Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 28 (2011) 485–498
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Existence of weak solutions to the three-dimensional steady compressible Navier–Stokes equations

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Received 23 September 2010; received in revised form 16 December 2010; accepted 28 February 2011

Available online 16 March 2011

Dedicated to Professor Rolf Leis on the occasion of his 80th birthday

Abstract

We prove the existence of a spatially periodic weak solution to the steady compressible isentropic Navier–Stokes equations in $\mathbb{R}^3$ for any specific heat ratio $\gamma > 1$. The proof is based on the weighted estimates of both pressure and kinetic energy for the approximate system which result in some higher integrability of the density, and the method of weak convergence.

Keywords: Steady compressible Navier–Stokes equations; Existence for $\gamma > 1$; Potential estimate; Effective viscous flux

1. Introduction

In this paper we prove the existence of a spatially periodic weak solution $(\rho, \mathbf{u})$ to the following steady isentropic compressible Navier–Stokes equations in $\mathbb{R}^3$ for any specific heat ratio $\gamma > 1$:

$$\operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

$$-\mu\triangle\mathbf{u} - \tilde{\mu}\nabla\operatorname{div}\mathbf{u} + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla P = \rho\mathbf{f}. \tag{1.2}$$

Here $\mathbf{u} = (u_1, u_2, u_3)$ is the velocity and ρ is the density, the viscosity constants μ and $\tilde{\mu}$ satisfy $\mu > 0$, $\tilde{\mu} = \mu + \lambda$ with $\lambda + 2\mu/3 \geqslant 0$, the pressure P for isentropic flows is given by $P(\rho) = a\rho^\gamma$ with a being a positive constant and $\gamma > 1$ being the specific heat ratio. $\mathbf{f} = (f_1, f_2, f_3)$ is the external force, and for simplicity, we assume that

$$\mathbf{f} \in L^\infty(\mathbb{R}^3).$$

Besides, we consider that $(\rho, \mathbf{u})$ and $\mathbf{f}$ are periodic in each x_i with period 2π for all $1 \leqslant i \leqslant 3$.

For simplicity, throughout this paper, we denote by Ω the periodic cell $(-\pi, \pi)^3$.

In general, there could be no solution for arbitrary $\mathbf{f}$, since for a (smooth) solution, which is periodic in x with period 2π, $\mathbf{f}$ has to satisfy the necessary condition:

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2011.02.008

$$\int_{\Omega} \rho f_i \, dx = 0 \quad \text{for } 1 \leqslant i \leqslant 3. \tag{1.3}$$

However, if we consider $\mathbf{f}$ with symmetry

$$f_i(x) = -f_i\big(Y_i(x)\big) \quad \text{and} \quad f_i(x) = f_i\big(Y_j(x)\big), \quad \text{if } i \neq j,\ i, j = 1, 2, 3, \tag{1.4}$$

where

$$Y_i(\ldots, x_i, \ldots) = (\ldots, -x_i, \ldots), \tag{1.5}$$

then $\mathbf{u}$ will have the same symmetry and ρ with the symmetry

$$\rho(x) = \rho\big(Y_i(x)\big) \quad \text{for } i = 1, 2, 3, \tag{1.6}$$

and condition (1.3) is satisfied automatically. Moreover, $\mathbf{u}$ satisfies

$$\int_{\Omega} u_i(x) \, dx = 0 \quad \text{for all } 1 \leqslant i \leqslant 3. \tag{1.7}$$

So, in this paper we will consider the external force $\mathbf{f}$ that has the symmetry (1.4).

In the last decades, the well-posedness of Eqs. (1.1), (1.2) for large $\mathbf{f}$ has been investigated by a number of researchers. In 1998, under the assumption that $\gamma > 1$ in two dimensions and $\gamma > 5/3$ in three dimensions, Lions [12] proved the existence of weak solutions to (1.1), (1.2). Roughly speaking, the condition on γ comes from the integrability of the density ρ in L^p. The higher integrability of ρ has, the smaller γ can be allowed. If $\mathbf{f}$ is potential, then weak solutions are shown to exist for any $\gamma > 3/2$, see [14]. Then, Frehse, Goj and Steinhauer [6], Plotnikov and Sokolowski [15] obtained an improved integrability bound for the density by deriving a new weighted estimate of the pressure, assuming a priori the L^1-boundedness of $\rho\mathbf{u}^2$ which, unfortunately, was not shown to hold. Recently, by combining the L^∞-estimate of $\triangle^{-1} P$ with the (usual) energy and density bounds, Březina and Novotný [2] were able to show the existence of weak solutions to the spatially periodic problem with symmetries (1.4)–(1.6) for any $\gamma > (3 + \sqrt{41})/8$ when $\mathbf{f}$ is potential, or for any $\gamma > 1.53$ when $\mathbf{f} \in L^\infty$, without assuming the boundedness of $\rho\mathbf{u}^2$ in L^1. More recently, Frehse, Steinhauer and Weigant [8] established the existence of weak solutions to the Dirichlet problem in three dimensions for any $\gamma > 4/3$ in the framework of [2]. Also, the existence of a weak solution to (1.1), (1.2) with periodic or mixed boundary conditions was obtained in the two-dimensional isothermal case ($\gamma = 1$) [7].

The aim of this paper, inspired by the works [8,2], is to improve the existence result in [2], namely, we shall prove the existence of spatially periodic weak solutions to the system (1.1), (1.2) for any $\gamma > 1$, thus extending the existence in [2] from $\gamma > 4/3$ to $\gamma > 1$. Roughly speaking, the basic idea in our proof is to employ a careful bootstrap argument to obtain the higher integrability of the density which eventually relaxes the restriction on γ in [2], see the paragraph below Theorem 1.2 for more details on the proof idea.

We mention that for a 3D model of steady compressible heat-conducting flows (i.e., the steady compressible Navier–Stokes–Fourier system), Mucha and Pokorný [13] recently studied the existence of weak solutions under some assumptions on the pressure and heat-conductivity, which unfortunately excludes the case of polytropic idea gases. For the corresponding non-steady system (to (1.1), (1.2)) with large initial data, Lions [12] first proved the global existence of weak solutions in the case of $\gamma \geqslant 3n/(n+2)$ ($n = 2, 3$: dimension). His result has been improved and generalized recently in [5,10,11] and among others, where the condition $\gamma > 3/2$ is required in three dimensions for general initial data.

Before defining a weak solution to (1.1), (1.2), we introduce the notation used throughout this paper.

Notation. Let G be a domain in $\mathbb{R}^3$ or the periodic cell. We denote by $L^p(G)$ the Lebesgue spaces, by $W^{k,p}(G)$ ($k \in \mathbb{N}$) the usual Sobolev spaces, by $C^k(G)$ (resp. $C^k(\overline{G})$) the space of k times continuously differentiable functions in G (resp. $\overline{G}$). We define

$$\mathcal{D}\big(\mathbb{R}^3\big) = \big\{\phi(x) \in C^\infty\big(\mathbb{R}^3\big),\ \phi(x) \text{ is periodic in } x_i \text{ of period } 2\pi \text{ for all } 1 \leqslant i \leqslant 3\big\}$$

and

$$\mathcal{D}(G) = \big\{\phi(x) \bigm| \exists\tilde{\phi}(x) \in \mathcal{D}\big(\mathbb{R}^3\big),\ \text{s.t. } \phi(x) = \tilde{\phi}(x),\ \text{for } x \in G\big\}.$$

By $\mathcal{D}'(\mathbb{R}^3)$ (resp. $\mathcal{D}'(G)$), we denote the dual space of $\mathcal{D}(\mathbb{R}^3)$ (resp. $\mathcal{D}(G)$). For example, $\mathcal{D}'(\mathbb{R}^3)$ is the space of periodic distributions in $\mathbb{R}^3$ (dual to $\mathcal{D}(\mathbb{R}^3)$). We also introduce the space of symmetric functions: $(W^{k,p}_{\mathrm{sym}}(\Omega))^3$ denotes the space of vector functions in $(W^{k,p}(\Omega))^3$ which possess the symmetry (1.4), while $L^p_{\mathrm{sym}}(\Omega)$ stands for the space of functions in $L^p(\Omega)$ with symmetry (1.6). $B_R(a) := \{x \in \mathbb{R}^3\colon |x-a| < R\}$ denotes the open ball centered at a with radius R.

Now, let us recall the definition of a bounded energy weak solution to (1.1), (1.2).

Definition 1.1 *(Renormalized bounded energy weak solution).* We call $(\rho, \mathbf{u})$ a renormalized bounded energy weak solution to the spatially periodic problem of the system (1.1) and (1.2), if the following is satisfied.

(1) $\rho \geqslant 0$, $\rho \in L^\gamma(\Omega)$, $\mathbf{u} \in (H^1(\Omega))^3$, $\int_\Omega \rho(x)\,dx = M > 0$.
(2) $(\rho, \mathbf{u})$ satisfies the energy inequality:

$$\int_\Omega \left(\mu|\nabla\mathbf{u}|^2 + \tilde{\mu}|\operatorname{div}\mathbf{u}|^2\right) dx \leqslant \int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\,dx. \tag{1.8}$$

(3) The system (1.1), (1.2) holds in the sense of $\mathcal{D}'(\Omega)$.
(4) The mass equation (1.1) holds in the sense of renormalized solutions, i.e.,

$$\operatorname{div}\big[b(\rho)\mathbf{u}\big] + \big[b'(\rho)\rho - b(\rho)\big]\operatorname{div}\mathbf{u} = 0 \quad \text{in } \mathcal{D}'(\Omega) \tag{1.9}$$

for any $b \in C^1(\mathbb{R})$, such that $b'(z) = 0$ when z is big enough.

Remark 1.1. In the periodic case, the periodic cell Ω in Definition 1.1 actually can be replaced by any cube in $\mathbb{R}^3$ with length 2π.

The main result of the current paper reads as

Theorem 1.2. *Let $\gamma > 1$ and $\mathbf{f} \in L^\infty(\mathbb{R}^3)$ satisfy* (1.4)*. Then, there exists a renormalized bounded energy weak solution $(\rho, \mathbf{u})$, satisfying* (1.6) *and* (1.4)*, to the spatially periodic problem of the system* (1.1), (1.2).

The proof of Theorem 1.2 is based on the uniform a priori estimates for the approximate solutions and the weak convergence method in the framework of Lions [12]. First, to get higher integrability of the density ρ_δ, we derive the weighted estimates for both pressure P_δ and kinetic energy $\rho_\delta|\mathbf{u}_\delta|^2$ which can be also understood as estimates in a Morrey space. These weighted estimates are inspired by the papers [8,2]. However, the new idea in the current paper is that both P_δ and $\rho_\delta|\mathbf{u}_\delta|^2$ are bounded simultaneously (cf. Lemma 2.2), while in [8,2] a weighted estimate only for the pressure P_δ was shown. This simultaneous boundedness of both P_δ and $\rho_\delta|\mathbf{u}_\delta|^2$ plays a crucial role in the derivation of the higher integrability of ρ_δ for any $\gamma > 1$, and moreover, implies some new uniform estimates for the velocity $\mathbf{u}_\delta$ and the pressure P_δ (cf. Lemmas 2.4 and 2.6). Then, with the help of the higher integrability of ρ_δ and the new estimates for $(\mathbf{u}_\delta, P_\delta)$, we use a bootstrap argument to obtain the a priori uniform estimates for the approximate solutions for any $\gamma > 1$ (cf. Theorem 2.1). To pass to the limit and obtain the existence of a weak solution, we cannot directly use the arguments in [12], since $\rho_\delta \in L^p(\Omega)$ $(p > 5/3)$ is required in [12] and this is not the case here. In fact, here we only have $\rho_\delta \in L^{\gamma r}(\Omega)$ with some $r > 1$ being very close to 1 when γ is close to 1. Instead, we exploit the proved uniform boundedness of $\rho_\delta\mathbf{u}_\delta$ and $\rho_\delta|\mathbf{u}_\delta|^2$ in $L^r(\Omega)$, and employ a careful analysis based on the classical method of weak convergence to circumvent this difficulty to prove the existence.

This paper is organized as follows. In Section 2, we first construct a sequence of approximate solutions $(\rho_\delta, \mathbf{u}_\delta)$ and then derive the uniform weighted estimates for both pressure P_δ and kinetic energy $\rho_\delta|\mathbf{u}_\delta|^2$. In Sections 2.2 and 2.3 we show the additional uniform estimates for the velocity $\mathbf{u}_\delta$ and the pressure P_δ in terms of the quantity $A = \|P_\delta|\mathbf{u}_\delta|^2 + \rho_\delta^\beta|\mathbf{u}_\delta|^{2+2\beta}\|_{L^1(\Omega)}$, $0 < \beta < 1$. This is crucial in the derivation of the higher integrability of ρ_δ for any $\gamma > 1$. In Section 2.4, by using of a bootstrap argument (see, for example, [2,6]), we prove that A is uniformly bounded which in turn implies the uniform H^1-boundedness of $\mathbf{u}_\delta$, and the L^r-boundedness of P_δ, $\rho_\delta\mathbf{u}_\delta$ and$\rho_\delta|\mathbf{u}_\delta|^2$. In Section 3, we prove the main theorem by using the weak convergence method in the framework of Lions [12].

2. Uniform estimates of the approximate solutions

2.1. The approximate system

To prove Theorem 1.2, we first work with the standard approximation by introducing an artificial pressure term:

$$P_\delta(\rho) := a\rho^\gamma + \delta\rho^6,$$

where $0 < \delta \leqslant 1$. Here we choose ρ^6 just for technical reason, and in fact we can take ρ^α for any $\alpha \geqslant 6$ instead of ρ^6. We consider the following approximate problem in Ω:

$$\operatorname{div}(\rho_\delta \mathbf{u}_\delta) = 0, \tag{2.1}$$

$$-\mu\triangle\mathbf{u}_\delta - \tilde{\mu}\nabla \operatorname{div}\mathbf{u}_\delta + \operatorname{div}(\rho_\delta\mathbf{u}_\delta \otimes \mathbf{u}_\delta) + \nabla P_\delta(\rho_\delta) = \rho_\delta \mathbf{f}. \tag{2.2}$$

According to [2], there is at least a weak solution $(\rho_\delta, \mathbf{u}_\delta)$ to the problem (2.1), (2.2) with the following properties ($\overline{\gamma} = \max(\gamma, 6)$):

$$(1)\quad \rho_\delta \in L^{2\overline{\gamma}}_{\mathrm{sym}}(\Omega), \qquad \mathbf{u}_\delta \in \big(W^{1,2}_{\mathrm{sym}}(\Omega)\big)^3, \qquad \int_\Omega \rho_\delta\, dx = M; \tag{2.3}$$

$$(2)\quad \operatorname{div}\big[b(\rho_\delta)\mathbf{u}_\delta\big] + \big[b'(\rho_\delta)\rho_\delta - b(\rho_\delta)\big]\operatorname{div}\mathbf{u}_\delta = 0 \quad \text{in } \mathcal{D}'(\Omega); \tag{2.4}$$

$$(3)\quad \int_\Omega \big[\mu|\nabla\mathbf{u}_\delta|^2 + \tilde{\mu}|\operatorname{div}\mathbf{u}_\delta|^2\big]\, dx \leqslant \int_\Omega \rho_\delta \mathbf{f}\cdot\mathbf{u}_\delta\, dx, \tag{2.5}$$

where b is the same as in (1.9).

In the rest of this section we will show some uniform-in-δ estimates for (ρ_δ, u_δ) which will be used in passing to the limit as $\delta \to 0$ in the next section to get a weak solution of the system (1.1), (1.2).

If we define

$$A = \big\| P_\delta|\mathbf{u}_\delta|^2 + \rho_\delta^\beta|\mathbf{u}_\delta|^{2+2\beta}\big\|_{L^1(\Omega)}, \quad 0 < \beta < 1, \tag{2.6}$$

then, we have

Theorem 2.1. *For A defined by* (2.6)*, it holds for any $1 < r < 2 - 1/\gamma$ that*

$$A + \|\mathbf{u}_\delta\|_{H^1(\Omega)} + \|P_\delta\|_{L^r(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^r(\Omega)} + \|\rho_\delta\mathbf{u}_\delta\|_{L^r(\Omega)} \leqslant C, \tag{2.7}$$

$$\delta\int_\Omega \rho_\delta^6\, dx \to 0 \quad \text{as } \delta \to 0, \tag{2.8}$$

where the constant C depends only on $\|\mathbf{f}\|_{L^\infty(\Omega)}$, μ, $\tilde{\mu}$, M, γ and β (but not on δ).

The proof of Theorem 2.1 is broken up into several lemmas given in Sections 2.2–2.4. We start with the following potential estimate.

2.2. A potential estimate

In this section we first derive the weighted estimate for P_δ and $\rho_\delta|\mathbf{u}_\delta|^2$ which can also be understood as an estimate in a Morrey space, and then use this estimate to get an estimate for A by the classical theory of elliptic equations of second order.

Lemma 2.2. *Let $(\rho_\delta, \boldsymbol{u}_\delta)$ be the solutions of the approximate problem* (2.1), (2.2). *Then the following estimate holds.*

$$\int_{B_1(x_0)} \frac{P_\delta + (\rho_\delta|\mathbf{u}_\delta|^2)^\beta}{|x - x_0|}\, dx \leqslant C\big(1 + \|P_\delta\|_{L^1(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^1(\Omega)} + \|\mathbf{u}_\delta\|_{H^1(\Omega)}\big) \tag{2.9}$$

for all $\beta \in (0, 1)$ and $x_0 \in \overline{\Omega}$, where the constant C depends only on $\|\mathbf{f}\|_{L^\infty(\Omega)}$, μ, $\tilde{\mu}$, M, γ and β, but not on x_0 and δ.

Proof. For $x_0 \in \overline{\Omega}$, we define $\phi = (\phi^1, \phi^2, \phi^3)$ with

$$\phi^i(x) = \frac{(x-x_0)^i}{|x-x_0|^\beta}\eta\big(|x-x_0|\big) \quad \text{in } b(x_0,\pi),\ i=1,2,3,$$

where $0 < \beta \leqslant 1$, $b(x_0,\pi) = \{x = (x^1,x^2,x^3) \in \mathbb{R}^3\colon |x^i - x_0^i| < \pi,\ i=1,2,3\}$ is a periodic cell, and $\eta \in C_0^\infty(\mathbb{R})$ is a cut-off function satisfying $0 \leqslant \eta(t) \leqslant 1$, $|D\eta| \leqslant 2$ and

$$\eta(t) = \begin{cases} 1 & |t| \leqslant 1, \\ 0 & |t| \geqslant 2. \end{cases}$$

If we extend ϕ to $\mathbb{R}^3$ periodically in x_i with period 2π for all $1 \leqslant i \leqslant 3$, then $\phi \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ can be a test function. Testing (2.2) with this ϕ, we find that

$$\int_{B_2(x_0)} P_\delta \operatorname{div}\phi\,dx + \int_{B_2(x_0)} \rho_\delta u_\delta^i u_\delta^j \partial_j \phi^i\,dx$$
$$= \mu \int_{B_2(x_0)} \nabla\mathbf{u}_\delta : \nabla\phi\,dx + \tilde{\mu} \int_{B_2(x_0)} \operatorname{div}\mathbf{u}_\delta \operatorname{div}\phi\,dx + \int_{B_2(x_0)} \rho_\delta \mathbf{f}\phi\,dx. \tag{2.10}$$

Since

$$\operatorname{div}\phi(x) = \frac{3-\beta}{|x-x_0|^\beta}\eta + \frac{(x-x_0)\cdot\nabla\eta}{|x-x_0|^\beta} \tag{2.11}$$

and

$$\int_{B_2(x_0)} \rho_\delta u_\delta^i u_\delta^j \partial_j \phi^i\,dx$$
$$= \int_{B_2(x_0)} \frac{\rho_\delta|\mathbf{u}_\delta|^2}{|x-x_0|^\beta}\eta\,dx - \beta\int_{B_2(x_0)} \frac{\rho_\delta[\mathbf{u}_\delta\cdot(x-x_0)]^2}{|x-x_0|^{2+\beta}}\eta\,dx + \int_{B_2(x_0)} \rho_\delta u_\delta^i u_\delta^j \frac{(x-x_0)^i}{|x-x_0|^\beta}\partial_j\eta$$
$$\geqslant (1-\beta)\int_{B_2(x_0)} \frac{\rho_\delta|\mathbf{u}_\delta|^2}{|x-x_0|^\beta}\eta\,dx + \int_{B_2(x_0)} \rho_\delta u_\delta^i u_\delta^j \frac{(x-x_0)^i}{|x-x_0|^\beta}\partial_j\eta\,dx, \tag{2.12}$$

we substitute (2.11) and (2.12) into (2.10) to obtain, after a straightforward calculation, that

$$\int_{B_1(x_0)} \frac{P_\delta}{|x-x_0|^\beta}\,dx + (1-\beta)\int_{B_1(x_0)} \frac{\rho_\delta|\mathbf{u}_\delta|^2}{|x-x_0|^\beta}\,dx \leqslant C\big(1 + \|P_\delta\|_{L^1(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^1(\Omega)} + \|\mathbf{u}_\delta\|_{H^1(\Omega)}\big),$$

from which it follows that

$$\int_{B_1(x_0)} \frac{P_\delta}{|x-x_0|}\,dx \leqslant C\big(1 + \|P_\delta\|_{L^1(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^1(\Omega)} + \|\mathbf{u}_\delta\|_{H^1(\Omega)}\big) \quad \text{for } \beta = 1,$$

and

$$\int_{B_1(x_0)} \frac{\rho_\delta|\mathbf{u}_\delta|^2}{|x-x_0|^\beta}\,dx \leqslant \frac{C}{1-\beta}\big(1 + \|P_\delta\|_{L^1(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^1(\Omega)} + \|\mathbf{u}_\delta\|_{H^1(\Omega)}\big) \tag{2.13}$$

for any $0 < \beta < 1$.

Using Hölder's inequality, we easily see that for any $0 < \beta < 1$,

$$\int_{B_1(x_0)} \frac{(\rho_\delta|\mathbf{u}_\delta|^2)^\beta}{|x-x_0|}\,dx \leqslant \frac{C}{1-\beta}\big(1 + \|P_\delta\|_{L^1(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^1(\Omega)} + \|\mathbf{u}_\delta\|_{H^1(\Omega)}\big). \tag{2.14}$$

Thus, (2.9) follows from (2.13) and (2.14) immediately. □

The following lemma is similar to the potential estimate in Section 3 of [8], and for the convenience of the reader, we give its proof in Appendix A.

Lemma 2.3. *Let A be defined by* (2.6)*, then we have*

$$A \leqslant C\|\mathbf{u}_\delta\|^2_{H^1(\Omega)}\left(1+\|P_\delta\|_{L^1(\Omega)}+\left\|\rho_\delta|\mathbf{u}_\delta|^2\right\|_{L^1(\Omega)}+\|\mathbf{u}_\delta\|_{H^1(\Omega)}\right), \tag{2.15}$$

where the constant C depends on $\|\mathbf{f}\|_{L^\infty(\Omega)}$, μ, $\tilde{\mu}$, M, γ and β, but not on δ.

2.3. Uniform estimates

The next lemma shows that the H^1-norm of $\mathbf{u}_\delta$ can be bounded by some power of A.

Lemma 2.4. *Let $(\rho_\delta, \mathbf{u}_\delta)$ be the solution of the approximate problem* (2.1), (2.2)*. Then,*

$$\|\mathbf{u}_\delta\|_{H^1(\Omega)} \leqslant CA^{\frac{\gamma-\beta}{4(\gamma\beta+\gamma-2\beta)}}, \tag{2.16}$$

where the constant C depends only on $\|\mathbf{f}\|_{L^\infty(\Omega)}$, μ, $\tilde{\mu}$, M, γ and Ω; β is the same as in Lemma 2.2.

Proof. First, we use the energy inequality (2.5) to obtain

$$\mu\int_\Omega |\nabla\mathbf{u}_\delta|^2\,dx+\tilde{\mu}\int_\Omega |\operatorname{div}\mathbf{u}_\delta|^2\,dx \leqslant \int_\Omega \rho_\delta f\cdot\mathbf{u}_\delta\,dx \leqslant \|\mathbf{f}\|_{L^\infty(\Omega)}\|\rho_\delta\mathbf{u}_\delta\|_{L^1(\Omega)}. \tag{2.17}$$

By Hölder's and Sobolev's inequalities, we see that

$$\begin{aligned}\|\rho_\delta\mathbf{u}_\delta\|_{L^1(\Omega)} &= \int_\Omega \left(P_\delta\mathbf{u}_\delta^2\right)^{\frac{1-\beta}{2(\gamma\beta+\gamma-2\beta)}}\left(\rho_\delta^\beta\mathbf{u}_\delta^{2\beta+2}\right)^{\frac{\gamma-1}{2(\gamma\beta+\gamma-2\beta)}}\rho_\delta^{\frac{2\gamma\beta+\gamma-3\beta}{2(\gamma\beta+\gamma-2\beta)}}\\ &\leqslant CA^{\frac{\gamma-\beta}{2(\gamma\beta+\gamma-2\beta)}},\end{aligned} \tag{2.18}$$

where we have used the fact that $\int_\Omega \rho_\delta = M$. Substituting (2.18) into (2.17), utilizing (1.7) and Poincaré's inequality, we obtain (2.16). This completes the proof. □

The following lemma is due to Bogovskii [1] and will be needed in Lemma 2.6.

Lemma 2.5. *Suppose that the bounded domain $\Omega \subset \mathbb{R}^n$ is starlike with respect to some ball contained in it, and that $1 < p < \infty$ and $n \geqslant 2$. Then, there exists a constant $C > 0$, depending only on n, p and Ω, such that for any $f \in L^p(\Omega)$ with $\int_\Omega f(x)\,dx = 0$, there is a vector field $\omega \in W^1_p(\Omega)$ satisfying*:

$$\begin{cases}\operatorname{div}\omega = f & \text{in } \Omega,\\ \omega = 0 & \text{on } \partial\Omega,\end{cases} \tag{2.19}$$

and

$$\|\omega\|_{W^1_p(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}. \tag{2.20}$$

Now, if we take

$$f = P_\delta^{s-1} - \frac{1}{|\Omega|}\int_\Omega P_\delta^{s-1}\,dx \quad \text{with } 1 < s < 2$$

in (2.19), then by virtue of Lemma 2.5, there is at least one solution ω_δ of (2.19) satisfying (2.20) and the estimate:

$$\|w_\delta\|_{W^{1,\frac{s}{s-1}}(\Omega)} \leqslant C\left\|p_\delta^{s-1} - \frac{1}{|\Omega|}\int_\Omega P_\delta^{s-1}\,dx\right\|_{L^{\frac{s}{s-1}}(\Omega)} \leqslant C(s,\Omega)\|P_\delta\|_s^{s-1}. \tag{2.21}$$

With the help of Lemma 2.5, we can show now

Lemma 2.6. *Let* $(\rho_\delta, \mathbf{u}_\delta)$ *be the solution of the approximate problem* (2.1), (2.2). *Then for* $s \in (1, \beta + 1 - \beta/\gamma]$, *we have*

$$\|P_\delta\|^s_{L^s(\Omega)} \leqslant C\big(1 + A^{\frac{\gamma s-\beta}{\gamma\beta+\gamma-2\beta}}\big), \tag{2.22}$$

where the constant C *depends only on* $\|\mathbf{f}\|_{L^\infty(\Omega)}$, μ, λ, M, γ *and* Ω.

Proof. We use the function ω_δ in (2.21) to test the momentum equation (2.2), and by a direct computation similar to Lemma 2.3 in [6] we obtain

$$\|P_\delta\|^s_{L^s(\Omega)} \leqslant C\big(1 + \|\mathbf{u}_\delta\|^s_{W^{1,2}(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|^s_{L^s(\Omega)}\big), \tag{2.23}$$

where the last term can be bounded as follows, using Hölder's and Sobolev's inequalities.

$$\begin{aligned}\big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|^s_{L^s(\Omega)} &= \int_\Omega \rho^s_\delta|\mathbf{u}_\delta|^{2s} = \int_\Omega \big(P_\delta|\mathbf{u}_\delta|^2\big)^{\frac{2s-\beta-1}{\gamma\beta+\gamma-2\beta}}\big(\rho^\beta|\mathbf{u}_\delta|^{2\beta+2}\big)^{\frac{\gamma s+1-2s}{\gamma\beta+\gamma-2\beta}}\rho_\delta^{\frac{\gamma\beta+\gamma-\beta-\gamma s}{\gamma\beta+\gamma-2\beta}} \\ &\leqslant C\big\|P_\delta|\mathbf{u}_\delta|^2\big\|^{\frac{2s-\beta-1}{\gamma\beta+\gamma-2\beta}}_{L^1(\Omega)}\big\|\rho^\beta_\delta|\mathbf{u}_\delta|^{2\beta+2}\big\|^{\frac{\gamma s+1-2s}{\gamma\beta+\gamma-2\beta}}_{L^1(\Omega)} \\ &\leqslant CA^{\frac{\gamma s-\beta}{\gamma\beta+\gamma-2\beta}}.\end{aligned} \tag{2.24}$$

The estimate (2.24), together with (2.23) and (2.16), gives then

$$\|P_\delta\|^s_{L^s(\Omega)} \leqslant C\big(1 + A^{\frac{s(\gamma-\beta)}{4(\gamma\beta+\gamma-2\beta)}} + A^{\frac{\gamma s-\beta}{\gamma\beta+\gamma-2\beta}}\big) \leqslant C\big(1 + A^{\frac{\gamma s-\beta}{\gamma\beta+\gamma-2\beta}}\big),$$

which proves the lemma. □

2.4. Proof of Theorem 2.1

In this section we will first show that the quantity A is uniformly bounded (with respect to δ), then with the help of this uniform boundedness, we can easily get (2.7) and (2.8).

Recalling that Lemma 2.6 is true for any $s \in (1, \beta + 1 - \beta/\gamma]$, we write $s = 1 + \varepsilon$, where ε will be chosen small enough later on, and use (A.1), (2.16), (2.22) and (2.24) to infer that

$$\begin{aligned}A &\leqslant C\|\mathbf{u}_\delta\|^2_{H^1(\Omega)}\big(1 + \|\mathbf{u}_\delta\|_{H^1(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^1(\Omega)} + \|P_\delta\|_{L^1(\Omega)}\big) \\ &\leqslant CA^{\frac{\gamma-\beta}{2(\gamma\beta+\gamma-2\beta)}}\big(1 + A^{\frac{\gamma-\beta}{4(\gamma\beta+\gamma-2\beta)}} + A^{\frac{\gamma s-\beta}{(\gamma\beta+\gamma-2\beta)}\cdot\frac{1}{1+\varepsilon}}\big) \\ &\leqslant C\big(1 + A^{\frac{3(\gamma-\beta)}{2(\gamma\beta+\gamma-2\beta)}+O(\varepsilon)}\big).\end{aligned} \tag{2.25}$$

Since (2.25) remains valid for any $\beta \in (0, 1)$, if we write $\beta = 1 - \sigma$ with $0 < \sigma < 1$, then

$$\gamma > \frac{1-\sigma}{1-2\sigma} \quad \Rightarrow \quad \frac{3(\gamma-\beta)}{2(\gamma\beta+\gamma-2\beta)} < 1,$$

where ε and σ can be arbitrary small. So, by our choice of the parameters ε and σ, the exponent $\frac{3(\gamma-\beta)}{2(\gamma\beta+\gamma-2\beta)} + O(\varepsilon)$ can be made less than 1, and therefore we conclude by (2.25) that

$$A \leqslant C,$$

which immediately implies that

$$\begin{aligned}&\|\mathbf{u}_\delta\|_{H^1(\Omega)} \leqslant C, \\ &\|p_\delta\|_{L^r(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^r(\Omega)} \leqslant C, \\ &\|\rho_\delta\mathbf{u}_\delta\|^r_{L^r(\Omega)} = \int_\Omega \big(\rho^r_\delta\mathbf{u}^{2r}_\delta\big)^{\frac12}\big(\rho^r_\delta\big)^{\frac12} \leqslant C\big\|\rho^r_\delta\mathbf{u}^{2r}_\delta\big\|^{\frac12}_{L^1(\Omega)}\big\|\rho^r_\delta\big\|^{\frac12}_{L^1(\Omega)} \leqslant C, \\ &\delta\int_\Omega \rho^6_\delta\,dx \leqslant C\delta^{\frac{\gamma(s-1)}{6+\gamma(s-1)}}\bigg(\int_\Omega \delta\rho_\delta^{6+\gamma(s-1)}\,dx\bigg)^{\frac{6}{6+\gamma(s-1)}} \leqslant C\delta^{\frac{\gamma(s-1)}{6+\gamma(s-1)}}.\end{aligned}$$

From the above four inequalities, we obtain (2.7) and (2.8). This completes the proof of Theorem 2.1.

3. Proof of Theorem 1.2

In this section we will take to the limit as $\delta \to 0$ for the approximate problem (2.1) and (2.2) to obtain a weak solution of (1.1), (1.2) for any $\gamma > 1$. As mentioned in the introduction, we cannot directly use the arguments in [12] to take to the limit, because we just have $\rho_\delta \in L^{\gamma r}(\Omega)$ with r being very close to 1 when γ is close to 1, while in [12] $\rho_\delta \in L^p(\Omega)$ $(p > 5/3)$ is required. To overcome this difficulty, we exploit the estimates established in Theorem 2.1 and Lemma 3.1 to show the weak compactness of the effective viscous flux (see Sections 3.2, 3.3). Then, by the standard procedure of the weak convergence method in Section 3.4, we can finish the proof of Theorem 1.2 (cf. [3, 12]).

3.1. Preliminaries

In this section we give the necessary preliminary lemmas which will be used in the proof of Theorem 1.2.

Lemma 3.1. *Let $1 < p_1, p_2, p < \infty$, $p \leqslant p_1$ and Ω be a bounded domain in $\mathbb{R}^3$. Suppose that*

$$f_n \rightharpoonup f \quad \text{weakly in } L^{p_1}(\Omega), \tag{3.1}$$

$$g_n \to g \quad \text{strongly in } L^{p_2}(\Omega), \tag{3.2}$$

and

$$f_n g_n \text{ are uniformly bounded in } L^p(\Omega). \tag{3.3}$$

Then there is a subsequence of $f_n g_n$ (still denoted by $f_n g_n$), such that

$$f_n g_n \rightharpoonup fg \quad \text{weakly in } L^p(\Omega).$$

Remark 3.1. We point out here that $\frac{1}{p_1} + \frac{1}{p_2} \leqslant 1$ is not needed in Lemma 3.1.

Proof of Lemma 3.1. Since $f_n g_n$ is uniformly bounded in $L^p(\Omega)$, there exists a subsequence of $f_n g_n$ (still denoted by $f_n g_n$) and $F \in L^p(\Omega)$, such that

$$f_n g_n \rightharpoonup F \quad \text{weakly in } L^p(\Omega). \tag{3.4}$$

On the other hand, (3.2) implies that for the subsequence g_n in (3.4), there is a subsequence (still denoted by g_n), such that

$$g_n \to g \quad \text{a.e. in } \Omega.$$

By Egoroff's theorem, for any $m \in \mathbb{N}$, there exits a subset $E_m \subset \Omega$ with $|E_m| < 1/(2m)$, such that

$$g_n \to g \quad \text{uniformly in } \Omega \setminus E_m.$$

Since $g \in L^{p_2}(\Omega)$, we see that g is finite a.e. in Ω, that is, for all $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \left|\left\{x \in \Omega,\ \left|g(x)\right| > k\right\}\right| = 0.$$

Hence, for any given m, there exists a sufficiently large $K_m \in \mathbb{N}$, such that

$$E_m^{K_m} := \left\{x \in \Omega,\ \left|g(x)\right| > K_m\right\} \quad \text{satisfy} \quad \left|E_m^{K_m}\right| < \frac{1}{2m}.$$

Now, we denote $\Omega_m := \Omega \setminus (E_m \cup E_m^{K_m})$, then $|\Omega \setminus \Omega_m| < 1/m$, $|g| \leqslant K_m$ and $g_n \to g$ in Ω_m. Thus, for any $\phi \in L^{p'}(\Omega)$, $1/p + 1/p' = 1$, one has

$$\lim_{n\to\infty} \int_{\Omega_m} (f_n g_n - fg)\phi\,dx = \lim_{n\to\infty} \int_{\Omega_m} (f_n - f)g\phi\,dx + \lim_{n\to\infty} \int_{\Omega_m} f_n(g_n - g)\phi\,dx = 0. \tag{3.5}$$

For any $\phi \in L^{p'}(\Omega)$, if we define

$$\tilde{\phi}(x) = \begin{cases} \phi(x), & x \in \Omega_m, \\ 0, & x \in \Omega \setminus \Omega_m, \end{cases}$$

we still have $\tilde{\phi} \in L^{p'}(\Omega)$. Thus, in view of (3.4), we have

$$\lim_{n\to\infty} \int_{\Omega} (f_n g_n - F)\tilde{\phi}\,dx = 0 = \lim_{n\to\infty} \int_{\Omega_m} (f_n g_n - F)\phi\,dx. \tag{3.6}$$

From (3.5) and (3.6) we easily get that

$$fg = F, \quad \text{a.e. in } \Omega_m.$$

Let $m \to \infty$, we arrive at

$$fg = F, \quad \text{a.e. in } \Omega,$$

which proves the lemma. □

Lemma 3.2 *(Div–Curl lemma). Let* $1 < p_1, p_2, q_1, q_2 < \infty$ $(1/p_1 + 1/p_2 = 1/p < 1)$*, and* Ω *be a domain in* $\mathbb{R}^3$*. Assume that*

$$\mathbf{f_n} \rightharpoonup \mathbf{f} \quad \textit{weakly in } \left(L^{p_1}(\Omega)\right)^3, \qquad \mathbf{g_n} \rightharpoonup \mathbf{g} \quad \textit{weakly in } \left(L^{p_2}(\Omega)\right)^3,$$

and

$$\operatorname{div}\mathbf{f_n} \to \operatorname{div}\mathbf{f} \quad \textit{strongly in } W^{-1,q_1}(\Omega), \qquad \operatorname{curl}\mathbf{g_n} \to \operatorname{curl}\mathbf{g} \quad \textit{strongly in } \left(W^{-1,q_2}(\Omega)\right)^3.$$

Then

$$\mathbf{f_n} \cdot \mathbf{g_n} \rightharpoonup \mathbf{f} \cdot \mathbf{g} \quad \textit{weakly in } L^p(\Omega).$$

The Div–Curl lemma is a classical statement of the compensated compactness due to Murat (1978) and Tartar (1975). Lemma 3.2 is one of its general formulation. For the proof of Lemma 3.2 we refer to, for example, the reference [14]. Actually, in the current paper we will apply Lemma 3.2 with

$$\operatorname{div}\mathbf{f_n} = 0, \qquad \operatorname{curl}\mathbf{g_n} = 0.$$

At the end of this section we introduce the operator $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ in $\mathbb{R}^3$ by

$$\mathcal{A}_j = \partial_j \triangle^{-1} = \mathcal{F}^{-1}\left(-\frac{i\xi_j}{|\xi|^2}\right), \quad j = 1, 2, 3, \ \operatorname{div}\mathcal{A} = id,$$

where $\triangle^{-1}$ is the inverse of the Laplace operator, and the Riesz operator defined by

$$\mathcal{R}_{ij}[v] = \partial_j \mathcal{A}_i[v], \qquad \mathcal{R}_{ij} = \mathcal{R}_{ji}, \quad i, j = 1, 2, 3.$$

Lemma 3.3. *Let* Ω *be a domain in* $\mathbb{R}^3$ *and the operator* $\mathcal{A}$ *is defined as above, then we have*

$$\|\mathcal{A}_i v\|_{W^{1,s}(\Omega)} \leqslant C(s, \Omega)\|v\|_{L^s(\mathbb{R}^3)}, \quad 1 < s < \infty, \ i = 1, 2, 3. \tag{3.7}$$

For the proof of Lemma 3.3 one can see, for example, the reference [16].

3.2. Vanishing limit as $\delta \to 0$

In this section we will study the limit for the problem (2.1), (2.2) as $\delta \to 0$.

By Theorem 2.1, and the compact embedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$, $p \in [1, 6)$, we have the following limits:

$$\begin{aligned}
&\delta\rho_\delta^6 \to 0 && \text{in } \mathcal{D}'(\Omega),\\
&\mathbf{u}_\delta \rightharpoonup \mathbf{u} && \text{weakly in } \big(W^{1,2}(\Omega)\big)^3,\\
&\mathbf{u}_\delta \to \mathbf{u} && \text{strongly in } \big(L^p(\Omega)\big)^3,\ 1 \leqslant p < 6,\\
&\rho_\delta \rightharpoonup \rho && \text{weakly in } L^{\gamma r}(\Omega),\\
&\rho_\delta^\gamma \rightharpoonup \overline{\rho^\gamma} && \text{weakly in } L^r(\Omega).
\end{aligned} \tag{3.8}$$

It is easy to see that Theorem 2.1, together with Lemma 3.1 and (3.8), yields that

$$\rho_\delta \mathbf{u}_\delta \rightharpoonup \rho\mathbf{u} \quad \text{and} \quad \rho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \rightharpoonup \rho\mathbf{u}\otimes\mathbf{u} \quad \text{in } L^r(\Omega). \tag{3.9}$$

Letting $\delta \to 0$ in (2.1) and (2.2), applying (3.8) and (3.9), we find that the weak limit $(\rho, \mathbf{u})$ of $(\rho_\delta, \mathbf{u}_\delta)$ satisfies

$$\operatorname{div}(\rho\mathbf{u}) = 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{3.10}$$

$$\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + a\nabla\overline{\rho^\gamma} - \mu\Delta\mathbf{u} - \tilde\mu\nabla\operatorname{div}\mathbf{u} = \rho\mathbf{f} \quad \text{in } \mathcal{D}'(\Omega); \tag{3.11}$$

$$\operatorname{div}\big[\overline{b(\rho_\delta)}\mathbf{u}_\delta\big] + \overline{\big[b'(\rho_\delta)\rho_\delta - b(\rho_\delta)\big]\operatorname{div}\mathbf{u}_\delta} = 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{3.12}$$

$$\int_\Omega \big[\mu|\nabla\mathbf{u}|^2 + \tilde\mu|\operatorname{div}\mathbf{u}|^2\big]\,dx \leqslant \int_\Omega \rho\mathbf{f}\cdot\mathbf{u}\,dx. \tag{3.13}$$

So, to show that $(\rho, \mathbf{u})$ is a weak solution of (1.1), (1.2), we have to prove

$$\overline{\rho^\gamma} = \rho^\gamma \quad \text{a.e. on } \Omega. \tag{3.14}$$

We will employ the weak convergence method in the framework of Lions (cf. [12,5]) to get (3.14).

If $\gamma r \geqslant 2$, then by use of the DiPerna–Lions transport theory, $(\rho, \mathbf{u})$ satisfies the renormalized continuity equation (1.9) in $\mathcal{D}'(\Omega)$.

If $1 < \gamma r < 2$, we will use the following cut-off function, due to Feireisl, Novotný and Petzeltová (cf. [5]), to prove that $(\rho, \mathbf{u})$ is a renormalized solution by replacing ρ by $T_k(\rho)$ and letting then $k \to 0$.

$$T_k(z) = kT\left(\frac{z}{k}\right), \quad k = 1, 2, \ldots,$$

where

$$T(z) = \begin{cases} z, & z \leqslant 1 \\ 2, & z \geqslant 3 \end{cases} \in C^\infty(\mathbb{R}), \quad \text{concave},\ z \in \mathbb{R}. \tag{3.15}$$

Since $(\rho_\delta, \mathbf{u}_\delta)$ is the renormalized solution (recalling $\rho_\delta \in L^2(\Omega)$), we can take $b(z) = T_k(z)$ in the definition of renormalized solutions to get

$$\operatorname{div}\big(T_k(\rho_\delta)\mathbf{u}_\delta\big) + \big[T_k'(\rho_\delta)\rho_\delta - T_k(\rho_\delta)\big]\operatorname{div}\mathbf{u}_\delta = 0 \quad \text{in } \mathcal{D}'(\Omega).$$

Letting $\delta \to 0$ in the above identity and making use of (3.8), we deduce that

$$\operatorname{div}\big(\overline{T_k(\rho)}\mathbf{u}\big) + \overline{\big[T_k'(\rho)\rho - T_k(\rho)\big]\operatorname{div}\mathbf{u}} = 0 \quad \text{in } \mathcal{D}'(\Omega), \tag{3.16}$$

where and in what follows, we denote by $\overline{f(\rho)}$ the weak limit of $f(\rho_\delta)$.

3.3. Effective viscous flux

The importance of the effective viscous flux for the existence of weak solutions has been addressed by a number of authors (see, for example, [12,5,3]). In this section we introduce the effective viscous flux and prove its weak compactness which will play a key role in the existence proof.

We define the effective viscous flux by

$$\tilde H_\delta := a\rho_\delta^\gamma - (\mu + \tilde\mu)\operatorname{div}\mathbf{u}_\delta \rightharpoonup \tilde H := a\overline{\rho^\gamma} - (\mu + \tilde\mu)\operatorname{div}\mathbf{u}, \quad \text{as } \delta \to 0,$$

and we have the following lemma.

Lemma 3.4. *For any $\phi \in C_0^\infty(\Omega)$, we have*

$$\lim_{\delta\to 0}\int_\Omega \phi(x)\tilde{H}_\delta T_k(\rho_\delta)\,dx = \int_\Omega \phi(x)\tilde{H}\overline{T_k(\rho)}\,dx. \tag{3.17}$$

Remark 3.2. By use of the density argument, one can actually take $\phi(x) \equiv 1$.

Proof of Lemma 3.4. First, testing (2.2) by $\Phi_\delta \equiv (\Phi_{1\delta}, \Phi_{2\delta}, \Phi_{3\delta}) = \phi(x)\mathcal{A}[\xi(x)T_k(\rho_\delta)]$ and (1.2) by $\Phi \equiv (\Phi_1, \Phi_2, \Phi_3) = \phi(x)\mathcal{A}[\xi(x)T_k(\rho)]$ with $\phi, \xi \in C_0^\infty(\Omega)$, we obtain

$$\begin{aligned}
&\int_\Omega \phi(x)\xi(x)\tilde{H}_\delta T_k(\rho_\delta)\,dx \\
&\quad= -\int_\Omega \rho_\delta u_\delta^i u_\delta^j \partial_i\phi \mathcal{A}_j\big[\xi T_k(\rho_\delta)\big] - \int_\Omega \rho_\delta u_\delta^i u_\delta^j \phi \mathcal{R}_{ij}\big[\xi T_k(\rho_\delta)\big] \\
&\qquad - \int_\Omega \big(a\rho_\delta^\gamma + \delta\rho_\delta^6\big)\nabla\phi\cdot\mathcal{A}\big[\xi(x)T_k(\rho_\delta)\big]\,dx + \tilde{\mu}\int_\Omega \operatorname{div}\mathbf{u}_\delta \nabla\phi\cdot\mathcal{A}\big[\xi(x)T_k(\rho_\delta)\big]\,dx \\
&\qquad + \mu\int_\Omega \big\{\mathbf{u}_\delta\cdot\nabla\phi\xi(x)T_k(\rho_\delta) + \nabla\phi\cdot\nabla u_\delta^i\mathcal{A}_i\big[\xi(x)T_k(\rho_\delta)\big] + u_\delta^i\partial_j\phi\mathcal{R}_{ij}\big[\xi(x)T_k(\rho_\delta)\big]\big\}\,dx \\
&\qquad - \int_\Omega \rho_\delta\mathbf{f}\phi(x)\mathcal{A}_j\big[\xi T_k(\rho_\delta)\big]
\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
&\int_\Omega \phi(x)\xi(x)\tilde{H}\overline{T_k(\rho)}\,dx \\
&\quad= -\int_\Omega \rho u^i u^j \partial_i\phi\mathcal{A}_j\big[\xi\overline{T_k(\rho)}\big] - \int_\Omega \rho u^i u^j\phi\mathcal{R}_{ij}\big[\xi\overline{T_k(\rho)}\big] \\
&\qquad - \int_\Omega a\overline{\rho^\gamma}\nabla\phi\cdot\mathcal{A}\big[\xi(x)\overline{T_k(\rho)}\big]\,dx + \tilde{\mu}\int_\Omega \operatorname{div}\mathbf{u}\nabla\phi\cdot\mathcal{A}\big[\xi(x)\overline{T_k(\rho)}\big]\,dx \\
&\qquad + \mu\int_\Omega \big\{\mathbf{u}\cdot\nabla\phi\xi(x)\overline{T_k(\rho)} + \nabla\phi\cdot\nabla u^i\mathcal{A}_i\big[\xi(x)\overline{T_k(\rho)}\big] + u^i\partial_j\phi\mathcal{R}_{ij}\big[\xi(x)\overline{T_k(\rho)}\big]\big\}\,dx \\
&\qquad - \int_\Omega \rho\mathbf{f}\phi(x)\mathcal{A}_j\big[\xi\overline{T_k(\rho)}\big].
\end{aligned} \tag{3.19}$$

Next, we will pass to the limit in (3.18) as $\delta \to 0$. For any k, it is easy to see that

$$\big\|T_k(\rho_\delta)\big\|_{L^\infty(\Omega)} \leqslant 2k, \quad \text{uniformly in } \delta,$$

from which, (3.7) and Sobolev's embedding theorem it follows that

$$\begin{aligned}
&\mathcal{R}_{ij}\big[\xi T_k(\rho_\delta)\big] \rightharpoonup \mathcal{R}_{ij}\big[\xi\overline{T_k(\rho)}\big] \quad \text{in } L^p(\Omega), \text{ for any } 1 < p < \infty, \\
&\mathcal{A}\big[\xi T_k(\rho_\delta)\big] \to \mathcal{A}\big[\xi\overline{T_k(\rho)}\big] \quad \text{in } C^0(\overline{\Omega}).
\end{aligned} \tag{3.20}$$

By (3.8), (3.9) and (3.20), we deduce that

$$\lim_{\delta\to 0}\int_\Omega \rho_\delta u_\delta^i u_\delta^j\partial_i\phi\mathcal{A}_j\big[\xi T_k(\rho_\delta)\big]\,dx = \int_\Omega \rho u^i u^j\partial_i\phi\mathcal{A}_j\big[\xi\overline{T_k(\rho)}\big]\,dx,$$

$$\lim_{\delta\to 0}\int_\Omega \big(a\rho_\delta^\gamma+\delta\rho_\delta^6\big)\nabla\phi\cdot\mathcal{A}\big[\xi T_k(\rho_\delta)\big]\,dx=\int_\Omega a\overline{\rho^\gamma}\nabla\phi\cdot\mathcal{A}\big[\xi\overline{T_k(\rho)}\big]\,dx,$$

$$\lim_{\delta\to 0}\int_\Omega \operatorname{div}\mathbf{u}_\delta\nabla\phi\cdot\mathcal{A}\big[\xi T_k(\rho_\delta)\big]\,dx=\int_\Omega \operatorname{div}\mathbf{u}\nabla\phi\cdot\mathcal{A}\big[\xi\overline{T_k(\rho)}\big]\,dx,$$

$$\lim_{\delta\to 0}\int_\Omega \rho_\delta\mathbf{f}\phi(x)\mathcal{A}_j\big[\xi T_k(\rho_\delta)\big]=\int_\Omega \rho\mathbf{f}\phi(x)\mathcal{A}_j\big[\xi\overline{T_k(\rho)}\big],$$

$$\lim_{\delta\to 0}\int_\Omega \big(\mathbf{u}_\delta\cdot\nabla\phi\xi T_k(\rho_\delta)+\nabla\phi\cdot\nabla u_\delta^i\mathcal{A}_i\big[\xi T_k(\rho_\delta)\big]+u_\delta^i\partial_j\phi\mathcal{R}_{ij}\big[\xi T_k(\rho_\delta)\big]\big)$$
$$=\int_\Omega \big(\mathbf{u}\cdot\nabla\phi\xi\overline{T_k(\rho)}+\nabla\phi\cdot\nabla u^i\mathcal{A}_i\big[\xi\overline{T_k(\rho)}\big]+u^i\partial_j\phi\mathcal{R}_{ij}\big[\xi\overline{T_k(\rho)}\big]\big). \tag{3.21}$$

Next, we apply Lemmas 3.1 and 3.2 to prove the convergence of the second term on the right-hand side of (3.18). First, for any given j, it can be easily verified that

$$\operatorname{curl}\mathcal{R}_{ij}\big[\xi T_k(\rho_\delta)\big]=0\quad\text{and}\quad\operatorname{div}(\rho_\delta\mathbf{u}_\delta)=0,$$

which combined with (3.9), (3.20) and Lemma 3.2 yields that

$$\rho_\delta u_\delta^i\mathcal{R}_{ij}\big[\xi T_k(\rho_\delta)\big]\rightharpoonup\rho u^i\mathcal{R}_{ij}\big[\xi\overline{T_k(\rho)}\big]\quad\text{in } L^{\tilde r}(\Omega)\text{ for any }1<\tilde r<r. \tag{3.22}$$

On the other hand, one has

$$\mathbf{u}_\delta\to\mathbf{u}\quad\text{in } L^p(\Omega),\ 1\leqslant p<6,\quad\text{and}\quad\big\|\rho_\delta u_\delta^i u_\delta^j\mathcal{R}_{ij}\big[\xi T_k(\rho)\big]\big\|_{L^{\tilde r}(\Omega)}\leqslant C. \tag{3.23}$$

Moreover, Lemma 3.1 gives

$$\rho_\delta u_\delta^i u_\delta^j\mathcal{R}_{ij}\big[\xi T_k(\rho_\delta)\big]\rightharpoonup\rho u^i u^j\mathcal{R}_{ij}\big[\xi\overline{T_k(\rho)}\big]\quad\text{in } L^{\tilde r}(\Omega). \tag{3.24}$$

Thus, by letting $\xi\equiv 1$, taking $\delta\to 0$ in (3.18) and utilizing (3.19)–(3.24), we obtain the lemma immediately. □

3.4. Strong convergence of the density

The main task for completing the proof of Theorem 1.2 consists now in establishing the strong convergence of ρ_δ to ρ in $L^1(\Omega)$. This task can be fulfilled by the following two lemmas, the proof of which can be found in [2,4].

Lemma 3.5 *(Control of the oscillation of the density). Let $T_k(z)$ be defined by* (3.15)*, then*

$$\overline{\lim_{\delta\to 0}}\big\|T_k(\rho_\delta)-T_k(\rho)\big\|_{L^{\gamma+1}(\Omega)}\leqslant C,$$

where the constant C is independent of k.

Lemma 3.6 *(Renormalized continuity equation). The weak limit* $(\rho,\mathbf{u})$ *is a renormalized solution of* (1.1)*, i.e.,* (ρ,u) *satisfies*

$$\operatorname{div}\big[b(\rho)\mathbf{u}\big]+\big[b'(\rho)\rho-b(\rho)\big]\operatorname{div}\mathbf{u}=0\quad\text{in }\mathcal{D}'(\Omega)$$

for any $b\in C^1(\mathbb{R})$, $b'(z)=0$ for sufficiently large z.

Now, by introducing a family of functions

$$L_k(z)=\begin{cases} z\log z, & 0\leqslant z\leqslant k\\ z\log k+z\int_k^z\frac{T_k(s)}{s^2}\,ds, & z\geqslant k\end{cases}\in C^1(\mathbb{R}_+)\cap C^0[0,\infty),$$

and making use of Lemmas 3.5 and 3.6, we argue, in the same manner as in [2], to conclude that

$$\lim_{\delta\to 0}\|\rho_\delta-\rho\|_{L^1(\Omega)}=0,$$

which, by (3.8) and the interpolation theory, implies that

$$\rho_\delta \to \rho \quad \text{strongly in } L^p(\Omega),\ \forall 1 \leqslant p < \gamma r.$$

Consequently, we have

$$\overline{\rho^\gamma} = \rho^\gamma, \quad \text{a.e.}$$

Thus, we complete the proof of Theorem 1.2.

Acknowledgements

The authors wish to thank the referees for the useful suggestions which improved the presentation of this paper. This work was partially supported by the National Basic Research Program (Grant No. 2005CB321700) and NSFC (Grant No. 40890154).

Appendix A. Proof of Lemma 2.3

Lemma 2.3. *Let A be defined by* (2.6)*, then we have*

$$A \leqslant C\|\mathbf{u}_\delta\|^2_{H^1(\Omega)}\big(1 + \|P_\delta\|_{L^1(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^1(\Omega)} + \|\mathbf{u}_\delta\|_{H^1(\Omega)}\big), \tag{A.1}$$

where the constant C depends on $\|\mathbf{f}\|_{L^\infty(\Omega)}$, μ, $\tilde{\mu}$, M, γ and β, but not on δ.

Proof. Since Ω is a $C^{0,1}$-domain in $\mathbb{R}^3$, for any open set $\Omega' \Supset \Omega$ there exists a bounded linear extension operator E from $W^{1,p}(\Omega)$ into $W_0^{1,p}(\Omega')$, such that $Eu = u$ in Ω and

$$\|Eu\|_{W^{1,p}(\Omega')} \leqslant C\|u\|_{W^{1,p}(\Omega)} \quad \text{for all } u \in W^{1,p}(\Omega), \tag{A.2}$$

where $C = C(k, \Omega, \Omega')$. We refer, for example, to [9, Theorem 7.25] for the proof.

Now, let $\Omega' \Supset \Omega$ be a bounded domain in $\mathbb{R}^3$, then $E\mathbf{u}_\delta \in (W_0^{1,2}(\Omega'))^3$. Since P_δ and $\mathbf{u}_\delta$ are periodic in x_i with period 2π for all $1 \leqslant i \leqslant 3$, we can get from Lemma 2.2 that

$$\int_{\Omega'} \frac{P_\delta + (\rho_\delta|\mathbf{u}_\delta|^2)^\beta}{|x - x_0|}\,dx \leqslant C\big(1 + \|P_\delta\|_{L^1(\Omega)} + \big\|\rho_\delta|\mathbf{u}_\delta|^2\big\|_{L^1(\Omega)} + \|\mathbf{u}_\delta\|_{H^1(\Omega)}\big) \tag{A.3}$$

for any $0 < \beta < 1$ and $x_0 \in \overline{\Omega'}$, where the constant C is independent of δ and x_0.

Let h be the unique weak solution of the elliptic problem:

$$\begin{cases} \triangle h = P_\delta + (\rho_\delta|\mathbf{u}_\delta|^2)^\beta \geqslant 0 & \text{in } \Omega', \\ h = 0 & \text{on } \partial\Omega'. \end{cases}$$

Then by the classical theory for elliptic equations, we have

$$\|h\|_{L^\infty(\Omega')} \leqslant C \sup_{x_0 \in \overline{\Omega'}} \int_{\Omega'} \frac{P_\delta + (\rho_\delta|\mathbf{u}_\delta|^2)^\beta}{|x - x_0|}\,dx. \tag{A.4}$$

For $E\mathbf{u}_\delta \in (W_0^{1,2}(\Omega'))^3$, we consider now

$$\begin{aligned} A' &= \int_{\Omega'} \big[P_\delta + \big(\rho_\delta|\mathbf{u}_\delta|^2\big)^\beta\big]|E\mathbf{u}_\delta|^2\,dx \\ &= \int_{\Omega'} \triangle h|E\mathbf{u}_\delta|^2\,dx \\ &\leqslant C\|E\mathbf{u}_\delta\|_{W_0^{1,2}(\Omega')}\big\||E\mathbf{u}_\delta||\nabla h|\big\|_{L^2(\Omega')}. \end{aligned} \tag{A.5}$$

On the other hand, integrating by parts, we infer that

$$\big\| |E\mathbf{u}_\delta||\nabla h| \big\|_{L^2(\Omega')}^2 \leqslant C \int_{\Omega'} \big(|h||\triangle h||E\mathbf{u}_\delta|^2 + |h||\nabla h||E\mathbf{u}_\delta||\nabla \mathbf{u}_\delta| \big)\, dx$$
$$\leqslant C\|h\|_{L^\infty(\Omega')} \big(A' + \big\| |E\mathbf{u}_\delta||\nabla h| \big\|_{L^2(\Omega')} \|E\mathbf{u}_\delta\|_{W_0^{1,2}(\Omega')} \big). \tag{A.6}$$

Thus, inequalities (A.5) and (A.6) imply that

$$A' \leqslant C\|E\mathbf{u}_\delta\|_{W_0^{1,2}(\Omega')}^2 \|h\|_{L^\infty(\Omega')}. \tag{A.7}$$

Finally, from (A.2)–(A.4) and (A.7) we get

$$A \leqslant A' \leqslant C\|\mathbf{u}_\delta\|_{H^1(\Omega)}^2 \big(1 + \|P_\delta\|_{L^1(\Omega)} + \big\| \rho_\delta |\mathbf{u}_\delta|^2 \big\|_{L^1(\Omega)} + \|\mathbf{u}_\delta\|_{H^1(\Omega)} \big),$$

which proves Lemma 2.3. □

Remark A.1. We point out here that Lemma 2.3 can be also obtained by using the arguments in [2].

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