A new insight into complexity from the local fractional calculus view point: modelling growths of populations

Xiao-Jun Yang and J. A. Tenreiro Machado

Communicated by A. Debbouche

In this paper, we model the growths of populations by means of local fractional calculus. We formulate the local fractional rate equation and the local fractional logistic equation. The exact solutions of local fractional rate equation and local fractional logistic equation with the Mittag-Leffler function defined on Cantor sets are presented. The obtained results illustrate the accuracy and efficiency for modeling the complexity of linear and nonlinear population dynamics (PD).

Keywords: exact solution; logistic equation; population dynamics; local fractional derivative

1. Introduction

The rate equation (RE) of the population dynamics (PD), structured by T. R. Malthus in 1798 [1, 2], is characterized by the ordinary differential equation (ODE):

$$\frac{d\mathrm{M}(\tau)}{d\tau} = \kappa \mathrm{M}(\tau), \tag{1}$$

where $\mathrm{M}(\tau)$ represents the number of elements of the population in time τ, and κ ($\kappa \neq 0$) is the Malthus constant. With the initial population $\mathrm{M}(0)$, the solution for the RE in time is $\mathrm{M}(\tau) = \mathrm{M}(0)\, e^{\kappa\tau}$, which expresses extinction or growth of the Malthus population.

The logistic equation (LE) of the PD, developed by P. R. Verhulst in 1838 [3, 4], is as follows:

$$\frac{d\mathrm{A}(\tau)}{d\tau} = \kappa \mathrm{A}(\tau)\left(1 - \frac{\mathrm{A}(\tau)}{\gamma}\right), \tag{2}$$

where $\mathrm{A}(\tau) = \mathrm{M}(\tau)/\mathrm{M}_{\max}$, for the population $\mathrm{M}(\tau)$ and the value of the maximum attainable population $\mathrm{M}_{\max}$, κ is the intrinsic growth parameter, and γ is the environmental carrying capacity.

The exact solution of the LE can be written in the form [4]:

$$\mathrm{A}(\tau) = \frac{\gamma}{1 + (\gamma/\mathrm{A}(0) - 1)\, e^{-\kappa\tau}}, \tag{3}$$

where $\mathrm{A}(0) = \mathrm{M}(0)/\mathrm{M}_{\max}$, and $\gamma = \lim_{\tau\to\infty} \mathrm{A}(\tau)$.

At the limit of environmental carrying capacity $\gamma = 1$, the LE becomes [5]

$$\frac{d\mathrm{A}(\tau)}{d\tau} = \kappa \mathrm{A}(\tau)\,(1 - \mathrm{A}(\tau)), \tag{4}$$

and its exact solution is [5]

$$\mathrm{A}(\tau) = \frac{\mathrm{A}(0)}{\mathrm{A}(0) + (1 - \mathrm{A}(0))\, e^{-\kappa\tau}}, \tag{5}$$

where $\mathrm{A}(0) = \mathrm{M}(0)/\mathrm{M}_{\max}$.

Fractional calculus (FC) was successfully used to describe the dynamics systems in engineering and applied science [6–12]. Recently, the applications of FC to the population growth models [5, 13–17] were proposed. The fractional-order RE and LE for fractional population dynamics in the sense of Caputo fractional derivative (FD) were determined by the expressions [5]

$$\frac{d^{\eta}\mathrm{M}(\tau)}{d\tau^{\eta}} = \kappa\mathrm{M}(\tau)\ (0 < \eta < 1) \tag{6}$$

and

$$\frac{d^{\eta}\mathrm{A}(\tau)}{d\tau^{\eta}} = \kappa\mathrm{A}(\tau)\,(1 - \mathrm{A}(\tau))\ (0 < \eta < 1), \tag{7}$$

respectively, where $\mathrm{M}(\tau)$ is the number of the element of the fractional population in time τ, $\mathrm{A}(\tau) = \mathrm{M}(\tau)/\mathrm{M}_{\max}$, κ ($\kappa \neq 0$) is the west constant of fractional population, and η is order of FD.

The solutions of the differentiable types is as follows [5]:

$$\mathrm{M}(\tau) = \mathrm{M}(0)\, E_{\eta}\left(\kappa\tau^{\eta}\right), \tag{8}$$

and

$$\mathrm{A}(\tau) = \sum_{i=0}^{\infty}\left(\frac{\mathrm{A}(0) - 1}{\mathrm{A}(0)}\right)^{i} E_{\eta}\left(-n\kappa\tau^{\eta}\right), \tag{9}$$

where the Mittag-Leffler function (MLF) is defined by using the series

$$E_{\eta}(\xi) = \sum_{i=0}^{n}\frac{\xi^{i}}{\Gamma(1 + i\eta)}. \tag{10}$$

More recently, local fractional calculus (LFC) was presented to explain the complexity of dynamics systems in the fields of mathematical physics, such as the oscillator of free damped vibrations [18], heat transfer [19], diffusion on Cantor sets [20], Navier-Stokes flow [21], Laplace equation (LE) [22], signal processing [23], and others [24–28]. The purpose of this article is to present for modelling complexity of the growth of populations using LFC.

The structure of this article is designed as follows. In Section 2, the recent results for the local fractional derivative (LFD) are presented. In Section 3, the local fractional RE (LFRE) for the PD and its solution of non-differentiable type are discussed. In Section 4, the local fractional LE (LFLE) for the PD and its solution are proposed. In Section 5, the discussions for LFRE and LFLE are given. Finally, the conclusions are outlined in Section 6.

2. Preliminaries

In this section, the concept and properties of LFD and local fractional Laplace transform (LFLT), which is applied in this article, are presented.

Let $\mathrm{C}_{\varsigma}(a, b)$ be a set of the non-differentiable functions (NDFs). The LFD of $\wp(\sigma)$ of order ς $(0 < \varsigma < 1)$ at the point $\sigma = \sigma_0$ is defined as follows ([18–28]):

$$D^{(\varsigma)}\wp(\sigma_0) = \frac{d^{\varsigma}\wp(\sigma_0)}{d\sigma^{\varsigma}} = \lim_{\sigma\to\sigma_0}\frac{\Delta^{\varsigma}(\wp(\sigma) - \wp(\sigma_0))}{(\sigma - \sigma_0)^{\varsigma}}, \tag{11}$$

where

$$\Delta^{\varsigma}(\wp(\sigma) - \wp(\sigma_0)) \cong \Gamma(1 + \varsigma)\left[\wp(\sigma) - \wp(\sigma_0)\right], \tag{12}$$

with $\wp(\sigma_0) \in \mathrm{C}_{\varsigma}(a, b)$.

The properties of the LFD are listed as follows [24, 25]:

1. $D^{(\varsigma)}\left[\Phi(\sigma) \pm \Theta(\sigma)\right] = D^{(\varsigma)}\Phi(\sigma) \pm D^{(\varsigma)}\Theta(\sigma)$, proved $\Phi(\sigma), \Theta(\sigma) \in \mathrm{C}_{\varsigma}(a, b)$;
2. $D^{(\varsigma)}\left[\Phi(\sigma)/\Theta(\sigma)\right] = \left\{\left[D^{(\varsigma)}\Phi(\sigma)\right]\Theta(\sigma) - \Phi(\sigma)\left[D^{(\varsigma)}\Theta(\sigma)\right]\right\}/\Theta^{2}(\sigma)$, provided $\Phi(\sigma), \Theta(\sigma) \in \mathrm{C}_{\varsigma}(a, b)$ and $\Theta(\sigma) \neq 0$.

The Mittag-Leffler function defined on Cantor sets (MLFCS) with the fractal dimension ς is defined as follows [18,24,25]:

$$E_{\varsigma}\left(\sigma^{\varsigma}\right) = \sum_{i=0}^{\infty}\frac{\sigma^{i\varsigma}}{\Gamma(1 + i\varsigma)}. \tag{13}$$

The formulas of the LFD of NDFs [18, 24, 25] are listed in Table I.

Table I. The formulas of the LFD of the NDFs.

NDFs	LFDs
$E_\varsigma(\sigma^\varsigma)$	$E_\varsigma(\sigma^\varsigma)$
$\sigma^\varsigma/\Gamma(1+\varsigma)$	1

LFD, local fractional derivative; NDF, non-differentiable function.

Table II. The operations of LFLTs of the NDFs.

$\wp(\sigma)$	$\wp_\varsigma(s)$
$E_\varsigma(a\sigma^\varsigma)$	$1/(s^\varsigma - a)$
$\sigma^\varsigma/\Gamma(1+\varsigma)$	$1/s^{2\varsigma}$

LFLT, local fractional Laplace transform; NDF, non-differentiable functions.

The LFLT of $\wp(\sigma)$, denoted by [24]

$$\Im_\varsigma[\wp(\sigma)] = \wp_\varsigma(s), \tag{14}$$

is applied to the LFD of the NDFs, namely,

$$\Im_\varsigma\left[\wp^{(\varsigma)}(\sigma)\right] = s^\varsigma \wp_\varsigma(s) - \wp(0). \tag{15}$$

The LFLTs of the NDFs (see, for example, [24]) are shown in Table II.

3. The growth rate for population within local fractional derivative

The conventional and fractional-order REs are differentiable for the linear PD by using the ODEs.

The LFRE of the non-differentiable PD is characterized by using the local fractional ODE:

$$\frac{d^\varsigma \mathrm{M}(\tau)}{d\tau^\varsigma} = \kappa \mathrm{M}(\tau)\,(0 < \varsigma < 1), \tag{16}$$

where $\mathrm{M}(\tau)$ is the number of the fractal population in time τ, κ is the parameter of the fractal population, and ς is the fractal dimension.

Using the LFLT of Eq.(16), we have the following:

$$s^\varsigma \mathrm{M}_\varsigma(s) - \mathrm{M}(0) = \kappa \mathrm{M}_\varsigma(s), \tag{17}$$

which yields

$$\mathrm{M}_\varsigma(s) = \frac{\mathrm{M}(0)}{s^\varsigma - \kappa}. \tag{18}$$

Thus, using the inverse LFLT, the non-differentiable solution for the LFRE is written in closed form:

$$\mathrm{M}(\tau) = \mathrm{M}(0)\,E_\varsigma\left(\kappa\tau^\varsigma\right). \tag{19}$$

4. The growth for population within local fractional derivative

Similarly to the Section 3, conventional and fractional-order LFLEs are differentiable for the nonlinear PD by using the nonlinear ODEs.

The LFLE of the fractal PD is formulated as follows:

$$\frac{d^\varsigma \mathrm{A}(\tau)}{d\tau^\varsigma} = \kappa \mathrm{A}(\tau)\,(1 - \mathrm{A}(\tau))\,(0 < \varsigma < 1), \tag{20}$$

where $\mathrm{A}(\tau) = \mathrm{M}(\tau)/\mathrm{M}_{\max}$ and κ is the intrinsic growth parameter for the fractal population.

The LFLE of the fractal PD may be determined by the following:

$$\frac{d^\varsigma \mathrm{A}(\tau)}{d\tau^\varsigma} = \kappa \mathrm{A}(\tau)\left(1 - \frac{\mathrm{A}(\tau)}{\gamma}\right), \tag{21}$$

where $\mathrm{A}(\tau) = \mathrm{M}(\tau)/\mathrm{M}_{\max}$, κ is the intrinsic growth parameter for the fractal population, and γ is the environmental carrying capacity, which satisfies with $\gamma = \lim_{\tau\to\infty} \mathrm{A}(\tau)$.

With the help of Eq.(5), we can formulate a general solution in the form

$$A_{*}(\tau) = \frac{\psi_A}{\psi_B + \psi_C E_\varsigma(-\kappa\tau^\varsigma)}, \tag{22}$$

where ψ_A, ψ_A, and ψ_A are the parameters for the fractal population dynamics.

Using the properties of the LFD (b), we find that

$$\begin{aligned} &\frac{d^\varsigma A_{*}(\tau)}{d\tau^\varsigma} \\ &= \frac{d^\varsigma}{d\tau^\varsigma}\left(\frac{\psi_A}{{}_{B}+\psi_C E_\varsigma(-\kappa\tau^\varsigma)}\right) \\ &= \frac{{}_{A}\psi_C\kappa E_\varsigma(-\kappa\tau^\varsigma)}{\left[\psi_B+\psi_C E_\varsigma(-\kappa\tau^\varsigma)\right]^2} \\ &= -\frac{-\psi_A\psi_B\kappa E_\varsigma(-\kappa\tau^\varsigma)}{\left[\psi_B+\psi_C E_\varsigma(-\kappa\tau^\varsigma)\right]^2} + \frac{{}_{A}\kappa E_\varsigma(-\kappa\tau^\varsigma)}{{}_{B}+\psi_C E_\varsigma(-\kappa\tau^\varsigma)}, \end{aligned} \tag{23}$$

which leads to the following:

$$\frac{d^\varsigma A_{*}(\tau)}{d\tau^\varsigma} = \kappa A_{*}(\tau)\left(1 - \frac{\psi_B}{\psi_A}A_{*}(\tau)\right), \tag{24}$$

where

$$-\frac{\kappa\psi_B}{\psi_A}A_{*}^2(\tau) = -\frac{-\psi_A\psi_B\kappa E_\varsigma(-\kappa\tau^\varsigma)}{\left[\psi_B+\psi_C E_\varsigma(-\kappa\tau^\varsigma)\right]^2}, \tag{25}$$

and

$$\kappa A_{*}(\tau) = \frac{{}_{A}\kappa E_\varsigma(-\kappa\tau^\varsigma)}{\psi_B+\psi_C E_\varsigma(-\kappa\tau^\varsigma)}. \tag{26}$$

Making use of $\psi_A/\psi_B = 1$ and $A(0) = \psi_A/(\psi_B + \psi_C)$, the non-differentiable solution for Eq.(20) may be written in the form:

$$A(\tau) = \frac{A(0)}{A(0) + (1 - A(0))E_\varsigma(-\kappa\tau^\varsigma)}, \tag{27}$$

where $A(0) = M(0)/M_{max}$.

In view of the expressions $\psi_A/\psi_B = \gamma$ and $A(0) = \psi_A/(\psi_B + \psi_C)$, the exact solution for Eq.(21) is of non-differentiable type:

$$A(\tau) = \frac{\gamma}{1 + (\gamma/A(0) - 1)E_\varsigma(-\kappa\tau^\varsigma)}, \tag{28}$$

where $A(0) = M(0)/M_{max}$.

5. Discussion

Using the limit of the MLFCS

$$\lim_{\varsigma\to 1} E_\varsigma\left(\kappa\sigma^\varsigma\right) = e^{-\kappa\tau}, \tag{29}$$

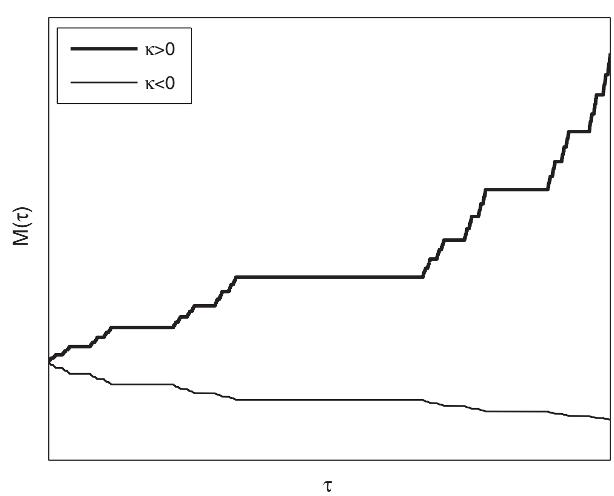

Figure 1. The non-differentiable solution for the local fractional rate equation when $\varsigma = ln2/ln3$.

the solution of the RE for the PD is obtained [1,2]. We remark the non-differentiable solution for the LFRE as follows. Figure 1 shows that the solution of the extremely large fractal population, when $\kappa < 0$, is the extinction of Mittag-Leffler function defined on Cantor sets, and that the solution of the extremely small fractal population, when $\kappa > 0$, is the growth of the Mittag-Leffler function defined on Cantor sets. The RE within conventional derivative (CDRE) [1,2] presents the population in time; The fractional-order RE within fractional derivative (FDRE) [5] depicts the fractional population in time; The LFRE in our manuscript displays the fractal population in time. The CDRE, FDRE, and LFRE and their solutions (TSs) are listed in Table III. The solutions for the CDRE, FDRE, and LFRE, when $\kappa > 0$ and $M(0) = 1$, are represented in Figure 2.

Similarly, using Eq.(29), Eq.(27), and Eq.(28) are rewritten as Eq.(5) and Eq.(3), respectively. The LE within conventional derivative (CDLE) [1,2] depicts the PD in time; The fractional-order LE within fractional derivative (FDLE) [5] exhibits the fractional PD in time; The LFLE in our manuscript shows the fractal PD in time. The exact solutions of the LFLE for nonlinear PD in different initial-values are illustrated in Figure 3.

The CDLE, FDLE, and LFLE and TSs are listed in Table IV.

Table III. REs of different types.

REs	TSs
$\frac{dM(\tau)}{d\tau} = \kappa M(\tau)$	$M(\tau) = M(0)\, e^{\kappa\tau}$
$\frac{d^{\eta}M(\tau)}{d\tau^{\eta}} = \kappa M(\tau)$	$M(\tau) = M(0)\, E_{\eta}(\kappa\tau^{\eta})$
$\frac{d^{\varsigma}M(\tau)}{d\tau^{\varsigma}} = \kappa M(\tau)$	$M(\tau) = M(0)\, E_{\varsigma}(\kappa\tau^{\varsigma})$

RE, rate equation.

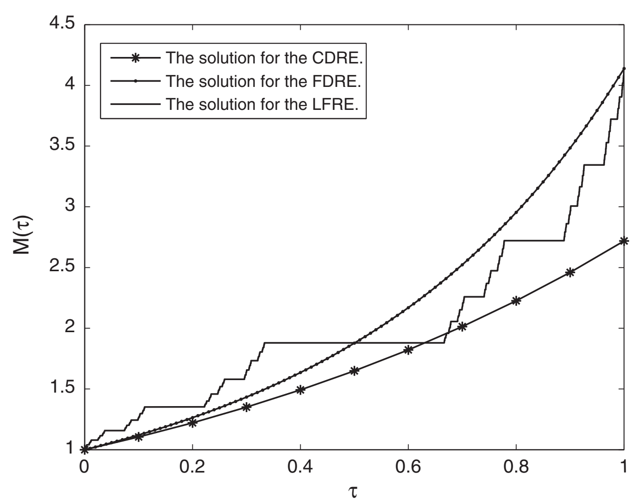

Figure 2. The solutions for the RE within conventional derivative, RE within fractional derivative, and local fractional rate equation when $\kappa > 0$ and $M(0) = 1$.

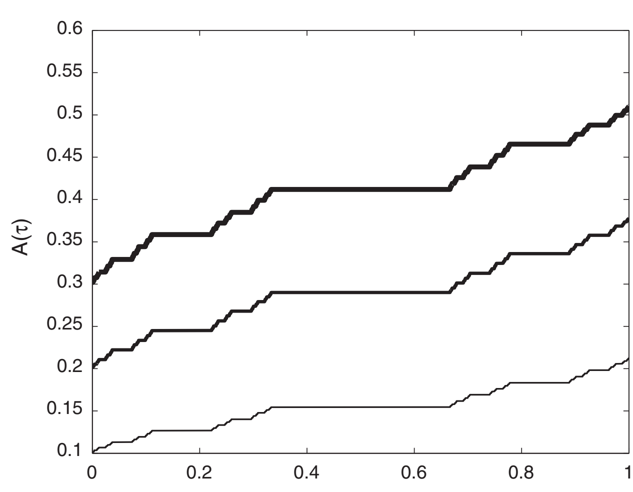

Figure 3. The closed solution of local fractional logistic equation when $\varsigma = ln2/ln3$.

Table IV. LEs of different types.

LEs	TSs
$\frac{d\mathrm{A}(\tau)}{d\tau} = \kappa \mathrm{A}(\tau)(1 - \mathrm{A}(\tau))$	$\mathrm{A}(\tau) = \frac{\mathrm{A}(0)}{\mathrm{A}(0)+(1-\mathrm{A}(0))e^{-\kappa\tau}}$
$\frac{d^{\eta}\mathrm{A}(\tau)}{d\tau^{\eta}} = \kappa \mathrm{A}(\tau)(1 - \mathrm{A}(\tau))$	$\mathrm{A}(\tau) = \sum_{i=0}^{\infty}\left(\frac{\mathrm{A}(0)-1}{\mathrm{A}(0)}\right)^{i} E_{\eta}(-n\kappa\tau^{\eta})$
$\frac{d^{\varsigma}\mathrm{A}(\tau)}{d\tau^{\varsigma}} = \kappa \mathrm{A}(\tau)(1 - \mathrm{A}(\tau))$	$\mathrm{A}(\tau) = \frac{\mathrm{A}(0)}{\mathrm{A}(0)+(1-\mathrm{A}(0))E_{\varsigma}(-\kappa\tau^{\varsigma})}$

LE, logistic equation.

6. Conclusions

This work presented a new application of LFC to model the complexity of linear and nonlinear PD. The LFRE and LFLE of the growth in human population and their closed solutions with non-differentiable graphs were used to explain the PD. Comparative results of two models with CD, FD, and LFD were discussed. It is shown that LFC is a relevant tool to analyze the PD.

References

1. Malthus TR. *Malthus—Population: The First Essay (1798)*. University of Michigan Press: Ann Arbor, MI, 1959.
2. Royama T. *Analytical Population Dynamics*. Springer: Netherlands, 2012.
3. Verhulst PF. Notice sur la loi que la population sint dons son accroissement. *Mathematical Physics* 1838; **10**:113–121.
4. Webb GF. *Theory of Nonlinear Age-dependent Population Dynamics*. CRC Press: New York, 1985.
5. West BJ. Exact solution to fractional logistic equation. *Physics A* 2015; **429**:103–108.
6. Klafter J, Lim SC, Metzler R. *Fractional Dynamics: Recent Advances*. World Scientific: London, 2011.
7. Povstenko Y. *Linear Fractional Diffusion-wave Equation for Scientists and Engineers*. Birkhäuser: Basel, 2015.
8. Mainardi F. *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models*. World Scientific: London, 2010.
9. Tarasov VE. *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*. Springer: Heidelberg, 2011.
10. Ortigueira MD. *Fractional Calculus for Scientists and Engineers*. Springer: New York, 2011.
11. Jesus IS, Machado JT. Fractional control of heat diffusion systems. *Nonlinear Dynamics* 2008; **54**(3):263–282.
12. Machado JT, Galhano AM, Trujillo JJ. On development of fractional calculus during the last fifty years. *Scientometrics* 2014; **98**(1):577–582.
13. Abbas S, Banerjee M, Momani S. Dynamical analysis of fractional-order modified logistic model. *Computers & Mathematics with Applications* 2011; **62**(3):1098–1104.
14. El-Sayed AMA, El-Mesiry AEM, El-Saka HAA. On the fractional-order logistic equation. *Applied Mathematics Letters* 2007; **20**(7):817–823.
15. Momani S, Qaralleh R. Numerical approximations and Padé approximants for a fractional population growth model. *Applied Mathematical Modelling* 2007; **31**(9):1907–1914.
16. Xu H. Analytical approximations for a population growth model with fractional order. *Communications in Nonlinear Science* 2009; **14**(5):1978–1983.
17. Suat Erturk V, Yildirim A, Momanic S, Khan Y. The differential transform method and Pade approximants for a fractional population growth model. *International Journal of Numerical Methods for Heat and Fluid Flow* 2012; **22**(6):791–802.
18. Yang XJ, Srivastava HM. An asymptotic perturbation solution for a linear oscillator of free damped vibrations in fractal medium described by local fractional derivatives. *Communications in Nonlinear Science and Numerical Simulation* 2015; **29**(1):499–504.
19. Yang XJ, Srivastava HM, He JH, Baleanu D. Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives. *Physics Letters A* 2013; **377**(28):1696–1700.
20. Yang XJ, Baleanu D, Srivastava HM. Local fractional similarity solution for the diffusion equation defined on Cantor sets. *Applied Mathematics Letters* 2015; **47**:54–60.
21. Liu HY, He JH, Li ZB. Fractional calculus for nanoscale flow and heat transfer. *International Journal of Numerical Methods for Heat and Fluid Flow* 2014; **24**(6):1227–1250.
22. Yan SP, Jafari H, Jassim HK. Local fractional Adomian decomposition and function decomposition methods for Laplace equation within local fractional operators. *Advances in Mathematical Physics* 2014; **2014, 1-7**:Article ID 161580.
23. Chen ZY, Cattani C, Zhong WP. Signal processing for nondifferentiable data defined on Cantor sets: a local fractional Fourier series approach. *Advances in Mathematical Physics* 2014; **2014, 1-7**:Article ID 561434.
24. Yang XJ, Baleanu D, Srivastava HM. *Local Fractional Integral Transforms and Their Applications*. Academic Press: New York, 2015.
25. Yang XJ. *Advanced Local Fractional Calculus and its Applications*. World Science: New York, 2012.
26. Yang XJ, Srivastava HM, Cattani C. Local fractional homotopy perturbation method for solving fractal partial differential equations arising in mathematical physics. *Romanian Reports in Physics* 2015; **67**(3):752–761.
27. Yang XJ, Baleanu D, Baleanu MC. Observing diffusion problems defined on cantor sets in different coordinate systems. *Thermal Science* 2015; **19**(S1):65–65.
28. Yang XJ, Baleanu D, Lazarevic MP, Cajic MS. Fractal boundary value problems for integral and differential equations with local fractional operators. *Thermal Science* 2015; **19**(3):959–966.