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Some existence results on the exterior Dirichlet problem for the minimal hypersurface equation

Nedir do Espírito-Santo [a], Jaime Ripoll [b,*]

[a] *UFRJ, Instituto de Matemática, Caixa Postal 68530, 21941-909 Rio de Janeiro, RJ, Brazil*
[b] *UFRGS, Instituto de Matemática, Av. Bento Gonçalves 9500, 91540-000 Porto Alegre, RS, Brazil*

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Abstract

It is proved the existence of solutions to the exterior Dirichlet problem for the minimal hypersurface equation in complete noncompact Riemannian manifolds either with negative sectional curvature and simply connected or with nonnegative Ricci curvature under a growth condition on the sectional curvature.

1. Introduction

With the development of the Riemannian geometry many PDE results proved in the Euclidean geometry for the class of geometric operators have been investigated in more general Riemannian manifolds. In the case of the Laplacian equation, S.T. Yau proved that Liouville's theorem (an entire nonconstant solution of the Laplace equation in $\mathbb{R}^n$ is necessarily unbounded) is also true in a complete Riemannian manifold M^n with nonnegative Ricci curvature and conjectured that if M is simply connected with sectional curvature satisfying $-\kappa_1^2 \leqslant K_M \leqslant -\kappa_2^2 < 0$, for some constants κ_1, κ_2, then there should exist bounded solutions of the Laplacian equation on M. This has been proved in the 80's by Anderson and Sullivan (see [12]). Actually, they proved that on the compactified manifold $\overline{M} := M \cup \partial_\infty M$ one can solve the Dirichlet problem for the Laplace equation for any given continuous data on $\partial_\infty M$.

For the case of the minimal surface equation, it is known that Bernstein's theorem is true in a complete noncompact 2-dimensional Riemannian manifold with nonnegative curvature: Any entire minimal graph in M^2 is totally geodesic in $M^2 \times \mathbb{R}$ [10]. As far as the authors know, if a similar result holds in higher dimensions is still an open question. In the case of simply connected Riemannian manifolds with negative curvature one should expect a similar existence result for minimal graphs as Anderson and Sullivan results for harmonic functions. In fact, in [4] the authors prove the existence of entire solutions of the minimal hypersurface equation with any given smooth data at the asymptotic boundary $\partial_\infty M$ of M if M is complete, simply connected, with sectional curvature satisfying $K_M \leqslant -k^2 < 0$ and such that isotropy subgroup of the isometry group of M at some point p of M acts transitively on the geodesic spheres centered at p (this result has a similar harmonic counterpart due to H.I. Choi, see [2, Theorem 3.6]). In [6]

* Corresponding author.
E-mail addresses: nedir@im.ufrj.br (N. do Espírito-Santo), jaime.ripoll@ufrgs.br (J. Ripoll).

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A. Gálvez and H. Rosenberg prove the existence of solutions when $\dim M = 2$ assuming only a negative upper bound for the sectional curvature of M. A partial extension of this last result and an improvement of the one of [4] (and of Theorem 3.6 of Choi [2] too) was obtained in [11], where an unified treatment for the harmonic and minimal case is given. In [8] it is investigated the extension to a 2-dimensional Riemannian manifold of the classical result of Jenkins and Serrin on the Dirichlet problem for the minimal surface equation on bounded domains with infinity boundary value data.

Another result on minimal graphs, due to N. Kutev and F. Tomi, asserts the existence of solutions to the *exterior* Dirichlet problem for the minimal surface equation, namely: If $\Omega \subset \mathbb{R}^2$ is a domain such that $\mathbb{R}^2 \setminus \Omega$ is bounded, then there is a nonzero solution of the minimal surface equation in Ω assuming at $\partial\Omega$ any given boundary data with small enough oscillation (see Theorem E of [7] for the precise statement). In the present article we investigate the extension of Kutev and Tomi's result to a Riemannian manifold in the case of vanishing boundary data. Precisely, we study the existence of solutions to the following Dirichlet problem:

$$\begin{cases} \mathcal{M}(u) := \operatorname{div}\left(\dfrac{\operatorname{grad} u}{\sqrt{1+|\operatorname{grad} u|^2}}\right) = 0 \quad \text{in } \Omega,\ u \in C^2(\Omega) \cap C^0(\overline{\Omega}), \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

where Ω is an unbounded domain (open and connected) in a complete noncompact Riemannian manifold M such that $\partial\Omega$ is compact; div and grad are the divergence and gradient in M.

We prove existence results of solutions of (1) in the cases that either M has nonnegative Ricci curvature or M is simply connected and with sectional curvature K_M satisfying $K_M \leqslant -k^2 < 0$ for some positive constant k. In the case of negative curvature, we prove:

Theorem 1. *Assume that M is simply connected and that the sectional curvature K_M of M satisfies $K_M \leqslant -k^2 < 0$ for some positive constant k. We require that Ω is a domain of M satisfying the exterior sphere condition, namely, given $p \in \partial\Omega$, there is a geodesic sphere of M passing through p, tangent to $\partial\Omega$ at p which is the boundary of a geodesic ball containing Ω.*

Given any nonnegative real number s, there exists a bounded solution $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ of (1) *such that*

$$\lim_{x \to \partial\Omega} \sup \left|\operatorname{grad} u(x)\right| = s$$

and

$$\max_{\Omega} |u| \leqslant \frac{2s}{k}.$$

We next consider the case that M has nonnegative Ricci curvature. We prove:

Theorem 2. *Let M be an n-dimensional, complete noncompact Riemannian manifold with nonnegative Ricci curvature and with sectional curvature satisfying the growth condition*

$$K_M(x) \leqslant \frac{4c^2}{(1+4c^2\rho^2(x))^2}, \quad x \in M,$$

for some constant $c > 0$, where ρ is the distance function to a totally geodesic submanifold S of M,

$$\rho(x) = \inf\left\{d(x,y) \;\middle|\; y \in S\right\},$$

$d =$ Riemannian distance in M, and $K_M(x)$ is the maximum of the sectional curvature of M on planes of T_xM containing $\operatorname{grad}\rho$. *Let Ω be an unbounded domain in M such that $\partial\Omega$ is compact and assume that ρ is smooth on $\overline{\Omega}$. Then, for any given nonnegative real number s, there exists a solution $u \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ of* (1) *such that*

$$\lim_{x \to \partial\Omega} \sup \left|\operatorname{grad} u(x)\right| = \sup_{\Omega} |\operatorname{grad} u| = s. \tag{2}$$

We observe that the existence of S is part of the hypothesis of Theorem 2. However, if M has nonnegative sectional curvature the existence of S is guaranteed by the Soul Theorem of Cheeger and Gromoll [1]. We remark that Cheeger and Gromoll's theorem does not apply if it is only assumed that the Ricci curvature of M is nonnegative.

We finally remark that the vanishing boundary data hypothesis of the exterior Dirichlet problem, at least in Theorem 2, cannot be dispensed, since Theorem N of [7] proves the existence, when $M = \mathbb{R}^2$, of continuous nonzero boundary data with arbitrarily small C^0 norm for which the exterior Dirichlet problem does not have a solution.

2. Some notations and preliminary results

It is assumed that M is a complete Riemannian manifold. We shall make use later of the following computation. Let N be any given smooth compact submanifold of M and denote by ζ the distance to N, that is,

$$\zeta(x) = d(x, N) = \min\{d(x, y) \mid y \in N\}$$

where d is the Riemannian distance in M. Given a function $\varphi \in C^2(\mathbb{R})$, set $w = \varphi(\zeta)$. If ζ is differentiable in a neighborhood of $x \in M$, considering an orthonormal basis in $T_x M$ containing $\operatorname{grad}\zeta(x)$, we obtain

$$\mathcal{M}(w) = \operatorname{div}\left(\frac{\operatorname{grad} w}{\sqrt{1 + |\operatorname{grad} w|^2}}\right) = \operatorname{div}\left(\frac{\varphi' \operatorname{grad}\zeta}{\sqrt{1 + (\varphi')^2}}\right),$$

and, after some computations

$$\mathcal{M}(w)(x) = \frac{(n-1)\varphi''(\zeta(x)) + \varphi'(\zeta(x))(1 + (\varphi'(\zeta(x)))^2)\Delta\zeta(x)}{(n-1)(1 + (\varphi'(\zeta(x)))^2)^{3/2}}. \tag{3}$$

The following result follows from Lemma 6 of [3]:

Lemma 3. *Let Λ be a $C^{2,\alpha}$ bounded open subset of M and $u \in C^{2,\alpha}(\overline{\Lambda})$ a solution of $\mathcal{M}(u) = 0$ in Λ. Assume that u is bounded in Λ and that $|\operatorname{grad} u|$ is bounded on $\Gamma = \partial\Lambda$. Then $|\operatorname{grad} u|$ is bounded in Λ by a constant that depends only on $\sup_\Lambda |u|$ and $\sup_\Gamma |\operatorname{grad} u|$.*

In the case that M has nonnegative Ricci curvature we have in fact a maximum principle for the gradient. This principle is fundamental for the proof of Theorem 2:

Lemma 4. *Assume that the Ricci curvature of M is nonnegative. Let Λ be a C^∞ bounded open subset of M. Then any solution $u \in C^\infty(\overline{\Lambda})$ of $\mathcal{M}(u) = 0$ in Λ satisfies the gradient maximum principle*

$$\max_\Lambda |\operatorname{grad} u| = \max_{\partial\Lambda} |\operatorname{grad} u|.$$

Proof. Let η be a unit normal vector field orthogonal to the graph G of u such that $\langle \eta, \partial_t \rangle \geqslant 0$, ∂_t being the unit vector in the $\mathbb{R}$ direction. Since $\operatorname{Ric}_{M\times\mathbb{R}} \geqslant 0$ it follows from Proposition 1 of [5] that

$$\Delta\langle \eta, \partial_t \rangle = -\left(\operatorname{Ric}_{M\times\mathbb{R}}(\eta) + \|B\|^2\right)\langle \eta, \partial_t \rangle \leqslant 0,$$

where $\|B\|$ is the norm of the second fundamental form of G. The function $\langle \eta, \partial_t \rangle$ is then superharmonic on G so that

$$\max_\Lambda |\operatorname{grad} u| = \min_G \langle \eta, \partial_t \rangle = \min_{\partial G} \langle \eta, \partial_t \rangle = \max_{\partial\Lambda} |\operatorname{grad} u|. \qquad \square$$

Notation. Under the hypothesis and notations of Theorem 1, we denote by γ the Riemannian distance in M to $\partial\Omega$ restrict to $\overline{\Omega}$, namely

$$\gamma(p) = \inf\{d(p, q) \mid q \in \partial\Omega\}, \quad p \in \overline{\Omega}.$$

Moreover, we set $F_r = \gamma^{-1}(r)$, $r \geqslant 0$ and denote by H_{F_r} the normalized mean curvature of F_r with respect to the unit vector field normal to F_r pointing to the bounded connected component of $M \setminus F_r$. Note that

$$\Delta\gamma(x) = (n-1)H_{F_{\gamma(x)}}(x), \tag{4}$$

$x \in M$.

In the next lemma we use the exterior sphere condition to estimate from below the mean curvature of the level hypersurfaces F_r which is fundamental to the construction of low barriers for problem (1) in the case of Theorem 1:

Lemma 5. *If M is simply connected and with sectional curvature satisfying $K_M \leqslant -k^2 < 0$, $k > 0$, then*

$$\inf_{F_r} H_{F_r} \geqslant k \tag{5}$$

for any $r > 0$.

Proof. Given $p \in F_r$ there is $q \in \partial\Omega$ and a minimizing geodesic $\alpha : [0, r] \to M$ such that $\alpha(0) = q$ and $\alpha(r) = p$. Let $S_l(p_0)$ be a geodesic sphere of M with some radius l and some center p_0 passing through q, which is tangent to $\partial\Omega$ at q and such that $\partial\Omega \subset B_l(p_0)$. Then, by Gauss Lemma, $S_{l+r}(p_0)$ passes to p, is tangent to F_r at p and $F_r \subset B_{l+r}(p_0)$. It follows from the tangency principle for constant mean curvature hypersurfaces and the Hessian Comparison Theorem that $H_{F_r} \geqslant H_{S_{l+r}(p_0)} \geqslant k$. $\square$

3. Proof of Theorem 1

We first assume that Ω is a $C^{2,\alpha}$ domain, $0 < \alpha < 1$. Given $s \geqslant 0$, we look for constants a, b and c such that $\varphi(r) = (ar + b)/(r + c)$ determines a subsolution $v_s(x) := \varphi(\gamma(x))$ satisfying $v_s|_{\partial\Omega} = 0$ and $|\operatorname{grad} v_s|_{\partial\Omega} = s$. Obviously $b = 0$ and $|\operatorname{grad} v_s|_{\partial\Omega} = \varphi'(0) = s$ implies that $a = sc$. Then $\varphi'(r) \geqslant 0$ and it follows from (3), (4) and (5) that v_s is a subsolution if

$$\varphi''(r) + k\varphi'(r)\bigl(1 + \bigl(\varphi'(r)\bigr)^2\bigr) = \frac{sc^2}{(c+r)^6}\bigl(kr^4 + 2(2ck-1)r^3 + 6c(ck-1)r^2 + 2c^2(2ck-3)r + c^3\bigl(kcs^2 + kc - 2\bigr)\bigr) \geqslant 0$$

which is valid if we choose $c = 2/k$ since then all the coefficients of the quartic polynomial on r become positive. We then proved that

$$v_s(x) = \varphi\bigl(\gamma(x)\bigr) = \frac{2s\gamma(x)}{k\gamma(x) + 2}$$

is a subsolution satisfying $v_s|_{\partial\Omega} = 0$ and $|\operatorname{grad} v_s|_{\partial\Omega} = s$. Moreover

$$\lim_{\gamma(x)\to\infty} v_s(x) = \lim_{r\to\infty} \varphi(r) = \frac{2s}{k}.$$

Let $m \in \mathbb{N}$ be given, $m \geqslant 1$. Setting $\Omega_m = \Omega \cap \{x \in \Omega \mid \gamma(x) < m\}$, we prove the existence of a solution $w_m \in C^{2,\alpha}(\overline{\Omega_m})$ of $\mathcal{M} = 0$ in Ω_m with $w_m|_{\partial\Omega} = 0$ and such that $\sup_{\partial\Omega}|\operatorname{grad} w_m| = s$. To this end, set

$$T_m = \Bigl\{t \geqslant 0 \Bigm| \exists u_t \in C^{2,\alpha}(\overline{\Omega_m}) \text{ such that } \mathcal{M}(u_t) = 0,\ u_t|_{\partial\Omega} = 0,\ u_t|_{\Gamma_m} = t,\ \sup_{\partial\Omega}|\operatorname{grad} u_t| \leqslant s\Bigr\}$$

where $\Gamma_m = \partial\Omega_m \setminus \partial\Omega$.

We have $T_m \neq \emptyset$, since $0 \in T_m$. We prove in the sequel that T_m is bounded, that the supremum of T_m is assumed and, what is the most fundamental point, that if $t_m = \sup T_m$ then $\sup_{\partial\Omega}|\operatorname{grad} u_{t_m}| = s$. We begin by proving that if $t \in T_m$ then

$$t \leqslant v_s|_{\Gamma_m} = \frac{2sm}{km + 2}. \tag{6}$$

Given $\varepsilon > 0$, we first prove that $t < v_{s+\varepsilon}|_{\Gamma_m}$. By contradiction, assume that $t \geqslant v_{s+\varepsilon}|_{\Gamma_m}$. Since

$$|\operatorname{grad} v_{s+\varepsilon}|_{\partial\Omega} = \inf_{\partial\Omega}|\operatorname{grad} v_{s+\varepsilon}| = s + \varepsilon > \sup_{\partial\Omega}|\operatorname{grad} u_t|$$

there is a neighborhood U of $\partial\Omega$ in Ω such that $u_t(x) < v_{s+\varepsilon}(x)$ for all $x \in U \setminus \partial\Omega$. It follows that there exists a domain $U \subset V \subset \Omega_m$ such that $u_t|_{\partial V} = v_{s+\varepsilon}|_{\partial V}$, which is an absurd since $v_{s+\varepsilon}|_V$ is a subsolution of $\mathcal{M}$ and hence $v_{s+\varepsilon}|_V \leqslant u_t|_V$. Letting $\varepsilon \to 0$ we have $t \leqslant v_s|_{\Gamma_m}$. It follows that T_m is bounded and we may set $t_m = \sup T_m < \infty$. We will prove that $t_m \in T_m$.

We first prove if $t \in T_m$, then

$$\sup_{\partial\Omega_m}|\operatorname{grad} u_t| \leqslant s. \tag{7}$$

Assume that $t \in T_m$. By definition of T_m we have $\sup_{\partial\Omega}|\operatorname{grad} u_t| \leqslant s$.

Setting $z_m = v_s|_{\Gamma_m}$ we have, as proved above, that $z_m > t$. Moreover, the function $w_s := v_s - (z_m - t)$ is a subsolution since v_s is one, $z_m - t$ is constant and $\mathcal{M}(w)$ does not depend on w, but just on its first and second derivatives. Also,

$$w_s|_{\partial\Omega} = -(z_m - t) \leqslant 0 = u_t|_{\Omega},$$
$$w_s|_{\Gamma_m} = t = u_t|_{\Gamma_m}.$$

It follows that

$$w_s \leqslant u_t \leqslant t,$$
$$w_s|_{\Gamma_m} = t = u_t|_{\Gamma_m}$$

and we may conclude that

$$\sup_{\Gamma_m} |\operatorname{grad} u_t| \leqslant \sup_{\Gamma_m} |\operatorname{grad} w_s| = \varphi'(m) = \frac{4s}{(km+2)^2} \leqslant s$$

proving (7).

Consider now a sequence $\{\tau_j\} \subset T_m$ converging to t_m as j goes to infinity. For each j, there is a function $u_j \in C^{2,\alpha}(\overline{\Omega_m})$ such that $\mathcal{M}(u_j) = 0$, $u_j|_{\partial\Omega} = 0$ and $u_j|_{\Gamma_m} = \tau_j$. Since, by the maximum principle,

$$\sup_{\Omega_m} |u_j| \leqslant t_m \tag{8}$$

it follows from (7) and Lemma 3 that the sequence $\{u_j\}$ has the C^1 norm uniformly bounded in $\overline{\Omega_m}$. Since $\overline{\Omega_m}$ is a compact $C^{2,\alpha}$ domain, from elliptic PDE theory we have C^2 compactness of $\{u_j\}$ in $\overline{\Omega_m}$ (see [9]). Hence, there is a subsequence of $\{u_j\}$ that converges uniformly C^2 in $\overline{\Omega_m}$ to a solution $w_m \in C^2(\overline{\Omega_m})$ of $\mathcal{M} = 0$ in Ω_m. From PDE elliptic regularity $w_m \in C^{2,\alpha}(\overline{\Omega_m})$.

The function w_m is a solution of $\mathcal{M} = 0$ in Ω_m that satisfies $w_m|_{\partial\Omega} = 0$, $w_m|_{\Gamma_m} = t_m$ and $\sup_{\partial\Omega_m} |\operatorname{grad} w_m| \leqslant s$. It follows that $t_m \in T_m$, that is, $w_m = u_{t_m}$.

We now note that $\sup_{\partial\Omega} |\operatorname{grad} u_{t_m}| = s$. In fact: By contradiction, assume that $\sup_{\partial\Omega} |\operatorname{grad} u_{t_m}| < s$.

Consider a function $\phi \in C^{2,\alpha}(\overline{\Omega_m})$ such that $\phi|_{\partial\Omega} = 0$ and $\phi|_{\Gamma_m} = t_m$, set

$$C_0^{2,a}(\overline{\Omega_m}) = \{\omega \in C^{2,\alpha}(\overline{\Omega_m}) \mid \omega|_{\partial\Omega_m} = 0\},$$

and define $T : [0, 2] \times C_0^{2,\alpha}(\overline{\Omega_m}) \to C^{\alpha}(\overline{\Omega_m})$ by

$$T(l, \omega) = \mathcal{M}(\omega + l\phi).$$

Then

$$T(1, \omega_m) = 0$$

where $\omega_m = u_{t_m} - \phi$. From elliptic PDE theory we have that the Fréchet derivative $\partial_2 T(1, \omega_m) = d\mathcal{M}_{w_m}$ is an isomorphism (this is clear since $d\mathcal{M}_{w_m}(h)$, $h \in C_0^{2,\alpha}(\overline{\Omega_m})$, depends only on the first and second derivatives of h and not on h and hence satisfies the maximum principle) so that, from the implicit function theorem, there exists a continuous function (on the $C^{2,\alpha}$ topology) $i : (1-\varepsilon, 1+\varepsilon) \to C_0^{2,\alpha}(\overline{\Omega_m})$, with $i(1) = \omega_m$ such that $T(l, i(l)) = 0$, $l \in (1-\varepsilon, 1+\varepsilon)$. Therefore, since $|\operatorname{grad} u_{t_m}|_{\partial\Omega} < s$ and $w_m = i(1) + \phi$, there exists $l \in (1, 1+\varepsilon)$ such that $\sup_{\partial\Omega} |i(l) + l\phi| < s$. Since

$$0 = T\big(l, i(l)\big) = \mathcal{M}\big(i(l) + l\phi\big),$$

$i(l) + l\phi = 0$ at $\partial\Omega$ and $i(l) + l\phi = lt_m$ at Γ_m, we have that $lt_m \in T_m$, contradiction since $lt_m > t_m = \sup T_m$. Hence, $\sup_{\partial\Omega} |\operatorname{grad} u_{t_m}| = s$.

Since, from (6), (7) and (8)

$$\max_{\partial\Omega_m} |\operatorname{grad} u_{t_m}| \leqslant \varphi'(m) \leqslant s,$$

$$\max_{\Omega_m} |u_{t_m}| = t_m \leqslant v_s|_{\Gamma_m} \leqslant \varphi(m) \leqslant \frac{2s}{k}$$

are estimates that do not depend on m it follows from Lemma 4 that the sequence $\{u_{t_m}\}$ has uniform C^1 estimates on compacts of $\overline{\Omega}$ which implies the existence of a subsequence of $\{u_{t_m}\}$ converging uniformly $C^{2,\alpha}$ on compacts of $\overline{\Omega}$ to a solution $u_s \in C^{2,\alpha}(\overline{\Omega})$ of $\mathcal{M} = 0$ in Ω satisfying $u_s|_{\partial\Omega} = 0$ and

$$\sup_{\partial\Omega} |\operatorname{grad} u_s| = s.$$

Moreover, it also follows from the above estimates that there is a constant $C(s)$ depending only on s such that $|u_s|_1 \leqslant C(s)$. If Ω is only a C^0 domain one considers a sequence of $C^{2,\alpha}$ domains Λ_m satisfying the exterior sphere condition such that $\overline{\Omega} \subset \Lambda_{m+1} \subset \Lambda_m$ and $\Omega = \bigcap \Lambda_m$. From what we have proved above, for each m there is a solution $v_m \in C^{2,\alpha}(\overline{\Lambda}_m)$ of $\mathcal{M}(v_m) = 0$ such that $v_m|_{\partial\Lambda_m} = 0$ and

$$\sup_{\partial\Omega} |\operatorname{grad} v_m| = s \leqslant |v_m|_1 \leqslant C(s). \tag{9}$$

From the Arzela–Ascoli theorem we may assume that v_m converges uniformly on compacts of $\overline{\Omega}$ to a function $u_s \in C^0(\overline{\Omega})$ such that $u_s|_{\partial\overline{\Omega}} = 0$. If $\overline{U} \subset \Omega$ is compact, U open, then we have uniform C^1 estimates of $v_m|_U$ and then, from elliptic PDE theory, $v_m|_U$ contains a subsequence converging C^∞ to u_s on U so that $u_s|_U \in C^\infty(U)$ and $\mathcal{M}(u) = 0$ in U. By the diagonal method, we obtain the existence of subsequence of v_m converging to a solution $u_s \in C^\infty(\Omega) \cap C^0(\overline{\Omega})$ of (1). It is immediate to prove from (9) that u_s satisfies (2). This concludes the proof of the theorem.

4. Proof of Theorem 2

We first consider the case that Ω is a C^∞ domain. Let $s \geqslant 0$ be given. For $m > 0$ set $B_m = \{\rho \leqslant m\}$ and let m_0 be such that $\partial\Omega \subset B_m$ for all $m \geqslant m_0$. Given $m > m_0$, set $\Omega_m = \Omega \cap B_m$ and $S_m = \partial\Omega_m \setminus \partial\Omega$.

We will prove the existence of $m_1 > m_0$ such that, given $m \geqslant m_1$, there is a solution $u_m \in C^\infty(\overline{\Omega_m})$ of $\mathcal{M} = 0$ in Ω_m with $u_m|_{\partial\Omega} = 0$ and $\sup_{\partial\Omega} |\operatorname{grad} u_m| = s$. To this end, given $m > m_0$, set

$$T_m = \left\{ t \geqslant 0 \;\middle|\; \exists u_t \in C^\infty(\overline{\Omega_m}) \text{ such that } \mathcal{M}(u_t) = 0,\ \sup_{\Omega_m} |\operatorname{grad} u_t| \leqslant s,\ u_t|_{\partial\Omega} = 0,\ u_t|_{S_m} = t \right\}.$$

It is clear that T_m is bounded and the uniform bound for the C^1 norm of the solutions of $\mathcal{M} = 0$ corresponding to points of T_m imply that the supremum $t_m = \sup T_m$ is attained. Moreover, we may make use of the implicit function theorem, as in the previous theorem, to assert that the solution $u_m \in C^\infty(\overline{\Omega_m})$ of $\mathcal{M} = 0$ in Ω_m such that $u_m|_{S_m} = t_m$ satisfies $\sup_{\Omega_m} |\operatorname{grad} u_{t_m}| = s$.

The main part of the proof consists in proving that the maximum of $|\operatorname{grad} u_{t_m}|$ is assumed at $\partial\Omega$. To this end, we construct barriers from above and from below to u_m which gradient less than or equal to $s/2$ at S_m.

Since $w_m := t_m$ is a solution of $\mathcal{M} = 0$ in Ω_m and $u|_{\partial\Omega_m} \leqslant w_m|_{\partial\Omega_m}$ we have $u_m \leqslant w_m$ on Ω_m and hence w_m is an upper barrier.

To construct a barrier from below for (1) we first consider the paraboloid P of $\mathbb{R}^{n+1}$ obtained by acting the rotational group of $\mathbb{R}^{n+1}$, that leaves fixed the x_{n+1}-axis, on the parabola $(t, 0, \ldots, 0, ct^2)$, $t \in \mathbb{R}$. The sectional curvature of P at $y = (y_1, y_2, \ldots, y_{n+1}) \in P$ with respect to any plane through the origin of $T_y P$ that contains the tangent vector of the geodesic passing through y and the vertex of P is given by

$$K_P(y) = \frac{4c^2}{(1 + 4c^2 r^2(y))^2}$$

where $r(y) = \sqrt{y_1^2 + \cdots + y_n^2}$.

Let $x \in M$ and $y \in P$ be such that $\rho(x) = d_P(y, 0)$, where d_P is the Riemannian distance in P and 0 is the origin of $\mathbb{R}^{n+1}$. Then

$$\rho(x) = \int_0^{r(y)} \sqrt{1 + 4c^2 t^2}\, dt \geqslant r(y),$$

and, from the hypothesis, it follows that

$$K_M(x) \leqslant \frac{4c^2}{(1+4c^2\rho^2(x))^2} \leqslant \frac{4c^2}{(1+4c^2r^2(y))^2} = K_P(y).$$

We then obtain, from the Hessian Comparison Theorem [12, Theorem 1.1], that $\Delta\rho(x) \geqslant \Delta\rho_P(y)$, where $\rho_P(y) = d_P(y,0)$. Therefore, if φ is such that $\varphi' \geqslant 0$ it follows from (3) that $v(x) = \varphi(\rho(x))$ is a subsolution of Q in Ω if

$$(n-1)\varphi''\big(\rho_P(y)\big) + \varphi'\big(\rho_P(y)\big)\big(1+\big(\varphi'\big(\rho_P(y)\big)\big)^2\big)\Delta\rho_P(y) \geqslant 0, \tag{10}$$

for all $y \in P$.

We have

$$\Delta\rho_P(y) = \frac{n-1}{r(y)\sqrt{1+4c^2r^2(y)}}. \tag{11}$$

Introducing the notation

$$\delta(r) = \int_0^r \sqrt{1+4c^2t^2}\,dt$$

and $\psi(r) = \varphi(\delta(r))$ we have

$$\varphi'\big(\delta(r)\big) = \frac{\psi'(r)}{\delta'(r)},$$

$$\varphi''\big(\delta(r)\big) = \frac{1}{(\delta'(r))^3}\big[\delta'(r)\psi''(r) - \psi'(r)\delta''(r)\big].$$

Setting $r = r(y)$ we have that the inequality (10) holds if and only if

$$\frac{1}{(\delta'(r))^3}\big\{(n-1)\big[\psi''(r)\delta'(r) - \psi'(r)\delta''(r)\big] + \psi'(r)\big[\big(\delta'(r)\big)^2 + \big(\psi'(r)\big)^2\big]\Delta\rho_P\big\} \geqslant 0. \tag{12}$$

Since $\delta'(r) = \sqrt{1+4c^2r^2}$, $\delta''(r) = 4c^2r/\sqrt{1+4c^2r^2}$, from (11) we have that (12) holds if, and only if,

$$(n-1)\left[\psi''(r)\sqrt{1+4c^2r^2} - \frac{4c^2r\psi'(r)}{\sqrt{1+4c^2r^2}}\right] + \psi'(r)\big[1+4c^2r^2+\big(\psi'(r)\big)^2\big](n-1)\frac{1}{r\sqrt{1+4c^2r^2}} \geqslant 0,$$

that is

$$\frac{(n-1)}{r\sqrt{1+4c^2r^2}}\big[r\big(1+4c^2r^2\big)\psi''(r) - 4c^2r^2\psi'(r) + \psi'(r)\big(1+4c^2r^2\big) + \big(\psi'(r)\big)^3\big] \geqslant 0$$

or

$$r\big(1+4c^2r^2\big)\psi''(r) + \psi'(r) + \big(\psi'(r)\big)^3 \geqslant 0.$$

We have that

$$\psi_a(r) = \psi(r) = \sqrt{a}\int_{\frac{\sqrt{a}}{2c\sqrt{4-a}}}^{r} \frac{\sqrt{4c^2t^2+1}}{\sqrt{4c^2t^2(4-a)-a}}\,dt$$

is a solution of

$$r\big(1+4c^2r^2\big)\psi''(r) + \psi'(r) + \big(\psi'(r)\big)^3 = 0$$

for all $a \in [0,4)$ and

$$r \geqslant \frac{\sqrt{a}}{2c\sqrt{4-a}}.$$

Moreover, ψ_a satisfies

$$\psi_a\left(\frac{\sqrt{a}}{2c\sqrt{4-a}}\right)=0,$$
$$\psi_a'\left(\frac{\sqrt{a}}{2c\sqrt{4-a}}\right)=+\infty. \tag{13}$$

Let r_0 be such that $\delta(r_0)=m_0$ and a_0 such that

$$\frac{\sqrt{a_0}}{2c\sqrt{4-a_0}}=r_0,$$

that is,

$$a_0=\frac{16c^2r_0^2}{1+4c^2r_0^2}.$$

Set $\varphi(\delta(r))=\psi_{a_0}(r)$. Given $m>m_0$, let r_m be such that $\delta(r_m)=m$. Since

$$\varphi'(m)=\frac{\psi_{a_0}'(r_m)}{\delta'(r_m)}=\frac{\sqrt{a_0}}{\sqrt{4c^2r_m^2(4-a_0)-a_0}}$$

we have $\varphi'(m)\leqslant s/2$ if and only if

$$\frac{\sqrt{a_0}}{\sqrt{4c^2r_m^2(4-a_0)-a_0}}\leqslant\frac{s}{2}\quad\Leftrightarrow\quad r_m\geqslant\frac{\sqrt{a_0(1+\frac{4}{s^2})}}{2c\sqrt{4-a_0}}.$$

Hence, if we choose $m_1>m_0$ such that

$$r_{m_1}\geqslant\frac{\sqrt{a_0(1+\frac{4}{s^2})}}{2c\sqrt{4-a_0}},$$

since r_m increases with m we have $\varphi'(m)\leqslant s/2$ for all $m\geqslant m_1$. It follows from (13) that, for all $m\geqslant m_1$, $v_m(x):=\varphi(\rho(x))$ is a "catenoid-like" subsolution of $\mathcal{M}$ on $\Omega\cap(B_m\setminus B_{m_0})$. It satisfies at $x\in M$ such that $\rho(x)=m_0$:

$$v_m(x)=0,$$
$$\left|\operatorname{grad} v_m(x)\right|=\infty$$

and, if $\rho(x)=m$,

$$\left|\operatorname{grad} v_m(x)\right|\leqslant\frac{s}{2}. \tag{14}$$

Clearly v_m+b is a subsolution of $\mathcal{M}$ on $B_m\setminus B_{m_0}$ for any constant b and there is b_0 such that $v_m(x)+b_0\leqslant u_m(x)$ for all $x\in\Omega\cap(B_m\setminus B_{m_0})$. Set

$$b_m=\max\left\{b\mid v_m(x)+b\leqslant u_m(x),\ \forall x\in B_m\setminus B_{m_0}\right\}.$$

Since $|\operatorname{grad} v_m(x)|=\infty$ for $x\in S_{m_0}=\partial B_{m_0}$ it follows from the maximum principle that $v_m(x)+b_m=t_m$ for $x\in S_m$ and $u_m\geqslant v_m$ on $B_m\setminus B_{m_0}$. Then

$$|\operatorname{grad} u_m|_{S_m}\leqslant\max\left\{|\operatorname{grad} v_m|_{S_m},|\operatorname{grad} w_m|_{S_m}\right\}=\max\left\{\frac{s}{2},0\right\}=\frac{s}{2}.$$

It follows by Lemma 4 that

$$\sup_{\partial\Omega}|\operatorname{grad} u_{t_m}|=s.$$

Now, letting $m\to\infty$, the uniform C^1 estimates of u_{t_m} on compacts of Ω implies the existence of a subsequence of $\{u_{t_m}\}$ converging uniformly on compacts of Ω to a solution $u_s\in C^\infty(\overline{\Omega})$ of $\mathcal{M}=0$ in Ω satisfying $u_s|_{\partial\Omega}=0$ and $\sup_{\partial\Omega}|\operatorname{grad} u_s|=s$. Note that for each m there is a point $q_m\in\partial\Omega$ such that $|\operatorname{grad} u_{t_m}(q_m)|=s$. Since $\partial\Omega$ is compact a subsequence of q_m converges to $q\in\partial\Omega$. It follows that $|\operatorname{grad} u_s(q)|=s$, so that u_s cannot be identically zero. If Ω is only a C^0 domain one considers a sequence of C^∞ domains converging to Ω similarly to what was done in the previous result. This concludes the proof of the theorem.

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