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Regularity and mass conservation for discrete coagulation–fragmentation equations with diffusion

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Abstract

We present a new a priori estimate for discrete coagulation–fragmentation systems with size-dependent diffusion within a bounded, regular domain confined by homogeneous Neumann boundary conditions. Following from a duality argument, this a priori estimate provides a global L^2 bound on the mass density and was previously used, for instance, in the context of reaction–diffusion equations.

In this paper we demonstrate two lines of applications for such an estimate: On the one hand, it enables to simplify parts of the known existence theory and allows to show existence of solutions for generalised models involving collision-induced, quadratic fragmentation terms for which the previous existence theory seems difficult to apply. On the other hand and most prominently, it proves mass conservation (and thus the absence of gelation) for almost all the coagulation coefficients for which mass conservation is known to hold true in the space homogeneous case.

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1. Introduction

We consider the time evolution of a physical system where a set of particles can aggregate into groups of two or more, called *clusters*, and where these clusters can diffuse in space with a diffusion constant which depends on their size. If we represent space by an open bounded set $\Omega \subseteq \mathbb{R}^N$ with regular boundary, the initial–boundary problem for the concentrations $c_i = c_i(t, x) \geqslant 0$ of clusters with integer size $i \geqslant 1$ at position $x \in \Omega$ and time $t \geqslant 0$ is given by the discrete coagulation–fragmentation system of equations with spatial diffusion and homogeneous Neumann boundary conditions:

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$$\partial_t c_i - d_i \Delta_x c_i = Q_i + F_i \quad \text{for } x \in \Omega,\ t \geqslant 0,\ i \in \mathbb{N}^*, \tag{1a}$$
$$\nabla_x c_i \cdot n = 0 \quad \text{for } x \in \partial\Omega,\ t \geqslant 0,\ i \in \mathbb{N}^*, \tag{1b}$$
$$c_i(0,x) = c_i^0(x) \quad \text{for } x \in \Omega,\ i \in \mathbb{N}^*, \tag{1c}$$

where $n = n(x)$ represents a unit normal vector at a point $x \in \partial\Omega$, d_i is the diffusion constant for clusters of size i, and

$$Q_i \equiv Q_i[c] := Q_i^+ - Q_i^- := \frac{1}{2}\sum_{j=1}^{i-1} a_{i-j,j} c_{i-j} c_j - \sum_{j=1}^{\infty} a_{i,j} c_i c_j,$$
$$F_i \equiv F_i[c] := F_i^+ - F_i^- := \sum_{j=1}^{\infty} B_{i+j}\beta_{i+j,i} c_{i+j} - B_i c_i. \tag{2}$$

The parameters B_i, $\beta_{i,j}$ and $a_{i,j}$, for integers $i, j \geqslant 0$, represent the total rate B_i of fragmentation of clusters of size i, the average number $\beta_{i,j}$ of clusters of size j produced due to fragmentation of a cluster of size i, and the coagulation rate $a_{i,j}$ of clusters of size i with clusters of size j. We refer to these parameters as *the coefficients* of the system of equations. They represent rates, so they are always nonnegative; single particles do not fragment further, and mass should be conserved when a cluster fragments into smaller pieces, so one always imposes

$$a_{i,j} = a_{j,i} \geqslant 0, \qquad \beta_{i,j} \geqslant 0 \quad \big(i, j \in \mathbb{N}^*\big), \tag{3a}$$
$$B_1 = 0, \qquad B_i \geqslant 0 \quad \big(i \in \mathbb{N}^*\big), \tag{3b}$$
$$i = \sum_{j=1}^{i-1} j\beta_{i,j} \quad (i \in \mathbb{N},\ i \geqslant 2). \tag{3c}$$

In fact, the last condition (3c) implies the conservation of the total mass $\int_\Omega \sum_{i=1}^\infty i c_i \, dx$, which becomes obvious from the following formal *fundamental identity* or *weak formulation* of the coagulation and fragmentation operators: Consider a sequence of nonnegative numbers $\{c_i\}$, and define Q_i, F_i as in Eqs. (2), then, for any sequence of numbers φ_i,

$$\sum_{i=1}^\infty \varphi_i Q_i = \frac{1}{2}\sum_{i=1}^\infty\sum_{j=1}^\infty a_{i,j} c_i c_j (\varphi_{i+j} - \varphi_i - \varphi_j),$$
$$\sum_{i=1}^\infty \varphi_i F_i = -\sum_{i=2}^\infty B_i c_i \left(\varphi_i - \sum_{j=1}^{i-1} \beta_{i,j}\varphi_j\right). \tag{4}$$

As a (still formal) consequence for solutions $\{c_i\}$ of (1)–(2), one can calculate the time derivative of the integral of the moment $\sum \varphi_i c_i$ to obtain

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^\infty \varphi_i c_i = \int_\Omega \sum_{i=1}^\infty \varphi_i (Q_i + F_i), \tag{5}$$

since the integral of the diffusion part vanishes due to the homogeneous Neumann boundary condition. By choosing $\varphi_i := i$ above and thanks to (3c), we have $\sum_{i=1}^\infty i Q_i = \sum_{i=1}^\infty i F_i = 0$, and the total mass is formally conserved:

$$\big\|\rho(t,\cdot)\big\|_{L^1} = \int_\Omega \sum_{i=1}^\infty i c_i(t,x)\,dx = \int_\Omega \sum_{i=1}^\infty i c_i^0(x)\,dx = \big\|\rho^0\big\|_{L^1} \quad (t \geqslant 0). \tag{6}$$

Our main aim in this work is to provide some new bounds on the regularity of weak solutions for system (1)–(2) by means of techniques developed in the context of reaction–diffusion equations [9,16,17], and to give three applications to those bounds, the main one proving rigorously (for almost all the coefficients where this is true in the homogeneous case) mass conservation (6) and thus the absence of gelation, a well-known phenomenon in coagulation–fragmentation models [11,10], where the formal conservation of mass is violated as clusters of infinite size are formed.

In this paper we will work with the global weak solutions constructed in [15] (see also [20] for the case $\Omega = \mathbb{R}^N$) under the assumption

$$\lim_{j\to+\infty} \frac{a_{i,j}}{j} = \lim_{j\to+\infty} \frac{B_{i+j}\beta_{i+j,i}}{i+j} = 0 \quad \text{(for fixed } i \geqslant 1\text{)}. \tag{7}$$

The notion of solution is the following, which we take from [15]:

Definition 1.1. A global weak solution $c = \{c_i\}_{i\geqslant 1}$ to (1)–(2) is a sequence of functions $c_i : [0, +\infty) \times \Omega \to [0, +\infty)$ such that for each $T > 0$,

$$c_i \in \mathcal{C}\big([0, T]; L^1(\Omega)\big), \quad i \geqslant 1, \tag{8}$$

$$\sum_{j=1}^{\infty} a_{i,j} c_i c_j \in L^1\big([0, T] \times \Omega\big), \tag{9}$$

$$\sup_{t\geqslant 0} \int_\Omega \left[\sum_{i=1}^{\infty} i c_i(t, x)\right] dx \leqslant \int_\Omega \left[\sum_{i=1}^{\infty} i c_i^0(x)\right] dx, \tag{10}$$

and for each $i \geqslant 1$, c_i is a mild solution to the i-th equation in (1a), that is,

$$c_i(t) = e^{d_i A_1 t} c_i^0 + \int_0^t e^{d_i A_1 (t-s)} \big(Q_i\big[c(s)\big] + F_i\big[c(s)\big]\big)\, ds, \quad t \geqslant 0, \tag{11}$$

where $Q_i[c]$ is defined by (2), A_1 denotes the closure in $L^1(\Omega)$ of the unbounded linear operator A of $L^2(\Omega)$ defined by

$$D(A) := \big\{w \in H^2(\Omega) \bigm| \nabla w \cdot n = 0 \text{ on } \partial\Omega\big\}, \qquad Aw = \Delta w, \tag{12}$$

and $e^{d_i A_1 t}$ is the C_0-semigroup generated by $d_i A_1$ in $L^1(\Omega)$.

The existence result of [15] reads:

Theorem 1.2 *(Laurençot–Mischler). Assume hypotheses* (3) *and* (7) *on the coagulation and fragmentation coefficients. Assume also that*

$$d_i > 0 \quad \textit{for all } i \geqslant 1,$$

and that the nonnegative initial datum has finite mass:

$$c_i^0 \geqslant 0 \quad \textit{on } \Omega \quad \textit{and} \quad \int_\Omega \sum_{i=1}^{\infty} i c_i^0 < +\infty.$$

Then, there exists a global weak solution to the initial–boundary problem (1)–(2) *in the sense of Definition* 1.1.

Under the extra assumptions on the diffusion constants and the initial data

$$0 < \inf_i \{d_i\} =: d, \qquad D := \sup_i \{d_i\} < +\infty, \tag{13}$$

$$\sum_{i=1}^{\infty} i c_i^0 \in L^2(\Omega), \tag{14}$$

we are in fact able to prove the following L^2 bound on the mass density $\rho(t, x) := \sum_{i=1}^{\infty} i c_i(t, x)$: Denoting by Ω_T the cylinder $[0, T] \times \Omega$, we have

Proposition 1.3. *Assume that* (3), (7), (13) *and* (14) *hold. Then, for all $T > 0$ the mass ρ of a weak solution to system* (1)–(2) (*given by Theorem* 1.2) *lies in $L^2(\Omega_T)$ and the following estimate holds*:

$$\|\rho\|_{L^2(\Omega_T)} \leqslant \left(1 + \frac{\sup_i\{d_i\}}{\inf_i\{d_i\}}\right) T \left\|\rho(0,\cdot)\right\|_{L^2(\Omega)}. \tag{15}$$

Remark 1.4. Note that the assumption (7) is only included in Proposition 1.3 in order to ensure the existence of a weak solution via Theorem 1.2. Without assumption (7), the bound (15) would still hold for smooth solutions of a truncated version of system (1)–(2) uniformly with respect to the truncation. See [15] for the details of such a truncation.

In addition to Proposition 1.3, we give a new proof of an L^1 bound of the various coagulation and fragmentation terms:

Proposition 1.5. *We still assume that* (3), (7), (13) *and* (14) *hold. Then, for all $T > 0$ and $i \in \mathbb{N}^*$ all the terms Q_i^+, Q_i^-, F_i^+ and F_i^- associated to a weak solution to system* (1)–(2) (*given by Theorem* 1.2) *lie in $L^1(\Omega_T)$ with a bound which depends in an explicit way on the coagulation and fragmentation coefficients, the diffusion coefficients, and the initial data c_i^0.*

Remark 1.6. The fact that the terms Q_i^+, Q_i^-, F_i^+ and F_i^- associated to a weak solution are in $L^1(\Omega_T)$ is included in the definition of weak solution; the main content of Proposition 1.5 is the explicit dependence of the bounds on the coefficients and initial data, which can be used to obtain uniform estimates for approximated solutions as we show for instance in Section 3. For details on the explicit L^1 bounds we refer to the proof of Proposition 1.5 in Section 2.

Remark 1.7. The L^1 bounds on Q_i^+, Q_i^-, F_i^+ and F_i^- require the assumption (7) only to ensure existence. They would hold at the formal level (that is, for smooth solutions of a truncated system) under the less stringent assumption

$$K_i := \sup_{j \in \mathbb{N}} \frac{B_{i+j}\beta_{i+j,i}}{i+j} < +\infty \quad \left(i \in \mathbb{N}^*\right). \tag{16}$$

Note that the above L^1 bound also holds when assumptions (3), (7) are replaced by the assumptions of Theorem 1.2 in [15], but the proof is then much more difficult as it requires an induction on i which can be removed under our extra assumptions.

In Section 3, as a first application of the bounds obtained in Propositions 1.3 and 1.5, we give a very simple proof of existence of weak solutions to (1)–(2) in dimension $N = 1$ (that is, the result of Theorem 1.2 in dimension 1) under the additional assumptions (13) and (14).

Our main application of Propositions 1.3 and 1.5 is however related to the problem of conservation of mass (6), which holds rigorously for solutions to a truncated system (see e.g [15]). Nevertheless, it is an important issue in coagulation–fragmentation theory whether (6) holds for weak solutions of system (1)–(2) itself, or if (6) is replaced by an inequality stating that mass is nonincreasing in time. If at some time t, the identity (6) does not hold any more, we say that gelation occurs, which means from a physical point of view that a macroscopic object has been created.

Our main result in Section 4 basically shows that (under the assumptions (3) and (7)) gelation does not occur when the coagulation coefficients $a_{i,j}$ are at most linear and, moreover, slightly sublinear far off the diagonal $i = j$. More precisely, we prove mass conservation under the following condition on the coefficients $a_{i,j}$:

Hypothesis 1.8. There is some bounded function $\theta : [0,+\infty) \to (0,+\infty)$ such that $\theta(x) \to 0$ when $x \to +\infty$ and

$$a_{i,j} \leqslant (i+j)\theta(j/i) \quad \text{for all } j \geqslant i. \tag{17}$$

(Or equivalently, by symmetry,

$$a_{i,j} \leqslant (i+j)\theta\left(\max\{j/i, i/j\}\right) \quad \text{for all } i, j \geqslant 1.)$$

Theorem 1.9. *Assume that* (3), (7), (13), *and* (14) *hold. Also, assume Hypothesis* 1.8. *Then, the weak solution to the system* (1) *given by Theorem* 1.2 *has a superlinear moment which is bounded on bounded time intervals*; *this is, there is some increasing function $C = C(T) > 0$, and some increasing sequence of positive numbers $\{\psi_i\}_{i\geqslant 1}$ with*

$$\lim_{i\to\infty} \psi_i \to +\infty \tag{18}$$

such that for all $T > 0$,

$$\int_\Omega \sum_{i=1}^{\infty} i\psi_i c_i \leqslant C(T) \quad \text{for all } t \in [0, T]. \tag{19}$$

As a consequence, under these conditions all weak solutions given by Theorem 1.2 *of* (1) *conserve mass*:

$$\int_\Omega \rho_0(x)\,dx = \int_\Omega \rho(t,x)\,dx \quad \text{for all } t \geqslant 0. \tag{20}$$

Remark 1.10 *(Admissible coagulation coefficients).* Let us comment on Hypothesis 1.8. First note that 1.8 includes coefficients of the form

$$a_{i,j} \leqslant \mathrm{Cst}\big(i^\alpha j^\beta + i^\beta j^\alpha\big)$$

for any $\alpha, \beta > 0$ such that $\alpha + \beta \leqslant 1$ (take $\theta(x) = x^{-\varepsilon}$ for $\varepsilon > 0$ small enough). It is also satisfied when

$$a_{i,j} \leqslant \mathrm{Cst}\left(\frac{i}{\phi(i)} + \frac{j}{\phi(j)}\right),$$

where $x \mapsto \phi(x)$ is any positive strictly increasing function (for x big enough), which goes to infinity at infinity, and such that $x \mapsto \frac{x}{\phi(x)}$ is also increasing (take $\theta(\lambda) = \phi(\lambda)^{-1/2}$). All the examples $\phi = \log(1+\cdot)$, $\phi = \log(1+\cdot)\circ\log(1+\cdot), \ldots, \phi = \log(1+\cdot)\circ\cdots\circ\log(1+\cdot)$ satisfy this condition. Likewise, condition (17) also holds when (for $i, j \geqslant 2$)

$$a_{ij} \leqslant \mathrm{Cst}\left(i\frac{R(\log j)}{\log i} + j\frac{R(\log i)}{\log j}\right) \tag{21}$$

for some nondecreasing function R such that $x \mapsto R(x)/x$ is nonincreasing and tends to 0 when $x \to +\infty$. Note indeed that when (21) holds,

$$\frac{a_{ij}}{i+j} \leqslant \frac{1}{1+j/i}\frac{R[\log(j/i)+\log i]}{\log i} + \frac{j/i}{1+j/i}\frac{R[\log i]}{\log(j/i)+\log i}. \tag{22}$$

Then, condition (17) is obtained by distinguishing the cases $i \geqslant j/i$ and $i \leqslant j/i$ in both terms of the right-hand side of (22).

Assumption (21) can even be replaced by

$$a_{ij} \leqslant Cst\left(i\frac{R(\log(\log j))}{\log(\log i)} + j\frac{R(\log(\log i))}{\log(\log j)}\right),$$

with the same requirements on R as previously.

Note however that the linear coefficient $a_{ij} = i + j$ (or the coefficient $a_{ij} = \frac{i}{\log i}\log j + \frac{j}{\log j}\log i$) does not satisfy Hypothesis 1.8, though one would expect that Theorem 1.9 still holds for such coefficients.

Before introducing a generalised coagulation–fragmentation model and thus, a third application of Propositions 1.3 and 1.5, let us briefly review previous results on existence theory and mass conservation for the coagulation–fragmentation system (1). With some further restrictions on the coefficients as compared to [15], existence of solutions by means of L^∞ bounds on the c_i has been proven in [3,7,13,18,19]. A different technique was used in [1] to prove that Eq. (1) is well posed, locally in time, and globally in time when the space dimension N is one, always assuming that the coagulation and fragmentation coefficients are bounded.

In a recent work [14], Hammond and Rezakhanlou considered Eq. (1) without fragmentation, and gave L^∞ bounds on moments of the solution (and as a consequence, L^∞ bounds on the c_i). This implies uniqueness and mass conservation for some coagulation coefficients that grow at most linearly as well as an alternative proof of the existence of L^∞ solutions by a priori bounds on the c_i; for instance, if $\Omega = \mathbb{R}^N$ and diffusion coefficients d_i are nonincreasing and satisfying (13) and if moreover

$$\sum_{i=1}^{\infty} i c_i^0 \in L^\infty(\mathbb{R}^N), \qquad \sum_{i=1}^{\infty} i^2 c_i^0 \in L^1(\mathbb{R}^N), \qquad a_{i,j} \leqslant C(i+j)$$

for some $C > 0$ and all $i, j \geqslant 1$, then they show that mass is conserved for all weak solutions of Eq. (1) without fragmentation. See [14, Theorems 1.3 and 1.4] and [14, Corollary 1.1] for more details.

In the spatially homogeneous case, mass conservation is known for general data with finite mass and coagulation coefficients including the critical linear case $a_{i,j} \leqslant \mathrm{Cst}(i+j)$ (see, for instance, [2,5]).

We finally give a third application of Propositions 1.3 and 1.5. As mentioned already in Remarks 1.4 and 1.7, Propositions 1.3 and 1.5 (despite true without restrictions on the coagulation coefficients $a_{i,j}$ for smooth approximating solutions) do not really improve the theory of existence of weak solutions for the usual models of coagulation–fragmentation like (1) as the full assumption (7) are needed in passing to the limit in the approximating solutions. At best they help provide simpler proofs in particular cases, as done in Section 3.

On the other hand, Propositions 1.3 and 1.5 are well suited for the existence theory of more exotic models, for instance, when fragmentation occurs due to binary collisions between clusters. Then, the break-up terms are quadratic, being proportional to the concentration of the two clusters which collide. This leads to coagulation–fragmentation models where all terms in the right-hand side are quadratic.

More precisely, we consider that clusters of size k and l collide with a rate $b_{k,l} \geqslant 0$, leading to fragmentation. As a consequence, clusters of size $i < \max\{k,l\}$ are produced, in average, at a rate $\beta_{i,k,l} \geqslant 0$ in such a way that the mass is conserved (that is, $\sum_{i<\max\{k,l\}} i\beta_{i,k,l} = k+l$). This leads to the following system (for $t \in \mathbb{R}_+$, $x \in \Omega$ a bounded regular open subset of $\mathbb{R}^N$):

$$\partial_t c_i - d_i \Delta_x c_i = \frac{1}{2} \sum_{k+l=i} a_{k,l} c_k c_l - \sum_{k=1}^{\infty} a_{i,k} c_i c_k + \frac{1}{2} \sum_{k,l=1}^{\infty} \sum_{i<\max\{k,l\}} b_{k,l} c_k c_l \beta_{i,k,l} - \sum_{k=1}^{\infty} b_{i,k} c_i c_k \quad \bigl(i \in \mathbb{N}^*\bigr), \tag{23}$$

together with the initial and boundary conditions (1b), (1c). For this model, the set of assumptions (3) is replaced by

$$a_{i,j} = a_{j,i} \geqslant 0 \quad \bigl(i, j \in \mathbb{N}^*\bigr), \tag{24a}$$
$$\beta_{i,k,l} = \beta_{i,l,k} \geqslant 0 \quad \bigl(i,k,l \in \mathbb{N}^*,\ i < \max\{k,l\}\bigr), \tag{24b}$$
$$b_{i,k} = b_{k,i} \geqslant 0, \quad b_{1,1} = 0 \quad \bigl(i,k \in \mathbb{N}^*,\ i < k\bigr), \tag{24c}$$
$$\sum_{i<\max\{k,l\}} i\beta_{i,k,l} = k + l \quad \bigl(k, l \in \mathbb{N}^*\bigr). \tag{24d}$$

Because of the quadratic character of the fragmentation terms, the inductive method for the proof of existence devised by Laurençot and Mischler [15] seems difficult to adapt in this case. The method presented in our first application can however be adapted, provided that the dimension is $N = 1$ and that the following assumptions are made on the coefficients:

Hypothesis 1.11. Assume (24), and suppose that the diffusion coefficients are uniformly bounded above and below (Eq. (13)) and that the initial mass lies in $L^2(\Omega)$ (Eq. (14)). In place of (7) we assume further that

$$\lim_{l\to\infty} \frac{a_{k,l}}{l} = 0, \qquad \lim_{l\to\infty} \frac{b_{k,l}}{l} = 0 \quad \bigl(\text{for fixed } k \in \mathbb{N}^*\bigr), \tag{25}$$
$$\lim_{l\to\infty} \sup_k \left\{ \frac{b_{k,l}}{kl} \beta_{i,k,l} \right\} = 0 \quad \bigl(\text{for fixed } i \in \mathbb{N}^*\bigr). \tag{26}$$

We define a solution to (23) along the same lines as in Definition 1.1:

Definition 1.12. A global weak solution $c = \{c_i\}_{i \geqslant 1}$ to (23), the boundary condition (1b) and the initial data (1c) is a sequence of functions $c_i : [0, +\infty) \times \Omega \to [0, +\infty)$ such that for each $T > 0$,

$$c_i \in \mathcal{C}\big([0,T]; L^1(\Omega)\big), \quad i \geqslant 1, \tag{27}$$

the four terms on the right-hand side of (23) are in $L^1([0,T] \times \Omega)$,

$$\sup_{t \geqslant 0} \int_\Omega \left[\sum_{i=1}^\infty i c_i(t,x) \right] dx \leqslant \int_\Omega \left[\sum_{i=1}^\infty i c_i^0(x) \right] dx, \tag{28}$$

and for each $i \geqslant 1$, c_i is a mild solution to the i-th equation in (23), that is,

$$c_i(t) = e^{d_i A_1 t} c_i^0 + \int_0^t e^{d_i A_1 (t-s)} Z_i\big[c(s)\big]\, ds, \quad t \geqslant 0,$$

where $Z_i[c]$ represents the right-hand side of (23) and A_1, $e^{d_i A_1 t}$ are the same as in Definition 1.1.

We are now able to prove the following theorem:

Theorem 1.13. *Under Hypothesis* 1.11 *on the coefficients and initial data of the equation, and in dimension* $N = 1$, *there exists a global weak solution to Eq.* (23) *satisfying*

$$c_i \in C\big([0,T], L^1(\Omega)\big) \cap L^{3-\varepsilon}(\Omega_T) \quad \big(\textit{for all } i \in \mathbb{N}^*,\ T > 0,\ \varepsilon > 0\big),$$

for which the four terms appearing in the right-hand side of (23) *lie in* $L^1(\Omega_T)$.

Remark 1.14. The method of proof unfortunately does not seem to provide existence in dimensions $N \geqslant 2$. Dimension $N = 2$ looks in fact critical as it doesn't allow a priori a bootstrap in the heat equation with right-hand side in L^1. A possible line of proof could follow [12] in the context of reaction–diffusion equations. In higher dimensions $N \geqslant 3$, assuming additionally a detailed balance relation between coagulation and fragmentation, an entropy based duality method as in [9] could be used to define global weak L^2 solutions (see also [16]).

Our paper is built in the following way: Section 2 is devoted to the proof of Propositions 1.3 and 1.5. Then Sections 3, 4, and 5 are each devoted to one of the three applications. In particular, Theorem 1.9 is proven in Section 4 first in a particular case (with a very short proof), and then in complete generality. Theorem 1.13 is proven in Section 5. Finally, an Appendix A is devoted to the proof of a lemma of duality due to M. Pierre and D. Schmitt (cf. [17]), which is the key to Proposition 1.3.

2. A new a priori estimate

The solutions given in [15] are constructed by approximating the system (1)–(2) by a truncated system (the procedure consists in setting the coagulation and fragmentation coefficients to zero beyond a given finite size, and smoothing the initial data) for which very regular solutions exist. Then, uniform estimates for the solutions of this approximate system are proven. Finally, it is shown that these solutions have a subsequence which converges to a solution to the original system. In the proofs below it must be understood that the bounds are obtained for the truncated system (in a uniform way) and then transferred to the weak solution by a passage to the limit: the fact that this transfer can be done (in the case of the total mass) without replacing the equality by an inequality is the heart of our second application.

We begin with

Proof of Proposition 1.3. Using the fact that

$$\partial_t \rho - \Delta(M\rho) = 0, \qquad \inf_{i\in\mathbb{N}^*}\{d_i\} \leqslant M(t,x) := \frac{\sum_{i=1}^\infty d_i i c_i}{\sum_{i=1}^\infty i c_i} \leqslant \sup_{i\in\mathbb{N}^*}\{d_i\},$$

we can deduce thanks to a lemma of duality [9, Appendix] that $\rho \in L^2(\Omega_T)$, and more precisely that

$$\|\rho\|_{L^2(\Omega_T)} \leqslant \left(1 + \frac{\sup_i\{d_i\}}{\inf_i\{d_i\}}\right) T \left\|\rho(0,\cdot)\right\|_{L^2(\Omega)},$$

for all $T > 0$. For the sake of completeness, the lemma is recalled with its proof in Appendix A (Lemma A.2). □

We now turn to

Proof of Proposition 1.5. For F_i^-, it is clear that

$$F_i^- \leqslant B_i\rho \in L^2\big([0,T]\times\Omega\big) \subseteq L^1\big([0,T]\times\Omega\big),$$

thanks to Proposition 1.3. For F_i^+ we use Eq. (16) to write

$$F_i^+ \leqslant \sum_{j=1}^\infty \left(\frac{B_{i+j}\beta_{i+j,i}}{i+j}\right)(i+j)c_{i+j} \leqslant K_i \sum_{j=1}^\infty (i+j)c_{i+j} \leqslant K_i\rho, \tag{29}$$

which is again in $L^2([0,T]\times\Omega)$, and hence in $L^1([0,T]\times\Omega)$.

For the coagulation terms, we have, since each c_i is less than ρ,

$$Q_i^+ \leqslant \frac{1}{4}\sum_{j=1}^{i-1} a_{i-j,j}\big(c_{i-j}^2 + c_j^2\big) \leqslant \frac{1}{2}\rho^2\left(\sum_{j=1}^{i-1} a_{i-j,j}\right), \tag{30}$$

which is in $L^1([0,T]\times\Omega)$ as the same is true for ρ^2 and the sum consists only of a finite number of terms. Finally, for Q_i^- we use the fact that Q_i^+ and F_i^+ are already known to be integrable: Thus, from Eq. (1) integrated over $[0,T]\times\Omega$,

$$\int_\Omega c_i(T,x)\,dx + \int_0^T\!\!\int_\Omega Q_i^-(t,x)\,dx\,dt \leqslant \int_\Omega c_i^0(x)\,dx + \int_0^T\!\!\int_\Omega Q_i^+(t,x)\,dx\,dt + \int_0^T\!\!\int_\Omega F_i^+(t,x)\,dx\,dt.$$

This proves our result. □

3. First application: A simplified proof of existence of solutions in dimension 1

We begin this section with the following corollary of Proposition 1.5, in the particular case of dimension $N = 1$.

Lemma 3.1. *Assume that the dimension $N = 1$, and that* (3), (13), (14) *and* (16) (*being more general than* (7)) *hold. Then, for all $T \geqslant 0$, $i \in \mathbb{N}^*$ the concentrations $c_i \in L^\infty([0,T]\times\Omega)$* (*where c_i are smooth solutions of a truncated version of* (1)–(2), *the L^∞ norm being independent of the truncation*).

Proof. We carry out a bootstrap regularity argument. Thanks to Proposition 1.5, we know that (for all $i \in \mathbb{N}^*$)

$$(\partial_t - d_i\Delta)c_i \in L^1\big([0,T]\times\Omega\big).$$

Using for example the results in [8], this implies that for any $\delta > 0$,

$$c_i \in L^{3-\delta}\big([0,T]\times\Omega\big) \quad \big(i \in \mathbb{N}^*\big). \tag{31}$$

Now, Eq. (31) shows that Q_i^+ is actually more regular: from (the first inequality in) (30),

$$Q_i^+ \in L^{\frac{3}{2}-\frac{\delta}{2}}\big([0,T]\times\Omega\big) \quad \text{for all } \delta>0,\ i\in\mathbb{N}^*. \tag{32}$$

In addition, we already knew from Eq. (29) that (for all $i\in\mathbb{N}^*$)

$$F_i^+ \in L^2\big([0,T]\times\Omega\big), \tag{33}$$

[for which we do not need to assume that the space dimension N is 1]. Consequently, omitting the negative terms (for all $i\in\mathbb{N}^*$, $\delta>0$), we can find h_i such that

$$(\partial_t - d_i\Delta)c_i \leqslant h_i \in L^{\frac{3}{2}-\frac{\delta}{2}}\big([0,T]\times\Omega\big).$$

As the c_i are positive, this implies that

$$c_i \in L^p\big([0,T]\times\Omega\big) \quad \text{for all } p\in[1,+\infty[,\ i\in\mathbb{N}^*.$$

Again from (30),

$$Q_i^+ \in L^p\big([0,T]\times\Omega\big) \quad \text{for all } p\in[1,+\infty[,\ i\in\mathbb{N}^*.$$

From this and (33), we can find h_i such that

$$(\partial_t - d_i\Delta)c_i \leqslant h_i \in L^2\big([0,T]\times\Omega\big),$$

which implies in turn that $c_i \in L^\infty([0,T]\times\Omega)$ (for all $i\in\mathbb{N}^*$). □

We now have the possibility to give a short proof of Theorem 1.2 in dimension 1 (and under the extra assumptions (13), (14)). Recall that a proof for any dimension can be found in [15].

Short proof of Theorem 1.2 in 1D assuming additionally (13) and (14). Consider a sequence c_i^M of (regular) solutions to a truncated version of system (1)–(2). Thanks to Lemma 3.1, we know that for each $i\in\mathbb{N}^*$, $\sup_M \|c_i^M\|_{L^\infty(\Omega_T)} < +\infty$. Then (for each $i\in\mathbb{N}^*$) there is a subsequence of the $(c_i^M)_{M\in\mathbb{N}}$ (which we still denote by $(c_i^M)_{M\in\mathbb{N}}$), and a function $c_i\in L^\infty(\Omega_T)$, such that

$$c_i^M \overset{*}{\rightharpoonup} c_i \quad \text{weak-}*\text{ in } L^\infty(\Omega_T). \tag{34}$$

Using Proposition 1.5, we also see that (for any fixed $i\in\mathbb{N}^*$), the $L^1(\Omega_T)$ norms of $C_i^{+,M}$, $C_i^{-,M}$, $F_i^{+,M}$, $F_i^{-,M}$ (the coagulation and fragmentation terms associated to $\{c_i^M\}$) are bounded independently of M. Using Eq. (1a) and the properties of the heat equation, one sees that for each $i\in\mathbb{N}^*$, the sequence $\{c_i^M\}$ lies in a strongly compact subset of $L^1(\Omega_T)$. Hence, by renaming our subsequence again, we may assume that

$$c_i^M \to c_i \quad \text{in } L^1(\Omega_T) \text{ strong , for all } i\in\mathbb{N}^*. \tag{35}$$

In order to prove that $\{c_i\}$ is indeed a solution to Eq. (1)–(2), let us prove that all terms $F_i^{+,M}$, $F_i^{-,M}$, $C_i^{+,M}$, $C_i^{-,M}$ converge to the corresponding expressions for c_i, which we denote by F_i^+, F_i^-, C_i^+, C_i^-, as usual.

1. Positive fragmentation term: for each fixed i, the sum

$$F_i^{+,M} = \sum_{j=1}^{\infty} B_{i+j}\beta_{i+j,i}c_{i+j}^M$$

converges to F_i^+ in $L^1(\Omega_T)$ because the tails of the sum converge to 0 uniformly in M (this is due to hypothesis (7)):

$$\int_0^T\int_\Omega \Big|\sum_j B_{i+j}\beta_{i+j,i}\big(c_{i+j}^M - c_{i+j}\big)\Big|\,dx\,dt \leqslant 2\Big(\sup_{j\geqslant J_0}\Big|\frac{B_{i+j}\beta_{i+j,i}}{i+j}\Big|\Big)\rho + \sup_{j\leqslant J_0}\big\|c_{i+j}^M - c_{i+j}\big\|_{L^1(\Omega_T)}.$$

2. The negative fragmentation term is just a multiple of c_i^M, so the convergence in $L^1(\Omega_T)$ is given by (35).
3. For each fixed i, the positive coagulation term is a finite sum of terms of the form $a_{i,j}c_i^M c_j^M$. Thanks to (34) and (35), this converges to $a_{i,j}c_i c_j$ in $L^1(\Omega_T)$.
4. The negative coagulation term is

$$Q_i^{-,M} = c_i^M \sum_{j=1}^{\infty} a_{i,j} c_j^M.$$

Since c_i^M converges to c_i weak-$*$ in $L^\infty(\Omega_T)$, it is enough to prove that $\sum_{j=1}^\infty a_{i,j}c_j^M$ converges to $\sum_{j=1}^\infty a_{i,j}c_j$ strongly in $L^1(\Omega_T)$. Observing that

$$\int_0^T \int_\Omega \left| \sum_j a_{i,j}\left(c_j^M - c_j\right)\right| dx\,dt \leqslant 2\left(\sup_{j \geqslant J_0} \left|\frac{a_{i,j}}{j}\right|\right)\rho + \sup_{j \leqslant J_0} \left\|c_j^M - c_j\right\|_{L^1(\Omega_T)},$$

we see thanks to (7) and (35) that this convergence indeed holds. □

4. Second application: Mass conservation

We begin this section with a very short proof of Theorem 1.9 in a particular case in order to show how estimate (15) works. More precisely, we consider the pure coagulation case with $a_{i,j} = \sqrt{ij}$ and $B_i = 0$ (no fragmentation), and with initial data satisfying additionally $\sum_{i=0}^\infty i \log i\, c_i(0,x)\,dx < +\infty$ (which is sightly more stringent than only assuming finite initial mass).

Then, using the weak formulation (4) with $\varphi_i = \log(i)$ (and remembering that $\log(1+x) \leqslant \mathrm{Cst}\sqrt{x}$)

$$\begin{aligned} \frac{d}{dt}\int_\Omega \sum_{i=1}^\infty i \log i\, c_i\,dx &= \int_\Omega \sum_{i=1}^\infty \sum_{j=1}^\infty \sqrt{ij}\, c_i c_j \left(i \log\left(1+\frac{j}{i}\right) + j \log\left(1 + \frac{i}{j}\right)\right) dx \\ &\leqslant 2\int_\Omega \sum_{i=1}^\infty \sum_{j=1}^\infty ij\, c_i c_j\,dx \leqslant 2\int_\Omega \rho(t,x)^2\,dx. \end{aligned} \tag{36}$$

As a consequence, we have for all $T > 0$

$$\int_\Omega \sum_{i=0}^\infty i \log i\, c_i(T,x)\,dx \leqslant \int_\Omega \sum_{i=0}^\infty i \log i\, c_i(0,x)\,dx + 2\int_0^T \int_\Omega \rho(t,x)^2\,dx\,dt,$$

which ensures the propagation of the moment $\int \sum_{i=0}^\infty i \log i\, c_i(\cdot,x)\,dx$, and therefore gives a rigorous proof of conservation of the mass for weak solutions of the system: no gelation occurs.

Our general result is obtained through a refinement of this argument under Hypothesis 1.8. Before giving the proof of Theorem 1.9 we need two technical lemmas, which will substitute the intermediate step in (36).

Lemma 4.1. *Let $\{\mu_i\}_{i \geqslant 1}$ and $\{\nu_i\}_{i \geqslant 1}$ be sequences of positive numbers such that $\{\mu_i\}$ is bounded,*

$$\sum_{i=1}^\infty \mu_i = +\infty \quad \text{and} \quad \lim_{i \to +\infty} \nu_i = +\infty.$$

Then we can find a sequence $\{\xi_i\}_{i \geqslant 1}$ of nonnegative numbers such that

$$\sum_{i=1}^\infty \xi_i = +\infty,$$

$$\xi_i \leqslant \mu_i \quad \text{and} \quad \psi_i := \sum_{j=1}^i \xi_j \leqslant \nu_i \quad \text{for all } i \geqslant 1.$$

Proof. We may assume that ν_i is nondecreasing, for otherwise we can consider $\tilde{\nu}_i := \inf_{j \geqslant i}\{\nu_j\}$ instead of ν_i. Then, in order to find ξ_i it is enough to define recursively $\xi_0 := 0$ and, for $i \geqslant 1$,

$$\xi_i := \begin{cases} \mu_i & \text{if } \mu_i + \sum_{j=0}^{i-1} \xi_j \leqslant \nu_i, \\ 0 & \text{otherwise.} \end{cases}$$

By construction, $\xi_i \leqslant \mu_i$ for all $i \geqslant 1$, and also $\sum_{j=1}^{i} \xi_j \leqslant \nu_i$ for $i \geqslant 1$, as we are assuming $\{\nu_i\}$ nondecreasing.

To see that $\{\xi_i\}$ cannot be summable, suppose otherwise that $\sum_{i=1}^{\infty} \xi_i = S < +\infty$. Take a bound $M > 0$ of $\{\mu_i\}$, and choose an integer k such that $\nu_i \geqslant S + M$ for all $i \geqslant k$. Then, by definition,

$$\xi_i = \mu_i \quad \text{for all } i \geqslant k,$$

which implies that $\{\xi_i\}$ is not summable, as $\{\mu_i\}$ is not, and gives a contradiction. □

Lemma 4.2. *Assume* (17). *There is a nondecreasing sequence of positive numbers $\{\psi_i\}_{i \geqslant 1}$ such that $\psi_i \to +\infty$ when $i \to +\infty$, and*

$$a_{i,j}(\psi_{i+j} - \psi_i) \leqslant Cj \quad \text{for all } i, j \geqslant 1, \tag{37}$$

for some constant $C > 0$.

In addition, for a given sequence of positive numbers λ_i with $\lim_{i \to +\infty} \lambda_i = +\infty$, we can choose ψ_i so that $\psi_i \leqslant \lambda_i$ for all i.

Proof. First, we may assume that the function θ given in Hypothesis 1.8 is nonincreasing on $[1, +\infty)$, as we can always take $\tilde{\theta}(x) := \sup_{y \geqslant x} \theta(y)$ instead.

We choose a sequence of nonnegative numbers $\{\xi_i\}$ by applying Lemma 4.1 with

$$\mu_i := \frac{1}{(1+i)\log(1+i)}, \tag{38}$$

$$\nu_i := \min\left\{\lambda_i, \frac{1}{\theta(\sqrt{i/2})}, \right\}. \tag{39}$$

Note that the conditions in Lemma 4.1 are met: the sequence in the right-hand side of (38) is not summable, and the right-hand side of (39) goes to $+\infty$ with i. If we define $\psi_i := \sum_{j=1}^{i} \xi_j$, then the following is given by Lemma 4.1:

$$\xi_i \leqslant \frac{1}{(1+i)\log(1+i)}, \qquad \psi_i \leqslant \frac{1}{\theta(\sqrt{i/2})}, \quad \psi_i \leqslant \lambda_i, \ i \geqslant 1,$$
$$\lim_{i \to +\infty} \psi_i = +\infty.$$

These conditions essentially say that ψ_i grows slowlier than $\log\log(i)$, slowlier than $\theta(\sqrt{i/2})^{-1}$, and slowlier than λ_i, yet still diverges as $i \to +\infty$.

We can now prove (37) to hold for these $\{\psi_i\}$ by distinguishing three cases:

1. For any $i, j \geqslant 1$, as $\log(1+k) \geqslant 1/2$ for all $k \geqslant 1$,

$$\psi_{i+j} - \psi_i = \sum_{k=i+1}^{i+j} \xi_k \leqslant 2 \sum_{k=i+1}^{i+j} \frac{1}{1+k} \leqslant 2\log(i+j+1) - 2\log(i+1) \leqslant \frac{2j}{i}.$$

Then, in case $j \leqslant i$ we use the fact that $\theta(x) \leqslant C_\theta$ for some constant $C_\theta > 0$ and all $x > 0$ and have

$$a_{i,j}(\psi_{i+j} - \psi_i) \leqslant 2C_\theta(i+j)\frac{j}{i} \leqslant 4C_\theta j, \quad \text{for } j \leqslant i.$$

2. Secondly, for $i < j \leqslant i^2$,

$$\psi_{i+j} - \psi_i \leqslant \sum_{k=i+1}^{2i^2} \xi_k \leqslant \sum_{k=i+1}^{2i^2} \frac{1}{(k+1)\log(k+1)}$$
$$\leqslant \log\log\big(2i^2+1\big) - \log\log(i+1) \leqslant \log\left(\frac{2\log(\sqrt{3}i)}{\log(i+1)}\right) \leqslant C_1,$$

for some number $C_1 > 0$. Thus,

$$a_{i,j}(\psi_{i+j} - \psi_i) \leqslant C_1 C_\theta (i+j) \leqslant 2C_1 C_\theta j.$$

3. Finally, for $j > i^2$,

$$\psi_{i+j} - \psi_i \leqslant \psi_{i+j} = \sum_{k=1}^{i+j} \xi_k \leqslant \frac{1}{\theta(\sqrt{(i+j)/2})} \leqslant \frac{1}{\theta(\sqrt{j})},$$

and as θ is nonincreasing on $[1, +\infty)$ (we may assume this; see the beginning of this proof), we have for all $j > i^2$

$$a_{i,j}(\psi_{i+j} - \psi_i) \leqslant (i+j)\theta(j/i)\frac{1}{\theta(\sqrt{j})} \leqslant (i+j)\theta(\sqrt{j})\frac{1}{\theta(\sqrt{j})} = i+j \leqslant 2j.$$

Together, these three cases show (37) for all $i, j \geqslant 1$. □

Now we are ready to finish the proof of our result on mass conservation:

Proof of Theorem 1.9. As remarked above (cf. beginning of Section 2), we will prove the estimate (19) for a regular solution to an approximating system, with a constant $C(T)$ that does not depend on the regularisation. Then, passing to the limit, the result is true for a weak solution thus constructed.

We consider a solution to an approximating system on $[0, +\infty)$, which we still denote by $\{c_i\}_{i\geqslant 1}$. Then, by a version of the de la Vallée–Poussin's Lemma (see, for instance, Proposition 9.1.1 in [4] or also proof of Lemma 7 in [6]), there exists a nondecreasing sequence of positive numbers $\{\lambda_i\}_{i\geqslant 1}$ (independent of the regularisation of the initial data) which diverges as $i \to +\infty$, and such that

$$\int_\Omega \sum_{i=1}^{\infty} i\lambda_i c_i^0 \, dx < +\infty. \tag{40}$$

If we define $r_i := \int_\Omega i c_i^0$, note that this is just the claim that one can find λ_i as above with $\sum_i \lambda_i r_i < +\infty$.

Taking $\{\psi_i\}$ as given by Lemma 4.2, such that $\psi_i \leqslant \lambda_i$ for all $i \geqslant 1$, we have thus $\int_\Omega \sum_{i=1}^\infty i\psi_i c_i^0(x)\, dx < +\infty$. Then, as integrating over Ω makes the diffusion term vanish due to the no-flux boundary conditions, we estimate

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^{\infty} i\psi_i c_i \, dx \leqslant \frac{1}{2}\int_\Omega \sum_{i,j=1}^{\infty} a_{i,j} c_i c_j \big((i+j)\psi_{i+j} - i\psi_i - j\psi_j\big)\, dx, \tag{41}$$

where we used that the contribution of the fragmentation term is nonpositive, as can be seen from (4) with $\varphi_i \equiv i\psi_i$, and the fact that

$$\sum_{j=1}^{i-1} \beta_{i,j} j\psi_j \leqslant \psi_i \sum_{j=1}^{i-1} \beta_{i,j} j = i\psi_i,$$

as ψ_i is nondecreasing and (3c) holds. Continuing from (41), by the symmetry of the $a_{i,j}$, and using the inequality (37) from Lemma 4.2, we have

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^{\infty} i\psi_i c_i \, dx \leqslant \int_\Omega \sum_{i,j=1}^{\infty} a_{i,j} c_i c_j i(\psi_{i+j} - \psi_i)\, dx \leqslant C\int_\Omega \rho^2 \, dx. \tag{42}$$

Thus, Proposition 1.3 showing $\rho \in L^2(\Omega_T)$ proves that $\int_\Omega \sum_{i=1}^\infty i\psi_i c_i\,dx$ is bounded on bounded time intervals. Mass conservation is a direct consequence of this. □

Remark 4.3 *(Absence of gelation via tightness).* It is interesting to sketch an alternative proof showing conservation of mass via a tightness argument and without establishing superlinear moments. By introducing the superlinear test sequence $i\phi_k(i)$ with $\phi_k(i) = \frac{\log i}{\log k}\mathbb{I}_{i<k} + \mathbb{I}_{i\geqslant k}$ for all $k \in \mathbb{N}^*$, we use the weak formulation (4) to see (as above) that the fragmentation part is nonnegative for superlinear test sequences, and use the symmetry of the $a_{i,j}$ to reduce summation over the indices $i \geqslant j \in \mathbb{N}^*$, which leads to the estimate

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^\infty c_i i\phi_k(i)\,dx$$
$$\leqslant \int_\Omega \sum_{i\geqslant j}^\infty \sum_{j=1}^\infty a_{i,j}[ic_i][c_j]\biggl(\frac{\log(1+\frac{j}{i})}{\log(k)}\mathbb{I}_{i<k} + \frac{j}{i}\biggl(\frac{\log(1+\frac{i}{j})}{\log(k)}\mathbb{I}_{i+j<k} + \frac{\log(\frac{k}{j})}{\log(k)}\mathbb{I}_{j<k\leqslant i+j}\biggr)\biggr)\,dx.$$

For the first term, we use $\log(1 + j/i) \leqslant j/i$. Then, for the second and third terms, we distinguish further the areas where $i/j \leqslant \log(k)$ and $i/j > \log(k)$. When $i/j \leqslant \log(k)$, we estimate $1 + i/j \leqslant 1 + \log(k)$ in the second term. In the third term, where $k \leqslant i + j$, we estimate $k/j \leqslant 1 + i/j \leqslant 1 + \log(k)$. On the other hand, when $i/j > \log(k)$, both the second and the third term are bounded by one. Altogether, we get thanks to Hypothesis 1.8, i.e. $\frac{a_{i,j}}{i} \leqslant \mathrm{Cst}\,\theta(i/j)$ for $i \leqslant j$:

$$\begin{aligned}\frac{d}{dt}\int_\Omega \sum_{i=1}^\infty c_i i\phi_k(i)\,dx &\leqslant \biggl(\frac{1}{\log(k)} + \frac{\log(1+\log k)}{\log(k)}\biggr)\sup_{i\geqslant j\in\mathbb{N}^*}\biggl\{\frac{a_{i,j}}{i}\biggr\}\int_\Omega \rho^2\,dx \\ &\quad + \int_\Omega \sum_{i\geqslant j}^\infty\sum_{j=1}^\infty [ic_i][jc_j]\frac{a_{i,j}}{i}\mathbb{I}_{i/j>\log(k);\, j<k}\,dx \\ &\leqslant \mathrm{Cst}\biggl(\frac{\log(1+\log k)}{\log(k)} + \sup_{i/j\geqslant\log(k)}\theta\biggl(\frac{i}{j}\biggr)\biggr)\int_\Omega \rho^2\,dx\end{aligned}$$

and the right-hand side tends to zero as $k \to \infty$. Hence, using Proposition 1.3 and integrating over a time interval $[0, T]$, we get thanks to a tightness argument that the mass is indeed conserved, and no gelation occurs.

5. Third application: Fragmentation due to collisions in dimension 1

Proof of Theorem 1.13. We introduce $(c_i^M)_M$ a sequence of smooth solutions for a truncated version of Eq. (23). We first observe that Proposition 1.3 still holds thanks to the duality estimate, that is $\rho := \sum_i ic_i \in L^2(\Omega_T)$ for all $T > 0$. Estimate (30), in which only the coagulation kernel appears, also holds. Moreover, thanks to (24d),

$$\sum_{k,l}\sum_{\max\{k,l\}>i} b_{k,l}c_k c_l\beta_{ikl} \leqslant \mathrm{Cst}_i \sum_k\sum_l (k+l)c_k c_l \leqslant \mathrm{Cst}_i\,\rho^2 \in L^1(\Omega_T).$$

The loss terms

$$\sum_{k=1}^\infty a_{i,k}c_i c_k, \qquad \sum_{k=1}^\infty b_{i,k}c_i c_k$$

lie then in $L^1(\Omega_T)$ by integration of the equation on $[0, T] \times \Omega$.

Using now Eq. (23), we see that (for all $i \in \mathbb{N}^*$) $\partial_t c_i^M - d_i\partial_{xx}c_i^M$ belongs to a bounded subset of $L^1(\Omega_T)$. As a consequence, c_i^M belongs (for all $i \in \mathbb{N}^*$) to a compact subset of $L^{3-\varepsilon}([0, T] \times \Omega)$ for all $T > 0$ and $\varepsilon > 0$. We denote (for all $i \in \mathbb{N}^*$) by c_i a limit (in $L^{3-\varepsilon}([0, T] \times \Omega)$ strong) of a subsequence of $(c_i^M)_{M\in\mathbb{N}}$ (still denoted by $(c_i^M)_{M\in\mathbb{N}}$).

We now pass to the limit in all terms of the right-hand side of Eq. (23). The first term can easily be dealt with, since it consists of a finite sum. Then, we pass to the limit in the second term:

$$\int_0^T \int_\Omega \left| \sum_{k=1}^{\infty} a_{i,k} c_i^n c_k^n - \sum_{k=1}^{\infty} a_{i,k} c_i c_k \right| dx\,dt \leqslant \int_0^T \int_\Omega \left| \sum_{k=1}^{K} a_{i,k} c_i^n c_k^n - \sum_{k=1}^{K} a_{i,k} c_i c_k \right| dx\,dt + 2\|\rho\|_{L^2}^2 \sup_{k>K} \left\{ \frac{a_{i,k}}{k} \right\}.$$

The second part of this expression is small when K is large enough thanks to assumptions (25), (26), while the first part tends to 0 for all given K.

The fourth term of the right-hand side of Eq. (23) can be treated exactly in the same way. We now turn to the third term:

$$\int_0^T \int_\Omega \left| \sum_{k,l=1}^{\infty} \sum_{i<\max\{k,l\}} b_{k,l} c_k^n c_l^n \beta_{i,k,l} - \sum_{k,l=1}^{\infty} \sum_{i<\max\{k,l\}} b_{k,l} c_k c_l \beta_{i,k,l} \right| dx\,dt$$

$$\leqslant \int_0^T \int_\Omega \left| \sum_{k,l=1}^{K} \sum_{i<\max\{k,l\}}^{k\leqslant K, l\leqslant K} b_{k,l} c_k^n c_l^n \beta_{i,k,l} - \sum_{k,l=1}^{K} \sum_{i<\max\{k,l\}}^{k\leqslant K, l\leqslant K} b_{k,l} c_k c_l \beta_{i,k,l} \right| dx\,dt + 4\|\rho\|_{L^2}^2 \sup_{l\geqslant K} \sup_{k\in\mathbb{N}} \left\{ \frac{b_{k,l}}{kl} \beta_{i,k,l} \right\}.$$

Once again, the second term is small when K is large enough thanks to assumptions (25), (26), while the first term tends to 0 for all given K. □

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Appendix A. A duality lemma

We recall here results from e.g. [17,9]. We start with

Lemma A.1. *Assume that* $z : \Omega_T \to [0, +\infty)$ *satisfies*

$$\begin{aligned} &\partial_t z + M\Delta z = -H \quad \text{on } \Omega, \\ &\nabla z \cdot n = 0 \quad \text{on } \partial\Omega, \\ &z(T, x) = 0 \quad \text{on } \Omega, \end{aligned} \tag{A.1}$$

where $H \in L^2(\Omega_T)$*, and* $d_1 \geqslant M \geqslant d_0 > 0$*. Then,*

$$\|z(0,\cdot)\|_{L^2(\Omega)} \leqslant \left(1 + \frac{d_1}{d_0}\right) T \|H\|_{L^2(\Omega_T)}. \tag{A.2}$$

Proof. Calculating the time derivative of $\int_\Omega |\nabla z|^2$, or alternatively multiplying Eq. (A.1) by Δz and integrating on Ω, we obtain

$$-\frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla z|^2\,dx + \int_\Omega M(\Delta z)^2\,dx = \int_\Omega -H\Delta z\,dx,$$

where the boundary condition on z was used. Integrating on $[0, T]$ and taking into account that $z(T, x) = 0$,

$$\frac{1}{2}\int_{\Omega} \left|\nabla z(0,\cdot)\right|^2 dx + \int_{\Omega_T} M(\Delta z)^2\,dx\,dt = \int_{\Omega_T} -H\Delta z\,dx\,dt \leqslant \|H\|_{L^2(\Omega_T)}\|\Delta z\|_{L^2(\Omega_T)}. \tag{A.3}$$

Using that $M \geqslant d_0$ we see that $\int_{\Omega_T} M(\Delta z)^2 \geqslant d_0\|\Delta z\|^2_{L^2(\Omega_T)}$, so (A.3) implies

$$d_0\|\Delta z\|_{L^2(\Omega_T)} \leqslant \|H\|_{L^2(\Omega_T)}.$$

From this and (A.1) we have

$$\begin{aligned}\|\partial_t z\|_{L^2(\Omega_T)} &\leqslant \|M\Delta z\|_{L^2(\Omega_T)} + \|H\|_{L^2(\Omega_T)}\\ &\leqslant d_1\|\Delta z\|_{L^2(\Omega_T)} + \|H\|_{L^2(\Omega_T)} \leqslant \left(1+\frac{d_1}{d_0}\right)\|H\|_{L^2(\Omega_T)}.\end{aligned}$$

Finally,

$$\left\|z(0,\cdot)\right\|_{L^2(\Omega)} \leqslant \int_0^T \|\partial_s z_s\|_{L^2(\Omega)}\,ds \leqslant \left(1+\frac{d_1}{d_0}\right)T\|H\|_{L^2(\Omega_T)}. \qquad \square$$

Lemma A.2. *Assume that $\rho : \Omega_T \to [0,+\infty)$ and satisfies*

$$\begin{aligned}&\partial_t\rho - \Delta(M\rho) \leqslant 0 \quad \text{on } \Omega,\\ &\nabla(\rho M)\cdot n = 0 \quad \text{on } \partial\Omega,\end{aligned} \tag{A.4}$$

where $M : \Omega_T \to \mathbb{R}$ is a function which satisfies $d_1 \geqslant M \geqslant d_0 > 0$ for some numbers d_1, d_0. Then,

$$\|\rho\|_{L^2(\Omega_T)} \leqslant \left(1+\frac{d_1}{d_0}\right)T\left\|\rho(0,\cdot)\right\|_2.$$

Proof. Consider the dual problem (A.1)–(A.2) for an arbitrary function $H \in L^2(\Omega_T)$, with $H \geqslant 0$. Then, $z \geqslant 0$, and integrating by parts in Eq. (A.1), one finds that

$$\begin{aligned}\int_{\Omega_T} \rho H\,dx\,dt &= -\int_{\Omega_T} \rho(\partial_t z + M\Delta z)\,dx\,dt\\ &= \int_{\Omega_T} z\big(\partial_t\rho - \Delta(\rho M)\big)\,dx\,dt + \int_{\Omega} \rho(0,\cdot)z(0,\cdot)\,dx\,dt\\ &\leqslant \int_{\Omega} \rho(0,\cdot)z(0,\cdot)\,dx\,dt,\end{aligned}$$

where we have used Eqs. (A.4), (A.2) and the boundary conditions on ρM and z. Hence, for any nonnegative function $H \in L^2(\Omega_T)$,

$$\int_{\Omega_T} \rho H\,dx\,dt \leqslant \left\|\rho(0,\cdot)\right\|_{L^2(\Omega)}\left\|z(0,\cdot)\right\|_{L^2(\Omega)},$$

and thanks to Lemma A.1,

$$\int_{\Omega_T} \rho H\,dx\,dt \leqslant (1+d_1/d_0)T\left\|\rho(0,\cdot)\right\|_{L^2(\Omega)}\|H\|_{L^2(\Omega_T)}.$$

Remembering that $\rho \geqslant 0$, we obtain by duality:

$$\|\rho\|_{L^2(\Omega_T)} \leqslant (1+d_1/d_0)T\left\|\rho(0,\cdot)\right\|_{L^2(\Omega)}.$$

This proves the lemma. $\square$

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