Completion of markets by variation processes

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Outline

Martingale Representation Theorem

- If we assume that $X_t \in R^n$ and that $\tilde{B}(t) \in R^n$ (recipe for completeness), then the value of portfolio $\Theta \in \mathbb{R}^n$ is $V(t) = V(0) + \int_0^t \Theta(s)dX(s)$.
- It can be shown that $e^{-\rho T}V(T) = z + \int_0^T \phi(t)d\tilde{B}(t) = z + \int_0^T \sum_{j=1}^n \phi_j(t)d\tilde{B}_j(t) \quad z \in \mathbb{R} \quad (**)$
- Thus $E_Q[e^{-\rho T}V(T)] = z$ and $\phi(t)$ is related to $\Theta(t)$ in a special way. It turns out that $z = E_Q[e^{-\rho T}F(\omega)$ in a complete market and $z = v(0, x)$ is the price at time 0 of the contingent claim.
- Therefore a market is complete iff there exists $z \in \mathbb{R}$ and $\Theta(.)$ such that $(**)$ is satisfied.
- What if we fail to find z or $\Theta(.)$?

Incomplete markets

If there is more than one risk neutral measure,

- The market is incomplete.
- Not every contingent claim $F(\omega)$ is attainable in the form of $(**)$
- There is an infinite number of prices for each contingent claim.
- This implies that buyers and sellers do not agree on a unique price
- It was proved that $p_0^b(F) \leq p_0(F) \leq p_0^s(F)$
- The problem is to find the “best" price and “perfect hedge"

Many ideas behind pricing

- relative entropy minimizer Q which minimizes
$$I(Q\backslash P) = \begin{cases} E_P\left[\frac{dQ}{dP}\ln\left(\frac{dQ}{dP}\right)\right] & if\ Q << P \\ +\infty & otherwise \end{cases}$$
- General $f-$ divergence
$$I_f(Q\backslash P) = \begin{cases} E_P\left[f\left(\frac{dQ}{dP}\right)\right] & if\ Q << P \\ +\infty & otherwise \end{cases}$$ where f is convex on $[0,\infty[$
- Essecher transform $\frac{dQ_{X(T),h}}{dP} = \frac{e^{hX(T)}}{E_P[e^{hX(T)}]}$
- Mean variance-measure $E_Q\left[V(T) - F\right]^2$
- Utility theory ...Nash equilibrium

Completion of a Lévy market: Nualart et al's idea!

- Completing an incomplete market due to jumps were studied by Nualart et al. For any Lévy process $Z_t = cB_t + X_t$, where $X_t = \int_{\mathbb{R}} z\tilde{N}(dt, dz) + \alpha t$ ($\alpha = E[X_1] - \int_{|z|\geq 1} z\nu(dz)$). Let $\Delta Z_s = Z_s - Z_{s-}$ be the jump process
- Set $Z_t^{(i)} = \sum_{0\leq s\leq t}(\Delta Z_s)^i$ and by default let $Z_t^{(1)} = Z_t$
- It turns out that $E[X_t] = E[X_t^{(1)}] = ta = tm_1 < \infty$ and $E[X_t^{(i)}] = E\left[\sum_{0<s\leq t}(\Delta X_s)^i\right] = t\int_{\mathbb{R}} x^i\nu(dx) = m_i t < \infty, \quad i \geq 2$
- Denote by $Y_t^{(i)} = Z_t^{(i)} - E[Z_t^{(i)}] = Z_t^{(i)} - m_i t, \quad i \geq 1$ (Teugels martingales of order i) and $T^{(i)} = c_{i,i}Y^{(i)} + c_{i,i-1}Y^{(i-1)} + \cdots + c_{i,1}Y^{(1)}, \quad i \geq 1$

A different direction: Complete the market!

- The market of stock ,Z_t and bond and normalized processes $\tilde{H}_t^{(i)} = e^{rt} T_t^{(i)}, \quad i \geq 2$ is complete.
- But if $X_t \equiv 0$, then $\Delta Z_s \equiv 0$ and this method fails!

Mathematical preliminaries

- Consider a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$
- bond price:

$$dS_0(t) = \rho(t)S_0(t)dt \quad (1)$$

- Stock:

$$dS_1(t) = S_1(t)\left[\alpha(t)dt + \sigma_1(t)dB_1(t) + \ldots \sigma_m(t)dB_m(t)\right] \quad (2)$$

- Girsanov theorems:
  $\vec{u}(t) = (u_1(t), u_2(t), \ldots, u_m(t))^{Tr}$
- so that $\sum_{j=1}^{m} \sigma_j . u_j(t) = \alpha(t) - \rho(t)$
- The equivalent martingale measure is Q, given by
  $\frac{dQ}{dP} = Z(T)$ where
  $Z(t) = \exp\left[-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds\right]$ is not

The f^q-variance minimizer

- Recall that the f^q-divergence is defined by $f^q(Q\backslash P) = E_P\left[f^q(Z(T))\right] = E_P\left[Z(T)^q\right]$.
- The f^q-variance minimizer is then the martingale measure Q^{q^*} such that $f^q(Q^{q^*}\backslash P) = \min_{Q\in\mathcal{M}} E_P\left[Z(T)^q\right]$, where $\mathcal{M}$ is the set of equivalent martingale measures and $q \in I$ is arbitrary and $I \in (-\infty, 0) \cup (1, \infty)$.
- $u_j = u_j^q = \dfrac{\sigma_j\left(\alpha - \rho\right)}{\Delta}$ where $\Delta = \displaystyle\sum_{j=1}^{m} \sigma_j^2$ which does not depend on q.
- Let Q^{q^*} be the equivalent martingale measure induced by $\vec{u} = \vec{u}^q = \left(u_1^q, u_2^q, \ldots, u_m^q\right)^{Tr}$
- Then with respect to Q^{q^*}, we have

$$dS_1(t) = S_1(t)\left[\rho dt + \sigma_1 d\tilde{B}_1(t) + \ldots \sigma_m d\tilde{B}_m(t)\right] \qquad (3)$$

Summary

- From now on, we consider that with respect to Q^{q^*} ,all asset parameters are constant. One can then use the measure Q^{q^*} to price any contingent claim F such that $V_t = E_{Q^{q^*}}\left[e^{-\rho(T-t)}F(\omega)|\mathcal{F}_t\right]$.
- However, what is not guaranteed here is for the hedging portfolio of F to exist, in other words, it is possible that for some contingent claim F, there is no self-financing portfolio such that the terminal value of the portfolio at least equals F with positive probability
- A quick example in this case is any payoff which depends on the terminal value of one of the standard Brownian motions, that is, if $F(\omega) = f(\tilde{B}_j(T)), \quad 0 \leq j \leq m$, say, for some measurable function f

Stock price and generated variation assets

We now construct the $i^{th}-$ variation processes as follows:

- We know that $e^{-\rho t} S_1(t)$ is a Q^{q^*}-martingale. Now, let $W_1(t) = e^{-\rho t} S_1(t) - S_0$, then

$$E_{Q^{q^*}}\left[W_1^2(t)\right] < \infty \tag{4}$$

- Let $\mathcal{M}_2$ be the set of all Q^{q^*}-martingales such that (4) holds. Then $W_1(t) \in \mathcal{M}_2$ and $W_1^2(t) - < W_1 >_t$ is a Q^{q^*}-martingale.
- Let $Z_1(t) = W_1^2(t) - < W_1 >_t$ and define $S_2(t)$ by $S_2(t) = e^{\rho t} Z_1(t)$ and $W_2(t) = e^{-\rho t} S_2$ is a Q^{q^*}-martingale
- We continue like this so that the process $Z_i(t) = W_i^2(t) - < W_i >_t$ is a Q^{q^*}-martingale for any $i \geq 1$.

Explicit solutions

- Note here that

$$dS_k(t) = \rho S_k(t)dt + 2^{(k-1)} e^{-(k-1)\rho t} \prod_{j=1}^{k-1} S_j(t).S_1(t) \sum_{j=1}^{m} \sigma_j d\tilde{B}_j(t) \qquad k \geq 1$$

- and

$$dZ_k(t) = 2^k e^{-(k+1)\rho t} \prod_{j=1}^{k} S_j(t).S_1(t) \sum_{j=1}^{m} \sigma_j d\tilde{B}_j(t), \quad k \geq 1$$

- so that

$$e^{-\rho t} S_k(t) = S_k(0) + \int_0^t 2^{k-1} e^{-\rho k u} \prod_{j=1}^{k-1} S_j(u) S_1(u) \sum_{j=1}^{m} \sigma_j d\tilde{B}_j(u) \quad (5$$

The Oksendals

- and

$$Z_k(t) = Z_k(0) + \int_0^t 2^k e^{-(k+1)\rho u} \prod_{j=1}^{k} S_j(u) S_1(u) \sum_{j=1}^{m} \sigma_j d\tilde{B}_j(u) \quad (6)$$

- we define the following processes Y_i as

$$Y_i = a_{i,i} Z_i + a_{i,i-1} Z_{i-1} + \dots \dots + a_{i,1} Z_1, \quad i \geq 1 \qquad (7)$$

- The coefficients $a_{i,j}, \quad i,j \geq 1$ through an orthogonalization process as in Nualart et al
- The processes $\tilde{Y}_k = e^{\rho t} Y_k, \quad k \geq 1$ (where Y_k, given in (7)) will be the orthonomal versions of the processes $S_k(t)$.
- We shall call the processes $\tilde{Y}_i(t)$, the Oksendals

Review of market completeness

- If a market is complete, then any contingent claim $F(\omega)$ can be replicated by a self-financing portfolio of stocks and bonds in that $F(\omega) = V(T)$.
- Moreover, by the martingale representation theorem (MRP) there exists a real number z and an adapted process $\phi(t,\omega) \in \mathbb{R}^{m\times 1}$ such that,

$$F(\omega) = z + \int_0^T \phi(t,\omega) d\tilde{B}(t) \tag{8}$$

where $\tilde{B}(t) = \left(\tilde{B}_1(t), \ldots, \tilde{B}_m(t)\right)^{Tr}$

- In incomplete markets, it is possible to find some contingent claims $F(\omega)$ such that there exists no $\phi(t,\omega)$ such that equation (8) holds

Proposition (Martingale Representation Theorem)

Let $F = e^{-\rho T} V(T) \in L^2\left(\Omega, Q^{q^*}\right)$ *where* $V(T)$ *is the terminal value of a portfolio of bond, stock and* i^{th}*-variation processes. Assume further that* $E\left[\sum_{i=2}^{\infty} \int_0^T h_i(s) dY_i(s)\right]^2 < \infty$*. Then there exist processes* $h(t)$ *and* $h_i(t), \ i \geq 2$ *such that* F *can be written as*

$$F = z + \int_0^T h(s) \sum_{j=1}^{m} \sigma_j d\tilde{B}_j(s) + \sum_{i=2}^{\infty} \int_0^T h_i(s) dY_i(s) \tag{9}$$

where $z \in \mathbb{R}$,and $h(t)$ and $h_i(t), \ i \geq 2$ are adapted processes such that

$$E\left[\int_0^t |h(s)|ds\right] < \infty \text{ and } E\left[\int_0^t \sum_{i=2}^{\infty} |h_i(s)|^2 ds\right] < \infty \qquad (10)$$

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Theorem

An incomplete market model with more noise terms than stocks can be completed by variation processes in the sense that any T-claim F such that $E_{Q^{q*}}[F] < \infty$ can be replicated by a portfolio of bond, stock and ith-variation processes.

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The Greeks

Theorem

Let

$$Y(t) = \left(S_1(t), \tilde{Y}_1(t), \ldots, \tilde{Y}_n(t)\right)^{Tr} = \left(Y_0(t), \tilde{Y}_1(t), \ldots, \tilde{Y}_n(t)\right)^{Tr}$$

with $Y(0) = y_0, \quad \tilde{Y}_i(0) = \tilde{y}_i, \quad \text{for } 1 \leq i \leq n$

and $Y_0(0) = S_1(0) = s_1 = y_0.$ [6]

Then

$$dY(t) = b\left(Y_0(t), \ldots, \tilde{Y}_n(t)\right) dt + \sigma\left(Y_0, \tilde{Y}_1(t), \ldots, \tilde{Y}_n(t)\right) d\tilde{B}(t)$$

where

$$b\left(Y_0(t), \ldots, \tilde{Y}_n(t)\right) =$$

$$\left(b_0(Y_0(t), \ldots, \tilde{Y}_n(t)), \ldots, b_n(Y_0(t), \ldots, \tilde{Y}_n(t))\right)^{Tr} =$$

$$\left(\rho S_1(t), \rho\tilde{Y}_1(t), \ldots, \tilde{Y}_n(t)\right)^{Tr} \text{ and}$$

[6] Note that we shall (with some abuse of notation) also use $y_0, \ \tilde{y}_i$ for respectively the process Y(t) and $\tilde{Y}_i(t)$ starting at any other point $t \geq 0$

$$\sigma\left(Y_0, \tilde{Y}_1(t), \ldots, \tilde{Y}_n(t)\right) =$$
$$\begin{pmatrix} S_1\sigma_1 & S_1\sigma_2 & \ldots & S_1\sigma_m \\ S_1\sigma_1\gamma_i & S_1\gamma_1\sigma_2 & \ldots & S_1\gamma_1\sigma_m \\ \vdots & \vdots & \vdots & \vdots \\ S_1\gamma_n\sigma_1 & S_1\gamma_n\sigma_2 & \ldots & S_1\gamma_n\sigma_m \end{pmatrix}.$$

Consider a function $h \in C_0^2$ then we have

$$h(t) = \frac{\partial}{\partial s_1}(t, Y(t))E^y_{Q^{q*}}\left[h(Y(T-t))\right]S_1(t) \quad (11)$$
$$i.e \;\; \varepsilon(t) = e^{\rho t}\frac{\partial}{\partial s_1}(t, Y(t))E^y_{Q^{q*}}\left[h(Y(T-t))\right] \quad and$$
$$h_i(t) = e^{\rho t}\frac{\partial}{\partial \tilde{y}_i}(t, Y(t))E^y_{Q^{q*}}\left[h(Y(T-t))\right],\; i = 1, 2, \ldots, n \quad (12)$$

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Summary

- Note that if $v(t, Y(t))$ is the value of portfolio $\Theta(t)$ of stock, bond and i^{th}-variation processes, and if $h(Y(t)) = e^{-\rho t} v(t, Y(t))$, then $\Delta(t) = \varepsilon(t) = \frac{\partial}{\partial s_1} E^y_{Q^{q*}}[v(t, Y(T-t))]$ and
- $\tilde{\Delta}_i(t) = \beta_i(t) = \frac{\partial}{\partial \tilde{y}_1} E^y_{Q^{q*}}[v(t, Y(T-t))] \quad i = 2, 3, \ldots, n$ and these are the “deltas" of the securities.

Definition (African option)

An option whose payoff is not attainable in a market of stocks and bonds but which can be hedged by a portfolio of stocks, bonds and Oksendals is called an African option.

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