideration has $N+1$ assets that can be traded continuously: one risk-free asset $S_0(t)$ (called the bond) and a fixed number of risky-assets $S_1(t), \ldots, S_N(t)$. The price process $S_0(\cdot)$ is continuous and increasing, whereas the processes $(S_1(t), S_2(t), \ldots, S_N(t))$ are continuous and strictly positive. Moreover, we assume that these processes satisfy the following stochastic differential equations:

$$
\begin{aligned}
\mathrm{d}S_0(t) &= r(t)S_0(t)\mathrm{d}t\ , && S_0(0) = 1,\\
\mathrm{d}S_i(t) &= S_i(t)\left[\mu_i(t)\mathrm{d}t + \sum_{j=1}^{M} \sigma_{ij}(t)\mathrm{d}W_j(t)\right], && S_i(0) = s_i > 0,
\end{aligned}
$$

for all $t \in [0, T]$ and $i = 1, \ldots, N$.

A portfolio process $(\pi_0(\cdot), \pi(\cdot))$ for the market $\mathcal{M}$ under consideration consists of a $\{\mathcal{F}_t\}$-progressively measurable real-valued process $\pi_0(\cdot)$ and a $\{\mathcal{F}_t\}$-progressively measurable $\mathbb{R}^N$-valued process $\pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_N(\cdot))$, such that the following integrability conditions hold almost surely

$$
\int_0^T \left|\pi_0(t) + \sum_{i=1}^{N} \pi_i(t)\right| |r(t)|\mathrm{d}t < \infty
$$

$$
\int_0^T \left|\pi(t)\alpha(t)\right|\mathrm{d}t < \infty \tag{1}
$$

$$
\int_0^T ||\sigma'(t)\pi(t)||^2\mathrm{d}t < \infty\ , \tag{2}
$$

where $\alpha(t) = (\mu_1(t) - r(t), \ldots, \mu_N(t) - r(t)) \in \mathbb{R}^N$ is the risk premium. The gains process $G(\cdot)$ associated with $(\pi_0(\cdot), \pi(\cdot))$ is given by

$$
\begin{aligned}
G(t) = &\int_0^t \left(\left(\pi_0(s) + \sum_{i=1}^{N} \pi_i(s)\right) r(s) + \pi(s)\alpha(s)\right)\mathrm{d}s\\
&+ \int_0^t \pi(s)\sigma(s)\mathrm{d}W(s),
\end{aligned} \tag{3}
$$

for $t \in [0, T]$. The portfolio process $(\pi_0(\cdot), \pi(\cdot))$ is said to be self-financed if

$$
G(t) = \pi_0(t) + \sum_{i=1}^{N} \pi_i(t),
$$

for all $t \in [0, T]$. The component $\pi_0(t)$ represents the number of units of the bond in the portfolio, whereas the component $\pi_j(t)$, $j \in \{1, \cdots, N\}$, represents the number of shares of stock j at any given time $t \in [0, T]$. We assume that each component of a trading strategy is predictable.

We now introduce a cumulative income process $I(t)$, $t \in [0, T]$, associated with the continuous income process $i(t)$, $t \in [0, T]$, in such a way that

$$
I(t) = \int_0^t i(t)\mathrm{d}t.
$$

We think of $I(t)$ as the cumulative wealth received by an investor up to time $t \in [0, T]$. We define the wealth process associated with $(I(\cdot), \pi_0(\cdot), \pi(\cdot))$ as being

$$X(t) = I(t) + G(t) \ , \tag{4}$$

where $G(\cdot)$ is the gains process in (3). We can rewrite the previous equation in differential form as

$$\mathrm{d}X(t) = \Big(i(t) + r(t)X(t) + \pi(t)\alpha(t)\Big)\mathrm{d}t + \pi(t)\sigma(t)\mathrm{d}W(t).$$

The SDE above describes the evolution of the wealth process $X(\cdot)$ associated with both the continuous-time income process $i(t)$ and portfolio $(\pi_0(\cdot), \pi(\cdot))$. Since the wealth process $X(\cdot)$ combines into a single quantity the income derived from investment in the financial market and the external income provided by the process $i(\cdot)$, the portfolio $(\pi_0(\cdot), \pi(\cdot))$ is not self-financing. Instead, the portfolio $(\pi_0(\cdot), \pi(\cdot))$ is $I(\cdot)$-financed (see Karatzas and Shreve 1998, Ch. I, Sec. 1.3) in the sense that

$$X(t) = \pi_0(t) + \sum_{i=1}^{N} \pi_i(t).$$

The existence of a self-financed portfolio process $\pi(\cdot)$ is an arbitrage opportunity in the financial market $\mathcal{M}$ if the corresponding gains process $G(\cdot)$ satisfies $G(T) \geq 0$ almost surely and $G(T) > 0$ with positive probability (see, e.g. Karatzas and Shreve 1998). We say that the financial market $\mathcal{M}$ is viable if no such arbitrage opportunities exist. It is well known that the financial market $\mathcal{M}$ is viable if there exists a progressively measurable process $\theta(\cdot) \in \mathbb{R}^N$, called the market price of risk, such that for Lebesgue-almost-every $t \in [0, T]$ the risk premium $\alpha(t)$ is related to $\theta(t)$ by the equation

$$\alpha(t) = \sigma(t)\theta(t) \qquad \text{a.s.}$$

and is such that the following two conditions hold

$$\int_0^T \|\theta(s)\|^2 \,\mathrm{d}s < \infty \qquad \text{a.s.}$$

$$E\left[\exp\left(-\int_0^T \theta'(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^T \|\theta(s)\|^2 \,\mathrm{d}s\right)\right] = 1.$$

Throughout this paper we will assume that the financial market $\mathcal{M}$ is viable, i.e. there are no arbitrage opportunities.

2.2 The pricing problem

In this section we introduce a problem where a pair of agents, a seller and a buyer (denoted by A and B, respectively), need to reach an agreement for a price p for the trade at time $T_0 < T$ of a contingent claim with payoff at time T given by a $\mathcal{F}_T$-measurable random variable F. Throughout this paper we will assume that the random variable F is non-constant and bounded.

We assume that each agent has some beliefs about the likelihood of occurrence of the future states of the world, which we associate with elements of the sample space $\omega \in \Omega$. Moreover, we assume that the agents' ability to assign probabilities to each set of future states of the world, i.e. subsets of Ω, is limited in the sense that we now pass to describe.

Let $\mathcal{P}(\Omega)$ denote the set of probability measures on Ω and denote by Δ^K the unit simplex in $\mathbb{R}^K$. Each agent fixes some finite partition of Ω, Π_β, of the form:

$$\Pi_\beta = \left\{\omega_1^\beta, \ldots, \omega_{K_\beta}^\beta\right\}, \qquad \beta \in \{A, B\},$$

where $K_\beta > 1$ and $\omega_i^\beta \in \mathcal{F}$ for every $i \in \{1, \ldots, K_\beta\}$. Then, each agent assigns probabilities a_i^β to each element in the corresponding partition of Ω, i.e. the beliefs of agent β, $\beta \in \{A, B\}$, belong the set of probability measures, $\mathcal{P}(\Omega)$, whose elements are of the form

$$Q(E) = \sum_{i=1}^{K_\beta} a_i^\beta P\left(E|\omega_i^\beta\right) \tag{5}$$

where $E \in \mathcal{F}$ and $(a_1^\beta, \ldots, a_{K_\beta}^\beta) \in \Delta^{K_\beta}$.

Throughout this paper, we will assume that each agent has beliefs given by probability measures as described above. Indeed, we claim that the restriction to a space of probability measures of this type is realistic from the modelling point of view. To support this statement, we note that the usual process under which the future value of financial assets is assessed or described has an inherently finite and discrete nature, as most other human behaviours. As an example, recall that financial advisers, investment companies, or other economic agents specialized in such type of financial counseling, very often classify the future values of financial assets in terms of a finite number of intervals on which the corresponding payoffs lie. For the sake of completeness, and to provide a particular example for the financial market under consideration herein, the behaviour described above amounts to fixing values

$$y_1^n(t_i) < \cdots < y_{K_{t_i}^n}^n(t_i), \qquad n \in \{1, \ldots, N\},$$

for each risky asset $S_1, \ldots, S_N$ at instants of time given by elements of a finite sequence $0 < t_1 < \ldots < t_\ell \le T$ of length $\ell \in \mathbb{N}$, in such a way that the future events considered by the economic agents referred to above are either of the form

$$S_n(t_i, \omega) \in (y_j^n(t_i), y_{j+1}^n(t_i)], \qquad j = 1, \ldots, K_{t_i}^n - 1,$$

or of one of the following two forms

$$S_n(t_i, \omega) \le y_1^n(t_i) \quad \text{or} \quad S_n(t_i, \omega) > y_{K_{t_i}^n}^n(t_i)$$

for every $n \in \{1, \ldots, N\}$ and $i \in \{1, \ldots, \ell\}$. Note that the set of events above defines a "natural" finite partition of the probability space Ω in terms of the prices of the risky assets $S_1, \ldots, S_N$ at the sequence of instants of time $\{t_i\}_{i=1}^\ell$.

In addition to the finiteness assumption detailed above, we will assume that such beliefs are not rigid, i.e. the agents are willing to update their beliefs about the future states of the world in order to reach an agreement leading to a common price for the contingent claim F. However, the agents are not flexible enough to instantaneously update their beliefs to reach the desired common price. Instead, they continuously adjust their beliefs, taking into consideration each other pricing updates, to reach an agreement on or before the trade date T_0. We model such continuous adjustment or bargaining process as follows. We identify $\mathcal{P}_\beta(\Omega)$ with the simplex $\Delta^{K_\beta} \subseteq \mathbb{R}^{K_\beta}$ and consider paths $\gamma_\beta : [0, T_0] \to \Delta^{K_\beta}$ such that for each $t \in [0, T_0]$ we have that

$$\gamma_\beta(t) = \left(a_1^\beta(t), \ldots, a_{K_\beta}^\beta(t)\right) \in \Delta^{K_\beta}$$

determines agent β beliefs at time $t \in [0, T_0]$, $Q_\beta(t) \in \mathcal{P}_\beta(\Omega)$, through the relation (5). The evolution of the agents' beliefs with time will then be realized as the minimum of an appropriate functional over a set of absolutely continuous paths. For our purposes to be achieved, such functional must satisfy the following modeling properties:

i) it must be increasing when seen as a function of the norm of the paths γ_β velocities (note that the path velocities exist for a.e. $t \in [0, T_0]$ by absolute continuity);
ii) it must be increasing when seen as a function of the agents disagreement concerning the price of the contingent claim at each instant of time $t \in [0, T_0]$.

Properties i) and ii) above influence the beliefs dynamics (modelled by the paths γ_β, $\beta \in \{A, B\}$) in two opposite ways, ultimately balancing each other. Property ii) promotes a fast agreement between the agents in what concerns the price of the contingent claim, while property i) prevents such agreement from taking place faster by introducing some disutility into a fast adjustment (i.e. large velocities of the paths γ_β, $\beta \in \{A, B\}$).

In what concerns the agents preferences, we assume that these are described by utility functions $U_\beta : \mathbb{R} \to \mathbb{R}$, $\beta \in \{A, B\}$, satisfying the usual Inada conditions (see Inada 1963), i.e. the utility functions have value zero when $x = 0$, are strictly increasing, strictly concave and continuously differentiable, and their first derivatives satisfy the following asymptotic conditions

$$\lim_{x \to -\infty} U'_\beta(x) = +\infty, \quad \lim_{x \to +\infty} U'_\beta(x) = 0, \quad \beta = A, B.$$

Assume that agents A and B have cumulative income processes $I_A(\cdot)$ and $I_B(\cdot)$, and let V_β, $\beta \in \{A, B\}$, be compact convex subsets of $\mathbb{R}^{N+1}$. We assume that the agents are allowed to allocate their wealth in the financial market by choosing trading strategies $\{\pi_\beta(t) : t \in [0, T]\}$ such that $\pi_\beta(t) \in V_\beta$ for every $t \in [0, T]$. The sets V_β may be interpreted as constraints on the portfolio strategies of the agents. For each $\beta \in \{A, B\}$, we will denote by $X^{\beta,\pi_\beta}_{t_0,x_\beta}(t, \omega)$ the stochastic process representing agent β wealth at time $t \in [0, T]$ and state of the world $\omega \in \Omega$, when choosing a trading strategy $\{\pi_\beta(t) : t \in [t_0, T]\}$ starting with wealth x_β at time t_0.

Assume for the time being that the agents agree to trade the contingent claim with payoff F at time T for the price p at time T_0. Then, it is easy to check that the agents wealth process $X^{\beta,\pi_\beta}_{t_0,x_\beta}(t)$ will be given by

$$X^{\beta,\pi_\beta}_{0,x_\beta}(t) = I_\beta(t) + l_\beta p \chi_{\{t \geq T_0\}}(t) - l_\beta F \chi_{\{t=T\}}(t) + \sum_{i=0}^{N} \int_0^t \pi_i^\beta(u) \mathrm{d}S_i(u) \tag{6}$$

where $l_A = 1$ and $l_B = -1$. Notice that, in accordance with the standard notion of a contingent claim (with maturity T), the buyer pays an amount p at time T_0 in order to receive a payoff of F at time T. Regarding the use of the signs l_β in equation (6), recall that the initial payment at time T_0 is made by the buyer to the seller, while the payoff F at time T corresponds to a payment from the seller to the buyer.

During the bargaining leading to the contingent claim trade at time T_0, and during the span of time between such trade and the payoff time T, the agents invest their wealth in such a way that maximizes their utility at the final time T, i.e. the agents choose investment strategies π^*_β, $\beta \in \{A, B\}$, such that

$$\pi^*_\beta = \operatorname*{argmax}_{\pi_\beta(t) \in V_\beta} E\Big[U_\beta\Big(X^{\beta,\pi_\beta}_{0,x_\beta}(T, \omega)\Big)\Big|\mathcal{F}_t\Big]. \tag{7}$$

We note that (7) is a standard finite horizon optimal control problem associated with a wealth process which is obtained by interlacing two Itô diffusions at time $t = T_0$, i.e. there is a discontinuity at T_0 caused by the trade of the contingent claim F. Such problem is well-posed and existence and uniqueness of solution follow by standard results from stochastic optimal control theory. For further details see Oksendal and Sulem (2005) and Yong and Zhou (1999).

For each $\beta \in \{A, B\}$, let $p_\beta : [0, T_0] \times \Delta^{K_\beta} \to \mathbb{R}$ be the time $t \in [0, T_0]$ valuation that agent β has for the contingent claim F under the beliefs $\gamma_\beta \in \Delta^{K_\beta}$ associated with the measure $Q_\beta \in \mathcal{P}_\beta(\Omega)$. Note that these pricing functions depend also on the random variable F.

Let $\psi_\beta : [0, T_0] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $\beta \in \{A, B\}$, be such that for every $t \in [0, T_0]$, $\psi_\beta(t, \cdot)$ is a continuous function with a unique mimimum at 0. Assume also that the function $\phi : \mathbb{R} \to \mathbb{R}$ is continuous and has a unique mimimum at 0. Taking into consideration the comments concerning the bargaining process provided earlier in this section, we will now introduce the functional to be minimized, determining the evolution of the agents beliefs. Such functional is given by

$$J(\gamma) = \int_0^{T_0} \alpha\psi_A\Big(t, ||\dot{\gamma}_A(t)||^2\Big) + (1-\alpha)\psi_B\Big(t, ||\dot{\gamma}_B(t)||^2\Big) + \phi(p_B(t, \gamma_B(t)) - p_A(t, \gamma_A(t)))\mathrm{d}t. \tag{8}$$

At each instant of time, the agents choose their beliefs $\gamma_A(t), \gamma_B(t)$ by minimizing the functional (8) over the set of absolutely continuous paths $\gamma : [0, T_0] \to \Delta^{K_A} \times \Delta^{K_B}$ subject to the initial beliefs

$$\gamma(0) = \left(\gamma_A^0, \gamma_B^0\right) \in \Delta^{K_A} \times \Delta^{K_B} \tag{9}$$

and the constraint that the transaction takes place at T_0, i.e.

$$p_A(T_0, \gamma_A(T_0)) \leq p_B(T_0, \gamma_B(T_0)). \tag{10}$$

In the next section we will resort to variational calculus techniques to prove the existence of a unique solution to this problem.

3 The variational problem

In this section we prove the existence of a solution to the minimization problem (8) under the constraints (9) and (10). Moreover, we provide a detailed description of the qualitative properties of the optimal paths.

3.1 A calculus of variations approach

We will use the direct method of the Calculus of Variations to prove the existence of minimizers of (8)–(10) within the class of absolutely continuous trajectories.

From now on we will use the notation $\|\cdot\|$ for the euclidean norm in $\mathbb{R}^{K_A+K_B}$. A path $\gamma : [0, T_0] \to \Delta^{K_A} \times \Delta^{K_B}$ is said to be absolutely continuous on $[0, T_0]$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that, for every disjoint family of subintervals $[s_i, t_i] \subseteq [0, T_0]$, $\sum_i ||\gamma(t_i) - \gamma(s_i)|| < \epsilon$ and $\sum_i |t_i - s_i| < \delta$. We will denote by $AC([0, T_0]; \Delta^{K_A} \times \Delta^{K_B})$ the set of absolutely continuous curves $\gamma : [0, T_0] \to \Delta^{K_A} \times \Delta^{K_B}$.

Consider the following set of assumptions:

A1. for each $\beta \in \{A, B\}$, $\psi_\beta : [0, T_0] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a C^2 function such that for every $t \in [0, T_0]$, $\psi_\beta(t, \cdot)$ is strictly convex and attains its minimum value at 0.

A2. there exist $D_1^\beta > 0$, $D_2^\beta \in \mathbb{R}$ and $\delta_\beta > 1$ such that

$$\psi_\beta(t, x) \geq D_1^\beta |x|^{\delta_\beta} + D_2^\beta,$$

for $\beta \in \{A, B\}$.

A3. the function $\phi : \mathbb{R} \to \mathbb{R}$ is strictly convex with a unique minimum at 0.

A4. the price functions $p_\beta : [0, T_0] \times \Delta^{K_\beta} \to \mathbb{R}$, $\beta \in \{A, B\}$, are continuous functions such that for each fixed $t \in [0, T_0]$, the map $p_A(t, \cdot)$ is strictly concave and the map $p_B(t, \cdot)$ is strictly convex.

A5. the initial conditions $\gamma_A^0 \in \Delta^{K_A}$ and $\gamma_B^0 \in \Delta^{K_B}$ are such that

$$p_A\left(0, \gamma_A^0\right) \geq p_B\left(0, \gamma_B^0\right)$$

for fixed levels of initial wealth $x_A, x_B \in \mathbb{R}$.

Theorem 1 *Let* $\alpha \in (0, 1)$ *be fixed and assume that* (A1)-(A5) *hold. Then, there exists a unique path* $\gamma^* = (\gamma_A^*, \gamma_B^*) \in AC([0, T_0]; \Delta^{K_A} \times \Delta^{K_B})$ *for which the constraints*

$$\begin{aligned} &p_A(T_0, \gamma_A(T_0)) \leq p_B(T_0, \gamma_B(T_0)), \\ &\gamma(0) = \left(\gamma_A^0, \gamma_B^0\right) \in \Delta^{K_A} \times \Delta^{K_B} \end{aligned} \tag{11}$$

hold and satisfying

$$J(\gamma^*) \leq J(\gamma)$$

for all $\gamma \in AC([0, T_0]; \Delta^{K_A} \times \Delta^{K_B})$.

The proof of Theorem 1 uses standard techniques from the Calculus of Variations (see, e.g. Fathi 2008 for further details). We will provide some details below for the sake of completeness. However, before proceeding to the proof, we need to establish some terminology and one auxiliary result.

Recall that a map $S : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be *superlinear* if for all $k \in \mathbb{R}$ there exists $C := C(k)$ such that

$$S(x) \geq kx + C$$

for all $x \in \mathbb{R}_0^+$. Equivalently, S is superlinear if and only if

$$\lim_{x \to +\infty} \frac{S(x)}{x} = +\infty.$$

Lemma 1 *Suppose that the sequence* $\gamma^k \in AC([0, T_0]; \Delta^{K_A} \times \Delta^{K_B})$ *is such that*

$$\int_0^{T_0} S(||\dot{\gamma}^k(t)||)\mathrm{d}t \leq C, \tag{12}$$

for some superlinear function $S : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. *Assume also that, for some* $t_0 \in [0, T_0]$, $||\gamma^k(t_0)|| \leq C$ *for every* $k \in \mathbb{N}$. *Then, there exists a subsequence* γ^{k_j} *satisfying:*

(a) $\gamma^{k_j} \to \gamma$, uniformly, for some $\gamma \in AC([0, T_0]; \Delta^{K_A} \times \Delta^{K_B})$;
(b) $\dot{\gamma}^{k_j} \to \dot{\gamma}$, weakly in $L^1([0, T_0]; \Delta^{K_A} \times \Delta^{K_B})$, i.e. for any function $\phi \in L^\infty([0, T_0]; \Delta^{K_A} \times \Delta^{K_B})$, we have

$$\int_0^{T_0} \phi(t)\dot{\gamma}^{k_j}(t)\mathrm{d}t \to \int_0^{T_0} \phi(t)\dot{\gamma}(t)\mathrm{d}t.$$

Proof We start by proving item (a). As a first step in the proof of absolute continuity of γ, we will prove that the sequence $\{\gamma^k\}$ is equicontinuous. Combining equicontinuity with the boundedness assumption in the Theorem statement and the Arzelà-Ascoli Theorem, one is able to ensure the existence of a subsequence $\{\gamma^{k_j}\}$ satisfying items (a) and (b) in the statement, as detailed below. Since S is superlinear, for any $k \geq 1$ there exists $C(k) \in \mathbb{R}$ such that

$$S(\tau) \geq k|\tau| - C(k)\,,$$

for every $\tau > 0$. Then, for any $t_1, t_2 \in [0, T_0]$, we have that since S is non-negative and

$$\int_{t_1}^{t_2} S\left(\|\dot{\gamma}(t)\|\right)\,\mathrm{d}t \leq \int_0^{T_0} S\left(\|\dot{\gamma}(t)\|\right)\,\mathrm{d}t$$

we obtain that

$$\begin{aligned} \left\|\gamma^k(t_2) - \gamma^k(t_1)\right\| &\leq \int_{t_1}^{t_2} \left\|\dot{\gamma}^k(t)\right\| \mathrm{d}t \\ &\leq \frac{1}{k}\int_{t_1}^{t_2} S\left(\left\|\dot{\gamma}^k(t)\right\|\right)\mathrm{d}t + \frac{C(k)}{k}|t_2 - t_1| \\ &\leq \frac{C}{k} + \frac{C(k)}{k}|t_2 - t_1|. \end{aligned} \tag{13}$$

Thus, given $\epsilon > 0$, if we choose k such that $C/k \leq \epsilon/2$, and $\delta > 0$ such that $C(k)\delta/k \leq \epsilon/2$, we obtain that whenever $|t_2 - t_1| < \delta$ we have $||\gamma^k(t_2) - \gamma^k(t_1)|| < \epsilon$. In particular, since the sequence γ^k is bounded in norm for some $t_0 \in [0, T_0]$, by Arzelà-Ascoli Theorem (see, e.g. Rudin 1973) there exists a subsequence $\{\gamma^{k_j}\}$ of $\{\gamma^k\}$ that converges uniformly to a continuous path $\gamma : [0, T_0] \to \Delta^{K_A} \times \Delta^{K_B}$.

We will now prove that the limit γ is an absolutely continuous path. Given a disjoint family of subintervals $[s_i, t_i] \subseteq [0, T_0]$, we have that

$$\begin{aligned} \sum_i \left\|\gamma^{k_j}(t_i) - \gamma^{k_j}(s_i)\right\| &\leq \sum_i \int_{s_i}^{t_i} \left\|\dot{\gamma}^{k_j}(t)\right\| \mathrm{d}t \\ &\leq \frac{1}{k_j}\sum_i \int_{s_i}^{t_i} S\left(\left\|\dot{\gamma}^{k_j}(t)\right\|\right)\mathrm{d}t + \frac{C(k_j)}{k_j}\sum_i |t_i - s_i| \\ &\leq \frac{C}{k_j} + \frac{C(k_j)}{k_j}\sum_i |t_i - s_i|. \end{aligned} \tag{14}$$

Taking $k_j \to +\infty$, for sufficiently large j we have that

$$\sum_i \|\gamma(t_i) - \gamma(s_i)\| \leq \frac{C}{k_j} + \frac{C(k_j)}{k_j}\sum_i |t_i - s_i|.$$

Moreover, for any $\epsilon > 0$, choosing $\delta > 0$ as above, we obtain that whenever $\sum_i |t_i - s_i| < \delta$ we have $\sum_i ||\gamma(t_i) - \gamma(s_i)|| < \epsilon$. We conclude that the limit path γ is absolutely continuous.

We will now prove item (b). It is enough to check that the result holds for characteristic functions of Borel sets. Indeed, such functions are dense in $L^\infty([0, T_0]; \Delta^{K_A} \times \Delta^{K_B})$. For that purpose, consider first the case where E is a finite union of disjoint intervals (a_i, b_i), i.e.

$$E = \bigcup_{i=1}^{N} (a_i, b_i)\,,$$

then

$$\int_E \dot{\gamma}^{k_j}(t)\mathrm{d}t = \sum_{i=1}^{N} \gamma^{k_j}(b_i) - \gamma^{k_j}(a_i) \to \sum_{i=1}^{N} \gamma(b_i) - \gamma(a_i) = \int_E \dot{\gamma}(t)\mathrm{d}t\,.$$

Consider now the case where E is an infinite union of disjoint intervals (a_i, b_i):

$$E = \bigcup_{i=1}^{\infty} (a_i, b_i).$$

Using superlinearity of S and assumption (12), we obtain that the sequence of derivatives $\dot{\gamma}^{k_j}$ is uniformly integrable (for the Lebesgue measure on $[0, T_0]$). Hence, for any $\epsilon > 0$ there exists $\delta > 0$ such that if $A \subset [0, T_0]$ is a Borel subset with Lebesgue measure smaller than δ, then $\int_A \left\|\dot{\gamma}^{k_j}(t)\right\| \mathrm{d}t < \epsilon$. Fix $\epsilon > 0$ and pick the corresponding value of δ determined by uniform integrability of the sequence $\dot{\gamma}^{k_j}$ to obtain the existence of $i_0 \in \mathbb{N}$ such that whenever $\sum_{i=i_0}^{\infty} |b_i - a_i| < \delta$ we have

$$\sum_{i=i_0}^{\infty} ||\gamma^{k_j}(b_i) - \gamma^{k_j}(a_i)|| < \epsilon$$

for every $j \in \mathbb{N}$. Taking the limit as $j \to +\infty$, we get that

$$\sum_{i=i_0}^{\infty} ||\gamma(b_i) - \gamma(a_i)|| \leq \epsilon.$$

Set

$$E_0 = \bigcup_{i=i_0}^{\infty} (a_i, b_i)$$

and notice that by absolute continuity we can write

$$\begin{aligned}\left|\int_{E_0} \dot{\gamma}^{k_j}(t)\mathrm{d}t\right| &= \left\|\sum_{i=i_0}^{\infty} \gamma^{k_j}(b_i) - \gamma^{k_j}(a_i)\right\| \\ &\leq \sum_{i=i_0}^{\infty} \left\|\gamma^{k_j}(b_i) - \gamma^{k_j}(a_i)\right\| \\ &< \epsilon\,,\end{aligned}$$

and also

$$\left|\int_{E_0} \dot{\gamma}(t)\mathrm{d}t\right| \leq \epsilon.$$

We now notice that since $E \backslash E_0$ is a finite union of disjoint intervals, by the previous case, we obtain that

$$\lim_{j\to\infty}\int_{E\setminus E_0}\dot{\gamma}^{k_j}(t)\mathrm{d}t=\int_{E\setminus E_0}\dot{\gamma}(t)\mathrm{d}t.$$

Thus, since we have that

$$\left|\int_E \dot{\gamma}^{k_j}(t)-\dot{\gamma}(t)\ \mathrm{d}t\right|\le\left|\int_{E\setminus E_0}\dot{\gamma}^{k_j}(t)-\dot{\gamma}(t)\ \mathrm{d}t\right|+\left|\int_{E_0}\dot{\gamma}^{k_j}(t)-\dot{\gamma}(t)\ \mathrm{d}t\right|$$

we conclude that

$$\limsup_{j\to\infty}\left|\int_E\dot{\gamma}^{k_j}(t)\mathrm{d}t-\int_E\dot{\gamma}(t)\mathrm{d}t\right|\le 2\epsilon$$

for any arbitrary positive ϵ. Hence, we arrive at

$$\int_E\dot{\gamma}^{k_j}(t)\mathrm{d}t\to\int_E\dot{\gamma}(t)\mathrm{d}t$$

as required.

Finally, if E is any Borel set, we approximate E by a decreasing sequence of open sets and use the Lebesgue monotone convergence Theorem (see, e.g. Rudin 1987) to conclude the proof. □

We now use Lemma 1 to prove Theorem 1.

Proof (*Proof of Theorem* 1) Let us start by considering a (minimizing) sequence $\gamma^k\in AC([0,T_0];\Delta^{K_A}\times\Delta^{K_B})$ with boundary conditions satisfying (11) and such that

$$J(\gamma^k)\to\inf J.$$

By assumptions (A1)–(A4) there exists a superlinear function $S:\mathbb{R}^+\to\mathbb{R}^+$ such that

$$\int_0^{T_0}S(||\dot{\gamma}^k||)\mathrm{d}t\le J(\gamma^k)\le C.$$

Hence, by Lemma 1 there exists $\gamma^*\in AC([0,T_0];\Delta^{K_A}\times\Delta^{K_B})$ such that the constraints (11) hold and a subsequence γ^{k_j} of γ^k such that $\gamma^{k_j}\to\gamma^*$ uniformly in $[0,T_0]$. Moreover, we have that $\dot{\gamma}^{k_j}\to\dot{\gamma}^*$ weakly in $L^1([0,T_0];\Delta^{K_A}\times\Delta^{K_B})$. Finally, by lower semicontinuity, we obtain that

$$J(\gamma^*)\le\liminf_k J(\gamma^k).$$

As a consequence, we obtain that

$$J(\gamma^*)=\inf_{\gamma\in AC\left([0,T_0];\Delta^{K_A}\times\Delta^{K_B}\right)}J(\gamma)$$

thus proving the existence of a minimizer. Uniqueness of the minimizer follows from convexity of the integrand in J with respect to $(\gamma,\dot{\gamma})$. □

We remark that the minimizer $\gamma^*:[0,T_0]\to\Delta^{K_A}\times\Delta^{K_B}$ of (8)–(9)–(10) is piecewise C^1. This follows from convexity of the integrand of J with respect to $(\gamma,\dot{\gamma})$ combined with strict convexity with respect to $\dot{\gamma}$ (see Clarke 1989, Cor. 2.5 and Prop. 2.1 for further details).

A remark regarding the nature of the problem under consideration herein is in order at this point. Notice that the optimization problem (8)–(9)–(10) is coupled with two (stochastic) optimal-investment problems of the form (7) via the evolution of each agent wealth process and its impact on the corresponding price functions on which the objective functional J

in (8) and the constraint (10) both depend on. If the price functions p_A and p_B are deterministic, corresponding to the modeling scenario where the agents disregard the financial market information on the determination of the contingent claim pricing functions, then the optimization problem (8)–(9)–(10) is deterministic and so are the optimal beliefs γ^* whose existence was proved in Theorem 1. However, a more realistic point of view is the one where the information collected regarding the financial market evolution is used by the agents to infer appropriate price mechanisms. In this case, the objective functional J can be seen as possessing a parametric dependence on $\omega \in \Omega$, i.e. J can be seen as a random variable taking values on a space of appropriate functionals. Nevertheless, for each $\omega \in \Omega$, Theorem 1 will hold, yielding appropriate optimal beliefs γ^*, which can be seen as a random variable taking values in the space of absolutely continuous paths $AC([0, T_0]; \Delta^{K_A} \times \Delta^{K_B})$. We also note that this feature has an impact on the results presented in Sect. 3.2, all of which hold pointwise with respect to $\omega \in \Omega$. However, for simplicity of notation and clarity of presentation, we drop the dependence on $\omega \in \Omega$ in what follows, while keeping the implicit relation in mind.

3.2 Qualitative properties

We can think of the minimizer of (8)–(9)–(10) as the orbit of a finite-horizon continuous-time dynamical system with initial condition $(\gamma_A(0), \gamma_B(0)) = (\gamma_A^0, \gamma_B^0) \in \Delta^{K_A} \times \Delta^{K_B}$ and terminal condition on the set

$$\left\{(\gamma_A, \gamma_B) \in \Delta^{K_A} \times \Delta^{K_B} : p_A(T_0, \gamma_A) = p_B(T_0, \gamma_B)\right\}.$$

Theorem 2 *Let $\alpha \in (0, 1)$ be fixed and assume that the hypotheses of Theorem 1 hold. Then, the following statements hold for the solution $\gamma^* = \{\big(\gamma_A^*(t), \gamma_B^*(t)\big) \in \Delta^{K_A} \times \Delta^{K_B} : t \in [0, T_0]\}$ of the variational problem (8)–(10):*

a) if there exists $t \in [0, T_0)$ such that $p_A(t, \gamma_A^*(t)) = p_B(t, \gamma_B^*(t))$, then for every u such that $u > t$ we have that

$$p_A\left(u, \gamma_A^*(u)\right) = p_B\left(u, \gamma_B^*(u)\right);$$

b) for every $t \in [0, T_0]$, the inequality below holds

$$p_A\left(t, \gamma_A^*(t)\right) \geq p_B\left(t, \gamma_B^*(t)\right);$$

c) at $t = T_0$, we have that

$$p_A\left(T_0, \gamma_A^*(T_0)\right) = p_B\left(T_0, \gamma_B^*(T_0)\right);$$

d) for every $t \in [0, T_0]$, the map $t \to p_A(t, \gamma_A^*(t))$ is non-increasing and the map $t \to p_B(t, \gamma_B^*(t))$ is non-decreasing.

Moreover, the minimizers $\gamma^* = (\gamma_A^*, \gamma_B^*)$ depend continuously on the relative bargaining power $\alpha \in (0, 1)$, as well as on the initial beliefs $(\gamma_A^0, \gamma_B^0) \in \Delta^{K_A} \times \Delta^{K_B}$ and the initial wealth levels $x_A, x_B \in \mathbb{R}$.

Proof We will start by proving item a). Let $\gamma^* = \{\gamma^*(t) : t \in [0, T_0]\}$ be a solution of the optimization problem (8)–(11). By contradiction, assume that γ^* is such that there exists $t \in [0, T_0)$ and $s \in (t, T_0)$ such that

$$p_A\left(t, \gamma_A^*(t)\right) = p_B\left(t, \gamma_B^*(t)\right) \quad \text{and} \quad p_A\left(s, \gamma_A^*(s)\right) > p_B\left(s, \gamma_B^*(s)\right).$$

Let $\overline{\gamma}(t)$ be the trajectory defined by

$$\overline{\gamma}(u) = \begin{cases} \gamma^*(u)\,, \; u \in [0, t] \\ \gamma^*(t)\,, \; u \in (t, T_0] \end{cases}. \tag{15}$$

Then, we have that

$$J(\overline{\gamma}) < J(\gamma^*)\,,$$

contradicting the minimality of γ^*.

To prove item b) assume, by contradiction, that γ^* is such that there exists $t \in [0, T_0)$ and $s \in (t, T_0)$ such that

$$p_A\left(t, \gamma_A^*(t)\right) \geq p_B\left(t, \gamma_B^*(t)\right) \quad \text{and} \quad p_A\left(s, \gamma_A^*(s)\right) < p_B\left(s, \gamma_B^*(s)\right).$$

Then, by continuity of p_A, p_B and γ^*, there exists $u \in [t, s)$ such that $p_A(u, \gamma_A^*(u)) = p_B(u, \gamma_B^*(u))$. The contradiction follows from the statement of item a).

Item c) follows from item b) and the constraint

$$p_A\left(T_0, \gamma_A^*(T_0)\right) \leq p_B\left(T_0, \gamma_B^*(T_0)\right).$$

In what concerns the proof of item d), we consider only the case of the price function p_A, the proof for the price function p_B being similar. By contradiction, assume that γ^* is such that there exists an interval $[t_a, t_b] \subseteq [0, T_0]$ such that $t \to p_A(t, \gamma_A^*(t))$ is increasing in $[t_a, t_b]$. We need to distinguish between the following two cases: $t_b = T_0$ and $t_b < T_0$. In the first case, we define

$$\overline{\gamma}_A(u) = \begin{cases} \gamma_A^*(u)\,, \; u \in [0, t_a] \\ \gamma_A^*(t_a)\,, \; u \in (t_a, T_0] \end{cases}. \tag{16}$$

Since $p_A(T_0, \gamma_A^*(T_0)) > p_A(T_0, \overline{\gamma}_A(T_0))$ and $p_B(T_0, \gamma_B^*(T_0)) = p_A(T_0, \gamma_A^*(T_0))$, then by continuity of $\gamma_B^*(t)$, $t \in [0, T_0]$, there exists $t_c \in [t_a, T_0)$ such that

$$p_B\left(t_c, \gamma_B^*(t_c)\right) = p_A\left(t_a, \gamma_A^*(t_a)\right).$$

Define

$$\overline{\gamma}_B(u) = \begin{cases} \gamma_B^*(u)\,, \; u \in [0, t_c] \\ \gamma_B^*(t_c)\,, \; u \in (t_c, T_0] \end{cases} \tag{17}$$

and let $\overline{\gamma}(t) = (\overline{\gamma}_A(t), \overline{\gamma}_B(t))$, $t \in [0, T_0]$. By construction of $\overline{\gamma}$, we have that

$$J(\overline{\gamma}) < J(\gamma^*)\,,$$

contradicting minimality of γ^*. We consider now the case where $t_b < T_0$. We need to consider the following two subcases: $p_A(T_0, \gamma_A^*(T_0)) \geq p_A(t_a, \gamma_A^*(t_a))$ and $p_A(T_0, \gamma_A^*(T_0)) < p_A(t_a, \gamma_A^*(t_a))$. For the first case, we let $\overline{\gamma}_A(t)$, $t \in [0, T_0]$ be as given in (16) and by the same reasoning used for the case $t_b = T_0$, we have that there exists $t_c \in [t_a, T_0]$ such that $p_B(t_c, \gamma_B^*(t_c)) = p_A(t_a, \gamma_A^*(t_a))$. Hence, letting $\overline{\gamma}_B(t)$, $t \in [0, T_0]$, we get that

$$J(\overline{\gamma}) < J(\gamma^*)\,,$$

contradicting minimality of γ^* once again. Finally, we consider the case where $t_b < T_0$ and $p_A(T_0, \gamma_A^*(T_0)) < p_A(t_a, \gamma_A^*(t_a))$. Since $p_A(t, \gamma_A^*(t))$ is increasing in $[t_a, t_b]$ and $p_A(T_0, \gamma_A^*(T_0)) < p_A(t_a, \gamma_A^*(t_a))$, by continuity of $\overline{\gamma}_A^*(t)$ there exists $t_c \in [t_b, T_0[$ such that $p_A(t_c, \gamma_A^*(t_c)) = p_A(t_a, \gamma_A^*(t_a))$. Define

$$\overline{\gamma}_A(u) = \begin{cases} \gamma_A^*(u)\,, & u \in [0, t_a] \\ \gamma_A^*(t_a)\,, & u \in (t_a, t_c] \\ \gamma_A^*(u)\,, & u \in (t_c, T_0] \end{cases}$$

and let $\overline{\gamma}(t) = (\overline{\gamma}_A(t), \gamma_B^*(t)),\ t \in [0, T_0]$. By construction of $\overline{\gamma}$, we obtain that

$$J(\overline{\gamma}) < J(\gamma^*)\,,$$

contradicting minimality of γ^*.

The continuity of the solution γ^* of the optimization problem (8) with respect to $(\gamma_A^0, \gamma_B^0, x_A, x_B, \alpha) \in \Delta^{K_A} \times \Delta^{K_B} \times \mathbb{R}^2 \times [0, 1]$ is a consequence of Berge's maximum Theorem (1997, Ch. VI, Sec. 3), which guarantees continuity of the minimal functional

$$J\left(\gamma^*; \gamma_A^0, \gamma_B^0, x_A, x_B, \alpha\right)$$

with respect $\gamma_A^0 \in \Delta^{K_A}$, $\gamma_B^0 \in \Delta^{K_B}$, $\alpha \in [0, 1]$, and $x_A, x_B \in \mathbb{R}$, and upper semicontinuity of the correspondence given by

$$\left(\gamma_A^0, \gamma_B^0, x_A, x_B, \alpha\right) \rightarrow \gamma^*\left(\gamma_A^0, \gamma_B^0, x_A, x_B, \alpha\right),$$

completing the proof. □

For the remaining of this section we will assume that the conditions of Theorem 2 are satisfied. Fix $x_A, x_B \in \mathbb{R}$, let $\mathcal{A}^+$ denote the set

$$\mathcal{A}^+ = \left\{\left(\gamma_A^0, \gamma_B^0\right) \in \Delta^{K_A} \times \Delta^{K_B} : p_A\left(0, \gamma_A^0\right) \geq p_B\left(0, \gamma_B^0\right)\right\},$$

and let $\mathcal{E}$ denote the set

$$\mathcal{E} = \mathcal{A}^+ \times [0, 1].$$

We define the *common price* map $p^* : \mathcal{E} \rightarrow \mathbb{R}$ by

$$\begin{aligned} p^*\left(\gamma_A^0, \gamma_B^0, \alpha\right) &= \left\{p_A\left(T_0, \gamma_A^*(T_0)\right) : \{\gamma^*(t)\}_{t \in [0, T_0]} \text{ is the minimizer of } (8)\right\} \\ &= \left\{p_B\left(T_0, \gamma_B^*(T_0)\right) : \{\gamma^*(t)\}_{t \in [0, T_0]} \text{ is the minimizer of } (8)\right\}. \end{aligned}$$

Note that the common price map p^* is a single valued map whose image corresponds to the actual price for which the asset is eventually traded.

Proposition 1 *The common price map p^* is continuous on $\mathcal{E}$.*

Proof Follows from the assumption $(A4)$ and the second part of Theorem 2. □

The next result describes the dependence of the common price map p^* with respect to the relative bargaining power $\alpha \in [0, 1]$.

Proposition 2 *Assume that the assumptions of Theorem 2 hold and fix the agents initial beliefs $(\gamma_A^0, \gamma_B^0) \in \mathcal{A}^+$. Let $p^*(\alpha)$ denote the dependence of the common price map on the agents relative bargaining power α. The following statements hold:*

(i) if $\alpha = 0$ then $\gamma_B^*(t) = \gamma_B^0$ for every $t \in [0, T_0]$ and $p(0) = p_B(0, \gamma_B^0)$;
(ii) if $\alpha = 1$ then $\gamma_A^*(t) = \gamma_A^0$ for every $t \in [0, T_0]$ and $p(1) = p_A(0, \gamma_A^0)$;
(iii) the graph of $p^*(\alpha)$ is non decreasing with α.

Proof The proofs of items (i) and (ii) are similar to Theorem 2 items b) and c) and we skip it. Item (iii) follows by Theorem 2, the form of functional J in (8) and the definition of the map $p^*(\alpha)$. □

In the next section we will provide an example satisfying the assumptions used in this section.

4 A family of pricing functions with appropriate convexity properties

We note that the functions to which assumptions (A1)–(A3) used in Theorem 1 refer to, play the role of "generalized" distance functions. A particular simple example is to take

$$\psi_\beta(t,x) = x^2 \,, \quad \beta = A, B \qquad \text{and} \qquad \phi(x) = x^2.$$

However, it should be clear that a very large class of functions satisfy assumptions (A1)–(A3). In this section we provide an example of a family of price functions satisfying the assumptions (A4)–(A5) in Theorem 1, namely that the price functions p_A and p_B are, respectively, strictly concave and strictly convex with respect to the agents beliefs $\gamma_A \in \Delta^{K_A}$ and $\gamma_B \in \Delta^{K_B}$, and that there exist initial conditions $\gamma_A^0 \in \Delta^{K_A}$ and $\gamma_B^0 \in \Delta^{K_B}$ such that $p_A(0, \gamma_A^0) \geq p_B(0, \gamma_B^0)$, for fixed levels of initial wealth $x_A, x_B \in \mathbb{R}$.

For each $\beta \in \{A, B\}$, $i \in \{1, \ldots, K_\beta\}$ and $t \in [0, T_0]$, let $F_i^\beta(t)$ be the quantity given by

$$F_i^\beta(t) = E\left[F\chi_{\omega_i^\beta} \middle| \mathcal{F}_t\right],$$

i.e. $F_i^\beta(t)$ is the mean value with respect to the probability measure P of the contingent claim payoff restricted to the set $\omega_i^\beta \in \Pi_\beta$ given the information $\mathcal{F}_t$ available to a market observer during the interval $[0, t]$. Since F is assumed to be bounded, then $F_i^\beta(t)$ is finite for every $\beta \in \{A, B\}$, $i \in \{1, \ldots, K_\beta\}$ and $t \in [0, T_0]$. Let $\tilde{F}^\beta(t)$ be the stochastic process with value $F_i^\beta(t)$ on each set $\omega_i^\beta \subseteq \Omega$, for each $t \in [0, T_0]$.

For each $t \in [0, T_0]$ and each $\gamma_\beta \in \Delta^{K_\beta}$, we define agent β price function $p_\beta : [0, T_0] \times \Delta^{K_\beta} \to \mathbb{R}$ as the unique real number P satisfying the implicit relation

$$U_\beta\left(\rho_{t,T} X_{0,x_\beta}^{\beta,\pi_\beta}(t)\right) = E_{Q_\beta}\left[U_\beta\left(\rho_{t,T} X_{0,x_\beta}^{\beta,\pi_\beta}(t) + l_\beta \rho_{T_0,T} P - l_\beta \tilde{F}^\beta(t)\right)\right], \tag{18}$$

where

$$E_{Q_\beta}\left[U_\beta\left(\rho_{t,T} X_{0,x_\beta}^{\beta,\pi_\beta}(t) + l_\beta \rho_{T_0,T} P - l_\beta \tilde{F}^\beta(t)\right)\right]$$
$$= \sum_{i=1}^{K_\beta} a_i^\beta U_\beta\left(\rho_{t,T} X_{0,x_\beta}^{\beta,\pi_\beta}(t) + l_\beta \rho_{T_0,T} P - l_\beta F_i^\beta(t)\right)$$

for the measure $Q_\beta \in \mathcal{P}_\beta(\Omega)$ associated with $\gamma_\beta = \left(a_1^\beta, \ldots, a_{K_\beta}^\beta\right) \in \Delta^{K_\beta}$ and

$$\rho_{t,T} = \exp\left(-\int_t^T r(s)\mathrm{d}s\right)$$

is a discount factor.

Using the properties of the utility functions U_β, $\beta \in \{A, B\}$, and of the expected value $E_{Q_\beta}[\cdot]$, it is possible to check that the price functions $p_\beta : [0, T] \times \Delta^{K_\beta} \to \mathbb{R}$ are well defined. The proof is similar to Azevedo et al. (2013, Lemma 5.8) and we skip it. It is also possible to prove that the price functions p_A and p_B are continuous, being that p_A is strictly concave with respect to $\gamma_A \in \Delta^{K_A}$ and p_B is strictly convex with respect to $\gamma_B \in \Delta^{K_B}$. The proof of this result is similar to that of Lemma 5.9 in Azevedo et al. (2013). Thus, we conclude that the family of pricing functions introduced here satisfies assumption $(A4)$ of Theorem 1. The only property that remains to be checked is the existence of $\gamma_A \in \Delta^{K_A}$ and $\gamma_B \in \Delta^{K_B}$ such that condition $(A5)$ also holds. This is a consequence of the following two lemmas.

Lemma 2 *For each $\beta \in \{A, B\}$ and initial wealth $x_\beta \in \mathbb{R}$, the price function $p_\beta : [0, T] \times \Delta^{K_\beta} \to \mathbb{R}$ satisfies the inequalities*

$$\min_{i \in \{1,\ldots,K_\beta\}} F_i^\beta(t) \leq p_\beta(t, \gamma) \leq \max_{i \in \{1,\ldots,K_\beta\}} F_i^\beta(t)\,,$$

for every $t \in [0, T_0]$ and $\gamma \in \Delta^{K_\beta}$.

The proof of Lemma 2 is analogous to the one of Azevedo et al. (2013, Cor. 5.10) and we skip it.

Lemma 3 *The following inequalities hold for every $t \in [0, T_0]$:*

$$\max_{i \in \{1,\ldots,K_A\}} F_i^A(t) > \min_{j \in \{1,\ldots,K_B\}} F_j^B(t)$$
$$\max_{j \in \{1,\ldots,K_B\}} F_j^B(t) > \min_{i \in \{1,\ldots,K_A\}} F_i^A(t).$$

Proof We only prove the first item, the proof of the second item being similar. Assume that the inequality in the first item does not hold. Then, we have that

$$F_i^A(t) \leq F_j^B(t)\,,$$

for every $i \in \{1, \ldots, K_A\}$, $j \in \{1, \ldots, K_B\}$ and $t \in [0, T_0]$, with strict inequality for at least one pair (i, j) such that $i \in \{1, \ldots, K_A\}$ and $j \in \{1, \ldots, K_B\}$. Hence, we obtain that

$$\sum_{i=1}^{K_A} F_i^A(t) < \sum_{i=1}^{K_B} F_i^B(t). \tag{19}$$

However, by definition of $F_i^\beta(t)$, $\beta \in \{A, B\}$, we have that

$$\sum_{i=1}^{K_\beta} F_i^\beta(t) = \sum_{i=1}^{K_\beta} E\left[F\chi_{\omega_i^\beta} \middle| \mathcal{F}_t\right] = E\left[F \sum_{i=1}^{K_\beta} \chi_{\omega_i^\beta} \middle| \mathcal{F}_t\right] = E\left[F \middle| \mathcal{F}_t\right]\,,$$

which contradicts (19). Thus, the inequality in the first item must hold. □

We remark that the setup described in this section can be applied to the modelling of realistic continuous time barganing examples such as, for instance, financial assets traded over the counter; company mergers and acquisitions; physical infrastructures; and real options, among others. We will now very briefly consider the special case of a bargaining session leading to the exchange of a real asset such as a mine or a factory. Recall the objective functional to be minimized, given by (8). Within the specificity of the particular example under consideration at this point, $\alpha \in (0, 1)$ can be “interpreted” as a measure of the relative bargaining power of the two agents. The disutility that the agents experience when updating their beliefs about the future states of the world is modeled by ψ_β, $\beta \in \{A, B\}$. These can be interpreted as distance functions quantifying the buyer’s and seller’s inertia to move from their currents beliefs, or else, their regret for being forced to update their beliefs in order for a unique price to be reached. In what concerns the funtion ϕ, it models the common disutility experienced by the two agents in terms of the distance between their prospective prices, ultimately leading the bargaining towards a common price for the two agents. Lastly, the constraint (10) ensures that at the transaction time, T_0, the valuation of the buyer is at least as high as the valuation of the seller, so that an agreement can be reached.

5 Conclusion

We study a variational problem modelling the interaction between two agents trading a contingent claim in a incomplete continuous-time multiperiod financial market. We combine techniques from stochastic optimal control theory and variational calculus to prove the existence and uniqueness of solutions to such problem. We conclude with an illustrative example of pricing functions satisfying the assumptions of our main results.

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