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The C^1 closing lemma for generic C^1 endomorphisms ☆

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Abstract

Given a compact m-dimensional manifold M and $1 \leqslant r \leqslant \infty$, consider the space $C^r(M)$ of self mappings of M. We prove here that for every map f in a residual subset of $C^1(M)$, the C^1 closing lemma holds. In particular, it follows that the set of periodic points is dense in the nonwandering set of a generic C^1 map. The proof is based on a geometric result asserting that for generic C^r maps the future orbit of every point in M visits the critical set at most m times.

MSC: 37Cxx; 58K05

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1. Introduction

Let M and N be real compact C^∞ manifolds without boundary. For every $1 \leqslant r \leqslant \infty$, $C^r(M, N)$ will stand for the set of class C^r maps endowed with the usual C^r topology. When $M = N$, the space $C^r(M, M)$ will be denoted by $C^r(M)$.

The problem known as *the Closing Lemma* states that: given a nonwandering point x of a C^1 map f, is it possible to find a C^1 perturbation g of f such that x is g-periodic? This was solved by C. Pugh for diffeomorphisms in 1967 [4]. Since then, many generalizations were obtained in different contexts besides of diffeomorphisms (see [5]). The case of C^1 endomorphisms was not included in the above contexts.

Nevertheless, for C^1 endomorphisms of a compact manifold and by a clever extension of Pugh's technique, L. Wen obtained a C^1 closing lemma valid for those having a finite number of critical points, see [7] and [8]. However, the set of C^1 maps for which this hypothesis holds is not dense. Here we will prove that:

Theorem 1. *There exists a residual set $\mathcal{R}$ in $C^1(M)$ such that given $f \in \mathcal{R}$, $\mathcal{U}$ a C^1 neighborhood of f and x a nonwandering point of f, there exists a map $g \in \mathcal{U}$ such that x is g-periodic. Moreover, there exists a residual subset $\mathcal{D}$ of $\mathcal{R}$ such that for every $f \in \mathcal{D}$ the set of periodic points is dense in the nonwandering set.*

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This result answers positively the questions posted by M. Shub in the final section of [6]. The first main ingredient in the proof of Theorem 1 is Wen's result on closing with finitely many singularities. The second one is a geometric result regarding the study of the singularities of a map. The study of the singularities of smooth mappings has been largely considered in the literature. It is well known, for example, that the singular set of a generic C^∞ map admits a stratification in submanifolds. The singular or critical set of a map $f \in C^1(M)$ is defined as the set of points where the differential of f is not invertible, and is denoted by S_f. Our interest here is to determine how frequently the future orbit of a point intersects the critical set of f, at least restricting the study to a residual set of maps.

Theorem 2. *Let M be a compact manifold of dimension m. Given any positive integer $r \geqslant 1$ it holds that*:

1. *There exists a residual subset $\mathcal{R}_r$ of $C^r(M)$ such that, for every $f \in \mathcal{R}_r$, and every $z \in M$, the set of integers $n \geqslant 0$ such that $f^n(z) \in S_f$ contains at most m elements.*
2. *There exists a residual subset $\mathcal{P}_r$ of $C^r(M)$ such that for every map $f \in \mathcal{P}_r$ it holds that $\bigcup_{n\geqslant 0} f^n(S_f)$ does not intersect the set of periodic points of f.*

2. Proof of Theorem 1 and some consequences

In this section we prove our generic C^1 closing lemma assuming that Theorem 2 holds. We first state Wen's result. To do that we need some definitions. A tree $\mathcal{F} = (Q, f)$ is an infinite sequence of mutually disjoint nonempty finite sets $Q_0, Q_1, \ldots, Q_n, \ldots$ where Q_0 consists of a single point q_0 together with a map $f : Q - \{q_0\} \to Q$ where $Q = \bigcup_{n=0}^{\infty} Q_n$ such that f maps Q_n into Q_{n-1} for each $n = 1, 2, \ldots.$ An infinite sequence $q_0, q_1, \ldots, q_n, \ldots$ is an infinite branch of $\mathcal{F}$ if $f(q_n) = q_{n-1}$ for each $n = 1, 2, \ldots.$ A tree $\mathcal{F}$ is called complete if f is onto.

We also denote by $\mathcal{O}^-(x) = \bigcup_{n\geqslant 0} f^{-n}(x)$ the negative orbit of x.

Theorem 3. *(See [8].) Let $f \in C^1(M)$ and let x be a nonwandering point of f. Assume that there exists $y \in \mathcal{O}^-(x) \cap \Omega(f)$ such that*

1. *$\mathcal{O}^-(y) \cap \Omega(f) \cap S_f = \emptyset$.*
2. *$(\mathcal{O}^-(y) \cap \Omega(f), f)$ is a complete tree.*

Then, given any C^1 neighbourhood $\mathcal{U}$ of f in $C^1(M)$ and U open set containing x, there exist $g \in \mathcal{U}$ and $p \in U$ such that p is a periodic point of g.

This theorem corresponds to the Case 1 in the proof of Theorem A in [8]. In that case it was assumed that $\mathcal{O}^-(y) \cap S_f = \emptyset$; the arguments used there can easily be adapted.

Lemma 1. *Let $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{P}_1$ the residual set in $C^1(M)$ as in Theorem 2. Let $f \in \mathcal{R}$ and let x be a nonwandering point of f. Then, one of following statements holds*:

1. *There exists $y \in \mathcal{O}^-(x) \cap \Omega(f)$ such that $f^{-1}(y) \cap \Omega(f) = \emptyset$.*
2. *There exists $y \in \mathcal{O}^-(x) \cap \Omega(f)$ such that*
   (a) *$\mathcal{O}^-(y) \cap \Omega(f) \cap S_f = \emptyset$.*
   (b) *$(\mathcal{O}^-(y) \cap \Omega(f), f)$ is a complete tree.*

Proof. Assume that (1) does not happen. Then, for any $y \in \mathcal{O}^-(x) \cap \Omega(f)$ we have that $f^{-1}(y) \cap \Omega(f) \neq \emptyset$. Thus, any point $y \in \mathcal{O}^-(x) \cap \Omega(f)$ is contained in an infinite branch $\Sigma_x = \{x_0 = x, x_1, x_2, \ldots\}$ with $f(x_n) = x_{n-1}$ for $n \geqslant 1$ and where $x_i \in \Omega(f)$. Since $f \in \mathcal{R}$ it follows that any branch Σ_x contains *at most m* singular points. Among the branches $\Sigma_x \subset \Omega(f)$ take one with a maximal cardinality of singular points in it, say $\Sigma = \{x_0 = x, x_1, x_2, \ldots\}$. Then, there exists n_0 such that $x_n \notin S_f$ for all $n \geqslant n_0$ Let $y = x_{n_0}$. It follows that $\mathcal{O}^-(y) \cap \Omega(f) \cap S_f = \emptyset$, otherwise we contradict the way we have chosen Σ. In particular, $f^{-n}(y) \cap \Omega(f)$ is finite for all $n \geqslant 0$. This and the fact that (1) does not hold imply that $(\mathcal{O}^-(y) \cap \Omega(f), f)$ is a complete tree. Thus, we have proved that (2) holds. □

By virtue of Theorem 3 and the previous Lemma 1, to prove the first part of Theorem 1 we have to handle the situation where there exists $y \in \mathcal{O}^-(x) \cap \Omega(f)$ such that $f^{-1}(y) \cap \Omega(f) = \emptyset$. We will prove this reasoning as in Case 2 of Theorem A in [8].

Let $f \in \mathcal{R}$ where $\mathcal{R}$ is as in Lemma 1, and let $\mathcal{U}$ be a neighborhood of f. We will use without proof that there exists ϵ such that for any r small and $x \in M$, if $u, w \in B(x, \epsilon r)$ then there exists $h : M \to M$ such that $h(u) = w$, $supp(h) \subset B(x, r)$ and that $h \circ f \in \mathcal{U}$. Here $supp(h)$ denotes the set of points y such that $h(y) \neq y$. Let x be a non-wandering point (and non-periodic, otherwise there is nothing to prove) of f and let U be any open set containing x. Let $y \in \mathcal{O}^-(x) \cap \Omega(f)$ such that $f^{-1}(y) \cap \Omega(f) = \emptyset$. Let k be such that $f^k(y) = x$ and consider V an open set containing y such that $f^i(V)$, $i = 0, \ldots, k$ are disjoint and $f^k(V) \subset U$. Consider a decreasing sequence $r_n \to 0$ such that $B(y, r_n) \subset V$ for all n. Since y is nonwandering, we can pick $z_n \in B(y, \epsilon r_n)$ such that $f^{m_n}(z_n) \in B(y, r_n)$ for some m_n (which is larger than k) and $f^i(z_n) \notin B(y, r_n)$ for $1 \leqslant i \leqslant m_n - 1$. Thus, we have that $f^{m_n}(z_n) \to y$ and, taking a subsequence if necessary, we have that $f^{m_n-1}(z_n) \to w$ with $f(w) = y$. Since w is wandering, there exists an open set W containing w such that $f^i(W) \cap W = \emptyset$ for all $i \geqslant 1$. Therefore, for n large enough, we can take h_1 supported on W such that $h_1(f^{m_n-1}(z_n)) = w$ and $h_1 \circ f \in \mathcal{U}$. (We would like to push $f^{m_n-1}(z_n)$ to a preimage of z_n but this is not possible, there is no preimage of z_n in W). Consider also h_2 supported on $B(y, r_n)$ such that $h_2(y) = z_n$ such that $h_2 \circ f \in \mathcal{U}$. Let $h : M \to M$ be $h = h_1$ in W, $h = h_2$ on $B(y, r_n)$ and otherwise the identity. It follows that $g = h \circ f \in \mathcal{U}$. Notice that $z_n, f(z_n), \ldots, f^{m_n-2}(z_n)$ does not intersects W since W is wandering. On the other hand, $f(z_n), \ldots, f^{m_n-1}(z_n)$ does not go through $B(y, r_n)$ by the way we have chosen m_n. Therefore $g^i(z_n) = f^i(z_n)$ for $i = 0, \ldots, m_n - 2$. Now, $g^{m_n-1}(z_n) = g(f^{m_n-2}(z_n)) = h_1(f^{m_n-1}(z_n)) = w$. And finally, $g^{m_n}(z_n) = g(w) = h(f(w)) = h_2(y) = z_n$. Therefore, z_n is g periodic. Moreover $g^k(z_n) = f^k(z_n) \in U$, and so we found a g periodic point in U. This completes the proof of the closing lemma for $f \in \mathcal{R}$.

To complete the proof of Theorem 1, we must prove that there exist a residual subset $\mathcal{D}$ of $\mathcal{R}$ such that for every $f \in \mathcal{D}$ the set of periodic points is dense in the nonwandering set. This follows by standard arguments. Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable basis of M. For all $n \in \mathbb{N}$ consider:

$$\mathcal{A}_n = \left\{f \in C^1(M) \big/ Per_h(f) \cap U_n \neq \emptyset\right\}$$

being $Per_h(f)$ the set of hyperbolic periodic points of f, and consider $\mathcal{B}_n = C^1(M) - \overline{\mathcal{A}_n}$. From the continuation of hyperbolic periodic points we know that $\mathcal{A}_n$ is an open set and therefore $\mathcal{A}_n \cup \mathcal{B}_n$ is open and dense $\forall\, n \in \mathbb{N}$. Then $\mathcal{D}_1 = \bigcap_{n\in\mathbb{N}}(\mathcal{A}_n \cup \mathcal{B}_n)$ is a residual set on $C^1(M)$. Let us see that if $f \in \mathcal{D} := \mathcal{D}_1 \cap \mathcal{R}$ then $\Omega(f) = \overline{Per(f)}$.

Suppose, by contradiction, that exist some $x \in \Omega(f) - \overline{Per(f)}$, and let $n \in \mathbb{N}$ such that $x \in U_n$ and $U_n \cap \overline{Per(f)} = \emptyset$ (in particular $f \notin \mathcal{A}_n$). From the first part of Theorem 1, we know that for every $\varepsilon > 0$ there exists $g_\varepsilon \in C^1(M)$ such that the C^1 distance between f and g_ε is less than ε and $x \in Per(g_\varepsilon)$. Now we perturb g_ε to transform x on a hyperbolic periodic point keeping the C^1 distance between f and g_ε less than ε. This implies that $g_\varepsilon \in \mathcal{A}_n$ and therefore $f \notin \mathcal{B}_n$, then $f \notin \mathcal{A}_n \cup \mathcal{B}_n$ which is absurd because $f \in \mathcal{D}$.

Now we state some straightforward consequence of our main result.

Corollary 1. *Let $f \in C^1(M)$ a $C^1 - \Omega$-stable map. Then $\Omega(f) = \overline{Per(f)}$.*

Now, lets define the set

$$\mathcal{E}^1(M) = int\left\{f \in C^1(M)\text{: every } p \in Per(f) \text{ is hyperbolic}\right\}$$

where *int* means "interior". In other words, $\mathcal{E}^1(M)$ is the set of endomorphisms such that there exists a neighborhood of it such that any periodic point of any endomorphism in this neighborhood es hyperbolic. For $f \in \mathcal{E}^1(M)$ we denote by $P_i(f)$ the set of periodic points of index i (i.e. $dim\, E^s_p = i$).

Corollary 2. *Let $f \in \mathcal{E}^1(M)$. Then,*

1. *There exists a neighborhood $\mathcal{V}(f)$ such that if U is a neighborhood of $\overline{Per(f)}$ and $g \in \mathcal{V}(f)$ satisfies $g = f$ in U, then $Per(f) = Per(g)$.*
2. *There exists a neighborhood $\mathcal{V}(f)$ such that if U is a neighborhood of $\overline{P_i(f)}$ and $g \in \mathcal{V}(f)$ satisfies $g = f$ in U, then $P_i(f) = P_i(g)$.*

The proof of this corollary is simple and it is the same as for diffeomorphisms. Indeed, just take $\mathcal{V} \subset \mathcal{E}^1(M)$ connected and the result follows from the fact that for each n the maps $\nu : \mathcal{V} \to \mathbb{Z}$, $\nu(g) = \sharp\{p \in Per(g)\colon \pi(p) = n\}$ and $\nu_i : \mathcal{V} \to \mathbb{Z}$, $\nu_i(g) = \sharp\{p \in P_i(g)\colon \pi(p) = n\}$ are continuous and hence constants.

3. Singularities of self mappings

The first assertion in Theorem 2 will be a consequence of the following transversality result:

Theorem 4. *Given a compact manifold M, and any $n > 0$, define $\mathcal{G}_n$ as the set of maps $f \in C^\infty(M)$ such that the two following conditions hold*:

1. *For every $j \leqslant n$, $f^{-j}(S_f)$ is a stratified submanifold.*
2. *The sequence of stratified submanifolds $\{f^{-j}(S_f)\ 0 \leqslant j \leqslant n\}$ is in general position.*

Then $\mathcal{G}_n$ is open and dense for every $n \geqslant 0$.

3.1. Preliminaries on critical sets and transversality

In this subsection we collect some definitions and standard results, in preparation to the proof of Theorem 4.

There exist different definitions of stratifications of sets. The more adequate to our purposes is the following.

Definition 1. Given a smooth manifold M, a pair (N, Σ) is a stratified submanifold of M if $\Sigma = \{N_1, \dots, N_k\}$ is a (finite) partition of a subset N of M, where each N_i is a (not necessarily closed) submanifold of M such that the following conditions hold:

1. The union $\bigcup_{i \geqslant j} N_i$ is a closed set for every $j \leqslant k$.
2. For every $j \leqslant k$, $x \in N_j$, and any vector v tangent to N_j at x, there exists a sequence (x_n, v_n) in TN_{j-1} that converges to (x, v) in TM.

Note that the second condition implies that the dimensions of the submanifolds N_i are decreasing. Each of the submanifolds $N_i \in \Sigma$ is called a stratum. It is also common to say that Σ is a stratification of N.

The most relevant example is the set $L^s(\mathbb{R}^m)$ of singular linear endomorphisms of $\mathbb{R}^m$. If L_j denotes the set of linear maps having kernel of dimension j, then $\{L_1, \dots, L_m\}$ is a stratification of $L^s(\mathbb{R}^m)$. Note that the codimension of L_j in the m^2-dimensional space of all linear self maps of $\mathbb{R}^m$ is j^2.

Definition 2. Given $f \in C^1(M)$ and $1 \leqslant j \leqslant m$, let $S_j(f)$ be the set of points $x \in M$ such that the kernel of Df_x has dimension j.

Theorem 5. *Given any compact manifold M there exists an open dense set $\mathcal{G}$ of $C^\infty(M)$ such that for every $f \in \mathcal{G}$ the following condition holds*:

> *either S_f is empty or there exists a maximum $k = k(f)$ such that $S_j(f)$ is nonempty for every $j \leqslant k$ and is empty for every $j > k$.*

If S_f is not empty, so that the second alternative holds, then (S_f, Σ_f), where $\Sigma_f = \{S_1(f), \dots, S_k(f)\}$, is a stratified submanifold of M.

There exists $\mathcal{G}_0$ open and dense in $\mathcal{G}$ such that for every $f \in \mathcal{G}_0$ no critical point of f is fixed.

The proof of the first statement consists in using jets to formalize the idea that for an open and dense set $f \in C^\infty(\mathbb{R}^m)$ it holds that Df is transverse to the stratified submanifold $L^s(\mathbb{R}^m)$ of the previous example. This proof can be found in Chapter VI of [1]. The proof of the last assertion is trivial, it will be used in the proof of the second assertion of Theorem 2.

There is another local property that will be used in the proof of Theorem 4. Given a map $f \in \mathcal{G}$ and a point $x \in S_f$, there exist neighborhoods $\mathcal{U}$ of f and U of x such that if $g \in \mathcal{U}$ and $S_g \cap U \subset S_f$, then $S_g \cap U = S_f \cap U$. This follows from the genericity of critical points.

Definition 3. Let M be a compact manifold.

1. Any set $\{N_i\colon i \in I\}$ of submanifolds of M is transverse (also called in general position) if for any finite subset J of I such that $N_J := \bigcap_{\{j\in J\}} Nj$ is nonempty, it holds that N_J is a submanifold whose codimension is equal to the sum of the codimensions of the N_j with $j \in J$. This is denoted by $\pitchfork \{N_i\colon i \in I\}$.
2. A set $\{(N_i, \Sigma_i)\colon i \in I\}$ of stratified submanifolds of M is transverse if $\pitchfork \{N_i\colon i \in I\}$ for each choice of submanifolds $N_i \in \Sigma_i$. The notation is $\pitchfork \{\Sigma_i\colon i \in I\}$.
3. Let M_0 be a manifold; a smooth map $f : M \to M_0$ is transverse to a submanifold N of M_0 if $Df_x(T_x(M)) + T_{f(x)}(N) = T_{f(x)}(M_0)$ for every x such that $f(x) \in N$.
4. A map $f \in C^\infty(M)$ is transverse to a stratified submanifold (N, Σ) if it is transverse to each $N_i \in \Sigma$. The notation is $f \pitchfork \Sigma$.

If V is an open subset of the manifold M we say that f is transverse to Σ in V if $f|V \pitchfork \Sigma$.

Theorem 6. *Let M and M_0 be manifolds, where only M is assumed to be compact.*
If $f : M \to M_0$ is a C^∞ map transverse to a stratified submanifold (N, Σ) of M_0, then

$$f^{-1}(\Sigma) := \{f^{-1}(N_i)\colon N_i \in \Sigma\}$$

is a stratification of $f^{-1}(N)$. The codimensions of N_i and $f^{-1}(N_i)$ are equal.
The set of smooth maps f such that $f \pitchfork \Sigma$ is open and dense in $C^\infty(M, M_0)$.

The last assertion of this statement deserves an explanation. Recall that the submanifolds belonging to Σ are not necessarily closed; so it is not trivial a priori that the set of maps f such that $f \pitchfork N$ is open when $N \in \Sigma$. However, if $f \pitchfork N_k$, where k is maximum such that the intersection of N_k with the image of f is not empty, then the last item in the definition of a stratified submanifold implies that every C^∞ perturbation of f is transverse to N_j for every $j \leqslant k$, and does not intersect N_j for every $j > k$.

This theorem is a consequence of the Thom Transversality Theorem, which is stated here in a mild version, sufficient for all our purposes.

Theorem 7. *Let Λ be an open set in an Euclidean space, M and N compact manifolds and assume that a smooth map $F : \Lambda \times M \to N$ is transverse to a compact submanifold V of N. Then the set of $\lambda \in \Lambda$ such that $F_\lambda \pitchfork V$ has total (Lebesgue) measure in Λ, where $F_\lambda : M \to N$ is the map defined by $F_\lambda(x) = F(\lambda, x)$.*

One also has continuous dependence of $f^{-1}(N)$ on f. Indeed, a possible way of defining this continuity is as follows:

Theorem 8. *Let $f : M \to M_0$ be a smooth map transverse to a stratified submanifold (N, Σ) of M_0. Given a submanifold $N_0 \subset M$ transverse to $f^{-1}(\Sigma)$, there exists a C^∞ neighborhood $\mathcal{U}$ of f such that $g^{-1}(\Sigma)$ is transverse to N_0 for every $g \in \mathcal{U}$.*

3.2. Proof of Part 1 of Theorem 2

Until now we treat with C^∞ maps, in preparation for the proof of Theorem 4. Now we give the proof of Part 1 of Theorem 2 and for this it is necessary to treat maps that are not C^∞.

For maps in $C^1(M)$, the variation of the critical sets is not continuous; however, the following semicontinuity is easy to establish:

Given $f \in C^1(M)$ and a neighborhood U of S_f, there exists a neighborhood $\mathcal{V}_0$ of f such that $S_g \subset U$ for every $g \in \mathcal{V}_0$.

Theorem 4 implies Part 1 of Theorem 2:

Given $n > 0$, let $\mathcal{U}_n$ be the set of $f \in C^r(M)$ such that

$$\sup_{z\in M} \operatorname{card}\{0 \leqslant j \leqslant n\colon\ f^j(z) \in S_f\} \leqslant m,$$

where m is the dimension of M.

Proof that $\mathcal{U}_n$ is open in $C^r(M)$.

Let $f \in \mathcal{U}_n$; given $z \in M$ there exist neighborhoods D_z of z, U_z of S_f and $\mathcal{V}_z \subset C^r(M)$ of f such that $g^j(D_z) \cap U_z = \emptyset$ for every $g \in \mathcal{V}_z$ and every $0 \leqslant j \leqslant n$ such that $f^j(z) \notin S_f$. Cover M with a finite number of such D_z, and let U and $\mathcal{V}'$ be the intersections of the corresponding U_z and $\mathcal{V}_z$. Let $\mathcal{V}$ be the intersection of $\mathcal{V}'$ with the neighborhood $\mathcal{V}_0$ associated to the neighborhood U of S_f given by the semicontinuity of S_f stated above. Then, for every $g \in \mathcal{V}$ and $z \in M$, it holds that $g^j(z) \in S_g$ for at most m iterates between 0 and n. This proves the claim.

Proof that $\mathcal{U}_n$ is dense in $C^r(M)$.

Given any C^r open set $\mathcal{U}$, use the denseness of $C^\infty(M)$ in $C^r(M)$ and Theorem 4 to get a map $g \in \mathcal{G}_n \cap \mathcal{U}$. To prove that $g \in \mathcal{U}_n$, let $x \in M$ and $\{j_1, \dots j_k\}$ be the set of indexes $j \leqslant n$ such that $g^j(x) \in S_g$. Let N_i be the stratum of $g^{-j_i}(S_g)$ that contains x. By Theorem 4 the set $\{N_1, \dots, N_k\}$ is transverse (in general position), and as it has nonempty intersection, it comes that:

$$m = dim\, M \geqslant codim\Big(\bigcap_i N_i\Big) = \sum_i codim(N_i) \geqslant k.$$

This proves the density of $\mathcal{U}_n$.

To conclude the proof of Part 1 of Theorem 2, let $\mathcal{R}_r$ be equal to the intersection of the $\mathcal{U}_n$ with $n \geqslant 1$.

3.3. Perturbation techniques

The first result needed for the proof of Theorem 4, allows us to make local perturbations without changing the critical set:

Lemma 2. *Let $f : \mathbb{R}^m \to \mathbb{R}^m$ be a C^∞ map in the open and dense set $\mathcal{G}$ obtained in Theorem* 5 *and let* $0 \in S_f$ *be a critical point. There exist arbitrary small numbers a and b, $0 < a < b$, and a C^∞ nonnegative function φ equal to* 1 *in B_a, equal to* 0 *outside B_b and whose gradient is orthogonal to the kernel of Df at each critical point of f in B_b. (B_a denotes the ball centered at* 0 *and having radius a.)*

Proof. Assume that $f(0) = 0$ and define $N(x) = \|f(x)\|^2$. There exists a neighborhood B of 0 such that $f^{-1}(0) \cap B = \{0\}$, so one can choose an arbitrary small positive number b such that $m = \min\{N(x)\colon\ x \in \partial B_b\}$ is positive, where ∂B denotes the boundary of the set B. Choose any $m' \in (0, m)$ and let $a \in (0, b)$ be such that $N(x) < m'$ for every $\|x\| < a$. Now let $h : [0, +\infty) \to \mathbb{R}$ be a C^∞ nonnegative function such that $h(t) = 0$ if $t \geqslant m$, and $h(t) = 1$ if $t \leqslant m'$. Finally define the function φ as being equal to 0 outside B_b and such that $\varphi(x) = h(N(x))$ in B_b.

Note that $\nabla\varphi(x) = h'(N(x))\nabla N(x)$ in B_b, and that $\nabla N(x)$ is a linear combination of the rows of the matrix associated to Df_x, hence orthogonal to the kernel of Df_x. □

Definition 4. Given $f \in C^\infty(M)$, $x \in M$ and a neighborhood $\mathcal{U}$ of f, say that a triple (V, U, f_1) is adequate to x, $f, \mathcal{U}$ if the following conditions hold:

1. Both U and V are open neighborhoods of x such that $\overline{V} \subset U$.
2. f_1 is a map in $\mathcal{U}$ that coincides with f outside U.
3. $S_f = S_{f_1}$.

For the proof of the theorem, we will still need two preliminary results where some local perturbations are performed. The first one says that we can perturb in order to make transverse the preimage of transverse manifolds, without changing the preimage of one of them. This will be used in the induction step.

Proposition 1. *Let $f \in \mathcal{G}_0$ ($\mathcal{G}_0$ as in the last assertion of Theorem 5), $\mathcal{U}$ a C^∞ neighborhood of f and $x \in S_f$ such that $f(x) \in N_1 \cap N_2$, and $f \pitchfork N_2$, where N_1 and N_2 are transverse submanifolds. There exists an adequate triple (V, U, f_1) for $x, f, \mathcal{U}$ such that $f_1^{-1}(N_2) = f^{-1}(N_2)$ and*:

1. $f_1 \pitchfork N_1$ *in* V.
2. $f_1^{-1}(N_1) \pitchfork f_1^{-1}(N_2)$ *in* V.

Before beginning with the proof note that the transversality of N_1 and N_2 does not imply the transversality of its preimages. It is easy to prove that if, in addition, f is transverse to $N_1 \cap N_2$, then $f^{-1}(N_1) \pitchfork f^{-1}(N_2)$.

Proof. Let (U', τ_1) be a local chart at x such that $f(U') \subset V'$, where (V', τ_2) is a local chart at $f(x)$. As x is not fixed for f one can choose U' such that $f(U') \cap U' = \emptyset$. By taking adequate local charts one can also assume that $\tau_2(N_i)$ is a subspace W_i for $i = 1, 2$. Of course, $W_1 \pitchfork W_2$. Let $\tilde{f} = \tau_2^{-1} f \tau_1 : \mathbb{R}^m \to \mathbb{R}^m$ and define, for the function φ of Lemma 2, and for each $w \in W_2$, the maps

$$F : \mathbb{R}^m \times W_2 \to \mathbb{R}^m \quad \text{by } F(X, w) = \tilde{f}(X) + \varphi(X)w$$

and for each $w \in W_2$, $\tilde{f}_w : \mathbb{R}^m \to \mathbb{R}^m$ by $\tilde{f}_w(X) = F(X, w)$.

Note that $DF_{(0,0)}$ generates W_2, from which it follows that $F \pitchfork W_1$ in a neighborhood of $(0, 0)$, since W_1 and W_2 are transverse. By the parametrized transversality theorem it comes that $\tilde{f}_w \pitchfork W_1$ for an open set of values $w \in W_2$. Therefore, it is no loss of generality to assume from the beginning that $\tilde{f}$ is transverse to W_1.

Note also that $DF_{(0,0)}$ is a submersion, because (by hypothesis) $\partial_x F = D\tilde{f}_x$ generates a subspace complementary to W_2 and $\partial_w F$ generates W_2.

Let $\tilde{f}_w(X) = F(X, w)$ and choose w small so that $\tilde{f}_w$ is transverse to $W_1 \cap W_2$ in a ball B_a. Observe that every critical point of $\tilde{f}$ is a critical point of $\tilde{f}_w$ because the gradient of $\nabla\varphi$ is orthogonal to the kernel of $D\tilde{f}$. This implies by genericity that $S_{\tilde{f}} = S_{\tilde{f}_w}$. Define f_1 as the map that coincides with f outside $U = \tau_1^{-1}(B_b)$ and is equal to $\tau_2^{-1}\tilde{f}_w\tau_1$ in $V = \tau_1^{-1}(B_a)$. Therefore, there exist vectors w of arbitrary small norm such that the triple (V, U, f_1) is adequate to $x, f, \mathcal{U}$.

By the choice of $w \in W_2$ it comes that the preimages of N_2 under f_1 and f_2 are equal. On the other hand, f_1 is transverse to N_1 and $f_1^{-1}(N_1) \pitchfork f_1^{-1}(N_2)$, by the remark preceding the proof of the proposition. □

The second ingredient in the proof of the induction step of Theorem 4 is the following proposition, intended to show that S_f can be made transverse to a new $f^{-n}(S_f)$ without changing $f^{-j}(S_f)$ for $j \leqslant n - 1$.

Proposition 2. *Suppose that f is a map in $C^\infty(M)$, N_1 and N_2 are transverse submanifolds of M and $f \pitchfork N_1 \cap N_2$. If N_0 is another submanifold of M, $\mathcal{U}$ is a neighborhood of f, and x is a point in M such that $x \in N_0 \cap f^{-1}(N_1 \cap N_2)$, then there exists a triple (V, U, f_1) adequate to x, f, $\mathcal{U}$, such that*:

1. $N_0 \pitchfork f_1^{-1}(N_1)$, *and*
2. $f_1^{-1}(N_2) = f^{-1}(N_2)$.

Proof. As in the proof of the above proposition, one can assume that f acts in $\mathbb{R}^m$ and that each N_i is a linear space. The map f_1 will be obtained from a perturbation of the type $f_1(x) = f(x) + w_2\varphi(x)$, for $w_2 \in W_2$ and φ as in Lemma 2. Define $L = Df_x$ and let V be a subspace where L is injective and such that $L(V) \oplus W_1 = \mathbb{R}^m$ this last exists because L is transverse to $W_1 \cap W_2$. Let $\pi : W_2 \to LV$ be the projection which assigns to each w_2 its component in LV relative to the decomposition $LV \oplus W_1 = \mathbb{R}^m$. Let $\pi_0 = (L|V)^{-1} \circ \pi$. Note that $\pi_0 : W_2 \to V$ is surjective since $W_2 + W_1 = \mathbb{R}^m$. Note that $L^{-1}(W_1 + w_2) = L^{-1}(W_1) + \pi_0(w_2)$ for $w_2 \in W_2$. As $V \oplus L^{-1}(W_1) = \mathbb{R}^m$, it follows that $L^{-1}(W_1) + v$ is transverse to W_0 for almost every $v \in V$. As π_0 is surjective, it comes that $L^{-1}(W_1 + w_2)$ is transverse to W_0 for almost every $w_2 \in W_2$.

This is equivalent to say that $L_{w_2}^{-1}(W_1)$ is transverse to W_0, where $L_w = L + w$. But this must hold also for f_{w_2}, that is arbitrarily close to L_{w_2}. □

3.4. Proof of Theorem 4

It will be proved that if n is a given nonnegative integer, then:

$$\pitchfork \{S_f, f^{-1}(S_f), \dots, f^{-n}(S_f)\}$$

for every f in an open and dense subset of $C^\infty(M)$. The proof will be done by induction.

The initial step requires that S_f is a stratified submanifold; this is an immediate consequence of Theorem 5.

Induction step: Assume that for every f in some open and dense set $\mathcal{V}_n$ it holds that for each $0 \leqslant j \leqslant n$, $f^{-j}(S_f)$ is a stratified submanifold and

$$\pitchfork \{S_f, f^{-1}(S_f), \dots, f^{-n}(S_f)\}.$$

We have to prove that there exists an open and dense set $\mathcal{V}_{n+1}$ such that for $g \in \mathcal{V}_{n+1}$ we have

1. $g^{-j}(S_g)$ is a stratified submanifold for every $j \leqslant n+1$.
2. $\pitchfork \{S_g, g^{-1}(S_g), \dots, g^{-(n+1)}(S_g)\}$.

Notice that the set of maps g satisfying each one of (1) and (2) is open. So, to complete the induction step it suffices to show that in any nonempty open set $\mathcal{U}$ contained in $\mathcal{V}_n$ there exists a map g satisfying (1) and (2).

Let us prove first that $f_1^{-(n+1)}(S_{f_1})$ is a stratified submanifold for some $f_1 \in \mathcal{U}$. For this, it is sufficient to find $f_1 \in \mathcal{U}$ such that $f_1 \pitchfork f_1^{-n}(S_{f_1})$.

Let $f \in \mathcal{U}$. For every $x \in S_f \cap f^{n+1}(S_f)$, let J_x be the set of indexes $j \in \{1, \dots, n\}$ such that $x \in f^{-j}(S_f)$, and for every $j \in J_x$, let M_j be the stratum of $f^{-j}(S_f)$ containing x. Take a neighborhood U of x such that $f(U) \cap U = \emptyset$ and the following two conditions hold: first, $U \cap f^{-k}(S_f) = \emptyset$ for every $1 \leqslant k \leqslant n$ that does not belong to J_x, and second, that $U \cap M' = \emptyset$ for every stratum M' of $f^{-j}(S_f)$ of dimension less than that of M_j. Define $N_2 = \bigcap_{j \in J_x} f(M_j)$ and let N_1 be the stratum of $f^{-n}(S_f)$ containing $f(x)$. It follows that $f(x) \in N_1 \cap N_2$. Note that by the hypothesis of the induction, $N_1 \pitchfork N_2$ and $f \pitchfork N_2$. So Proposition 1 can be applied to obtain a triple (V, U, f_1) adequate to $x, f, \mathcal{U}$ such that $f_1 \pitchfork N_1$ in V, while $f_1^{-1}(N_2) = f^{-1}(N_2)$. The point here is that the perturbation of f is supported in U, and as the preimage of N_2 is not changed when the perturbation is made, then N_1 can be thought as a fixed manifold that does not depend on the map. It follows that $f_1 \pitchfork f_1^{-n}(S_f)$ in V, which provides the perturbation that makes $f_1 \pitchfork f_1^{-n}(S_f)$ locally. The way these local perturbations are accomplished in order to obtain a map f_1 that is transverse to $f_1^{-n}(S_{f_1})$ is standard and can be done by means of partitions of unity, as in the proof of the Transversality Theorem, see references [2, Theorem 3.2], [3, Section 3.2] or [1, Theorem 4.9].

Once it is known that $f_1^{-(n+1)}(S_{f_1})$ is a stratified submanifold, this property persists in a neighborhood of f_1 (contained in $\mathcal{U}$). For each $x \in M$, define $J(x)$ as the set of indexes $0 \leqslant j \leqslant n+1$ such that $f_1^j(x) \in S_{f_1}$. Let also M_j be the stratum of $f_1^{-j}(S_{f_1})$ that contains x. If $0 \notin J_x$, then there is nothing to prove since f_1 is locally a diffeomorphism around x and the induction hypothesis implies that the images of these submanifolds form a transverse set. On the other hand, if $n+1 \notin J_x$, the induction hypothesis applies directly. For the other cases take a neighborhood U of x as above.

Define $N_1 = f_1(M_{n+1})$, and

$$N_2 = \bigcap_{j \in J_x \setminus \{0, n+1\}} f_1(M_j)$$

By hypothesis, N_1 and N_2 are transverse submanifolds and f_1 is transverse to their intersection. We will show that we can perturb f_1 to obtain g so that S_g is transverse to $g^{-1}(N_1 \cap N_2)$. First show that we can perturb to obtain f_2 such that S_{f_2} is transverse to the preimage of N_1 by applying Proposition 2. Indeed, let $N_0 = M_0$ be the stratum of S_{f_1} that contains x; then N_0 can be considered as a fixed submanifold because the perturbations leave S_{f_1} unchanged by Lemma 2, as well as N_1 and N_2, that do not change by the choice of the perturbations. We will now perturb this map in order to prove that the intersection of the corresponding $\{M_j\colon\ j \in J_x\}$ is transverse at x. To do this apply Proposition 2 again, taking N_1 and N_2 as above but defining $N_0 = \bigcap_{j \in J_x \setminus \{n+1\}} M_j$. As before, by means of partition of unity we obtain $g \in \mathcal{U}$ satisfying (1) and (2). This finishes the proof of Theorem 4.

3.5. *Proof of Part 2 of Theorem 2*

Proof. Given integers n and k, where $n \geqslant 0$ and $k \geqslant 1$, denote by $P_k^n(f)$ the set of points $x \in M$ such that $f^n(x)$ is a periodic point of f of period at most k. Given any fixed $k > 0$, there exists an open and dense set $\mathcal{KS}_k$ such that, for every $f \in \mathcal{KS}_k$, every periodic point of f of period at most k is hyperbolic. In particular, for every $f \in \mathcal{KS}_k$, the following properties hold: $P_k^0(f)$ is a finite set, it varies continuously with f, and it does not intersect the critical set of f. Now fix some $r \geqslant 1$ and define:

$$\mathcal{A}_k^n = \mathcal{KS}_k \cap \{f\colon\ P_k^n(f) \cap S_f = \emptyset\}.$$

Note that $\mathcal{A}_k^n$ is open in $C^r(M)$; indeed, given disjoint neighborhoods U of $P_k^n(f)$ and V of S_f, it is clear that for every perturbation g of f it holds that $P_k^n(g) \subset U$ and $S_g \subset V$.

Fix some positive k. We will prove now by induction on n that every $\mathcal{A}_k^n$ is dense in $C^r(M)$. This implies the theorem.

To prove the claim, let $f \in C^r(M)$ and $\mathcal{U}$ a neighborhood of f. We can begin taking $f \in \mathcal{KS}_k \cap \mathcal{G}$ as both are residual sets. The initial step of the induction is then consequence of $f \in \mathcal{KS}_k$. So assume that $f \in \mathcal{A}_k^{n-1}$. Note that as $f \in \mathcal{G}$, the preimage of any point is a finite set, so it is sufficient to prove that if $x \in S_f \cap P_k^n(f)$ then there exists a perturbation g of f such that $P_k^n(g) \cap S_g \cap U = \emptyset$ for some neighborhood U of x. So begin with an $x \in S_f \cap P_k^n(f)$; there exists a neighborhood U of x such that $U \cap P_k^{n-1}(f) = \emptyset$ and such that x is the unique point of $S_f \cap P_k^n(f)$ in U. As $P_k^{n-1}(f)$ is a (zero-dimensional) submanifold, one can use again the Thom Transversality Theorem to find a map $g \in \mathcal{U}$ such that the restriction of g to U is transverse to $P_k^{n-1}(f)$ and $f = g$ outside the closure of U. This last assertion implies that $P_k^{n-1}(g) = P_k^{n-1}(f)$ by the choice of U. It follows that $g_{|U}$ is transverse to $P_k^{n-1}(g)$ which is equivalent to say that every point of $P_k^{n-1}(g)$ is a regular value of g, or that $P_k^n(g) \cap S_g \cap U = \emptyset$. This provides the proof of the induction step. $\square$

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