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Existence, uniqueness and stability of equilibrium states for non-uniformly expanding maps ☆

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Abstract

We prove existence of finitely many ergodic equilibrium states for a large class of non-uniformly expanding local homeomorphisms on compact metric spaces and Hölder continuous potentials with not very large oscillation. No Markov structure is assumed. If the transformation is topologically mixing there is a unique equilibrium state, it is exact and satisfies a non-uniform Gibbs property. Under mild additional assumptions we also prove that the equilibrium states vary continuously with the dynamics and the potentials (statistical stability) and are also stable under stochastic perturbations of the transformation.

Résumé

Nous prouvons l'existence d'un nombre fini d'états d' équilibre ergodiques pour une classe assez grande d'homéomorphismes locaux non-uniformément dilatants sur des espaces métriques compacts et pour les potentiels de Hölder continus à oscillation pas trop grande. Aucune structure de Markov n'est supposée. Si la transformation est topologiquement mélangeante alors il existe un unique état d' équilibre, il est une mesure exacte et vérifie une propriété de Gibbs non-uniforme. Avec quelques hypothèses supplémentaires, nous prouvons aussi que les états d' équilibre varient de façon continue avec la dynamique et le potentiel (stabilité statistique) et sont également stables sous des perturbations stochastiques de la transformation.

1. Introduction

The theory of equilibrium states of smooth dynamical systems was initiated by the pioneer works of Sinai, Ruelle, Bowen [51,7,6,47]. For uniformly hyperbolic diffeomorphisms and flows they proved that equilibrium states exist and are unique for every Hölder continuous potential, restricted to every basic piece of the non-wandering set. The basic strategy to prove this remarkable fact was to (semi)conjugate the dynamics to a subshift of finite type, via a Markov partition.

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Several important difficulties arise when trying to extend this theory beyond the uniformly hyperbolic setting and, despite substantial progress by several authors, a global picture is still far from complete. For one thing, existence of generating Markov partitions is known only in a few cases and, often, such partitions can not be finite. Moreover, equilibrium states may actually fail to exist if the system exhibits critical points or singularities (see Buzzi [13]).

A natural starting point is to try and develop the theory first for smooth systems which are hyperbolic in the non-uniform sense of Pesin theory, that is, whose Lyapunov exponents are non-zero "almost everywhere". This was advocated by Alves, Bonatti, Viana [3], who assume non-uniform hyperbolicity at *Lebesgue* almost every point and deduce existence and finiteness of physical (Sinai–Ruelle–Bowen) measures. In this setting, physical measures are absolutely continuous with respect to Lebesgue measure along expanding directions.

It is not immediately clear how this kind of hypothesis may be useful for the more general goal we are addressing, since one expects most equilibrium states to be singular with respect to Lebesgue measure. Nevertheless, in a series of recent works, Oliveira, Viana [37–39] managed to push this idea ahead and prove existence and uniqueness of equilibrium states for a fairly large class of smooth transformations on compact manifolds, inspired by [3]. Roughly speaking, they assumed that the transformation is expanding on most of the phase space, possibly with some relatively mild contracting behavior on the complement. Moreover, the potential should be Hölder continuous and its oscillation $\sup\phi - \inf\phi$ not too big. On the other hand, they need a number of additional conditions on the transformation, most notably the existence of (non-generating) Markov partitions, that do not seem natural.

Important contributions to the theory of equilibrium states outside the uniformly hyperbolic setting have been made by several other authors: Denker, Keller, Nitecki, Przytycki, Urbański [25,21,23,26,27,52], Bruin, Keller, Todd [8,10,9], and Pesin, Senti, Zhang [42,43], for one-dimensional maps, real and complex. Wang, Young [55] for Hénon-like maps. Buzzi, Maume, Paccaut, Sarig, [11,16,15,18] for piecewise expanding maps in higher dimensions. Buzzi, Sarig [18,48–50,58] for countable Markov shifts. Denker, Urbański [20,22,24] and Yuri [56–58] for maps with indifferent periodic points. Leplaideur, Rios [32,33] for horsehoes with tangencies at the boundary of hyperbolic systems. This list is certainly not complete. Some results, including [17] and [38] are specific for measures of maximal entropy. An important notion of entropy-expansiveness was introduced by Buzzi [12], which influenced [14,38] among other papers.

In this paper we carry out the program set by Alves, Bonatti, Viana towards a theory of equilibrium states for the class of non-uniformly expanding maps originally proposed in [3, Appendix]. We improve upon previous results of [39] in a number of ways. For one thing, we completely remove the need for a Markov partition (generating or not). In fact, one of the technical novelties with respect to previous recent works in this area is that we prove, in an abstract way inspired by Ledrappier [31], that every equilibrium state must be absolutely continuous with respect to a certain conformal measure. When the map is topologically mixing, the equilibrium state is unique, and a non-lacunary Gibbs measure. In this regard let us mention that Pinheiro [44] has recently announced an inducing scheme for constructing (countable) Markov partitions for a class of non-invertible transformations closely related to ours. Another improvement is that our results are stated for local homeomorphisms on compact metric spaces, rather than local diffeomorphisms on compact manifolds (compare [39, Remark 2.6]). In addition, we also prove stability of the equilibrium states under random noise (stochastic stability) and continuity under variations of the dynamics (statistical stability).

Our basic strategy to prove these results goes as follows. First we construct an expanding conformal measure ν as a special eigenmeasure of the dual of the Ruelle–Perron–Frobenius operator. Then we show that every accumulation point μ of the Cesaro sum of the push-forwards $f_*^n\nu$ is an invariant probability measure that is absolutely continuous with respect to ν with density bounded away from infinity, and that there are finitely many distinct such ergodic measures. In addition, we prove that these absolutely continuous invariant measures are equilibrium states, and that any equilibrium state is necessarily an expanding measure. Finally, we establish an abstract version of Ledrappier's theorem [31] and characterize equilibrium states as invariant measures absolutely continuous with respect to ν.

This paper is organized as follows. The precise statement of our results is given in Section 2. We included in Section 3 preparatory material that will be necessary for the proofs. Following the approach described above, we construct an expanding conformal measure and prove that there are finitely many invariant and ergodic measures absolutely continuous with respect it through Sections 4 and 5. In Section 6 we prove Theorems A and B. Finally, in Section 7 we prove the stochastic and statistical stability results stated in Theorems D and E.

2. Statement of the results

2.1. Hypotheses

We say that X is a *Besicovitch metric space* if it is a metric space where the Besicovitch covering lemma (see e.g. [19]) holds. These metric spaces are characterized in [28] and include e.g. any subsets of Euclidean metric spaces and manifolds.

We consider N to be compact Besicovitch metric space of topological dimension m with distance d. Let $M \subset N$ be a compact set, $f : M \to N$ be a *local homeomorphism* and assume that there exists a bounded function $x \mapsto L(x)$ such that, for every $x \in M$ there is a neighborhood U_x of x so that $f_x : U_x \to f(U_x)$ is invertible and

$$d\big(f_x^{-1}(y), f_x^{-1}(z)\big) \leqslant L(x)\, d(y,z), \quad \forall y, z \in f(U_x).$$

Assume also that every point has finitely many preimages and that the level sets for the degree $\{x\colon \#\{f^{-1}(x)\} = k\}$ are closed. Given $x \in M$ set $\deg_x(f) = \# f^{-1}(x)$. Define $h_n(f) = \min_{x\in M} \deg_x(f^n)$ for $n \geqslant 1$, and consider the limit

$$h(f) = \liminf_{n\to\infty} \frac{1}{n} \log h_n(f).$$

It is clear that

$$\log\Big(\max_{x\in M} \#\big\{f^{-1}(x)\big\}\Big) \geqslant h(f) \geqslant \log\Big(\min_{x\in M} \#\big\{f^{-1}(x)\big\}\Big).$$

If M is connected, every point has the same number $\deg(f)$ of preimages by f, which coincides with the spectral radius for the Ruelle–Perron–Frobenius operator. Hence, if this is the case, $h(f) = \log\deg(f)$ is the topological entropy of f (see Lemma 6.5 below). The limit above also exists e.g. when the dynamics is (semi)conjugated to a subshift of finite type. By definition, there exists $N \geqslant 1$ such that $\deg_x(f^n) \geqslant e^{h(f)n}$ for every $x \in M$ and every $n \geqslant N$. Up to consider the iterate f^N instead of f we will assume that every point in M has at least $e^{h(f)}$ preimages by f.

For all our results we assume that f and ϕ satisfy conditions (H1), (H2), and (P) stated in what follows. Assume that there exist constants $\sigma > 1$ and $L > 0$, and an open region $\mathcal{A} \subset M$ such that

(H1) $L(x) \leqslant L$ for every $x \in \mathcal{A}$ and $L(x) \leqslant \sigma^{-1}$ for all $x \in M \setminus \mathcal{A}$, and L is close to 1: the precise conditions are given in (3.2) and (3.3) below.
(H2) There exists $k_0 \geqslant 1$ and a covering $\mathcal{P} = \{P_1, \ldots, P_{k_0}\}$ of M by domains of injectivity for f such that $\mathcal{A}$ can be covered by $q < e^{h(f)}$ elements of $\mathcal{P}$.

The first condition means that we allow expanding and contracting behavior to coexist in M: f is uniformly expanding outside $\mathcal{A}$ and not too contracting inside $\mathcal{A}$. The second one requires essentially that in average every point has at least one preimage in the expanding region. The interesting part of the dynamics is given by the restriction of f to the compact metric space

$$K = \bigcap_{n\geqslant 0} f^{-n}(M),$$

that can be connected or totally disconnected. We give some examples below where $K = M$ is a manifold and where $K \subset M$ is a Cantor set.

In addition we assume that $\phi : M \to \mathbb{R}$ is Hölder continuous and that its variation is not too big. More precisely, assume that:

(P) $\sup\phi - \inf\phi < h(f) - \log q$.

Notice this is an open condition on the potential, relative to the uniform norm, and it is satisfied by constant functions. It can be weakened somewhat. For one thing, all we need for our estimates is the supremum of ϕ over the union of the elements of $\mathcal{P}$ that intersect $\mathcal{A}$. With some extra effort (replacing the q elements of $\mathcal{P}$ that intersect $\mathcal{A}$ by the same number of smaller domains), one may even consider the supremum over $\mathcal{A}$, that is, $\sup \phi|_{\mathcal{A}} - \inf \phi < h(f) - \log q$. However, we do not use nor prove this fact here.

Let us comment on this hypothesis. A related condition, $P_{\mathrm{top}}(f, \phi) > \sup \phi$, was introduced by Denker, Urbański [21] in the context of rational maps on the sphere. Another related condition, $P(f, \phi, \partial \mathcal{Z}) < P(f, \phi)$, is used by Buzzi, Paccaut, Schmitt [16], in the context of piecewise expanding multidimensional maps, to control the map's behavior at the boundary $\partial \mathcal{Z}$ of the domains of smoothness: without such a control, equilibrium states may fail to exist [13]. Condition (P) seems to play a similar role in our setting.

2.2. Examples

Here we give several examples and comment on the role of the hypotheses (H1), (H2) and (P), specially in connection with the supports of the measures we construct, the existence and finitude of equilibrium states.

Example 2.1. Let $f_0 : \mathbb{T}^d \to \mathbb{T}^d$ be a linear expanding map. Fix some covering $\mathcal{P}$ for f_0 and some $P_1 \in \mathcal{P}$ containing a fixed (or periodic) point p. Then deform f_0 on a small neighborhood of p inside P_1 by a pitchfork bifurcation in such a way that p becomes a saddle for the perturbed local homeomorphism f. By construction, f coincides with f_0 in the complement of P_1, where uniform expansion holds. Observe that we may take the deformation in such a way that f is never too contracting in P_1, which guarantees that (H1) holds, and that f is still topologically mixing. Condition (P) is clearly satisfied by $\phi \equiv 0$. Hence, Theorems A and B imply that there exists a unique measure of maximal entropy, it is supported in the whole manifold $\mathbb{T}^d$ and it is a non-lacunary Gibbs measure.

Now, we give an example where the union of the supports of the equilibrium states does not coincide with the whole manifold.

Example 2.2. Let f_0 be an expanding map in $\mathbb{T}^2$ and assume that f_0 has a periodic point p with two complex conjugate eigenvalues $\tilde{\sigma} e^{i\varpi}$, with $\tilde{\sigma} > 3$ and $k\varpi \notin 2\pi\mathbb{Z}$ for every $1 \leqslant k \leqslant 4$. It is possible to perturb f_0 through a Hopf bifurcation at p to obtain a local homeomorphism f, C^5-close to f_0 and such that p becomes a periodic attractor for f (see e.g. [29] for details). Moreover, if the perturbation is small then (H1) and (H2) hold for f. Thus, there are finitely many ergodic measures of maximal entropy for f. Since these measures are expanding their support do not intersect the basin of attraction the periodic attractor p.

An interesting question concerns the restrictions on f imposed by (P). For instance, if $\phi = -\log|\det Df|$ satisfies (P) then there can be no periodic attractors. In fact, the expanding conformal measure ν coincides with the Lebesgue measure which is an expanding measure and positive on open sets. An example where the potential $\phi = -\log|\det Df|$ satisfies (P) is given by Example 2.1 above, since condition (P) can be rewritten as

$$\frac{\sup_{x \in \mathbb{T}^2} |\det Df(x)|}{\inf_{x \in \mathbb{T}^2} |\det Df(x)|} < \deg(f), \tag{2.1}$$

and clearly satisfied if the perturbation is small enough.

The next example shows that some control on the potential ϕ is needed to have uniqueness of the equilibrium state: in absence of the hypothesis (P), uniqueness may fail even if we assume (H1) and (H2).

Example 2.3 *(Manneville–Pomeau map).* If $\alpha \in (0, 1)$, let $f : [0, 1] \to [0, 1]$ be the local homeomorphism given by

$$f_\alpha(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} < x \leqslant 1. \end{cases}$$

Observe that conditions (H1) and (H2) are satisfied. It is well known that f has a finite invariant probability measure μ absolutely continuous with respect to Lebesgue. Using Pesin formula and Ruelle inequality, it is not hard to check that both μ and the Dirac measure δ_0 at the fixed point 0 are equilibrium states for the potential $\phi = -\log|\det Df|$.

Thus, *uniqueness fails* in this topologically mixing context. For the sake of completeness, let us mention that in this example f is not a local homeomorphism, but one can modify it to a local homeomorphisms in $S^1 = [0,1]/\sim$ by

$$f_\alpha(x) = \begin{cases} x(1+2^\alpha x^\alpha) & \text{if } 0 \leqslant x \leqslant \frac{1}{2}, \\ x - 2^\alpha(1-x)^{1+\alpha} & \text{if } \frac{1}{2} < x \leqslant 1, \end{cases}$$

where $\sim$ means that the extremal points in the interval are identified. Note that the potential ϕ is not (Hölder) continuous.

The previous phenomenon concerning the lack of uniqueness of equilibrium states can appear near the boundary of the class of maps and potentials satisfying (H1) and (H2) and (P).

Example 2.4. Let f_α be the map given by the previous example and let $(\phi_\beta)_{\beta>0}$ be the family of Hölder continuous potentials given by $\phi_\beta = -\log(\det|Df| + \beta)$. On the one hand, observe that ϕ_β converge to $\phi = -\log(|\det Df|)$ as $\beta \to 0$. On the other hand, similarly to (2.1), one can write condition (P) as

$$\frac{\beta+2+\alpha}{\beta+1} < 2, \quad \text{or simply } \beta > \alpha.$$

For every $\alpha > 0$, since f_α is topologically mixing and satisfies (H1),(H2) and $\phi_{2\alpha}$ satisfies (P) for every $\alpha > 0$ there is a unique equilibrium state μ_α for f_α with respect to $\phi_{2\alpha}$. Moreover, $\phi_{2\alpha}$ approaches ϕ, which seems to indicate that the condition (P) on the potential should be close to optimal in order to get uniqueness of equilibrium states. Furthermore, the potential $\psi_t = -t\log|Df_\alpha|$ satisfies (P) if and only if $\psi_t(0) - \psi_t(1/2) < \log 2$ or, equivalently, $t < \frac{\log 2}{\log(2+\alpha)}$.

Since $h_{\text{top}}(f) = \log 2$, condition (P) can be rewritten also as $\sup\phi - \inf\phi < h_{\text{top}}(f)$. In [9, Proposition 2], the authors proved that for every Hölder continuous potential that does not satisfy (P) has no equilibrium state obtained from some 'natural' inducing schemes.

The next example illustrates that our results also apply when the set K is totally disconnected.

Example 2.5. Let N denote the unit interval $[0,1]$ and let $f : N \to \mathbb{R}$ be the unimodal map $f(x) = -8x(x-1) \times (x+1/8)$. Since the critical value is outside of the unit interval $[0,1]$, $K = \bigcap_n f^{-n}([0,1)]$ is clearly a Cantor set. Although the existence of a critical point, the restriction of f to the set $M = f^{-1}[0,1])$ is a local homeomorphism. It is not hard to check that (H1) and (H2) hold in this setting and that $f \mid K$ is topologically mixing. As a consequence of the results below we show that there is a unique measure of maximal entropy for f, whose support is K.

2.3. Existence of equilibrium states

Throughout the paper we assume, with some abuse of notation, that K coincides with M. There is no restriction in this since $M \subset N$ was taken as an arbitrary compact set such that $f|_M$ is a local homeomorphism. We say that f is *topologically mixing* if, for each open set U there is a positive integer N so that $f^N(U) = M$. Let $\mathcal{B}$ denote the Borel σ-algebra of M. An f-invariant probability measure η is *exact* if the σ-algebra $\mathcal{B}_\infty = \bigcap_{n\geqslant 0} f^{-n}\mathcal{B}$ is η-trivial, meaning that it contains only zero and full η-measure sets. Given a continuous map $f : M \to M$ and a potential $\phi : M \to \mathbb{R}$, the variational principle for the pressure asserts that

$$P_{\text{top}}(f,\phi) = \sup\left\{h_\mu(f) + \int \phi\, d\mu\colon\ \mu \text{ is } f\text{-invariant}\right\}$$

where $P_{\text{top}}(f,\phi)$ denotes the topological pressure of f with respect to ϕ and $h_\mu(f)$ denotes the metric entropy. An *equilibrium state* for f with respect to ϕ is an invariant measure that attains the supremum in the right-hand side above.

Theorem A. *Let $f : M \to M$ be a local homeomorphism with Lipschitz continuous inverse and $\phi : M \to \mathbb{R}$ a Hölder continuous potential satisfying* (H1), (H2)*, and* (P)*. Then, there is a finite number of ergodic equilibrium states $\mu_1, \mu_2, \ldots, \mu_k$ for f with respect to ϕ such that any equilibrium state μ is a convex linear combination of $\mu_1, \mu_2, \ldots, \mu_k$. In addition, if the map f is topologically mixing then the equilibrium state is unique and exact.*

Our strategy for the construction of equilibrium states is, first to construct a certain conformal measure ν which is expanding and a non-lacunary Gibbs measure. Then we construct the equilibrium states, which are absolutely continuous with respect to this reference measure ν. Both steps explore a weak hyperbolicity property of the system. In what follows we give precise definitions of the notions involved.

A probability measure ν, not necessarily invariant, is *conformal* if there exists some function $\psi : M \to \mathbb{R}$ such that

$$\nu\big(f(A)\big) = \int_A e^{-\psi}\,d\nu$$

for every measurable set A such that $f \mid A$ is injective.

Let $S_n\phi = \sum_{j=0}^{n-1} \phi \circ f^j$ denote the nth Birkhoff sum of a function ϕ. The *dynamical ball* of center $x \in M$, radius $\delta > 0$, and length $n \geqslant 1$ is defined by

$$B(x,n,\delta) = \big\{y \in M\colon\ d\big(f^j(y), f^j(x)\big) \leqslant \delta,\ \forall 0 \leqslant j \leqslant n\big\}.$$

An integer sequence $(n_k)_{k\geqslant 1}$ is *non-lacunary* if it is increasing and $n_{k+1}/n_k \to 1$ when $k \to \infty$.

Definition 2.6. A probability measure ν is a *non-lacunary Gibbs measure* if there exist uniform constants $K > 0$, $P \in \mathbb{R}$ and $\delta > 0$ so that, for ν-almost every $x \in M$ there exists some non-lacunary sequence $(n_k)_{k\geqslant 1}$ such that

$$K^{-1} \leqslant \frac{\nu(B(x,n_k,\delta))}{\exp(-Pn_k + S_{n_k}\phi(y))} \leqslant K$$

for every $y \in B(x,n_k,\delta)$ and every $k \geqslant 1$.

The weak hyperbolicity property of f is expressed through the notion of hyperbolic times, which was introduced in [1,3] for differentiable transformations. We say that n is a *c-hyperbolic time* for $x \in M$ if

$$\prod_{j=n-k}^{n-1} L\big(f^j(x)\big) < e^{-ck} \quad \text{for every } 1 \leqslant k \leqslant n. \tag{2.2}$$

Often we just call them hyperbolic times, since the constant c will be fixed, as in (3.2). We denote by H the set of points $x \in M$ with infinitely many hyperbolic times and by H_j the set of points having $j \geqslant 1$ as hyperbolic time. A probability measure ν, not necessarily invariant, is *expanding* if $\nu(H) = 1$.

The *basin of attraction* of an f-invariant probability measure μ is the set $B(\mu)$ of points $x \in M$ such that

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)} \quad \text{converges weakly to } \mu \text{ when } n \to \infty.$$

Theorem B. *Let $f : M \to M$ be a local homeomorphism and $\phi : M \to \mathbb{R}$ be a Hölder continuous potential satisfying* (H1), (H2)*, and* (P)*. Let $\mu_1, \mu_2, \ldots, \mu_k$ be the ergodic equilibrium states of f for ϕ. Then every μ_i is absolutely continuous with respect to some conformal, expanding, non-lacunary Gibbs measure ν. The union of all basins of attraction $B(\mu_i)$ contains ν-almost every point $x \in M$. If, in addition, f is topologically mixing then the unique absolutely continuous invariant measure μ is a non-lacunary Gibbs measure.*

As a byproduct of the previous results we can obtain the existence of equilibrium states for *continuous* potentials satisfying (P). Without some extra condition no uniqueness of equilibrium states is expected to hold even if f is topologically mixing.

Corollary C. *Let $f : M \to M$ be a local homeomorphism satisfying* (H1) *and* (H2)*. If $\phi : M \to \mathbb{R}$ is a continuous potential satisfying* (P) *then there exists an equilibrium state for f with respect to ϕ. Moreover, there is a residual set $\mathcal{R}$ of potentials in $C(M)$ that satisfy* (P) *such that there is unique equilibrium state for f with respect to ϕ.*

2.4. Stability of equilibrium states

Let $\mathcal{F}$ be a family of local homeomorphisms with Lipschitz inverse and $\mathcal{W}$ be some family of continuous potentials ϕ. A pair $(f,\phi) \in \mathcal{F} \times \mathcal{W}$ is *statistically stable* (relative to $\mathcal{F} \times \mathcal{W}$) if, for any sequences $f_n \in \mathcal{F}$ converging to f in the uniform topology, with L_n converging to a L in the uniform topology, and $\phi_n \in \mathcal{W}$ converging to ϕ in the uniform topology, and for any choice of an equilibrium state μ_n of f_n for ϕ_n, every weak* accumulation point of the sequence $(\mu_n)_{n\geqslant 1}$ is an equilibrium state of f for ϕ. In particular, when the equilibrium state is unique, statistical stability means that it depends continuously on the data (f,ϕ).

Theorem D. *Suppose every $(f,\phi) \in \mathcal{F} \times \mathcal{W}$ satisfies* (H1), (H2), *and* (P), *with uniform constants (including the Hölder constants of ϕ). Assume that the topological pressure $P_{\mathrm{top}}(f,\phi)$ varies continuously in the parameters $(f,\phi) \in \mathcal{F} \times \mathcal{W}$. Then every pair $(f,\phi) \in \mathcal{F} \times \mathcal{W}$ is statistically stable relative to $\mathcal{F} \times \mathcal{W}$.*

The assumption on continuous variation of the topological pressure might hold in great generality in this setting. See the comment at the end of Section 7.1 for a discussion.

Now let $\mathcal{F}$ be a family of local homeomorphisms satisfying (H1) and (H2) with uniform constants. A *random perturbation* of $f \in \mathcal{F}$ is a family θ_ε, $0 < \varepsilon \leqslant 1$ of probability measures in $\mathcal{F}$ such that there exists a family $V_\varepsilon(f)$, $0 < \varepsilon \leqslant 1$ of neighborhoods of f, depending monotonically on ε and satisfying

$$\operatorname{supp}\theta_\varepsilon \subset V_\varepsilon(f) \quad \text{and} \quad \bigcap_{0<\varepsilon\leqslant 1} V_\varepsilon(f) = \{f\}.$$

Consider the skew product map

$$\begin{aligned} F : \mathcal{F}^{\mathbb{N}} \times M &\to \mathcal{F} \times M, \\ (\underline{f}, x) &\mapsto \big(\sigma(\underline{f}), f_1(x)\big) \end{aligned}$$

where $\underline{f} = (f_1, f_2, \ldots)$ and $\sigma : \mathcal{F}^{\mathbb{N}} \to \mathcal{F}^{\mathbb{N}}$ is the shift to the left. For each $\varepsilon > 0$, a measure μ^ε on M is *stationary* (respectively, *ergodic*) for the random perturbation if the measure $\theta_\varepsilon^{\mathbb{N}} \times \mu^\varepsilon$ on $\mathcal{F}^{\mathbb{N}} \times M$ is invariant (respectively, ergodic) for F.

We assume the random-perturbation to be *non-degenerate*, meaning that, for every $\varepsilon > 0$, the push-forward of the measure θ_ε under any map

$$\mathcal{F} \ni g \mapsto g(x)$$

is absolutely continuous with respect to some probability measure ν, with density uniformly (on x) bounded from above, and its support contains a ball around $f(x)$ with radius uniformly (on x) bounded from below. The first condition implies that any stationary measure is absolutely continuous with respect to ν. In Theorem 7.3 we shall use also the second condition to conclude that, assuming ν is expanding and conformal, for any $\varepsilon > 0$ there exists a finite number of ergodic stationary measures $\mu_1^\varepsilon, \mu_2^\varepsilon, \ldots, \mu_l^\varepsilon$. We say that f is *stochastically stable* under random perturbation if every accumulation point, as $\varepsilon \to 0$, of stationary measures $(\mu^\varepsilon)_{\varepsilon>0}$ absolutely continuous with respect to ν is a convex combination of the ergodic equilibrium states $\mu_1, \mu_2, \ldots, \mu_k$ of f for ϕ.

A *Jacobian* of f with respect to a probability measure η is a measurable function $J_\eta f$ such that

$$\eta\big(f(A)\big) = \int_A J_\eta f \, d\eta \tag{2.3}$$

for every measurable set A (in some full measure subset) such that $f \mid A$ is injective. A Jacobian may fail to exist, in general, and it is essentially unique when it exists. If f is at most countable-to-one and the measure η is invariant, then Jacobians do exist (see [40]).

Theorem E. *Let $(\theta_\varepsilon)_\varepsilon$ be a non-degenerate random perturbation of $f \in \mathcal{F}$ and ν be the reference measure in Theorem* B. *Assume ν admits a Jacobian for every $g \in \mathcal{F}$, and the Jacobian varies continuously with g in the uniform norm. Then f is stochastically stable under the random perturbation $(\theta_\varepsilon)_\varepsilon$.*

The conditions on the Jacobian are automatically satisfied in some interesting cases, for instance when ν is the Riemannian volume or f is an expanding map. This is usually associated to the potential $\phi = -\log|\det(Df)|$. Example 2.1 describes a situation where this potential satisfies the condition (P).

3. Preliminary results

Here, we give a few preparatory results needed for the proof of the main results. The content of this section may be omitted in a first reading and the reader may choose to return here only when necessary.

3.1. Combinatorics of orbits

Since the region $\mathcal{A}$ is contained in q elements of the partition $\mathcal{P}$ we can assume without any loss of generality that $\mathcal{A}$ is contained in the first q elements of $\mathcal{P}$. Given $\gamma \in (0,1)$ and $n \geqslant 1$, let us consider the set $I(\gamma, n)$ of all itineraries $(i_0, \dots, i_{n-1}) \in \{1, \dots, k_0\}^n$ such that $\#\{0 \leqslant j \leqslant n-1: i_j \leqslant q\} > \gamma n$. Then let

$$c_\gamma = \limsup_{n\to\infty} \frac{1}{n} \log \# I(\gamma, n). \tag{3.1}$$

Lemma 3.1. *Given $\varepsilon > 0$ there exists $\gamma_0 \in (0,1)$ such that $c_\gamma < \log q + \varepsilon$ for every $\gamma \in (\gamma_0, 1)$.*

Proof. It is clear that

$$\# I(\gamma, n) \leqslant \sum_{k=[\gamma n]}^{n} \binom{n}{k} p^{n-k} q^k \leqslant \sum_{k=[\gamma n]}^{n} \binom{n}{k} p^{(1-\gamma)n} q^n,$$

where $p = k_0 - q$ denotes the number of elements in $\mathcal{P}$ that do not intersect $\mathcal{A}$. Assume that $\gamma > 1/2$. A standard computation using Stirling's formula implies that

$$\sum_{k=[\gamma n]}^{n} \binom{n}{k} \leqslant \frac{n}{2} \binom{n}{[\gamma n]} \leqslant C_1 e^{2t(1-\gamma)n}$$

for some uniform constants $C_1 > 0$ and $t > 0$. Hence $c_\gamma < \log q + \varepsilon$ provided that γ is sufficiently close to 1, which proves the lemma. □

We are in a position to state our precise condition on the constant L in assumption (H1) and the constant c in the definition of hyperbolic time. By (P), we may find $\varepsilon_0 > 0$ small such that $\sup\phi - \inf\phi + \varepsilon_0 < h(f) - \log q$. By Lemma 3.1, we may find $\gamma < 1$ such that $c_\gamma < \log q + \varepsilon_0/4$. Assume L is close enough to 1 and c is close enough to zero so that

$$\sigma^{-(1-\gamma)} L^\gamma < e^{-2c} < 1 \tag{3.2}$$

and

$$\sup\phi - \inf\phi < h(f) - \log q - \varepsilon_0 - m \log L. \tag{3.3}$$

3.2. Hyperbolic times

The next lemma, whose proof is based on a lemma due to Pliss (see e.g. [36]), asserts that, for points satisfying a certain condition of asymptotic expansion, there are infinitely many hyperbolic times: even more, the set of hyperbolic times has positive density at infinity.

Lemma 3.2. *Let $x \in M$ and $n \geqslant 1$ be such that*

$$\frac{1}{n} \sum_{j=1}^{n} \log L\big(f^j(x)\big) \leqslant -2c < 0.$$

Then, there is $\theta > 0$, depending only on f and c, and a sequence of hyperbolic times $1 \leqslant n_1(x) < n_2(x) < \cdots < n_l(x) \leqslant n$ for x, with $l \geqslant \theta n$.

Proof. Analogous to Corollary 3.2 of [3]. □

Corollary 3.3. *Let η be a probability measure relative to which*

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log L\big(f^j(x)\big) \leqslant -2c < 0$$

holds almost everywhere. If A is a positive measure set then

$$\liminf_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \frac{\eta(A\cap H_j)}{\eta(A)} \geqslant \frac{\theta}{2}.$$

Proof. By Lemma 3.2, for η-almost every point $x \in M$ there is $N(x) \in \mathbb{N}$ so that $n^{-1}\sum_{j=0}^{n-1}\chi_{H_j}(x) \geqslant \theta$ for every $n \geqslant N(x)$. Fix an integer $N \geqslant 1$ and choose $\tilde{A} \subset A$ so that $\eta(\tilde{A}) \geqslant \eta(A)/2$ and $N(x) \leqslant N$ for every $x \in \tilde{A}$. If we integrate the expression above with respect to η on A we obtain that

$$\frac{1}{n}\sum_{j=0}^{n-1} \eta(H_j \cap A) \geqslant \theta\eta(\tilde{A}) \geqslant \frac{\theta}{2}\eta(A)$$

for every integer n larger than N, completing the proof of the lemma. □

Lemma 3.4. *There exists $\delta = \delta(c, f) > 0$ such that, whenever n is a hyperbolic time for a point x, the dynamical ball $V_n(x) = B(x, n, \delta)$ is mapped homeomorphically by f^n onto the ball $B(f^n(x), \delta)$, with*

$$d\big(f^{n-j}(y), f^{n-j}(z)\big) \leqslant e^{-\frac{c}{2}j} d\big(f^n(y), f^n(z)\big)$$

for every $1 \leqslant j \leqslant n$ and every $y, z \in V_n(x)$.

Proof. Analogous to the proof of [3, Lemma 2.7], just replacing $\log \|Df(\cdot)^{-1}\|$ by $\log L(\cdot)$, and using the definition of hyperbolic time and the Lipschitz property of the inverse branches of f. □

If n is a hyperbolic time for a point $x \in M$, the neighborhood $V_n(x)$ given by the lemma above is called *hyperbolic pre-ball*. As a consequence of the previous lemma we obtain the following property of bounded distortion on pre-balls.

Corollary 3.5. *Assume $J_\eta f = e^\psi$ for some Hölder continuous function ψ. There exists a constant $K_0 > 0$ so that, if n is a hyperbolic time for x then*

$$K_0^{-1} \leqslant \frac{J_\eta f^n(y)}{J_\eta f^n(z)} \leqslant K_0$$

for every $y, z \in V_n(x)$.

Proof. Let n be a hyperbolic time for a point x in M and (C, α) be the Hölder constants of ψ. Using Lemma 3.4 it is not hard to see that

$$\big|S_n\psi(y) - S_n\psi(z)\big| \leqslant C\sum_{j=0}^{+\infty} e^{-c\alpha/2j} d\big(f^n(x), f^n(y)\big)^\alpha \leqslant C\delta^\alpha \sum_{j=0}^{+\infty} e^{-c\alpha j/2}$$

for any given $y, z \in V_n(x)$. Choosing K_0 as the exponential of this last term and noting $J_\eta f^n$ is the exponential of $S_n\psi$, the result follows immediately. □

3.3. Non-lacunary sequences

The set H of points with infinitely many hyperbolic times plays a central role in our strategy. We are going to see that for such a point the sequence of hyperbolic times has some special properties. The first one is described in the following remark:

Remark 3.6. If n is a hyperbolic time for x then, clearly, $n-s$ is a hyperbolic time for $f^s(x)$, for any $1 \leqslant s < n$. The following converse is a simple consequence of (2.2): if $k < n$ is a hyperbolic time for x and there exists $1 \leqslant s \leqslant k$ such that $n-s$ is a hyperbolic time for $f^s(x)$ then n is a hyperbolic time for x. Thus, if $n_j(x)$, $j \geqslant 1$ denotes the sequence of values of n for which x belongs to H_n then, for every j and l

$$n_j(x) + n_l\big(f^{n_j(x)}(x)\big) = n_{j+l}(x).$$

We will refer to this property as *concatenation* of hyperbolic times. Moreover, if n is a hyperbolic time for x and k is a hyperbolic time for $f^n(x)$, the intersection $V_n(x) \cap f^{-k}(V_k(f^k(x)))$ coincides with the hyperbolic pre-ball $V_{n+k}(x)$.

The next lemma, which we borrow from [39], provides an abstract criterium for non-lacunarity *at almost every point* of certain sequences of functions.

Lemma 3.7. *(See [39, Proposition 3.8].) Let $T : M \to \mathbb{N}$ and $T_i : M \to \mathbb{N}$, $i \in \mathbb{N}$ be measurable functions and η be a probability measure such that*

$$T\big(f^{T_i(x)}(x)\big) \geqslant T_{i+1}(x) - T_i(x)$$

at η-almost every $x \in M$. Assume η is invariant under f and T is integrable for η. Then $(T_i(x))_i$ is non-lacunary for η-almost every x.

The application we have in mind is when $T_i = n_i$ is the sequence of hyperbolic times, with $T = n_1$. In this case the assumption of the lemma follows from the concatenation property in Remark 3.6. Thus, we obtain

Corollary 3.8. *If η is an invariant expanding measure and $n_1(\cdot)$ is η-integrable then the sequence $n_j(\cdot)$ is non-lacunary at η-almost every point.*

3.4. Relative pressure

We recall the notion of topological pressure on non-necessarily compact invariant sets, and quote some useful properties. In fact, we present two alternative characterizations of the relative pressure, both from a dimensional point of view. See Chapter 4 §11 and Appendix II of [41] for proofs and more details.

Let M be a compact metric space, $f : M \to M$ be a continuous transformation, $\phi : M \to \mathbb{R}$ be a continuous function, and Λ be an f-invariant set.

Relative pressure using partitions. Given any finite open covering $\mathcal{U}$ of Λ, denote by $\mathcal{I}_n$ the space of all n-strings $\underline{i} = \{(U_0, \ldots, U_{n-1})\colon U_i \in \mathcal{U}\}$ and put $n(\underline{i}) = n$. For a given string $\underline{i}$ set

$$\underline{U} = \underline{U}(\underline{i}) = \big\{x \in M\colon\ f^j(x) \in U_{i_j},\ \text{for } j = 0, \ldots, n(\underline{i})\big\}$$

to be the cylinder associated to $\underline{i}$ and $n(\underline{U}) = n$ to be its depth. Furthermore, for every integer $N \geqslant 1$, let $\mathcal{S}_N\mathcal{U}$ be the space of all cylinders of depth at least N. Given $\alpha \in \mathbb{R}$ define

$$m_\alpha(f, \phi, \Lambda, \mathcal{U}, N) = \inf_{\mathcal{G}} \left\{ \sum_{\underline{U} \in \mathcal{G}} e^{-\alpha n(\underline{U}) + S_{n(\underline{U})}\phi(\underline{U})} \right\},$$

where the infimum is taken over all families $\mathcal{G} \subset \mathcal{S}_N\mathcal{U}$ that cover Λ and we write $S_n\phi(\underline{U}) = \sup_{x \in \underline{U}} S_n\phi(x)$. Let

$$m_\alpha(f, \phi, \Lambda, \mathcal{U}) = \lim_{N \to \infty} m_\alpha(f, \phi, \Lambda, \mathcal{U}, N)$$

(the sequence is monotone increasing) and

$$P_\Lambda(f,\phi,\mathcal{U}) = \inf\{\alpha\colon\ m_\alpha(f,\phi,\Lambda,\mathcal{U}) = 0\}.$$

Definition 3.9. The *pressure of* (f,ϕ) *relative to* Λ is

$$P_\Lambda(f,\phi) = \lim_{\operatorname{diam}(\mathcal{U})\to 0} P_\Lambda(f,\phi,\mathcal{U}).$$

Theorem 11.1 in [41] states that the limit does exist, that is, given any sequence of coverings $\mathcal{U}_k$ of L with diameter going to zero, $P_L(f,\phi,\mathcal{U}_k)$ converges and the limit does not depend on the choice of the sequence.

Relative pressure using dynamical balls. Fix $\varepsilon > 0$. Set $\mathcal{I}_n = M \times \{n\}$ and $\mathcal{I} = M \times \mathbb{N}$. For every $\alpha \in \mathbb{R}$ and $N \geqslant 1$, define

$$m_\alpha(f,\phi,\Lambda,\varepsilon,N) = \inf_{\mathcal{G}}\left\{\sum_{(x,n)\in\mathcal{G}} e^{-\alpha n + S_n\phi(B(x,n,\varepsilon))}\right\}, \tag{3.4}$$

where the infimum is taken over all finite or countable families $\mathcal{G} \subset \bigcup_{n\geqslant N}\mathcal{I}_n$ such that the collection of sets $\{B(x,n,\varepsilon)\colon (x,n)\in\mathcal{G}\}$ cover Λ. Then let

$$m_\alpha(f,\phi,\Lambda,\varepsilon) = \lim_{N\to\infty} m_\alpha(f,\phi,\Lambda,\mathcal{U},N)$$

(once more, the sequence is monotone increasing) and

$$P_\Lambda(f,\phi,\varepsilon) = \inf\{\alpha\colon\ m_\alpha(f,\phi,\Lambda,\varepsilon) = 0\}.$$

According to Remark 1 in [41, p. 74] there is a limit when $\varepsilon \to 0$ and it coincides with the relative pressure:

$$P_\Lambda(f,\phi) = \lim_{\varepsilon\to 0} P_\Lambda(f,\phi,\varepsilon).$$

Remark 3.10. Since ϕ is uniformly continuous, the definition of the relative pressure is not affected if one replaces, in (3.4), the supremum $S_n\phi(B(x,n,\varepsilon))$ by the value $S_n\phi(x)$ at the center point.

The following properties on relative pressure, will be very useful later. See Theorem 11.2 and Theorem A2.1 in [41], and also [54, Theorem 9.10].

Proposition 3.11. *Let M be a compact metric space, $f: M \to M$ be a continuous transformation, $\phi: M \to \mathbb{R}$ be a continuous function, and Λ be an f-invariant set. Then*

(1) $P_\Lambda(f,\phi) \geqslant \sup\{h_\mu(f) + \int \phi\, d\mu\}$ *where the supremum is over all invariant measures μ such that $\mu(\Lambda) = 1$. If Λ is compact, the equality holds.*
(2) $P_{\mathrm{top}}(f,\phi) = \sup\{P_\Lambda(f,\phi), P_{M\setminus\Lambda}(f,\phi)\}$.

The next proposition is probably well-known. We include a proof since we could not find one in the literature.

Proposition 3.12. *Let M be a compact metric space, $f: M \to M$ be a continuous transformation, $\phi: M \to \mathbb{R}$ be a continuous function, and Λ be an f-invariant set. Then $P_\Lambda(f^\ell, S_\ell\phi) = \ell P_\Lambda(f,\phi)$ for every $\ell \geqslant 1$.*

Proof. Fix $\ell \geqslant 1$. By uniform continuity of f, given any $\rho > 0$ there exists $\varepsilon > 0$ such that $d(x,y) < \varepsilon$ implies $d(f^j(x), f^j(y)) < \rho$ for all $0 \leqslant j < \ell$. It follows that

$$B_f(x,\ell n,\varepsilon) \subset B_{f^\ell}(x,n,\varepsilon) \subset B_f(x,\ell n,\rho), \tag{3.5}$$

where $B_g(x,n,\varepsilon)$ denotes the dynamical ball for a map g. This is the crucial observation for the proof.

First, we prove the $\geqslant$ inequality. Given $N \geqslant 1$ and any family $\mathcal{G}_\ell \subset \bigcup_{n\geqslant N}\mathcal{I}_n$ such that the balls $B_{f^\ell}(x,j,\varepsilon)$ with $(x,j)\in\mathcal{G}_\ell$ cover Λ, denote

$$\mathcal{G} = \{(x,j\ell)\colon (x,j)\in\mathcal{G}_\ell\}.$$

The second inclusion in (3.5) ensures that the balls $B_f(x,k,\rho)$ with $(x,k)\in\mathcal{G}$ cover Λ. Clearly,

$$\sum_{(x,j)\in\mathcal{G}_\ell} e^{-\alpha\ell j+\sum_{i=0}^{j-1} S_\ell\phi(f^{i\ell}(x))} = \sum_{(x,k)\in\mathcal{G}} e^{-\alpha k+\sum_{i=0}^{k-1}\phi(f^i(x))}.$$

Since $\mathcal{G}_\ell$ is arbitrary, and recalling Remark 3.10, this proves that

$$m_{\alpha\ell}\big(f^\ell, S_\ell\phi, \Lambda, \varepsilon, N\big) \geqslant m_\alpha(f,\phi,\Lambda,\rho,N\ell).$$

Therefore, $m_{\alpha\ell}(f^\ell, S_\ell\phi, \Lambda,\varepsilon) \geqslant m_\alpha(f,\phi,\Lambda,\rho)$. Then $P_\Lambda(f^\ell, S_\ell\phi,\varepsilon)\geqslant \ell P_\Lambda(f,\phi,\rho)$. Since $\varepsilon\to 0$ when $\rho\to 0$, it follows that $P_\Lambda(f^\ell,S_\ell\phi)\geqslant \ell P_\Lambda(f,\phi)$.

For the $\leqslant$ inequality, we observe that the definition of the relative pressure is not affected if one restricts the infimum in (3.4) to families $\mathcal{G}$ of pairs (x,k) such that k is always a multiple of ℓ. More precisely, let $m_\alpha^\ell(f,\phi,\Lambda,\varepsilon,N)$ be the infimum over this subclass of families, and let $m_\alpha^\ell(f,\phi,\Lambda,\varepsilon)$ be its limit as $N\to\infty$.

Lemma 3.13. *We have $m_\alpha^\ell(f,\phi,\Lambda,\varepsilon)\leqslant m_{\alpha-\rho}(f,\phi,\Lambda,\varepsilon)$ for every $\rho>0$.*

Proof. We only have to show that, given any $\rho>0$,

$$m_\alpha^\ell(f,\phi,\Lambda,\varepsilon,N)\leqslant m_{\alpha-\rho}(f,\phi,\Lambda,\varepsilon,N) \tag{3.6}$$

for every large N. Let ρ be fixed and N be large enough so that $N\rho>\ell(\alpha+\sup|\phi|)$. Given any $\mathcal{G}\subset\bigcup_{n\geqslant N}\mathcal{I}_n$ such that the balls $B_f(x,k,\varepsilon)$ with $(x,k)\in\mathcal{G}$ cover Λ, define $\mathcal{G}'$ to be the family of all (x,k'), $k'=\ell[k/\ell]$ such that $(x,k)\in\mathcal{G}$. Notice that

$$-\alpha k'+S_{k'}\phi(x)\leqslant -\alpha k+\alpha\ell+S_k\phi(x)+\ell\sup|\phi|\leqslant(-\alpha+\rho)k+S_k\phi(x)$$

given that $k\geqslant N$. The claim follows immediately. □

Let $\mathcal{G}'$ be any family of pairs (x,k) with $k\geqslant N\ell$ and such that every k is a multiple of ℓ. Define $\mathcal{G}_\ell$ to be the family of pairs (x,j) such that $(x,j\ell)\in\mathcal{G}'$. The first inclusion in (3.5) ensures that if the balls $B_f(x,k,\varepsilon)$ with $(x,k)\in\mathcal{G}'$ cover Λ then so do the balls $B_{f^\ell}(x,j,\varepsilon)$ with $(x,j)\in\mathcal{G}_\ell$. Clearly,

$$\sum_{(x,k)\in\mathcal{G}'} e^{-\alpha k+\sum_{i=0}^{k-1}\phi(f^i(x))} = \sum_{(x,j)\in\mathcal{G}_\ell} e^{-\alpha\ell j+\sum_{i=0}^{j-1} S_\ell\phi(f^{i\ell}(x))}.$$

Since $\mathcal{G}_\ell$ is arbitrary, and recalling Remark 3.10, this proves that

$$m_\alpha^\ell(f,\phi,\Lambda,\varepsilon,N\ell)\geqslant m_{\alpha\ell}\big(f^\ell,S_\ell\phi,\Lambda,\varepsilon,N\big).$$

Taking the limit when $N\to\infty$ and using Lemma 3.13,

$$m_{\alpha-\rho}(f,\phi,\Lambda,\varepsilon)\geqslant m_\alpha^\ell(f,\phi,\Lambda,\varepsilon)\geqslant m_{\alpha\ell}\big(f^\ell,S_\ell\phi,\Lambda,\varepsilon\big).$$

It follows that $\ell(P_\Lambda(f,\phi,\varepsilon)+\rho)\geqslant P_\Lambda(f^\ell,S_\ell\phi,\varepsilon)$. Since ρ is arbitrary, we conclude that $\ell P_\Lambda(f,\phi,\varepsilon)\geqslant P_\Lambda(f^\ell,S_\ell\phi,\varepsilon)$ and so $P_\Lambda(f^\ell,S_\ell\phi)\geqslant \ell P_\Lambda(f,\phi)$. □

The next lemma will be used later to reduce some estimates for the relative pressure to the case when $\phi\equiv 0$. Denote $h_\Lambda(f)=P_\Lambda(f,0)$ for any invariant set Λ.

Lemma 3.14. $P_\Lambda(f,\phi)\leqslant h_\Lambda(f)+\sup\phi$.

Proof. Let $\mathcal{U}$ be any open covering of M and $N\geqslant 1$. By definition,

$$m_\alpha(f,\phi,\Lambda,\mathcal{U},N)=\inf_{\mathcal{G}}\bigg\{\sum_{\underline{\mathrm{U}}\in\mathcal{G}} e^{-\alpha n(\underline{\mathrm{U}})+S_{n(\underline{\mathrm{U}})}\phi(\underline{\mathrm{U}})}\bigg\},$$

where the infimum is taken over all families $\mathcal{G}\subset\mathcal{S}_N\mathcal{U}$ that cover Λ. Therefore,

$$m_\alpha(f,\phi,\Lambda,\mathcal{U},N)\leqslant\inf_{\mathcal{G}}\bigg\{\sum_{\underline{\mathrm{U}}\in,\mathcal{G}} e^{(-\alpha+\sup\phi)n(\underline{\mathrm{U}})}\bigg\}=m_{\alpha-\sup\phi}(f,0,\Lambda,\mathcal{U},N).$$

Since N and $\mathcal{U}$ are arbitrary, this gives that $P_\Lambda(f,\phi)\leqslant h_\Lambda(f)+\sup\phi$, as we wanted to prove. □

3.5. Natural extension and local unstable leaves

Here we present the natural extension associated to a non-invertible transformation and recall some results on the existence of local unstable leaves in the context of non-uniform hyperbolicity.

Let $(M, \mathcal{B}, \eta)$ be a probability space and let f denote a measurable non-invertible transformation. Consider the space

$$\hat{M} = \{(\dots, x_2, x_1, x_0) \in M^{\mathbb{N}}:\ f(x_{i+1}) = x_i,\ \forall i \geqslant 0\},$$

endowed with the metric $\hat{d}(\underline{x}, \underline{y}) = \sum_{i\geqslant 0} 2^{-i} d(x_i, y_i)$, $\underline{x}, \underline{y} \in \hat{M}$ and with the sigma-algebra $\hat{\mathcal{B}}$ that we now describe. Let $\pi_i : \hat{M} \to M$ denote the projection in the ith coordinate. Note also that $f^{-i}(\mathcal{B}) \subset \mathcal{B}$ for every $i \geqslant 0$, because f^i is a measurable transformation. Let $\hat{\mathcal{B}}_0$ be the smallest sigma-algebra that contain the elements $\pi_i^{-1}(f^{-i}(\mathcal{B}))$. The measure $\hat{\eta}$ defined on the algebra $\bigcup_{i=0}^{\infty} \pi_i^{-1}(f^{-i}\mathcal{B})$ by

$$\hat{\eta}(E_i) = \eta(\pi_i(E_i)) \quad \text{for every } E_i \in \pi_i^{-1}(f^{-i}(\mathcal{B})),$$

admits an extension to the sigma-algebra $\hat{\mathcal{B}}_0$. Let $\hat{\mathcal{B}}$ denote the completion of $\hat{\mathcal{B}}_0$ with respect to $\hat{\eta}$. The *natural extension* of f is the transformation

$$\hat{f} : \hat{M} \to \hat{M}, \quad \hat{f}(\dots, x_2, x_1, x_0) = (\dots, x_2, x_1, x_0, f(x_0)),$$

on the probability space $(\hat{M}, \hat{\mathcal{B}}, \hat{\eta})$. The measure $\hat{\eta}$ is the unique $\hat{f}$-invariant probability measure such that $\pi_*\hat{\eta} = \eta$. Furthermore, $\hat{\eta}$ is ergodic if and only if η is ergodic, and its entropy $h_{\hat{\eta}}(\hat{f})$ coincides with $h_\eta(f)$. We refer the reader to [46] for more details and proofs. For simplicity reasons, when no confusion is possible we denote by π the projection in the zeroth coordinate and by x_0 the point $\pi(\hat{x})$.

Given a local homeomorphism f as above, the natural extension $\hat{f}^{-1}$ is Lipschitz continuous: every $\hat{x}$ admits a neighborhood $U_{\hat{x}}$ such that

$$d(\hat{f}^{-1}(\hat{y}), \hat{f}^{-1}(\hat{z})) \leqslant \hat{L}(\hat{x})\, d(\hat{y}, \hat{z}), \quad \forall \hat{y}, \hat{z} \in \hat{f}(U_{\hat{x}}),$$

where $\hat{L} = L \circ \pi$. In the presence of asymptotic expanding behavior it is possible to prove the existence of local unstable manifolds passing through almost every point and varying measurably. In fact, since L is continuous bounded away from zero and infinity, given an $\hat{f}$-invariant probability measure $\hat{\eta}$, Birkhoff's ergodic theorem asserts that the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \hat{L}(\hat{f}^{\pm j}\hat{x})$$

exist and coincide $\hat{\eta}$-almost everywhere. Given $\lambda > 0$, denote by $\hat{B}_\lambda$ the set of points such that the previous limit is well defined and smaller than $-\lambda$.

Proposition 3.15. *Assume that η is an f-invariant probability measure such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log L(f^j(x)) < -\lambda < 0$$

almost everywhere. Given $\varepsilon > 0$ small, there are measurable functions δ_ε and γ from $\hat{B}_\lambda$ to $\mathbb{R}_+$ and, for every $\hat{x} \in \hat{B}_\lambda$, there exists an embedded topological disk $W^u_{\mathrm{loc}}(\hat{x})$ that varies measurably with the point $\hat{x}$ and

(1) *For every $y_0 \in W^u_{\mathrm{loc}}(\hat{x})$ there is a unique $\hat{y} \in \hat{M}$ such that $\pi(\hat{y}) = y_0$ and*

$$d(x_{-n}, y_{-n}) \leqslant \gamma(\hat{x}) e^{-(\lambda-\varepsilon)n}, \quad \forall n \geqslant 0;$$

(2) *If a point $\hat{z} \in \hat{M}$ satisfies $d(x, z) \leqslant \delta_\varepsilon(\hat{x})/\gamma(\hat{f}^{-1}(\hat{x}))$ and*

$$d(x_{-n}, z_{-n}) \leqslant \delta_\varepsilon(\hat{x}) e^{-(\lambda-\varepsilon)n}, \quad \forall n \geqslant 0$$

then z_0 belongs to $W^u_{\mathrm{loc}}(\hat{x})$;

(3) *If* $\hat{W}^u_{\mathrm{loc}}(\hat{x})$ *is the set of points* $\hat{y} \in \hat{M}$ *given by* (2) *above then it holds that*

$$d(y_{-n}, z_{-n}) \leqslant \gamma(\hat{x}) e^{-(\lambda-\varepsilon)n} d(y, z)$$

for every $\hat{y}, \hat{z} \in \hat{W}^u_{\mathrm{loc}}(\hat{x})$ *and every* $n \geqslant 0$.

Proof. Let $\varepsilon > 0$ be small enough such that the restriction of f to any ball of radius ε is injective. Given $\hat{x} \in \hat{B}_\lambda$, consider the local unstable set

$$W^u_{\mathrm{loc}}(\hat{x}) = \bigl\{y \in M\colon \exists \hat{y} \in \hat{M},\ \pi(\hat{y}) = y,\ d(y_{-n}, x_{-n}) \leqslant \varepsilon,\ \forall n \geqslant 0\bigr\}.$$

By construction $W^u_{\mathrm{loc}}(\hat{x})$ is non-empty, since it contains x. Moreover, define $\hat{W}^u_{\mathrm{loc}}(\hat{x})$ as the set of points $\hat{y}$ considered in the definition of $W^u_{\mathrm{loc}}(\hat{x})$. It is clear that $\hat{f}^{-1}(\hat{W}^u_{\mathrm{loc}}(\hat{x})) \subset \hat{W}^u_\varepsilon(\hat{f}^{-1}(\hat{x}))$. We claim that $W^u_{\mathrm{loc}}(\hat{x})$ contains an open neighborhood of x in M and that there exists a constant $\gamma(\hat{x}) > 0$ such that

$$d(y_{-n}, z_{-n}) \leqslant \gamma(\hat{x}) e^{-(\lambda-\varepsilon)n}, \quad \forall n \geqslant 1,$$

for every $\hat{y}, \hat{z} \in \hat{W}^u_{\mathrm{loc}}(\hat{x})$. By hypothesis, there exists $N = N_{\hat{x}} \geqslant 1$ such that

$$\prod_{j=0}^{n-1} \hat{L}\bigl(\hat{f}^{-j}(\hat{x})\bigr) \leqslant e^{-\lambda n}, \quad \forall n \geqslant N.$$

Take $0 < \delta_\varepsilon(\hat{x}) < \varepsilon$ such that f^N is invertible in a neighborhood of x_{-N} and that $B(x, \delta_\varepsilon(\hat{x})) \subset f^N(B(x_{-N}, \varepsilon))$. Moreover, by uniform continuity, there exists $0 < \varepsilon_1 < \varepsilon$ such that $L(z) \leqslant L(z')e^\varepsilon$ for every $z' \in B(z, \varepsilon_1)$. So, given $y, z \in B(x, \delta_\varepsilon(\hat{x}))$ there are $\hat{y}, \hat{z} \in \hat{M}$ such that $d(y_{-n}, z_{-n}) \leqslant \varepsilon$ for every $n \geqslant 0$, since

$$d(y_{-n}, z_{-n}) \leqslant e^{-(\lambda-\varepsilon)n} d(y, z)$$

for every $n \geqslant N$. This shows that $W^u_\varepsilon(\hat{x})$ contains the ball $B(x, \delta_\varepsilon(\hat{x}))$ of radius $\delta_\varepsilon(\hat{x})$ around x in M and that

$$d(y_{-n}, z_{-n}) \leqslant \gamma(\hat{x}) e^{-(\lambda-\varepsilon)n} d(y, z)$$

for every $\hat{y}, \hat{z} \in W^u_\varepsilon(\hat{x})$ and $n \geqslant 0$, where $\gamma(\hat{x}) = L^{N_{\hat{x}}}$. Our choice on ε guarantees that any $y \in W^u_\varepsilon(\hat{x})$ admits a unique $\hat{y} \in \hat{W}^u_\varepsilon(\hat{x})$ such that $\pi(\hat{y}) = y$. This shows that the projection $\pi_{\hat{x}} : \hat{W}^u_{\mathrm{loc}}(\hat{x}) \to W^u_\varepsilon(\hat{x})$ is an homeomorphism between topological disks and completes the proof of items (1) and (3) in the proposition. On the other hand, if $\hat{z}$ satisfies the requirements in (2) then clearly $d(x_{-n}, z_{-n}) \leqslant \varepsilon$ for all $n \geqslant 00$, which imply that $z \in W^u_{\mathrm{loc}}(\hat{x})$. Since the measurability of γ and δ_ε follows from the one of $N_{\hat{x}}$, the proof of the proposition in now complete. □

We shall omit the dependence of $W^u_{\mathrm{loc}}(\hat{x})$ on λ and ε for notational simplicity. Since local unstable leaves vary measurably with the point, there are compact sets of arbitrary large measure, referred as *hyperbolic blocks*, restricted to which the local unstable leaves passing through those points vary continuously. More precisely,

Corollary 3.16. *There are countably many compact sets* $(\hat{\Lambda}_i)_{i\in\mathbb{N}}$ *whose union is a* $\hat{\eta}$*-full measure set and such that the following holds*: *for every* $i \geqslant 1$ *there are positive numbers* $\varepsilon_i \ll 1$, λ_i, r_i, γ_i *and* R_i *such that for every* $\hat{x} \in \hat{\Lambda}_i$ *there exists an embedded submanifold* $W^u_{\mathrm{loc}}(\hat{x})$ *in* M *of dimension* m, *and*

(1) *If* $y_0 \in W^u_{\mathrm{loc}}(\hat{x})$ *then there is a unique* $\hat{y} \in \hat{M}$ *such that for every* $n \geqslant 1$

$$d(x_{-n}, y_{-n}) \leqslant r_i e^{-\varepsilon_i n} \quad \text{and} \quad d(x_{-n}, y_{-n}) \leqslant \gamma_i e^{-\lambda_i n};$$

(2) *For every* $0 < r \leqslant r_i$ *the set* $W^u_{\mathrm{loc}}(\hat{y}) \cap B(x_0, r)$ *is connected and the map*

$$B(\hat{x}, \varepsilon_i r) \cap \hat{\Lambda}_i \ni \hat{y} \mapsto W^u_{\mathrm{loc}}(\hat{y}) \cap B(x_0, r)$$

is continuous (*in the Hausdorff topology*);

(3) *If* $\hat{y}$ *and* $\hat{z}$ *belong to* $B(\hat{x}, \varepsilon_i r) \cap \hat{\Lambda}_i$ *then either* $W^u_{\mathrm{loc}}(\hat{y}) \cap B(x_0, r)$ *and* $W^u_{\mathrm{loc}}(\hat{z}) \cap B(x_0, r)$ *coincide or are disjoint*; *in the later case, if* $\hat{y} \in \hat{W}^u(\hat{z})$ *then* $d(y_0, z_0) > 2r_i$;

(4) *If* $\hat{y} \in \hat{\Lambda}_i \cap B(\hat{x}, \varepsilon_i r)$ *then* $W^u_{\mathrm{loc}}(\hat{y})$ *contains the ball of radius* R_i *around* $W^u_{\mathrm{loc}}(\hat{y}) \cap B(x_0, r)$.

4. Conformal measures

The *Ruelle–Perron–Frobenius transfer operator* $\mathcal{L}_\phi : C(M) \to C(M)$ associated to $f : M \to M$ and $\phi : M \to \mathbb{R}$ is the linear operator defined on the space $C(M)$ of continuous functions $g : M \to \mathbb{R}$ by

$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)} g(y).$$

Notice that $\mathcal{L}_\phi g$ is indeed continuous if g is continuous, because f is a local homeomorphism. It is also easy to see that $\mathcal{L}_\phi$ is a bounded operator, relative to the norm of uniform convergence in $C(M)$:

$$\|\mathcal{L}_\phi\| \leqslant \max_{x\in M} \# f^{-1}(x) e^{\sup|\phi|}.$$

The dual operator $\mathcal{L}_\phi^*$ acts on the Borel measures of M by Consider the dual operator $\mathcal{L}_\phi^* : \mathcal{M}(M) \to \mathcal{M}(M)$ acting on the space $\mathcal{M}(M)$ of Borel measures in M by

$$\int g\, d\big(\mathcal{L}_\phi^*\eta\big) = \int (\mathcal{L}_\phi g)\, d\eta$$

for every $g \in C(M)$. Let $\lambda_0 = r(\mathcal{L}_\phi)$ be the spectral radius of $\mathcal{L}_\phi$. In this section we prove the following result:

Theorem 4.1. *There exists $k \geqslant 1$, $r(\mathcal{L}_\phi) = \lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant e^{h(f)+\inf\phi}$ real numbers and expanding conformal probability measures $\nu_0, \nu_1, \dots, \nu_k$ such that*

$$\mathcal{L}_\phi^* \nu_i = \lambda_i \nu_i, \quad \forall 0 \leqslant i \leqslant k, \quad \text{and} \quad \bigcup_{i=0}^{k} \operatorname{supp}(\nu_i) = \overline{H}.$$

Moreover, each ν_i is a non-lacunary Gibbs measure and has a Jacobian with respect to f given by $J_{\nu_i} f = \lambda_i e^{-\phi}$. If f is topologically mixing then ν_0 is an expanding conformal measure such that $\operatorname{supp}\nu_0 = \overline{H} = M$.

4.1. Eigenmeasures of the transfer operator

The following lemma asserts that any positive eigenmeasure for the dual of the Ruelle–Perron–Fröbenius operator is a conformal measure. Its proof is quite standard: see, for instance, [39, Lemma 4.1].

Lemma 4.2. *Suppose ν is a Borel probability such that $\mathcal{L}_\phi^*\nu = \lambda\nu$ for some $\lambda > 0$. Then the Jacobian of ν with respect to f exists and is given by $J_\nu f = \lambda e^{-\phi}$.*

The proof of the next lemma is analogous to [39, Lemma 4.2].

Lemma 4.3. *The spectral radius λ_0 of the operator $\mathcal{L}_\phi$ is at least $e^{h(f)+\inf\phi}$ and it is an eigenvalue for the dual operator $\mathcal{L}_\phi^*$.*

Proof. Observe that, for every positive integer n and every $x \in M$,

$$\mathcal{L}_\phi^n 1(x) = \sum_{f^n(y)=x} e^{S_n\phi(y)} \geqslant \deg_x\big(f^n\big) e^{n\inf\phi}.$$

So, the spectral radius is at least $e^{h(f)+\inf\phi}$, as claimed in the first part of the lemma. The second part follows from general results in functional analysis. Let C^+ be the open convex cone of positive continuous functions on M and consider the linear subspace

$$N = \big\{\mathcal{L}_\phi g - \lambda_0 g\colon\ g \in C(M)\big\}.$$

Notice that these sets are disjoint. Indeed, assuming otherwise then there exists some continuous function $g \in C(M)$ such that $\mathcal{L}_\phi g - \lambda_0 g$ is a strictly positive continuous function. By compactness and continuity, there is $\varepsilon > 0$ such that $\mathcal{L}_\phi g \geqslant (\lambda_0 + \varepsilon) g$. Since $\mathcal{L}_\phi$ is a positive operator, it is clear that

$$\mathcal{L}_\phi^n g \geqslant (\lambda_0 + \varepsilon)^n g \quad \text{for every } n \geqslant 1.$$

This shows that the spectral radius of $\mathcal{L}_\phi$ is at least $\lambda_0 + \varepsilon$, contradicting the definition of λ_0. This contradiction proves that $C^+ \cap N = \emptyset$, as we claimed. Then, as a consequence of geometric Hahn–Banach theorem there exists some continuous linear functional $\nu_0 : C(M) \to \mathbb{R}$ such that

$$\int g\,d\nu_0 > 0 \quad \text{for every } g \in C^+ \quad \text{and} \quad \int g\,d\nu_0 = 0 \quad \text{for every } g \in N.$$

The first property means that ν_0 is a measure and so, up to normalization, we may suppose it is a probability. The second property means that

$$\int g\,d\big(\mathcal{L}_\phi^* \nu_0\big) = \int \mathcal{L}_\phi g\,d\nu_0 = \lambda_0 \int g\,d\nu_0 \quad \text{for every } g \in C(M),$$

that is, $\mathcal{L}_\phi^* \nu = \lambda_0 \nu_0$. Thus, λ_0 is indeed an eigenvalue for the dual operator $\mathcal{L}_\phi^*$. □

Throughout, let λ denote a fixed eigenvalue of $\mathcal{L}_\phi^*$ larger than $e^{h(f)+\inf\phi}$, let ν be any eigenmeasure of $\mathcal{L}_\phi^*$ associated to λ and set $P = \log\lambda$. The only property of λ that we shall use is that $\lambda > e^{\log q + \sup\phi + \varepsilon_0}$. From Lemma 4.2 we get that

$$J_\nu f(x) = \lambda_0 e^{-\phi(x)} > e^{\log q + \varepsilon_0} > q \quad \text{for all } x \in M. \tag{4.1}$$

This property will allow us to prove that ν-almost every point spends at most a fraction γ of time inside the domain $\mathcal{A}$ where f may fail to be expanding. As we will see later, in Lemma 6.5, $\log\lambda = P_{\mathrm{top}}(f,\phi)$. This determines completely the spectral radius of $\mathcal{L}_\phi$ as the *unique* eigenvalue of $\mathcal{L}_\phi^*$ larger than the lower bound above. Consequently all the eigenvalues λ_i given by Theorem 4.1 are equal and coincide with $\lambda_0 = r(\mathcal{L}_\phi)$ and $\frac{1}{k}\sum_{j=0}^{k} \nu_i$ is an expanding conformal measure whose support coincides with the closure of the set H. The later is the conformal measure referred at Theorem B.

4.2. Expanding structure

Here we prove that any eigenmeasure ν as above is expanding and has integrable first hyperbolic time. Given $n \geqslant 1$, let $B(n)$ denote the set of points $x \in M$ whose frequency of visits to $\mathcal{A}$ up to time n is at least γ, that is,

$$B(n) = \left\{ x \in M\colon \ \frac{1}{n} \# \left\{ 0 \leqslant j \leqslant n-1\colon \ f^j(x) \in \mathcal{A} \right\} \geqslant \gamma \right\}.$$

Proposition 4.4. *The measure $\nu(B(n))$ decreases exponentially fast as n goes to infinity. Consequently, ν-almost every point belongs to $B(n)$ for at most finitely many values of n.*

Proof. The strategy is to cover $B(n)$ by elements of the covering $\mathcal{P}^{(n)} = \bigvee_{j=0}^{n-1} f^{-j}\mathcal{P}$ which, for convenience, will be referred to as cylinders. Then, the estimate relies on an upper bound for the measure of each cylinder, together with an upper bound on the number of cylinders corresponding to large frequency of visits to $\mathcal{A}$.

Since f^n is injective on every $P \in \mathcal{P}^{(n)}$ then we may use (4.1) to conclude that

$$1 \geqslant \nu\big(f^n(P)\big) = \int_P J_\nu f^n\,d\nu = \int_P \prod_{j=0}^{n-1} \big(J_\nu f \circ f^j\big)\,d\nu \geqslant e^{(\log q + \varepsilon_0)n} \nu(P).$$

This proves that $\nu(P) \leqslant e^{-(\log q + \varepsilon_0)n}$ for every $P \in \mathcal{P}^n$. Since $B(n)$ is contained in the union of cylinders $P \in \mathcal{P}^n$ associated to itineraries in $I(\gamma, n)$, we deduce from our choice of γ after Lemma 3.1 that

$$\nu\big(B(n)\big) \leqslant \#I(\gamma,n) e^{-(\log q + \varepsilon_0)n} \leqslant e^{-\varepsilon_0 n/2},$$

for every large n. This proves the first statement in the lemma. The second one is a direct consequence, using the Borel–Cantelli lemma. □

Corollary 4.5. *The measure ν is expanding and satisfies $\int n_1\,d\nu < \infty$.*

Proof. By Proposition 4.4, almost every point x is outside $B(n)$ for all but finitely many values of n. Then, in view of our choice (3.2),

$$\sum_{j=0}^{n-1} \log L\big(f^j(x)\big) \leqslant \gamma \log L + (1-\gamma)\log \sigma^{-1} \leqslant -2c$$

if n is large enough. In view of Lemma 3.2, this proves that ν-almost every point has infinitely many hyperbolic times (positive density at infinity). In other words, ν is expanding. Moreover, using Proposition 4.4 once more,

$$\int n_1\,d\nu = \sum_{n=0}^{\infty} \nu\big(\{x\colon n_1(x) > n\}\big) \leqslant 1 + \sum_{n=1}^{\infty} \nu\big(B(n)\big) < \infty,$$

as we claimed. □

4.3. Gibbs property

Now we prove that ν satisfies a Gibbs property at hyperbolic times. Later we shall see that hyperbolic times form a non-lacunary sequence, almost everywhere, and then it will follow that ν is a non-lacunary Gibbs measure.

Lemma 4.6. *The support of ν is an f-invariant set contained in the closure of H. For any $\rho > 0$ there exists $\xi > 0$ such that $\nu(B(x,\rho)) \geqslant \xi$ for every $x \in \operatorname{supp}(\nu)$.*

Proof. Since ν is expanding, it is clear $\operatorname{supp}(\nu) \subset \overline{H}$. Let $x \in M$. Since f is a local homeomorphism, the relation $V = f(W)$ is a one-to-one correspondence between small neighborhoods W of x and small neighborhood V of $f(x)$. Moreover,

$$\nu(V) = \int_W J_\nu f\,d\nu.$$

is positive if and only if $\nu(W) > 0$, because the Jacobian is bounded away from zero and infinity. This proves that the support is invariant by f. The second claim in the lemma is standard. Assume, by contradiction, that there exists $\rho > 0$ and a sequence $(x_n)_{n\geqslant 1}$ in $\operatorname{supp}(\nu)$ such that $\nu(B(x_n,\rho)) \to 0$ as $n \to \infty$. Since $\operatorname{supp}(\nu)$ is compact set, the sequence must accumulate at some point $z \in \operatorname{supp}(\nu)$. Then

$$\nu\big(B(z,\rho)\big) \leqslant \liminf_{n\to\infty} \nu\big(B(x_n,\rho)\big) = 0,$$

which contradicts $z \in \operatorname{supp}(\nu)$. This completes the proof of the lemma. □

Lemma 4.7. *There exists $K > 0$ such that, if n is a hyperbolic time for $x \in \operatorname{supp}(\nu)$ then*

$$K^{-1} \leqslant \frac{\nu(B(x,n,\delta))}{e^{-Pn+S_n\phi(y)}} \leqslant K,$$

for every $y \in B(x,n,\delta)$.

Proof. Since $f^n \mid B(x,n,\delta)$ is injective, we get from the previous lemma that

$$\xi(\delta) \leqslant \nu\big(B\big(f^n(x),\delta\big)\big) = \int_{B(x,n,\delta)} J_\nu f^n\,d\nu \leqslant 1$$

for every $x \in \operatorname{supp}(\nu)$. Then, the bounded distortion property in Corollary 3.5 applied to the Hölder continuous function $J_\nu f = \lambda e^{-\phi}$ gives that

$$K_0^{-1}\xi(\delta) \leqslant \nu\big(B(x,n,\delta)\big)\lambda^n e^{-S_n\phi(y)} \leqslant K_0$$

for every $y \in B(x,n,\delta)$. Recalling that $P = \log\lambda$, this gives the claim with $K = K_0\xi(\delta)^{-1}$. □

Remark 4.8. The same proof gives a somewhat stronger result: for ν-almost every x and any $0 < \varepsilon \leqslant \delta$, there exists $K(\varepsilon) > 0$ such that

$$K^{-1}(\varepsilon) \leqslant \frac{\nu(B(x,n,\varepsilon))}{e^{-Pn+S_n\phi(x)}} \leqslant K(\varepsilon)$$

if n is a hyperbolic time for x. It suffices to take $K(\varepsilon) = K_0\xi(\varepsilon)^{-1}$.

We proceed with the proof of Theorem 4.1. We have proven that any eigenmeasure ν for $\mathcal{L}_\phi$ associated to an eigenvalue $\lambda \geqslant e^{h(f)+\inf\phi}$ is necessarily expanding, satisfies the Gibbs property at hyperbolic times and has a Jacobian $J_\nu f = \lambda e^{-\phi}$. Furthermore, Lemma 4.3 guarantees that the spectral radius λ_0 is an eigenvalue of the operator $\mathcal{L}_\phi$. Let ν_0 denote any such eigenmeasure. If f is topologically mixing then $\operatorname{supp}\nu_0 = \overline{H} = M$. Indeed, given an open set U there exists $N \geqslant 1$ such that $f^N(U) = M$. Since $J_{\nu_0} f$ is bounded from zero and infinity then clearly $\nu_0(U) > 0$, which proves our claim. Hence, to prove Theorem 4.1 we are left to show that there are finitely many eigenmeasures of $\mathcal{L}_\phi^*$ associated to eigenvalues greater or equal to $e^{h(f)+\inf\phi}$ whose union of their supports coincide with $\overline{H}$. Given an f-invariant compact set Λ we denote by $\mathcal{L}_\Lambda : C(\Lambda) \to C(\Lambda)$ the restriction of the operator $\mathcal{L}_\phi$ to the space of continuous functions $C(\Lambda)$.

Lemma 4.9. *There are finitely many $\lambda_0 \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_k \geqslant e^{h(f)+\inf\phi}$ and probability measures $\nu_0, \nu_1, \ldots, \nu_k$ such that $\mathcal{L}_\phi^* \nu_i = \lambda_i \nu_i$, for every $0 \leqslant i \leqslant k$, and that the union of their supports coincides with the closure of the set H.*

Proof. We obtain the desired finite sequence of conformal measures using the ideas involved in the proof of Lemma 4.3 recursively. Indeed, Lemma 4.3, Corollary 4.5 and Lemma 4.7 assert that there exists an expanding conformal measure ν_0 such that $\mathcal{L}_\phi^* \nu_0 = \lambda_0 \nu_0$ and satisfies the Gibbs property at hyperbolic times. Clearly $\operatorname{supp}(\nu_0)$ is an invariant set contained in $\overline{H}$.

If $\operatorname{supp}(\nu_0) = \overline{H}$ then we are done. Otherwise we proceed as follows. As we shall see in Lemma 5.3, the interior of the support of any expanding conformal measure ν is non-empty and contains almost every point in a ball of radius δ (depending only on f and c). Consider the non-empty compact invariant set $K_1 = M \setminus \mathit{interior}(\operatorname{supp}(\nu_0))$ and set $\lambda_1 = r(\mathcal{L}_{K_1}) \leqslant \lambda_0$. It is easy to check that $\lambda_1 \geqslant e^{h(f)+\inf\phi}$. Then we may argue as in the proof of Lemma 4.3: the cone of strictly positive functions in K_1 is disjoint from the subspace $\{\mathcal{L}_\phi g - \lambda g\colon\ g \in C(K_1)\}$ and so there exists a probability measure ν_1 such that $\mathcal{L}_\phi^* \nu_1 = \lambda_1 \nu_1$ whose support $\operatorname{supp}(\nu_1)$ is contained in K_1. Since $\lambda_1 \geqslant e^{h(f)+\inf\phi}$ then ν_1 is also expanding and its support must also contain a ball of radius δ in its interior.

Since M is compact this procedure will finish after a finite number of times. Hence there are finitely many compact sets $K_0, \ldots, K_k$ and expanding measures $\nu_0, \ldots, \nu_k$ such that $\operatorname{supp}(\nu_i) \subset K_i$ and $\overline{H} = \bigcup_i \operatorname{supp}(\nu_i)$. This completes the proof of the lemma. □

For any conformal measure ν_i as above, we prove in Proposition 5.1 that there are finitely many invariant ergodic measures that are absolutely continuous with respect to ν_i, that their densities are bounded from above and that their basins cover ν_i-almost every point. Hence, the non-lacunarity of the sequence of hyperbolic times will be a consequence of Lemma 3.7. So, up to the proof of Proposition 5.1, this shows that each ν_i is a non-lacunary Gibbs measure and completes the proof of Theorem 4.1.

5. Absolutely continuous invariant measures

In this section we analyze carefully the Cesaro averages

$$\nu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j \nu,$$

and prove that every weak* accumulation point is absolutely continuous with respect to ν. It is well known, and easy to check, that the accumulation points are invariant probabilities. In the topologically mixing setting we also prove that there is a unique absolutely continuous invariant measure and that it satisfies the non-lacunar Gibbs property. The precise statement is

Proposition 5.1. *There are finitely many invariant, ergodic probability measures $\mu_1, \mu_2, \ldots, \mu_k$ that are absolutely continuous with respect to ν and such any absolutely continuous invariant measure is a convex linear combination of $\mu_1, \mu_2, \ldots, \mu_k$. In addition, the measures μ_i are expanding and the densities $d\mu_i/d\nu$ are bounded away from infinity. Moreover, the union of the basins $B(\mu_i)$ cover ν-almost every point in M. If f is topologically mixing then there is a unique absolutely continuous invariant measure and it is a non-lacunary Gibbs measure.*

5.1. Existence and finitude

First we prove that every accumulation point of $(\nu_n)_{n\geqslant 1}$ is absolutely continuous invariant measure with bounded density. For every $n \in \mathbb{N}$ it holds that

$$H_n^c \subset \{n_1(\cdot) > n\} \cup \left[\bigcup_{k=0}^{n-1} H_k \cap f^{-k}\big(\{n_1(\cdot) > n-k\}\big)\right].$$

In particular, we can use the inclusion above to write

$$\nu_n \leqslant \mu_n + \frac{1}{n}\sum_{j=0}^{n-1} \eta_j,$$

where

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\nu \mid H_j) \quad \text{and} \quad \eta_j = \sum_{l=0}^{\infty} f_*^l\big(f_*^j(\nu \mid H_j) \bigm| \{n_1 > l\}\big).$$

Lemma 5.2. *There exists $C_2 > 0$ such that for every positive integer n the measures $f_*^n(\nu \mid H_n)$, μ_n and ν_n are absolutely continuous with respect to ν with densities bounded from above by C_2. Moreover, the same holds for every weak* accumulation point μ of $(\nu_n)_{n\geqslant 1}$.*

Proof. Let A be any measurable set of small diameter, say $\operatorname{diam}(A) < \delta/2$, and such that $\nu(A) > 0$. First we claim that there is $C_2 > 0$ such that

$$f_*^n(\nu \mid H_n)(A) \leqslant C_2 \nu(A), \quad \forall n \geqslant 1.$$

Observe that either $f_*^n(\nu \mid H_n)(A) = 0$, or A is contained in a ball $B = B(f^n(x), \delta)$ of radius δ for some $x \in H_n$. In the first case we are done. In the later situation we compute

$$f_*^n(\nu \mid H_n)(A) = \nu\big(f^{-n}(A) \cap H_n\big) = \sum_i \nu\big(f_i^{-n}(A \cap B)\big),$$

where the sum is over all hyperbolic inverse branches $f_i^{-n} : B \to V_i$ for f^n. Recall that the ν-measure of any positive measure ball of radius δ is at least $\xi(\delta) > 0$ by Lemma 4.6. Thus, by bounded distortion

$$f_*^n(\nu \mid H_n)(A) \leqslant K_0 \sum_i \frac{\nu(A)}{\nu(B)} \nu(V_i) \leqslant K_0 \xi(\delta)^{-1} \nu(A),$$

which proves our claim with $C_2 = K_0\xi(\delta)^{-1}$. It follows from the arbitrariness of A that both $f_*^n(\nu \mid H_n)$ and μ_n are absolutely continuous with respect to ν with density bounded from above by C_2.

Similar estimates on the density of η_n hold using that $\{n_1 > n\} \subset B(n)$, there are at most $e^{c_\gamma n}$ cylinders in $B(n)$, and that $J_\nu f^n > e^{(\log q + \varepsilon_0)n}$ on each of one of them. Indeed,

$$\big((f_*^l \nu) \bigm| \{n_1 > l\}\big)(A) \leqslant \sum_{\substack{P \in \mathcal{P}^{(l)} \\ P \cap B(l) \neq \emptyset}} \nu\big(f^{-l}(A) \cap P\big) \leqslant \#B(l) e^{-(\log q + \varepsilon_0) l} \nu(A)$$

for every $l \geqslant 1$ and every measurable set A. Using that $df_*^n(\nu \mid H_n)/d\nu \leqslant K_0\xi(\delta)^{-1}$ and summing up the previous terms one concludes that

$$\eta_j(A) \leqslant K_0 \xi(\delta)^{-1} \sum_{l=0}^{\infty} e^{-\frac{\varepsilon_0}{4} l} \nu(A), \quad \forall j \geqslant 1.$$

This shows that (up to replace C_2 by a larger constant) the measures ν_n are also absolutely continuous with respect to ν and that $d\nu_n/d\nu$ is bounded from above by C_2. The second assertion in the lemma is an immediate consequence by weak* convergence. □

The following lemma, whose proof explores the generating property of hyperbolic pre-balls, plays a key role in proving finitude of equilibrium states.

Lemma 5.3. *If G is an f-invariant set such that $\nu(G) > 0$ then there is a disk Δ of radius $\delta/4$ so that $\nu(\Delta \setminus G) = 0$.*

Proof. In the case that ν coincides with the Lebesgue measure this corresponds to [3, Lemma 5.6]. Since the argument will be used later on, we give a brief sketch of the proof.

Let $\varepsilon > 0$ be small. Take a compact K and an open set O such that $K \subset G \cap H \subset O$ and $\nu(O \setminus K) < \varepsilon\nu(K)$. Set $n_0 \in \mathbb{N}$ such that $B(x, n, \delta) \subset O$ for any $x \in K \cap H_n$. If $n(x)$ denotes the first hyperbolic time of x larger than n_0 then

$$K \subset \bigcup_{x\in K} B\big(x, n(x), \delta/4\big) \subset O.$$

Set $V(x) = B(x, n(x), \delta)$ and $W(x) = B(x, n(x), \delta/4)$. Since K is compact it is covered by finite open sets $(W(x))_{x\in X}$ for some family $X = \{x_1, \dots, x_k\}$. Now we proceed recursively and define

$$n_1 = \inf\big\{n(x)\colon x \in X\big\} \quad \text{and} \quad X_1 = \big\{x \in X\colon n(x) = n_1\big\}$$

and, assuming that n_i and X_i are well defined for $1 \leqslant i \leqslant m-1$, set

$$n_m = \inf\big\{n(x)\colon x \in X \setminus (X_1 \cup \dots \cup X_{m-1})\big\} \quad \text{and} \quad X_m = \big\{x \in X\colon n(x) = n_m\big\}$$

up to some finite positive integer s. Let $\tilde X_1 \subset X_1$ be a maximal family of points with pairwise disjoint $W(\cdot)$ elements. Moreover, given $\tilde X_i \subset X_i$ for $1 \leqslant i \leqslant m-1$ let $\tilde X_m \subset X_m$ maximal such that every $W(x)$, $x \in \tilde X_m$, does not intersect any element $W(y)$ for some $y \in \tilde X_1 \cup \dots \cup \tilde X_m$. If $\tilde X = \bigcup\{\tilde X_i\colon 1 \leqslant i \leqslant s\}$ then the dynamical balls $W(x)$, $x \in \tilde X$, are pairwise disjoint (by construction). It is also easy to see that for every $y \in X$ there exists $x \in \tilde X$ such that $W(y) \subset V(x)$. Hence

$$\nu\bigg(\bigcup_{x\in\tilde X} W(x) \setminus K\bigg) \leqslant \nu(O \setminus K) < \varepsilon\nu(K)$$

and, by the bounded distortion property,

$$\nu\bigg(\bigcup_{x\in\tilde X} W(x)\bigg) \geqslant \tau\nu\bigg(\bigcup_{x\in\tilde X} V(x)\bigg)$$

for some $\tau > 0$. We conclude immediately that there exists $x \in \tilde X$ such that

$$\frac{\nu(W(x) \setminus G)}{\nu(W(x))} \leqslant \frac{\nu(W(x) \setminus K)}{\nu(W(x))} < \tau^{-1}\varepsilon.$$

Using the bounded distortion of f^n restricted to the dynamical ball $W(x)$ once more it follows that

$$\nu\big(B \setminus f^n(G)\big) < \tau^{-1}K_0\varepsilon,$$

where B is a ball of radius $\delta/4$ around $f^n(x)$. Since ε was arbitrary and G is invariant then there exists a sequence Δ_n of balls of radius $\delta/4$ such that $\nu(\Delta_n \setminus G) \to 0$ as $n \to \infty$. By compactness, the sequence $(\Delta_n)_n$ accumulate on a ball Δ that satisfies the requirements of the lemma. □

We are now in a position to show that there are finitely many distinct ergodic measures $\mu_1, \mu_2, \dots, \mu_k$ absolutely continuous with respect to ν. Indeed, let μ be any invariant measure that is absolutely continuous. Then, either μ is ergodic or there are disjoint invariant sets I_1 and I_2 of positive ν-measure such that $\mu(\cdot) = a_1\mu(\cdot \cap I_1)/\mu(I_1) + a_2\mu(\cdot \cap I_2)/\mu(I_2)$, where $a_i = \mu(I_i)$. In the later case it is also clear that each of the measures involved in the sum is absolutely continuous with respect to ν. Repeating the process one obtains that μ can be written as linear convex combination of ergodic absolutely continuous invariant measures $\mu_1, \mu_2, \dots, \mu_k$. Indeed, since M is compact the previous lemma implies that this process will stop after a finite number of steps (depending only on δ) with each μ_i ergodic. It is also clear from the construction that each μ_i is expanding and that their basins cover almost every point.

5.2. Invariant non-lacunary Gibbs measure

Through the rest of this section assume that f is topologically mixing. Here we prove that there is a unique invariant measure μ absolutely continuous with respect to ν and that it is a non-lacunary Gibbs measure. This will complete the proof of Proposition 5.1. We begin with a couple of auxiliary lemmas. Let $\theta > 0$ and $\delta > 0$ be given by Lemmas 3.2 and 3.4.

Lemma 5.4. *There exists a constant $\tau_0 > 0$, and for any n there is a finite subset $\hat{H}_n$ of H_n such that the dynamical balls $B(x, n, \delta/4)$, $x \in \hat{H}_n$, are pairwise disjoint and their union W_n satisfies $\nu(W_n) \geqslant \tau_0 \nu(H_n)$.*

Proof. This lemma is a direct consequence of Lemma 3.4 in [3]. Indeed, if $\omega = f_*^n(\nu \mid \bigcup\{B(n, x, \delta/4)\colon\ x \in H_n\})$, $\Omega = f^n(H_n) = M$ and $r = \delta$ in that lemma then there exists a finite set $I \subset f^n(H_n)$ such that the pairwise disjoint union Δ_n of balls of radius $\delta/4$ around points in I satisfies

$$\omega\big(\Delta_n \cap f^n(H_n)\big) \geqslant \tau_0 \omega\big(f^n(H_n)\big).$$

Set $\hat{H}_n = H_n \cap f^{-n}(I)$. As the restriction of f^n to any dynamical ball $B(x, n, \delta/4)$, $x \in \hat{H}_n$ is a bijection it is easy to see that these dynamical balls are pairwise disjoint. Furthermore, their union W_n satisfies $\nu(W_n) \geqslant \tau_0 \nu(H_n)$. This completes the proof of the lemma. □

In the remaining of the section, let μ be an arbitrary accumulation point of the sequence $(\nu_n)_n$ and $(n_k)_k$ be a subsequence of the integers such that

$$\mu = \lim_{k\to\infty} \nu_{n_k}.$$

In the next lemmas we prove that the density $d\mu/d\nu$ is bounded away from zero in some small disk and use this to deduce the uniqueness of the equilibrium state and the non-lacunar Gibbs property.

Lemma 5.5. *There exists $C_1 > 0$ and a small disk $D(x)$ around a point x in M such that the density $d\mu/d\nu$ in the disk $D(x)$ is bounded from below by C_1.*

Proof. Given a small $\varepsilon > 0$ we construct a disk $D(x)$ of radius smaller than ε where the assertion above holds. Let W_j and $\hat{H}_j$ be given by the previous lemma and let $W_{j,\varepsilon} \subset W_j$ denote the preimages by f^j of the disks $\Delta_{j,\varepsilon}$ of radius $\delta/4 - \varepsilon$ around points in $f^j(\hat{H}_j)$. Lemma 3.5 implies that

$$\frac{\nu(W_{j,\varepsilon})}{\nu(W_j)} \geqslant K_0^{-1} \frac{\nu(\Delta_{j,\varepsilon})}{\nu(\Delta_j)},$$

where the right-hand side is larger than some uniform positive constant τ_1 that depends only on the radius of the disks $\Delta_{j,\varepsilon}$ (recall Lemma 4.6). Observe also that Corollary 3.3 with $A = M$ implies that

$$\frac{1}{n} \sum_{j=0}^{n-1} \nu(H_j) \geqslant \theta/2$$

for every large n. This shows that there is a positive constant τ_2 such that the measures ν_n^ε satisfy $\nu_n^\varepsilon(M) \geqslant \tau_2$ for every large n, where

$$\nu_n^\varepsilon = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j(\nu \mid W_{j,\varepsilon}).$$

Thus, there exists a subsequence of $(\nu_{n_k}^\varepsilon)_k$ that converge to some measure ν_∞^ε and

$$\operatorname{supp}(\nu_\infty^\varepsilon) \subset \bigcap_{n\geqslant 1} \Big(\overline{\bigcup_{j\geqslant n} \Delta_{j,\varepsilon}}\Big).$$

Choose $x \in \operatorname{supp}(\nu_\infty^\varepsilon)$ and a disk $D(x)$ of radius smaller than ε around x such that $\nu_\infty^\varepsilon(\partial D(x)) = 0$. By construction, $D(x)$ is contained in every disk of Δ_j such that the corresponding disk of $\Delta_{j,\varepsilon}$ intersects $D(x)$. Let $\tilde{\Delta}_j$ denote the pairwise disjoint union of disks in Δ_j that contain $D(x)$ and $\tilde{W}_j$ be defined accordingly as the preimages of $\tilde{\Delta}_j$. It is clear that $\nu_n \geqslant \nu_n^0$, where

$$\nu_n^0 = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j(\nu \mid \tilde{W}_j).$$

Moreover, since $df_*^j(\nu \mid \tilde{W}_j)/d\nu$ is Hölder continuous, the bounded distortion at Lemma 3.5 implies that it is bounded from below by its L^1 norm up to the multiplicative constant K_0^{-1}. So,

$$\frac{d\nu_n^0}{d\nu}(y) = \frac{1}{n}\sum_{j=0}^{n-1}\frac{df_*^j(\nu \mid \tilde{W}_j)}{d\nu}(y) = \frac{1}{n}\sum_{j=0}^{n-1}\Bigg[\sum_{\substack{f^j(z)=y\\ z\in\tilde{W}_j}} \lambda^{-j} e^{S_j\phi(z)}\Bigg] \geqslant K_0^{-1}\frac{1}{n}\sum_{j=0}^{n-1}\nu(\tilde{W}_j)$$

for every $y \in D(x)$. Furthermore, by construction the set $W_{j,\varepsilon} \cap f^{-j}(D(x))$ is contained in $\tilde{W}_j$. This guarantees that

$$\frac{d\nu_n^0}{d\nu}(y) \geqslant K_0^{-1}\frac{1}{n}\sum_{j=0}^{n-1}\nu(\tilde{W}_j) \geqslant K_0^{-1}\nu_n^\varepsilon\big(D(x)\big) \geqslant K_0^{-1}\frac{\nu_\infty^\varepsilon(D(x))}{2}$$

for every large $n \geqslant 1$ in the subsequence of $(n_k)_k$ chosen above. By weak* convergence it holds that $d\mu/d\nu \geqslant C_1$ in the disk $D(x)$. □

We finish this section by proving the uniqueness of the equilibrium state, which completes the proof of Proposition 5.1.

Lemma 5.6. *If f is topologically mixing there is a unique invariant measure μ absolutely continuous with respect to ν. Moreover, the density $d\mu/d\nu$ is bounded away from zero and infinity and the sequences of hyperbolic times $\{n_j(x)\}$ are non-lacunary μ-almost everywhere. Furthermore, μ is a non-lacunary Gibbs measure.*

Proof. We have proven that any accumulation point μ of $(\nu_n)_n$ is absolutely continuous with respect to ν and that the density $h = d\mu/d\nu$ is bounded from above by C_2 and is bounded from below by C_1 on some disk $D(x)$. Since f is topologically mixing there is $N \geqslant 1$ be such that $f^N(D(x)) = M$, that is, any point has some preimage by f^N in $D(x)$. It is not difficult to check that $h \in L^1(\nu)$ satisfies $\mathcal{L}_\phi h = \lambda h$. Then

$$h(y) = \lambda^{-N}\sum_{f^N(z)=y} e^{S_N\phi(z)}h(z) \geqslant C_1\lambda^{-N}e^{N\inf\phi}$$

for almost every $y \in M$, which allows to deduce that the measures μ and ν are equivalent.

We claim that μ is ergodic. Indeed, if G is any f-invariant set such that $\mu(G) > 0$ then it follows from Lemma 5.3 that there is a disk Δ of radius $\delta/4$ such that $\nu(\Delta \setminus G) = 0$. Furthermore, using that $J_\nu f$ is bounded from above and from below, the invariance of G and that there is $\tilde{N} \geqslant 1$ such that $f^{\tilde{N}}(\Delta) = M$ it follows that $\nu(M \setminus G) = 0$, or equivalently, that $\mu(G) = 1$, proving our claim. So, if $\mu_1 \ll \nu$ is any f-invariant probability measure then $\mu_1 \ll \mu$. By invariance of $d\mu_1/d\mu$ and ergodicity of μ it follows that $d\mu_1/d\mu$ is almost everywhere constant and that $\mu_1 = \mu$. This proves the uniqueness of the absolutely continuous invariant measure. Lemma 5.2 also implies that

$$C_3\nu\big(B(x,n,\delta)\big) \leqslant \mu\big(B(x,n,\delta)\big) \leqslant C_2\nu\big(B(x,n,\delta)\big)$$

for ν-almost every x and every $n \geqslant 1$, where $C_3 = C_1\lambda^{-N}e^{N\inf\phi}$. In particular μ is expanding and, if n is a hyperbolic time for x and $y \in B(x,n,\delta)$ then

$$K^{-1}C_3 \leqslant \frac{\mu(B(x,n,\delta))}{e^{-Pn+S_n\phi(y)}} \leqslant KC_2.$$

Corollary 4.5 implies that the first hyperbolic time map n_1 is μ_i-integrable. Hence, the sequence of hyperbolic times is almost everywhere non-lacunary (see Corollary 3.8) and both μ and ν are non-lacunary Gibbs measures. This completes the proof of the lemma. □

6. Proof of Theorems A and B

In this section we manage to estimate the topological entropy of f for the potential ϕ using the characterizations of relative pressure given in Section 3.4: $P_{H^c}(f,\phi) < \log\lambda$ and $P_H(f,\phi) \leqslant \log\lambda$. Then, using that the measure theoretical pressure $P_\mu(f,\phi) = h_\mu(f) + \int \phi\, d\mu$ of every absolutely continuous invariant measure given by Proposition 5.1 is at least $\log\lambda$, we deduce that $P_{\text{top}}(f,\phi) = \log\lambda$ and that equilibrium states do exist. Finally, the variational property of equilibrium states yields that they coincide with the absolutely continuous invariant measures. This will complete the proofs of Theorems A and B.

6.1. Existence of equilibrium states

We give two estimates on the relative pressure and deduce the existence of equilibrium states for f with respect to ϕ.

Proposition 6.1. $P_{H^c}(f,\phi) < \log\lambda$.

Since we deal with a potential ϕ whose oscillation is not very large, the main point in the proof of Proposition 6.1 is to control the relative entropy $h_{H^c}(f)$. The key idea is that $h_{H^c}(f)$ can be bounded above using the maximal distortion and growth rate of the inverse branches that cover H^c. We will begin with some preparatory lemmas.

Lemma 6.2. *Let M be a compact Besicovitch metric space of dimension m. There exists $C > 0$ and a sequence of finite open coverings $(\mathcal{Q}_k)_{k\geqslant 1}$ of M such that* $\operatorname{diam}(\mathcal{Q}_k) \to 0$ *as $k \to \infty$, and every set $E \subset M$ satisfying* $\operatorname{diam}(E) \leqslant D \operatorname{diam} \mathcal{Q}_k$ *intersects at most $C D^m$ elements of $\mathcal{Q}_k$.*

Proof. First we construct a special family $\mathcal{T}_k$ of partitions in M. Let (r_k) be a decreasing sequence of positive numbers converging to zero. Given $k \geqslant 1$, let X_k be a maximal r_k separated set: any two balls of radius r_k centered at distinct points in X_k are pairwise disjoint and X_k is a maximal set with this property. In particular, it follows that $\{B(x, 2r_k)\colon x \in X_k\}$ is a covering of M. Since there exists no covering of M by a smaller number of balls as above, by Besicovitch covering lemma there exists a constant C_1 (depending only on the dimension m) that any point in M is contained in at most C_1 balls. Consider a partition $\mathcal{T}_k$ in M such that every element $T_k \in \mathcal{T}_k$ contains a ball of radius r_k and such that $\operatorname{diam}(\mathcal{T}_k) \leqslant 2r_k$.

Fix a sequence of positive numbers $(\varepsilon_k)_{k\geqslant 1}$ such that $0 < \varepsilon_k \ll r_k$ for every $k \geqslant 1$. We claim that the family $\mathcal{Q}_k$ of open neighborhoods of size ε_k around elements of $\mathcal{T}_k$ satisfies the requirements of the lemma. It is immediate that $\operatorname{diam}(\mathcal{Q}_k) \to 0$ as $k \to \infty$. Since, by construction, every point in M is contained in at most C_1 elements of $\mathcal{T}_k$, any set $E \subset M$ satisfying $\operatorname{diam}(E) < D \operatorname{diam}(\mathcal{Q}_k) \leqslant 2D(r_k + \varepsilon_k)$ can intersect at most $[2C_1 D(1 + \varepsilon_k/r_k))]^m$ elements of $\mathcal{Q}_k$. This shows that E can intersect at most $C D^m$ elements of $\mathcal{Q}_k$ for some constant C depending only on the dimension m, completing the proof of the lemma. □

The next result is the most technical lemma in the article and provides the key estimate to prove Proposition 6.1.

Lemma 6.3. *Given any $\ell \geqslant 1$ the following property holds*:

$$h_{H^c}\big(f^\ell\big) \leqslant (\log q + m \log L + \varepsilon_0/2)\ell + \log C.$$

Proof. Fix $\ell \geqslant 1$ and let $(\mathcal{Q}_k)_k$ be the family of finite open coverings given by the previous lemma. Since $\operatorname{diam}(\mathcal{Q}_k) \to 0$ as $k \to \infty$ then

$$P_{H^c}(f,\phi) = \lim_{k\to\infty} P_{H^c}(f,\phi,\mathcal{Q}_k),$$

by Definition 3.9. Recall $\mathcal{P}$ is the finite covering given by (H2) and $B(n,\gamma)$ is the set of points whose frequency of visits to $\mathcal{A}$ up to time n is at least γ. The starting point is the next observation:

Claim 1. *For every $0<\varepsilon<\gamma$ there exists $j_0\geqslant 1$ such that for every $j\geqslant j_0$ the following holds*:

$$B(n,\gamma)\subset B(\ell j,\gamma-\varepsilon)\quad \textit{for every } j\ell\leqslant n<(j+1)\ell.$$

Proof of Claim 1. Given $\varepsilon>0$, let j_0 be a positive integer larger than $(1-\gamma)/\varepsilon$. Given an arbitrary large n we can write $n=\ell j+r$, where $0\leqslant r<\ell$ and $j\geqslant j_0$. Moreover, if x belongs to $B(n,\gamma)$ then $\#\{0\leqslant i\leqslant n-1\colon f^i(x)\in\mathcal{A}\}\geqslant\gamma n$ and consequently

$$\frac{1}{\ell j}\#\left\{0\leqslant i\leqslant \ell j-1\colon\ f^i(x)\in\mathcal{A}\right\}\geqslant\gamma+\frac{\gamma r-r}{\ell j}.$$

Our choice of j_0 implies that the right-hand side above is bounded from below by $\gamma-\varepsilon$. This shows that x belongs to $B(\ell j,\gamma-\varepsilon)$ and proves the claim. □

We proceed with the proof of the lemma. Observe that the set H^c is covered by

$$\bigcup_{n\geqslant N}\bigcup_{P\in\mathcal{P}^{(n)}}\left\{P\in\mathcal{P}^{(n)}\colon\ P\cap B(n,\gamma)\neq\emptyset\right\}$$

for every $N\geqslant 1$. Let $\varepsilon>0$ be small such that $\#I(n,\gamma-\varepsilon)\leqslant\exp(\log q+\varepsilon_0/2)n$ for every large n. Such an ε do exist because c_γ varies monotonically on γ (see the proof of Lemma 3.1). Then, the previous claim allow us to cover H^c using only cylinders whose depth is a multiple of ℓ: for any $N\geqslant 1$

$$H^c\subset\bigcup_{j\geqslant\frac{N}{\ell}}\bigcup_{P\in\mathcal{P}^{(\ell j)}}\left\{P\in\mathcal{P}^{(\ell j)}\colon\ P\cap B(\ell j,\gamma-\varepsilon)\neq\emptyset\right\}.\tag{6.1}$$

Thus, from this moment on we will only consider iterates $n=j\ell$. Denote by $\mathcal{R}^{(n)}$ the collection of cylinders in $\mathcal{P}^{(n)}$ that intersect $B(n,\gamma-\varepsilon)$. Our aim is now to cover any element in $\mathcal{R}^{(n)}$ by cylinders relatively to the transformation f^ℓ. Given $k\geqslant 1$, denote by $\mathcal{S}_{f^\ell,j}\mathcal{Q}_k$ the set of j-cylinders of f^ℓ by elements in $\mathcal{Q}_k$, that is

$$\mathcal{S}_{f^\ell,j}\mathcal{Q}_k=\left\{Q_0\cap f^{-\ell}(Q_1)\cap\cdots\cap f^{-\ell(j-1)}(Q_{j-1})\colon\ Q_i\in\mathcal{Q}_k,\ i=0,\dots,j-1\right\}.$$

Furthermore, let $\mathcal{G}_{n,k}$ be the set of cylinders in $\mathcal{S}_{f^\ell,j}\mathcal{Q}_k$ that intersect any element of $\mathcal{R}^{(n)}$.

Claim 2. *Let $k\geqslant 1$ be large and fixed. Then*

$$\#\mathcal{G}_{j\ell,k}\leqslant\#\mathcal{Q}_k\times\left[CL^{\ell m}\right]^j\times e^{(\log q+\varepsilon_0/2)j\ell}$$

for every large j.

Proof of Claim 2. Recall $n=j\ell$ and fix $P_n\in\mathcal{R}^{(n)}$. Since f is a local homeomorphism then the inverse branch $f^{-n}: f^n(P_n)\to P_n$ extends to the union of all $Q\in\mathcal{Q}_k$ so that $Q\cap f^n(P_n)\neq\emptyset$, provided that k is large. Notice that $\operatorname{diam}(f^{-\ell}(Q))\leqslant L^\ell\operatorname{diam}(Q)$ for every $Q\in\mathcal{Q}_k$ because $\log\|Df(x)^{-1}\|\leqslant L$ for every $x\in M$. By Lemma 6.2, $f^{-\ell}(Q)$ intersects at most $CL^{\ell m}$ elements of the covering $\mathcal{Q}_k$. This proves that there are at most $\#\mathcal{Q}_k\times[CL^{\ell m}]^j$ cylinders in $\mathcal{S}_{f^\ell,j}\mathcal{Q}_k$ that intersect P_n. The claim is a direct consequence of our choice of ε since $\#\mathcal{R}^{(n)}\leqslant e^{(\log q+\varepsilon_0/2)n}$ for large n. □

Finally we complete the proof of the lemma. Indeed, it is immediate from (6.1) that

$$m_\alpha\left(f^\ell,0,H^c,\mathcal{Q}_k,N\right)\leqslant\sum_{j\geqslant N/\ell}\sum_{\underline{\mathrm{U}}\in\mathcal{G}_{\ell j,k}}e^{-\alpha n(\underline{\mathrm{U}})}=\sum_{j\geqslant N/\ell}e^{-\alpha j}\,\#\mathcal{G}_{\ell j,k}$$

for every large k. Moreover, Claim 2 implies that the sum in the right-hand side above converges to zero as $N\to\infty$ (independently of k) whenever $\alpha>(\log q+\varepsilon_0/2+m\log L),\ell+\log C$. This shows that $h_{H^c}(f^\ell)\leqslant(\log q+m\log L+\varepsilon_0/2)\ell+\log C$ and completes the proof of the lemma. □

Proof of Proposition 6.1. Recall that $h_{H^c}(f^\ell) = \ell\, h_{H^c}(f)$, by Proposition 3.12. Then, as a consequence of the previous lemma we obtain

$$h_{H^c}(f) \leqslant \log q + m \log L + \varepsilon_0/2 + \frac{\log C}{\ell}$$

for every $\ell \geqslant 1$. Finally, it follows from (3.3) and Lemma 3.14 that

$$P_{H^c}(f,\phi) \leqslant \log q + m \log L + \sup \phi + \varepsilon_0 < h(f) + \inf \phi \leqslant \log \lambda. \qquad \square$$

In the present lemma we give an upper bound on the relative pressure of ϕ relative to the set H. More precisely,

Lemma 6.4. $P_H(f,\phi) \leqslant \log\lambda$.

Proof. Recall the characterization of relative pressure using dynamical balls in Section 3.4. Pick $\alpha > \log\lambda$. For any given $N \geqslant 1$, H is contained in the union of the sets H_n over $n \geqslant N$. Thus, given $0 < \varepsilon \leqslant \delta$

$$H \subset \bigcup_{n\geqslant N} \bigcup_{x \in H_n} B(x,n,\varepsilon).$$

Now we claim that there exists $D > 0$ (depending only on $m = \dim(M)$) so that for every $n \geqslant N$ there is a family $\mathcal{G}_n \subset H_n$ in such a way that every point in H_n is covered by at most D dynamical balls $B(x,n,\varepsilon)$ with $x \in \mathcal{G}_n$. In fact, Besicovitch's covering lemma asserts that there is a constant $D > 0$ (depending on m) and an at most countable family $\mathcal{G}_n \subset H_n$ such that every point of $f^n(H_n)$ is contained in at most D elements of the family $\{B(f^n(x),\varepsilon)\colon x \in \mathcal{G}_n\}$. Using that each dynamical ball $B(x,n,\varepsilon)$, $x \in H_n$, is mapped homeomorphically onto $B(f^n(x),\varepsilon)$, it follows that every point in H_n is contained in at most D dynamical balls $B(x,n,\varepsilon)$ with $x \in \mathcal{G}_n$, proving our claim. Given any positive integer $N \geqslant 1$, it follows by bounded distortion and the Gibbs property of ν at hyperbolic times that

$$m_\alpha(f,\phi,H,\varepsilon,N) \leqslant K(\varepsilon) \sum_{n\geqslant N} e^{-(\alpha-P)n} \biggl\{ \sum_{x\in\mathcal{G}_n} \nu\bigl(B(x,n,\varepsilon)\bigr) \biggr\}.$$

Consequently $m_\alpha(f,\phi,H,\varepsilon,N) \leqslant K(\varepsilon) \frac{D}{1-e^{-(\alpha-P)}} e^{-(\alpha-P)N}$, which tends to zero as $N \to \infty$ independently of ε. This shows that $P_H(f,\phi) \leqslant \log\lambda$ and completes the proof of the lemma. $\square$

We know that every ergodic component of an absolutely continuous invariant measure is also absolutely continuous. Now we prove that the absolutely continuous invariant measures are indeed an equilibrium states.

Lemma 6.5. *If μ is an ergodic measure absolutely continuous with respect to ν then $P_\mu(f,\phi) \geqslant \log\lambda$. Moreover, μ is an equilibrium state for f with respect to ϕ and the following equalities hold*

$$P_{\mathrm{top}}(f,\phi) = P_H(f,\phi) = \log\lambda.$$

Proof. The previous estimates and Proposition 3.11 guarantee that

$$P_{\mathrm{top}}(f,\phi) = \sup\bigl\{P_H(f,\phi), P_{H^c}(f,\phi)\bigr\} \leqslant \log\lambda.$$

Using that $d\mu/d\nu \leqslant C_2$, that ν satisfies the Gibbs property at hyperbolic times and μ-almost every point x admits a sequence $\{n_k(x)\}$ of hyperbolic times then

$$\mu\bigl(B(x,n_k,\varepsilon)\bigr) \leqslant C_2 K(\varepsilon) e^{-P n_k + S_{n_k}\phi(y)}$$

for every $0 < \varepsilon \leqslant \delta$, every $k \geqslant 1$ and every $y \in B(x,n_k,\varepsilon)$. Thus, Brin–Katok's local entropy formula for ergodic measures and Birkhoff's ergodic theorem (see e.g. [36]) immediately imply that

$$h_\mu(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} -\frac{1}{n} \log \mu\bigl(B(x,n,\varepsilon)\bigr) \geqslant P - \int \phi \, d\mu,$$

where the first equality holds μ-almost everywhere. In particular

$$\log\lambda \geqslant P_{\mathrm{top}}(f,\phi) \geqslant P_H(f,\phi) \geqslant \sup_{\mu(H)=1} \biggl\{ h_\mu(f) + \int \phi \, d\mu \biggr\} \geqslant \log\lambda,$$

which proves that μ is an equilibrium state and that the three quantities in the statement of the lemma do coincide. This completes the proof of the lemma. □

6.2. Finitude of ergodic equilibrium states

In this subsection we will complete the proof of Theorems A and B and Corollary C. First we combine that every equilibrium state is an expanding measure with some ideas involved in the proof of the variational properties of SRB measures in [31] to deduce that every equilibrium state is absolutely continuous with respect to some conformal measure supported in the closure of the set H, and to obtain finitude of ergodic equilibrium states. Finally, we show that under the topologically mixing assumption there is a unique equilibrium state, and that it is exact and a non-lacunary Gibbs measure. We begin with the following abstract result:

Theorem 6.6. *Let $f : M \to M$ be a local homeomorphism, $\phi : M \to \mathbb{R}$ be a Hölder continuous potential and ν be a conformal measure such that $J_\nu f = \lambda e^{-\phi}$, where $\lambda = \exp(P_{\mathrm{top}}(f,\phi))$. Assume that η is an equilibrium state for f with respect to ϕ gives full weight to* $\mathrm{supp}(\nu)$ *and that*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} L\big(f^j(x)\big) < 0$$

almost everywhere. Then η is absolutely continuous with respect to ν.

Let us stress out that this theorem holds in a more general setting. Since this fact will not be used here, we will postpone the discussion to Remark 6.15 near the end of the section. The finitude of equilibrium states is a direct consequence of the previous result. Indeed,

Corollary 6.7. *Let f be a local homeomorphism and let ϕ be a Hölder continuous potential satisfying* (H1), (H2) *and* (P). *There exists an expanding conformal probability measure ν such that every equilibrium state for f with respect to ϕ is absolutely continuous with respect to ν with density bounded from above. If, in addition, f is topologically mixing then there is unique equilibrium state and it is a non-lacunary Gibbs measure.*

Proof. Let ν be the expanding conformal measure given by Theorem 4.1 and η be an ergodic equilibrium state for f with respect to ϕ. We claim that η is an expanding measure. Indeed, assume by contradiction that one can decompose η as a linear convex combination of two measures $\eta = t\eta_1 + (1-t)\eta_2$ with $\eta_2(H^c) = 1$ for some $0 \leqslant t < 1$. But Lemma 6.5, the first part of Proposition 3.11 and the convexity of the pressure yield

$$P_\eta(f,\phi) = tP_{\eta_1}(f,\phi) + (1-t)P_{\eta_2}(f,\phi) \leqslant tP_{\mathrm{top}}(f,\phi) + (1-t)P_{H^c}(f,\phi) < P_{\mathrm{top}}(f,\phi),$$

which contradicts that η is an equilibrium state and proves our claim. Moreover, $\eta(\mathrm{supp}(\nu)) = 1$ because the support of ν coincides with the closure of H. Finally, since

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log L\big(f^j(x)\big) \leqslant -2c < 0$$

at η-almost every point (Corollary 6.7), the assumptions of Theorem 6.6 are verified. This result is a direct consequence of the previous theorem and Proposition 5.1. □

In the sequel we prove Theorem 6.6. Since f is a non-invertible transformation we use the natural extension, introduced in Section 3.5, to deal with unstable manifolds.

Proof of Theorem 6.6. It is easy to check, using the variational principle, that almost every ergodic component of an equilibrium state is an equilibrium state. Thus, by ergodic decomposition it is enough to prove the result for ergodic measures.

Let η be an ergodic equilibrium state and $(\hat{f}, \hat{\eta})$ be the natural extension of η introduced in Section 3.5. Then $\pi_* \hat{\eta} = \eta$ and that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \hat{L}\big(\hat{f}^j(\hat{x})\big) < 0$$

$\hat{\eta}$-almost everywhere.

We proceed with the construction of a special partition $\hat{\mathcal{Q}}$ of $\hat{M}$ that is closely related with Ledrappier's geometric construction in Proposition 3.1 of [31] and provides a key ingredient for the proof of Theorem 6.6. The main differences from the original result due to Ledrappier are that the natural extension $\hat{M}$ is not in general a manifold and that there is no well defined unstable foliation in M. Given a partition $\hat{\mathcal{Q}}$ denote by $\hat{\mathcal{Q}}(\hat{x})$ the element of $\hat{\mathcal{Q}}$ that contains $\hat{x} \in \hat{M}$. We say that $\hat{\mathcal{Q}}$ is an *increasing partition* if $(\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x}) \subset \hat{\mathcal{Q}}(\hat{x})$ for $\hat{\eta}$-almost every $\hat{x}$, in which case we write $\hat{f}^{-1}\hat{\mathcal{Q}} \succ \hat{\mathcal{Q}}$.

Proposition 6.8. *There exists an invariant and full $\hat{\eta}$-measure set $\hat{S} \subset \hat{M}$, and a measurable partition $\hat{\mathcal{Q}}$ of $\hat{S}$ such that*:

(1) $\hat{f}^{-1}\hat{\mathcal{Q}} \succ \hat{\mathcal{Q}}$,
(2) $\bigvee_{j=0}^{+\infty} \hat{f}^{-j}\hat{\mathcal{Q}}$ *is the partition into points,*
(3) *The sigma-algebras $\mathcal{M}_n$ generated by the partitions $\hat{f}^{-n}\hat{\mathcal{Q}}$, $n \geqslant 1$, generate the σ-algebra in $\hat{S}$, and*
(4) *For almost every $\hat{x}$ the element $\hat{\mathcal{Q}}(\hat{x}) \subset \hat{W}^u(\hat{x})$ contains a neighborhood of $\hat{x}$ in $\hat{W}^u(\hat{x})$ and the projection $\pi(\hat{\mathcal{Q}}(\hat{x}))$ contains a neighborhood of x_0 in M.*

Proof. Since $\hat{\eta}$ is an expanding measure, Proposition 3.15 guarantees the existence of local unstable manifolds at $\hat{\eta}$-almost every point. Take $i \geqslant 1$ such that $\hat{\eta}(\hat{\Lambda}_i) > 0$ and let r_i, ε_i, γ_i and R_i be given by Corollary 3.16. Fix also $0 < r \leqslant r_i$ and $\hat{x} \in \operatorname{supp}(\hat{\eta}|_{\hat{\Lambda}_i})$. Recall that $\hat{y} \mapsto W^u_{\mathrm{loc}}(\hat{y}) \cap B(x_0, r)$ is a continuous function on $B(\hat{x}, \varepsilon_i r) \cap \hat{\Lambda}_i$. Consider the sets

$$\hat{V}(\hat{y}, r) = \big\{\hat{z} \in \hat{W}^u_{\mathrm{loc}}(\hat{y}) \colon z_0 \in B(x_0, r)\big\},$$

defined for any $\hat{y} \in B(\hat{x}, \varepsilon_i r) \cap \hat{\Lambda}_i$. Define also

$$\hat{S}(\hat{x}, r) = \bigcup \big\{\hat{V}(\hat{y}, r) \colon \hat{y} \in B(\hat{x}, \varepsilon_i r) \cap \hat{\Lambda}_i\big\}$$

and the partition $\hat{\mathcal{Q}}_0(r)$ of $\hat{M}$ whose elements are the connected components $\hat{V}(\hat{y}, r)$ of unstable manifolds just constructed and their complement $\hat{M} \setminus \hat{S}(\hat{x}, r)$. Furthermore, consider the set $\hat{S}_r$ and the partition $\hat{\mathcal{Q}}(r)$ given by

$$\hat{S}_r = \bigcup_{n=0}^{+\infty} \hat{f}^n\big(\hat{S}(\hat{x}, r)\big) \quad \text{and} \quad \hat{\mathcal{Q}}(r) = \bigvee_{n=0}^{+\infty} \hat{f}^n\big(\hat{\mathcal{Q}}_0(r)\big).$$

Then, the partition $\hat{\mathcal{Q}}$ coincides with the partition $\hat{\mathcal{Q}}(r)$ and the set $\hat{S}$ is given by $\bigcap_{j\geqslant 0} \hat{f}^{-j}(\hat{S}_r)$ for a particular choice of the parameter r. In what follows, for notational convenience and when no confusion is possible we shall omit the dependence of the partition $\hat{\mathcal{Q}}$ on r.

It is clear from the construction that every partition $\hat{\mathcal{Q}}(r)$ is increasing, that is the content of (6.8). In addition, since $\hat{\eta}$ is ergodic and $\hat{\eta}(\hat{S}(\hat{x}, r)) > 0$ then the set of points that return infinitely often to $\hat{S}(\hat{x}, r)$, which we called $\hat{S}_r$, is a full measure set by Poincaré's Recurrence Theorem. In other words, if a point $\hat{y}$ belongs to $\hat{S}_r$ there are positive integers $(n_j)_j$ such that $\hat{f}^{n_j}(\hat{y}) \in \hat{V}(\hat{f}^{n_j}(\hat{y}), r)$. Hence, the backward distance contraction along unstable leaves guarantees that the diameter of the partition $\bigvee_{n=0}^{n} \hat{f}^{-j}\hat{\mathcal{Q}}$ tend to zero as $n \to \infty$, proving (6.8). By construction, there is a full measure set such that any two distinct points $\hat{y}$ and $\hat{z}$ lie in different elements of $\hat{f}^{-n}\hat{\mathcal{Q}}$ for some $n \in \mathbb{N}$. Indeed, if $\hat{f}^{-n}\hat{\mathcal{Q}}(\hat{y}) = \hat{f}^{-n}\hat{\mathcal{Q}}(\hat{z})$ for every $n \geqslant 0$ then $\hat{f}^n(\hat{y})$ and $\hat{f}^n(\hat{z})$ lie infinitely often in the same local unstable manifold. But (6.8) implies that $\hat{y}$ and $\hat{z}$ should coincide, which is a contradiction and proves our claim. In particular, the decreasing family of σ-algebras $\mathcal{M}_n$, $n \geqslant 1$, generate the σ-algebra in $\hat{S}_r$, which proves (6.8).

We proceed to show that the partition $\hat{\mathcal{Q}}(r)$ satisfies (6.8) for Lebesgue almost every parameter r. Given $0 < r \leqslant r_i$ and $\hat{y} \in \hat{S}_r$ define

$$\beta_r(\hat{y}) = \inf_{n \geqslant 0} \left\{ R_i, \frac{r}{\gamma_i}, \frac{1}{2\gamma_i} e^{\lambda_i n} d\big(y_{-n}, \partial B(x_0, r)\big) \right\},$$

that it clearly non-negative. It is enough to obtain the following:

(a) If $z_0 \in W^u_{\mathrm{loc}}(\hat{y})$ and $d(y_0, z_0) < \beta_r(\hat{y})$ then there exists $\hat{z} \in \hat{\mathcal{Q}}(\hat{y})$ such that $\pi(\hat{z}) = z_0$;
(b) There exists a full Lebesgue measure set of parameters $0 < r \leqslant r_i$ such that the function $\beta_r(\cdot)$ is strictly positive almost everywhere and $\hat{\eta}(\partial \hat{\mathcal{Q}}(r)) = 0$.

Take $\hat{y} \in \hat{S}_r$ and assume that $z_0 \in W^u_{\mathrm{loc}}(\hat{y})$ is such that $d(y_0, z_0) < \beta_r(\hat{y})$. If $\hat{y} \in \hat{S}(\hat{x}, r)$ then there exists $\hat{w} \in B(\hat{x}, \varepsilon_i r)$ such that $\hat{y} \in \hat{W}^u_{\mathrm{loc}}(\hat{w})$. Furthermore, since $d(y_0, z_0) < \beta_r(\hat{y}) < R_i$ then there exists $\hat{z} \in \hat{W}^u_{\mathrm{loc}}(\hat{w})$ such that $\pi(\hat{z}) = z_0$. Hence

$$d(y_{-n}, z_{-n}) \leqslant \gamma_i e^{-n\lambda_i} d(y_0, z_0), \quad \forall n \in \mathbb{N},$$

which implies that $d(y_{-n}, z_{-n}) \leqslant r$ and $d(y_{-n}, z_{-n}) \leqslant 1/2\, d(y_{-n}, \partial B(x_0, r))$ for every $n \in \mathbb{N}$. Together with Corollary 3.16, this shows that y_{-n} and z_{-n} belong to the same element of the partition $\hat{\mathcal{Q}}_0$ for every $n \geqslant 1$ and, assuming (b) for the moment, that $\pi(\mathcal{Q}(\hat{y}))$ contains a neighborhood of y_0 in $W^u_{\mathrm{loc}}(\hat{y})$. On the other hand, if $\hat{y} \in \hat{S}_r \setminus \hat{S}(\hat{x}, r)$ then there exists $k \geqslant 1$ such that $\hat{f}^{-k}(\hat{y}) \in \hat{S}(\hat{x}, r)$ and consequently the projection of the set

$$\hat{\mathcal{Q}}(\hat{y}) = \hat{f}^k\big(\hat{\mathcal{Q}}\big(\hat{f}^{-k}(\hat{y})\big)\big)$$

contains an open neighborhood of y_0 in $W^u_{\mathrm{loc}}(\hat{y})$. This completes the proof of (a).

The proof of (b) is slightly more involving. We begin with the following remark from measure theory: if $r_0 > 0$, ϑ is a Borel measure in $[0, r_0]$ and $0 < a < 1$ then Lebesgue almost every $r \in [0, r_0]$ satisfies

$$\sum_{k=0}^{\infty} \vartheta\big(\big[r - a^k, r + a^k\big]\big) < \infty. \tag{6.2}$$

Indeed, the set

$$B_{a,k} = \left\{ r \in [0, r_0]\colon\ \vartheta\big(\big[r - a^k, r + a^k\big]\big) > \frac{\vartheta([0, r_0])}{k^2} \right\}$$

can be covered by a family I_k of balls of radius a^k centered at points of $B_{a,k}$ in such a way that any point is contained in at most two intervals of I_k. Since

$$\# I_k \frac{\vartheta([0, r_0])}{k^2} \leqslant \sum_{I \in I_k} \vartheta(I) \leqslant 2\vartheta\big([0, r_0]\big)$$

then $\# I_k \leqslant 2k^2$ and it is clear that $\mathrm{Leb}(B_{a,k}) \leqslant 2a^k \,\#\, I_k$ is summable. Borel–Cantelli's lemma implies that Lebesgue almost every $r \in [0, r_0]$ belongs to finitely many sets $B_{a,k}$, which proves the summability condition in (6.2).

Back to the proof of (b), let ϑ be the measure of the interval $[0, r_i]$ defined by $\vartheta(E) = \hat{\eta}(\hat{y} \in \hat{M}\colon\ d(x_0, y_0) \in E)$. The previous assertion guarantees that for Lebesgue almost every $r \in [0, r_i]$ it holds

$$\sum_{k=0}^{\infty} \hat{\eta}\big(\hat{y} \in \hat{M}\colon\ \big|d(x_0, y_0) - r\big| < e^{-\lambda_i k}\big) < \infty. \tag{6.3}$$

On the other hand, there exists $D > 0$ such that $|d(z_0, x_0) - r| < D\tau$ whenever $d(z_0, \partial B(x_0, r)) < \tau$ and $0 < \tau < r \leqslant r_i$. Therefore

$$\sum_{k=0}^{\infty} \hat{\eta}\big(\hat{y} \in \hat{M}\colon\ \big|d\big(y_{-n}, \partial B(x_0, r)\big)\big| < D^{-1} e^{-\lambda_i k}\big) \leqslant \sum_{k=0}^{\infty} \hat{\eta}\big(\hat{y} \in \hat{M}\colon\ \big|d(x_0, y_{-n}) - r\big| < e^{-\lambda_i k}\big),$$

which is finite because of the invariance of $\hat\eta$ and the former condition (6.3). Using Borel–Cantelli's lemma once more it follows that $\hat\eta$-almost every $\hat y$ satisfies

$$\left|d\left(y_{-n}, \partial B(x_0, r)\right)\right| < D^{-1} e^{-\lambda_i k}$$

for at most finitely many positive integers k, proving that $\beta_r(\hat y) > 0$. Furthermore, since $\eta(\bigcup_{n\geqslant 0} f^n(\partial B(x_0, r))) = 0$ for all but a countable set of parameters $0 < r \leqslant r_i$ then $\hat{\mathcal{Q}}(\hat y)$ contains a neighborhood of $\hat y$ in $\hat W^u_{\mathrm{loc}}(\hat y)$ for $\hat\eta$-almost every $\hat y \in \hat M$. This shows that (b) holds and, in consequence, for Lebesgue almost every $r \in [0, r_i]$ the partition $\hat{\mathcal{Q}}(r)$ satisfies the requirements of the proposition. □

Let $(\hat\eta_x)_x$ be the disintegration of the measure $\hat\eta$ on the measurable partition $\hat{\mathcal{Q}}$, given by Rokhlin's theorem. Recall that for $\hat\eta$-almost every $\hat x$ the map $\pi|_{\hat W^u_{\mathrm{loc}}(\hat x)} : \hat W^u_{\mathrm{loc}}(\hat x) \to W^u_{\mathrm{loc}}(\hat x)$ is a bijection. For any such $\hat x$ let $\hat\nu_x$ be the measure on $\hat W^u_{\mathrm{loc}}(\hat x)$ obtained as the pull-back of $\nu|_{W^u_{\mathrm{loc}}(\hat x)}$ by the bijection $\pi|_{\hat W^u_{\mathrm{loc}}(\hat x)}$. Let $\hat\nu$ denote the measure defined on $\hat M$ by the disintegration $(\hat\nu_{\hat x})_{\hat x}$, that is to say that

$$\hat\nu(\hat E) = \int \hat\nu_{\hat x}(\hat E)\, d\hat\eta(\hat x)$$

for every measurable set $\hat E$ in $\hat M$. As a byproduct of the previous result we obtain

Corollary 6.9. $0 < \hat\nu_{\hat x}(\hat{\mathcal{Q}}(\hat x)) < \infty$, *for $\hat\eta$-almost every $\hat x$.*

Proof. For every $\hat x$ in a full $\hat\eta$-measure set one has that

$$\hat\nu_{\hat x}\left(\hat{\mathcal{Q}}(\hat x)\right) = \nu\left(\pi\left(\hat{\mathcal{Q}}(\hat x)\right) \cap W^u_{\mathrm{loc}}(\hat x)\right).$$

Since $\hat\eta$ is an expanding measure then $\hat W^u_{\mathrm{loc}}(\hat x)$ is a neighborhood $\hat x$ and $W^u_{\mathrm{loc}}(\hat x) \cap \pi(\hat{\mathcal{Q}}(\hat x))$ contains a neighborhood of x_0 in M. In addition, since $\eta(\operatorname{supp}\nu) = 1$, for every $\hat x$ in a full $\hat\eta$-measure set it holds that $x_0 \in \operatorname{supp}(\nu)$. Then it is clear that $0 < \hat\nu_{\hat x}(\hat{\mathcal{Q}}(\hat x)) < \infty$, $\hat\eta$-almost everywhere, which proves the corollary. □

The next preparatory lemma shows that $\hat\nu$ has a Jacobian with respect to $\hat f$ and establishes Rokhlin's formula for the natural extension.

Lemma 6.10. *The measure $\hat\nu$ has a Jacobian $J_{\hat\nu}\hat f = J_\nu f \circ \pi$ with respect to $\hat f$. In addition,*

$$h_{\hat\eta}(\hat f) = \int \log J_{\hat\nu}\hat f\, d\hat\eta.$$

Furthermore, for $\hat\eta$-almost every $\hat x$ and every $\hat y \in \hat{\mathcal{Q}}(\hat x)$ the product

$$\Delta(\hat x, \hat y) = \prod_{j=1}^{\infty} \frac{J_{\hat\nu}\hat f(\hat f^{-j}(\hat x))}{J_{\hat\nu}\hat f(\hat f^{-j}(\hat y))}$$

is positive and finite.

Proof. Since the sigma-algebra $\hat{\mathcal{B}}$ is the completion of the sigma-algebra generated by the cylinders $\pi_i^{-1}(f^{-i}\mathcal{B})$, $i \geqslant 1$, then the first claim in the lemma is a consequence from the fact that

$$\hat\nu_{\hat f(\hat x)}\left(\hat f(\hat E)\right) = \int_{\hat E \cap (\hat f^{-1}\hat{\mathcal{Q}})(\hat x)} J_\nu f \circ \pi\, d\hat\nu_{\hat x} \tag{6.4}$$

for almost every $\hat x$ and every small cylinder $\hat E = \pi^{-1}(E)$. Indeed, if $\hat E$ is a small cylinder then it is clear that

$$\hat\nu\left(\hat f(\hat E)\right) = \int \hat\nu_{\hat f(\hat x)}\left(\hat f(\hat E)\right) d\hat\eta(\hat x) = \int \int_{\hat E \cap (\hat f^{-1}\hat{\mathcal{Q}})(\hat x)} J_\nu f \circ \pi\, d\hat\nu_{\hat x}\, d\hat\eta(\hat x). \tag{6.5}$$

Let $\tilde{\nu}_{\hat{x}}$ denote the restriction of the measure $\hat{\nu}_{\hat{x}}$ to the set $(\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x}) \subset \hat{\mathcal{Q}}(\hat{x})$. Then $\hat{\nu}$ has a disintegration $\hat{\nu} = \int \tilde{\nu}_{\hat{x}}\, d\hat{\eta}$ with respect to the measurable partition $\hat{f}^{-1}\hat{\mathcal{Q}}$. Together with (6.5) this gives

$$\hat{\nu}\big(\hat{f}(\hat{E})\big) = \int \int_{\hat{E}} J_\nu f \circ \pi \, d\tilde{\nu}_{\hat{x}}\, d\hat{\eta}(\hat{x}) = \int_{\hat{E}} J_\nu f \circ \pi \, d\hat{\nu},$$

which proves that $\hat{\nu}$ has a Jacobian and $J_{\hat{\nu}}\hat{f} = J_\nu f \circ \pi$. Hence, to prove the first assertion in the lemma we are reduced to prove (6.4) above. If $f \mid E$ is injective and $\hat{E} = \pi^{-1}(E)$ then

$$\begin{aligned}\hat{\nu}_{\hat{f}(\hat{x})}\big(\hat{f}(\hat{E})\big) &= \hat{\nu}_{\hat{f}(\hat{x})}\big(\hat{f}\big[\hat{E} \cap \big(\hat{f}^{-1}\hat{\mathcal{Q}}\big)(\hat{x})\big]\big) = \nu\big(f\big(E \cap \pi\big(\big(\hat{f}^{-1}\hat{\mathcal{Q}}\big)(\hat{x})\big)\big)\big) \\ &= \int_{E\cap\pi((\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x}))} J_\nu f \, d\nu = \int_{\hat{E}\cap(\hat{f}^{-1}\hat{\mathcal{Q}})(\hat{x})} J_\nu f \circ \pi \, d\hat{\nu}_{\hat{x}},\end{aligned}$$

which proves (6.4). On the other hand, $h_\eta(f) = \int J_\nu f \, d\eta$ because η is an equilibrium state, $P_{\mathrm{top}}(f, \phi) = \log \lambda$ and $J_\nu f = \lambda e^{-\phi}$. So, using $\pi_* \hat{\eta} = \eta$ we obtain

$$h_{\hat{\eta}}(\hat{f}) = h_\eta(f) = \int \log J_\nu f \, d\eta = \int \log(J_\nu f \circ \pi)\, d\hat{\eta} = \int \log J_{\hat{\nu}} \hat{f} \, d\hat{\eta},$$

which proves the second assertion in the lemma. Finally, the Hölder continuity of the Jacobian $J_{\hat{\nu}}\hat{f} = J_\nu f \circ \pi$, the fact that $\hat{\mathcal{Q}}$ is subordinated to unstable leaves and the backward distance contraction for points in the same unstable leaf yield that the product

$$\Delta(\hat{x}, \hat{y}) = \prod_{j=1}^{\infty} \frac{J_{\hat{\nu}}\hat{f}(\hat{f}^{-j}(\hat{x}))}{J_{\hat{\nu}}\hat{f}(\hat{f}^{-j}(\hat{y}))}$$

is convergent for almost every $\hat{x}$ and every $\hat{y} \in \hat{\mathcal{Q}}(\hat{x})$. The proof of the lemma is now complete. □

The last main ingredient to the proof of Theorem 6.6 is the following generating property of the partition $\hat{\mathcal{Q}}$.

Proposition 6.11. $h_{\hat{\eta}}(\hat{f}) = H_{\hat{\eta}}(\hat{f}^{-1}\hat{\mathcal{Q}} \mid \hat{\mathcal{Q}})$.

The proof of this result involves two preliminary lemmas. Let $i \geqslant 1$ and $\hat{\Lambda}_i$ be given as in the proof of Proposition 6.8 and r_i given by Corollary 3.16. The following lemma gives a dynamical characterization of the local unstable manifolds.

Lemma 6.12. *Given $\varepsilon > 0$ there is a measurable function $\hat{D}_\varepsilon : \hat{B}_\lambda \to \mathbb{R}_+$ satisfying $\log \hat{D}_\varepsilon \in L^1(\hat{\eta})$ and such that, if $d(x_{-n}, y_{-n}) \leqslant \hat{D}_\varepsilon(\hat{f}^{-n}(\hat{x}))$ $\forall n \geqslant 0$ then $\hat{y} \in \hat{W}^u_{\mathrm{loc}}(\hat{x})$ and $d(x_0, y_0) < 2r_i$.*

Proof. Since $\hat{\eta}(\hat{\Lambda}_i) > 0$ and $\hat{\eta}$ is assumed to be ergodic then some iterate of almost every point will eventually belong to $\hat{\Lambda}_i$ by Poincaré's recurrence theorem. So, the first hitting time $R(\hat{x})$ is well defined almost everywhere in $\hat{\Lambda}_i$ and $\int_{\hat{\Lambda}_i} R\, d\hat{\eta} = 1/\hat{\eta}(\hat{\Lambda}_i)$, by Kac's lemma. This proves that the logarithm of the function $\hat{D}_\varepsilon : \hat{M} \to \mathbb{R}$ given by

$$\hat{D}_\varepsilon(\hat{x}) = \begin{cases} \min\{2r_i, \delta_i, \delta_i/\gamma_i\} e^{-(\lambda+\varepsilon)R(\hat{x})}, & \text{if } \hat{x} \in \hat{\Lambda}_i, \\ \min\{2r_i, \delta_i, \delta_i/\gamma_i\}, & \text{otherwise} \end{cases}$$

is $\hat{\eta}$-integrable. On the other hand, if $\hat{x} \in \hat{\Lambda}_i$ then $R(\hat{f}^{-n}(\hat{x})) = n$. Any $\hat{y} \in \hat{M}$ such that $d(x_{-n}, y_{-n}) \leqslant \hat{D}_\varepsilon(\hat{f}^{-n}(\hat{x}))$ for every $n \geqslant 0$ clearly satisfies $d(x_0, y_0) < 2r_i$ and, by Proposition 3.15(2), belongs to $W^u_{\mathrm{loc}}(\hat{x})$. This concludes the proof of the lemma. □

This result allow us to construct an auxiliary measurable partition of finite entropy that will be useful to compute the metric entropy $h_{\hat{\eta}}(\hat{f})$.

Lemma 6.13. *There exists a measurable partition $\hat{\mathcal{P}}$ of $\hat{S}$ such that $H_{\hat{\eta}}(\hat{\mathcal{P}}) < \infty$, $\operatorname{diam}(\hat{\mathcal{P}}(\hat{x})) \leqslant \hat{D}_\varepsilon(\hat{x})$ at $\hat{\eta}$-almost every $\hat{x}$, and that the partition*

$$\hat{\mathcal{P}}^{(\infty)} = \bigvee_{n=0}^{+\infty} \hat{f}^n \hat{\mathcal{P}}$$

is finer than $\hat{\mathcal{Q}}$.

Proof. Let $\hat{D}_\varepsilon$ be the measurable function given by the previous lemma. By Lemma 2 in [35], there exists a measurable and countable partition $\hat{\mathcal{P}}_0$ such that $H_{\hat{\eta}}(\hat{\mathcal{P}}_0) < \infty$ and $\operatorname{diam} \hat{\mathcal{P}}(\hat{x}) \leqslant \hat{D}_\varepsilon(\hat{x})$ for a.e. $\hat{x} \in \hat{M}$. Let $\hat{\mathcal{P}}$ be the finite entropy partition obtained as the refinement of $\hat{\mathcal{P}}_0$ and $\{\hat{M} \setminus \hat{S}(\hat{x}, r), \hat{S}(\hat{x}, r)\}$. Notice that there is a full measure set where any two points $\hat{x}$ and $\hat{y}$ belong to the same element of $\hat{f}^n \hat{\mathcal{P}}$ for every $n \geqslant 0$ if and only

$$d(x_{-n}, y_{-n}) \leqslant \hat{D}_\varepsilon\big(\hat{f}^{-n} \hat{x}\big) \quad \text{for every } n \geqslant 0.$$

In particular, Lemma 6.12 above implies that each element of $\hat{\mathcal{P}}$ is a piece of some local unstable manifold. Hence, since $\hat{\mathcal{P}}$ was chosen to refine $\{\hat{M} \setminus \hat{S}(\hat{x}, r), \hat{S}(\hat{x}, r)\}$ then it is easy to see that

$$\bigcap_{n \geqslant 0} \hat{f}^n \hat{\mathcal{P}}\big(\hat{f}^{-n}(\hat{x})\big) \subset \hat{\mathcal{Q}}(\hat{x})$$

for almost every $\hat{x}$. So, the partition $\hat{\mathcal{P}}$ just constructed satisfies the conclusions of the lemma. □

Proof of Proposition 6.11. Let $\varepsilon > 0$ be arbitrary small. Up to a refinement of the partition $\hat{\mathcal{P}}$ we may assume without loss of generality that $h_{\hat{\eta}}(\hat{f}, \hat{\mathcal{P}}) \geqslant h_{\hat{\eta}}(\hat{f}) - \varepsilon$. Since the partition $\hat{\mathcal{P}}^{(\infty)}$ is finer than $\hat{\mathcal{Q}}$ then

$$h_{\hat{\eta}}(\hat{f}, \hat{\mathcal{P}}) = h_{\hat{\eta}}\big(\hat{f}, \hat{\mathcal{P}}^{(\infty)}\big) = h_{\hat{\eta}}\big(\hat{f}, \hat{\mathcal{P}}^{(\infty)} \vee \hat{\mathcal{Q}}\big) = h_{\hat{\eta}}\big(\hat{f}, \hat{f}^n \hat{\mathcal{P}}^{(\infty)} \vee \hat{\mathcal{Q}}\big)$$

for every $n \geqslant 1$. Using that $h_{\hat{\eta}}(\hat{f}, \hat{\xi}) = H_{\hat{\eta}}(\hat{f}^{-1} \hat{\xi}, \hat{\xi})$ for every increasing partition $\hat{\xi}$, the right-hand side term in the previous equalities coincides with the relative entropy $H_{\hat{\eta}}(\hat{f}^n \hat{\mathcal{P}}^{(\infty)} \vee \hat{\mathcal{Q}} \mid \hat{f}^{n+1} \hat{\mathcal{P}}^{(\infty)} \vee \hat{f} \hat{\mathcal{Q}})$ and, consequently,

$$h_{\hat{\eta}}(\hat{f}, \hat{\mathcal{P}}) = H_{\hat{\eta}}\big(\hat{\mathcal{Q}} \mid \hat{f} \hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}\big) + H_{\hat{\eta}}\big(\hat{\mathcal{P}}^{(\infty)} \mid \hat{f}^{-n} \hat{\mathcal{Q}} \vee \hat{f} \hat{\mathcal{P}}^{(\infty)}\big).$$

The second term in the right-hand side above is bounded by $H_{\hat{\eta}}(\hat{\mathcal{P}})$, which is finite. Then Proposition 6.8(3) implies that it tends to zero as $n \to \infty$. On the other hand, the diameter of almost every element in $\hat{f}^{-n+1} \hat{\mathcal{Q}}$ tend to zero as $n \to \infty$, proving that there exists a sequence of sets $(\hat{D}_n)_{n \geqslant 1}$ in $\hat{M}$ satisfying $\lim_n \hat{\eta}(\hat{D}_n) = 1$ and such that $\hat{f} \mathcal{Q}(\hat{x}) \subset \hat{f}^n \hat{\mathcal{P}}^{(\infty)}(\hat{x})$ for every $\hat{x} \in \hat{D}_n$. Then

$$\begin{aligned} H_{\hat{\eta}}\big(\hat{\mathcal{Q}} \mid \hat{f} \hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}\big) &= \int -\log \hat{\eta}_{(\hat{f} \hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)})(\hat{x})}\big(\hat{\mathcal{Q}}(\hat{x})\big)\, d\hat{\eta}(\hat{x}) \\ &\geqslant \int_{\hat{D}_n(\hat{x})} -\log \hat{\eta}_{(\hat{f} \hat{\mathcal{Q}})(\hat{x})}\big(\hat{\mathcal{Q}}(\hat{x})\big)\, d\hat{\eta}(\hat{x}), \end{aligned}$$

where the measures $\hat{\eta}_{\hat{f} \hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}}$ and $\hat{\eta}_{\hat{f} \hat{\mathcal{Q}}}$ denote respectively the conditional measures of η with respect to the partitions $\hat{f} \hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}$ and $\hat{f} \hat{\mathcal{Q}}$. This proves that $\lim_n H_{\hat{\eta}}(\hat{\mathcal{Q}} \mid \hat{f} \hat{\mathcal{Q}} \vee \hat{f}^n \hat{\mathcal{P}}^{(\infty)}) \geqslant H_{\hat{\eta}}(\hat{\mathcal{Q}} \mid \hat{f} \hat{\mathcal{Q}})$. Since the other inequality is always true we deduce that $h_{\hat{\eta}}(\hat{f}, \hat{\mathcal{P}}) = H_{\hat{\eta}}(\hat{\mathcal{Q}} \mid \hat{f} \hat{\mathcal{Q}})$. Since $\varepsilon > 0$ was chosen arbitrary this proves that $h_{\hat{\eta}}(\hat{f}) = H_{\hat{\eta}}(\hat{\mathcal{Q}} \mid \hat{f} \mathcal{Q})$, as claimed. □

It follows from Lemma 6.10 and Proposition 6.11 that

$$H_{\hat{\eta}}\big(\hat{f}^{-1} \hat{\mathcal{Q}} \mid \hat{\mathcal{Q}}\big) = \int \log J_{\hat{\nu}} \hat{f}\, d\hat{\eta}. \tag{6.6}$$

With this in mind we obtain the following:

Lemma 6.14. *$\hat\eta$ admits a disintegration $(\hat\eta_{\hat x})_{\hat x}$ along the measurable partition $\hat{\mathcal Q}$ such that*

$$\hat\eta_{\hat x}(B) = \frac{1}{Z(\hat x)} \int_{\hat{\mathcal Q}(\hat x)\cap B} \Delta(\hat x, \hat y)\, d\hat\nu_{\hat x}(\hat y), \quad \textit{where} \quad Z(\hat x) = \int_{\hat{\mathcal Q}(\hat x)} \Delta(\hat x, \hat y)\, d\hat\nu_{\hat x}(\hat y) \tag{6.7}$$

for every measurable set B and $\hat\eta$-almost every $\hat x$. In consequence $\hat\eta_{\hat x}$ is absolutely continuous with respect to $\hat\nu_{\hat x}$ for almost every $\hat x$.

Proof. Recall that $\Delta(\hat x, \hat y)$ is well defined for almost every $\hat x$ and every $\hat y \in \hat{\mathcal Q}(\hat x)$ according to Lemma 6.10. In particular Corollary 6.9 implies that $0 < Z(\hat x) < \infty$ almost everywhere. Let $\rho_{\hat x}$ denote the measure in the right-hand side of the first equality in (6.7). Since $\hat f^{-1}\hat{\mathcal Q} \succ \hat{\mathcal Q}$ a simple computation involving a change of coordinates gives that

$$\rho_{\hat x}\big((\hat f^{-1}\hat{\mathcal Q})(\hat x)\big) = \frac{1}{Z(\hat x)} \int_{(\hat f^{-1}\hat{\mathcal Q})(\hat x)} \Delta(\hat x, \hat y)\, d\hat\nu_{\hat x}(\hat y) = \frac{Z(\hat f(\hat x))}{Z(\hat x) J_{\hat\nu} \hat f(\hat x)}.$$

We claim that

$$-\int \log \rho_{\hat x}\big((\hat f^{-1}\hat{\mathcal Q})(\hat x)\big)\, d\hat\eta = \int \log J_{\hat\nu}\hat f\, d\hat\eta.$$

Since $\rho_{\hat x}$ is a probability measure then $-\log\rho_{\hat x}((\hat f^{-1}\hat{\mathcal Q})(\hat x))$ is a positive function and clearly the negative part of this function belongs to $L^1(\hat\eta)$. Using that $J_\nu f$ is bounded away from zero and infinity the same is obviously true also for $\log\frac{Z(\hat f(\hat x))}{Z(\hat x)}$. So, Birkhoff's ergodic theorem yields that the limit

$$\omega(\hat x) := \lim_{n\to\infty} \frac{1}{n} \log Z\big(\hat f^n(\hat x)\big) = \lim_{n\to\infty} \frac{1}{n} \log \frac{Z(\hat f^n(\hat x))}{Z(\hat x)} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \frac{Z\circ \hat f(\hat f^j(\hat x))}{Z(\hat f^j(\hat x))}$$

do exist (although possibly infinite) and that

$$\int \omega(\hat x)\, d\hat\eta(\hat x) = \int \log \frac{Z(\hat f(\hat x))}{Z(\hat x)}\, d\hat\eta(\hat x).$$

Since Z is almost everywhere positive and finite, the sequence $1/n \log Z(\hat f^n(\hat x))$ converge to zero in probability and, consequently, it is almost everywhere convergent to zero along some subsequence $(n_j)_j$. This shows that $\omega(\hat x) = 0$ for $\hat\eta$-almost every $\hat x$ and proves our claim. On the other hand using relation (6.6) and the equality

$$H_{\hat\eta}\big(\hat f^{-1}\hat{\mathcal Q} \mid \hat{\mathcal Q}\big) = -\int \log \hat\eta_{\hat x}\big(\hat f^{-1}\hat{\mathcal Q}(\hat x)\big)\, d\hat\eta(\hat x)$$

we obtain

$$\int \log\bigg(\frac{d\hat\rho}{d\hat\eta}\bigg|_{\hat f^{-1}\hat{\mathcal Q}}\bigg)\, d\hat\eta = 0.$$

Since the logarithm is a strictly concave function then

$$0 = \int \log\bigg(\frac{d\hat\rho_{\hat x}}{d\hat\eta_{\hat x}}\bigg|_{\hat f^{-1}\hat{\mathcal Q}}\bigg) d\hat\eta \leqslant \log\bigg(\int \frac{d\hat\rho_{\hat x}}{d\hat\eta_{\hat x}}\bigg|_{\hat f^{-1}\hat{\mathcal Q}} d\hat\eta\bigg) = 0,$$

and the equality holds if and only if the Radon–Nykodym derivative $\frac{d\hat\rho_{\hat x}}{d\hat\eta_{\hat x}}$ restricted to the sigma-algebra generated by $\hat f^{-1}\hat{\mathcal Q}$ is almost everywhere constant and equal to one. Replacing $\hat f$ by any power $\hat f^n$ in the previous computations it is not difficult to check that $\hat\eta_{\hat x}$ and $\hat\rho_{\hat x}$ coincide in the increasing family of sigma-algebras generated by the partitions $\hat f^{-n}(\hat{\mathcal Q})$, $n \geqslant 1$. Proposition 6.8(3) readily implies that $\hat\eta_{\hat x} = \rho_{\hat x}$ at $\hat\eta$-almost every $\hat x$, which completes the proof of the lemma. □

We know from the previous lemma that $\hat\eta_{\hat x} \ll \hat\nu_{\hat x}$ almost everywhere. Then, using that $W^u_{\mathrm{loc}}(\hat x)$ is a neighborhood of x_0 in M and the bijection

$$\pi|_{\hat W^u_{\mathrm{loc}}(\hat x)} : \hat W^u_{\mathrm{loc}}(\hat x) \to W^u_{\mathrm{loc}}(\hat x)$$

it follows that $\pi_* \hat{\eta}_{\hat{x}} \ll \nu$ for $\hat{\eta}$-almost every $\hat{x}$. Since $(\hat{\eta}_{\hat{x}})$ is a disintegration of $\hat{\eta}$ and $\pi_* \hat{\eta} = \eta$ it is immediate that $\eta \ll \nu$. This completes proof of the theorem. □

Remark 6.15. We point out there is an analogous version of Theorem 6.6 that holds for piecewise differentiable maps f that behave like a power of the distance to a possible critical or singular locus, as considered in [3]. Indeed, assume that ϕ is an Hölder continuous potential and ν is an expanding conformal measure such that $J_\nu f = \lambda e^{-\phi}$ is Hölder continuous, where $\lambda = \exp P_{\text{top}}(f, \phi)$. Assume also that η is an equilibrium state for f with respect to ϕ and $\eta(\operatorname{supp} \nu) = 1$. If η has non-uniform expansion and satisfies a slow recurrence condition then there is a local unstable leaf passing through almost every point, in the same way as in Proposition 3.15. The construction of an increasing partition as in Proposition 6.8 and the proof of the absolute continuity of η with respect to ν remains unaltered. This is of independent interest and can be applied, e.g. when f is a quadratic map with positive Lyapunov exponent, $\phi = -\log|\det Df|$ and ν is the Lebesgue measure to prove the uniqueness of the SRB measure. Some of these ideas can be traced back to [30,34,45] but since these papers do not use hyperbolic times, the results are less precise than here.

Through the remaining of the section assume that f is topologically mixing. Since equilibrium states coincide with the invariant measures that are absolutely continuous with respect to ν then there is only one equilibrium state μ for f with respect to ϕ. Thus, Theorem B is a direct consequence of Proposition 5.1 and the previous statement. To finish the proof of Theorem A it remains only to show exactness of the equilibrium state:

Lemma 6.16. *μ is exact.*

Proof. Let $E \in \mathcal{B}_\infty$ be such that $\mu(E) > 0$ and let $\varepsilon > 0$ be arbitrary. There are measurable sets $E_n \in \mathcal{B}$ such that $E = f^{-n}(E_n)$. On the other hand, since μ is regular there exists a compact set K and an open set O such that $K \subset E \cap H \subset O$ and $\mu(O \setminus K) < \varepsilon \mu(K)$, where H denotes as before the set of points with infinitely many hyperbolic times and $\varepsilon > 0$ is small. The same argument used in the proof of Lemma 5.3 shows that there exists $\tau > 0$ $n \geqslant 1$ and $x \in H_n$ such that

$$\frac{\mu(B(x,n,\delta/4) \setminus E)}{\mu(B(x,n,\delta/4))} < \tau^{-1}\varepsilon.$$

Since n is a hyperbolic time then $f^n|_{B(x,n,\delta)}$ is a homeomorphism that satisfies the bounded distortion property. Hence

$$\frac{\mu(B(f^n(x),\delta/4) \setminus f^n(E))}{\mu(B(f^n(x),\delta/4))} < K_0 \tau^{-1}\varepsilon.$$

The topologically mixing assumption guarantees the existence of a uniform $N \geqslant 1$ (depending only on δ) such that every ball of radius $\delta/4$ is mapped onto M by f^N. Furthermore, since $\mu \ll \nu$ with density $h = \frac{d\mu}{d\nu}$ bounded away from zero and infinity then $J_\mu f = J_\nu f\ (h \circ f)/h$ satisfies $C^{-1} \leqslant J_\mu f \leqslant C$ for some constant $C > 1$. In particular, since d^N is an upper bound for the number of inverse branches of f^N, C bounds the maximal distortion of the Jacobian at each iterate and μ is f-invariant we obtain that

$$\mu(M \setminus E) = \mu(M \setminus E_{n+N}) < K_0 d^N C^N \tau^{-1}\varepsilon.$$

The arbitrariness of $\varepsilon > 0$ shows that $\mu(E) = 1$. This proves that μ is exact. □

We finish this section with the

Proof of Corollary C. If ϕ is a continuous potential satisfying (P), the existence of an equilibrium state for f with respect to ϕ will follow from upper semi-continuity of the metric entropy. Let $\{\phi_n\}$ be a sequence of Hölder continuous potentials satisfying (P) and converging to ϕ in the uniform topology. Take μ_n to be an equilibrium state for f with respect to ϕ_n, given by Theorem B, and let μ be an accumulation point of the sequence $(\mu_n)_n$. Note that the constants c and δ given by Lemma 3.4 are uniform for every μ_n. So, any partition $\mathcal{R}$ of diameter smaller than δ that satisfies $\mu(\partial\mathcal{R}) = 0$ is generating with respect to μ_n, and

$$h_\mu(f, \mathcal{R}) \geqslant \limsup h_{\mu_n}(f, \mathcal{R}).$$

Using the continuity of $\phi \mapsto P_{\mathrm{top}}(f,\phi)$ and $\phi \mapsto \int \phi\, d\mu$ it follows that

$$h_\mu(f,\mathcal{R}) = \limsup_{n\to\infty}\left[P_{\mathrm{top}}(f,\phi_n) - \int \phi_n\, d\mu_n\right] = P_{\mathrm{top}}(f,\phi) - \int \phi\, d\mu \geqslant h_\mu(f).$$

This proves that μ is an equilibrium state for f with respect to ϕ. Furthermore, the function

$$(\eta,\phi) \mapsto h_\eta(f) + \int \phi\, d\eta$$

is upper-semicontinuous on the product space of c-expanding measures and convex set of continuous potentials satisfying (P). Hence, proceeding as in [54, Corollary 9.15.1] there exists a residual $\mathcal{R} \subset C(M)$ of potentials satisfying (P) such that there is a unique equilibrium state for f with respect to ϕ. The proof of the corollary is now complete. □

7. Stability of equilibrium states

7.1. Statistical stability

Here we prove upper semi-continuity of the metric entropy and use the continuity assumption on the topological pressure to prove that the equilibrium states vary continuously with respect to the data f and ϕ.

Proof of Theorem D. Let $\mathcal{W}$ be the set of Hölder continuous potentials and $\mathcal{F}$ the set of local homeomorphisms introduced in Section 2.4. The strategy is to construct a generating partition for *all* maps in $\mathcal{F}$. A similar argument was considered in [4]. Fix $(f,\phi) \in \mathcal{F}\times\mathcal{W}$ and arbitrary sequences $\mathcal{F} \ni f_n \to f$ in the uniform topology, with $L_n \to L$ in the uniform topology, and $\mathcal{W} \ni \phi_n \to \phi$ in the uniform topology, let μ_n be an equilibrium state for f_n with respect to ϕ_n and η be an f-invariant measure obtained as an accumulation point of the sequence $(\mu_n)_n$.

We begin with the following observation. Since the constants c and δ given by Lemma 3.4 are uniform in $\mathcal{F}$, any partition $\mathcal{R}$ of diameter smaller than $\delta/2$ satisfying $\eta(\partial\mathcal{R}) = 0$ generates the Borel sigma-algebra for every $g \in \mathcal{F}$. Then, Kolmogorov–Sinai theorem implies that $h_{\mu_n}(f_n) = h_{\mu_n}(f_n,\mathcal{R})$ and $h_\eta(f) = h_\eta(f,\mathcal{R})$, that is,

$$h_{\mu_n}(f_n) = \inf_{k\geqslant 1}\frac{1}{k}H_{\mu_n}\big(\mathcal{R}_n^{(k)}\big) \quad \text{and} \quad h_\eta(f) = \inf_{k\geqslant 1}\frac{1}{k}H_\eta\big(\mathcal{R}^{(k)}\big),$$

where $H_\eta(\mathcal{R}) = \sum_{R\in\mathcal{R}} -\eta(R)\log\eta(R)$ and we considered the dynamically defined partitions

$$\mathcal{R}_n^{(k)} = \bigvee_{j=0}^{k-1} f_n^{-j}(\mathcal{R}) \quad \text{and} \quad \mathcal{R}^{(k)} = \bigvee_{j=0}^{k-1} f^{-j}(\mathcal{R}).$$

Since η gives zero measure to the boundary of $\mathcal{R}$ then $H_{\mu_n}(\mathcal{R}_n^{(k)})$ converge to $H_\eta(\mathcal{R}^{(k)})$ as $n\to\infty$ by weak* convergence. Furthermore, for every $\varepsilon > 0$ there is $N \geqslant 1$ such that

$$h_{\mu_n}(f_n) \leqslant \frac{1}{N}H_{\mu_n}\big(\mathcal{R}_n^{(N)}\big) \leqslant \frac{1}{N}H_\eta\big(\mathcal{R}^{(N)}\big) + \varepsilon \leqslant h_\eta(f) + 2\varepsilon.$$

Recalling the continuity assumption of the topological pressure $P_{\mathrm{top}}(f,\phi)$ on the data (f,ϕ), that μ_n is an equilibrium state for f_n with respect to ϕ_n, and that $\int \phi_n\, d\mu_n \to \int \phi\, d\eta$ as $n\to\infty$, it follows that

$$h_\eta(f) + \int \phi\, d\eta \geqslant P_{\mathrm{top}}(f,\phi).$$

This shows that η is an equilibrium state for f with respect to ϕ. Since every equilibrium state belongs to the convex hull of ergodic equilibrium states and these coincide with finitely many ergodic measures absolutely continuous with respect to ν (recall Theorem B), this completes the proof of Theorem D. □

We finish this subsection with some comments on the assumption involving the continuity of the topological pressure. The map $\phi \mapsto P_{\mathrm{top}}(f,\phi)$ varies continuously, provided that f is a continuous transformation (see for instance [54, Theorem 9.5]). On the other hand, in this setting the topological pressure $P_{\mathrm{top}}(f,\phi)$ coincides with $\log\lambda_{f,\phi}$, where $\lambda_{f,\phi}$ is the spectral radius of the transfer operator $\mathcal{L}_{f,\phi}$, *for every* $f \in \mathcal{F}$ *and every* $\phi \in \mathcal{W}$. Moreover, the operators $\mathcal{L}_{f,\phi}$ vary continuously with the data (f,ϕ). So, the continuous variation of the topological pressure should be a consequence of the most likely spectral gap for the transfer operator $\mathcal{L}_{f,\phi}$ in the space of Hölder continuous observables. Some spectral gap properties were obtained by Arbieto, Matheus [5] and Varandas [53] in a related context.

7.2. *Stochastic stability*

The results in this section are inspired by some analogous in [2]. First we introduce some definitions and notations. Given $\underline{f} \in \mathcal{F}^{\mathbb{N}}$, define $\underline{f}^j = f_j \circ \cdots \circ f_2 \circ f_1$. Let $(\theta_\varepsilon)_{0<\varepsilon\leqslant 1}$ be a family of probability measures in $\mathcal{F}$. Given a (not necessarily invariant) probability measure ν, we say that (f,ν) is *non-uniformly expanding along random orbits* if there exists $c>0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} \log \bigl\| Df\bigl(\underline{f}^j(x)\bigr)^{-1}\bigr\| \leqslant -2c < 0$$

for $(\theta_\varepsilon^{\mathbb{N}} \times \nu)$-almost every $(\underline{f},x) \in \mathcal{F}^{\mathbb{N}} \times M$. If this is the case, Pliss's lemma guarantees the existence of infinitely many hyperbolic times for almost every point where, in this setting, $n \in \mathbb{N}$ is a *c-hyperbolic time for* $(\underline{f},x) \in \mathcal{F}^{\mathbb{N}} \times M$ if

$$\prod_{j=n-k}^{n-1} \bigl\| Df\bigl(\underline{f}^j(x)\bigr)^{-1}\bigr\| < e^{-ck} \quad \text{for every } 0 \leqslant k \leqslant n-1.$$

We refer the reader to [2, Proposition 2.3] for the proof. Given $\varepsilon>0$, let $n_1^\varepsilon : \mathcal{F}^{\mathbb{N}} \times M \to \mathbb{N}$ denote the first hyperbolic time map. Set also $H_n(\underline{f}) = \{x \in M\colon n \text{ is a } c\text{-hyperbolic time for } (\underline{f},x)\}$. In the remaining of the section let $f \in \mathcal{F}$ and ν be an expanding conformal measure such that $\operatorname{supp}\nu = H$. The next result shows that f has random non-uniform expansion. More precisely,

Lemma 7.1. *Let $(\theta_\varepsilon)_{0<\varepsilon\leqslant 1}$ be a family of probability measures in $\mathcal{F}$ such that* $\operatorname{supp}\theta_\varepsilon$ *is contained in a small neighborhood $V_\varepsilon(f)$ of f and $\bigcap_\varepsilon V_\varepsilon(f) = \{f\}$. If $\mathcal{F} \ni g \mapsto J_\nu g$ is a continuous function and ε is small enough then (f,ν) is non-uniformly expanding along every random orbit of $(\hat{f},\theta_\varepsilon)$. Furthermore,*

$$\bigl(\theta_\varepsilon^{\mathbb{N}} \times \nu\bigr)\bigl(\bigl\{(\underline{f},x) \in \mathcal{F}^{\mathbb{N}} \times M\colon n_1^\varepsilon(\underline{f},x) > k\bigr\}\bigr)$$

decays exponentially fast and, consequently, $\int n_1^\varepsilon \, d(\theta_\varepsilon^{\mathbb{N}} \times \nu) < \infty$.

Proof. Given $g \in \mathcal{F}$, let $\mathcal{A}_g \subset M$ be the region described in (H1) and (H2). Denote by $\tilde{\mathcal{A}}$ the enlarged set obtained as the union of the regions $\mathcal{A}_g$ taken over all $g \in \operatorname{supp}\theta_\varepsilon$. If $\varepsilon>0$ is small enough then we can assume that $\tilde{\mathcal{A}}$ is contained in the same q elements of the covering $\mathcal{P}$ as the set $\mathcal{A}_f$.

Now we claim that, if γ is chosen as before and $\underline{f} \in \mathcal{F}^{\mathbb{N}}$ the measure of the set

$$B(n,\underline{f}) = \left\{x \in M\colon \frac{1}{n}\#\bigl\{0 \leqslant j \leqslant n-1\colon \underline{f}^j(x) \in \tilde{\mathcal{A}}\bigr\} \geqslant \gamma\right\}$$

decays exponentially fast. Indeed, the same proof of Lemma 3.1 yields that $B(n,\underline{f})$ is covered by at most $e^{(\log q+\varepsilon_0/2)n}$ elements of $\mathcal{P}^{(n)}(\underline{f}) = \bigvee_{j=0}^{n-1} \underline{f}^{-j}(\mathcal{P})$, for every large n. On the other hand, since $\operatorname{supp}(\theta_\varepsilon)$ is compact the function $\operatorname{supp}\theta_\varepsilon \ni g \mapsto J_\nu g$ is uniformly continuous: for every $\varepsilon>0$ there exists $a(\varepsilon)>0$ (that tends to zero as $\varepsilon \to 0$) such that

$$e^{-a(\varepsilon)} \leqslant \frac{J_\nu f(x)}{J_\nu g(x)} \leqslant e^{a(\varepsilon)}$$

for every $g \in \operatorname{supp}(\theta_\varepsilon)$ and every $x \in M$. As in the proof of Proposition 4.4, this implies that

$$1 \geqslant \nu\bigl(\underline{f}^n(P)\bigr) = \int_P \prod_{j=0}^{n-1} J_\nu f_j \circ \underline{f}^j \, d\nu \geqslant e^{-a(\varepsilon)n} \int_P J_\nu f^n \, d\nu > e^{(\log q+\varepsilon_0-a(\varepsilon))n} \nu(P)$$

and, consequently, $\nu(P) \leqslant e^{-(\log q+\varepsilon_0-a(\varepsilon))n}$ for every $P \in \mathcal{P}^{(n)}(\underline{f})$ and every large n. Hence

$$\nu\big(B(n,\underline{f})\big) \leqslant \#\big\{P \in \mathcal{P}^{(n)}(\underline{f})\colon\ P \cap B(n,\underline{f}) \neq \emptyset\big\} \times e^{-(\log q+\varepsilon_0-a(\varepsilon))n}$$

which decays exponentially fast and proves the claim. Then, the set

$$\underline{B}(n) = \left\{(\underline{f},x) \in \mathcal{F}^{\mathbb{N}} \times M\colon\ \frac{1}{n}\#\big\{0 \leqslant j \leqslant n-1\colon\ \underline{f}^j(x) \in \tilde{\mathcal{A}}\big\} \geqslant \gamma\right\}$$

is such that $(\theta_\varepsilon \times \nu)(B(n)) = \int \nu(B(n,\underline{f}))\, d\theta_\varepsilon^{\mathbb{N}}(\underline{f})$ also decays exponentially fast. Borel–Cantelli guarantee that the frequency of visits of the random orbit $\{\underline{f}^j(x)\}$ to $\tilde{\mathcal{A}}$ is smaller than γ for $\theta_\varepsilon^{\mathbb{N}} \times \nu$-almost every $(\underline{f},x)$. Moreover, since every $g \in \mathcal{F}$ satisfy (H1) and (H2) with uniform constants this proves that f is non-uniformly expanding along random orbits. Moreover, the first hyperbolic time map n_1^ε is integrable because

$$\int n_1\, d\big(\theta_\varepsilon^{\mathbb{N}} \times \nu\big) = \sum_{n\geqslant 0}\big(\theta_\varepsilon^{\mathbb{N}} \times \nu\big)\big(\{n_1 > n\}\big) \leqslant \sum_{n\geqslant 0}\big(\theta_\varepsilon^{\mathbb{N}} \times \nu\big)\big(B(n)\big) < \infty.$$

This completes the proof of the lemma. □

Remark 7.2. Before proceeding with the proof, let us discuss briefly the continuity assumption on $\mathcal{F} \ni g \to J_\nu g$. First notice that in our setting this is automatically satisfied when ν coincides with the Lebesgue measure since it reduces to the continuity of $g \mapsto \log|\det Dg|$. Given $g \in \mathcal{F}$, let ν_g denote the expanding conformal measure and set $P_g = P_{\mathrm{top}}(f,\phi)$. Observe that if k is a c-hyperbolic time for x with respect to f then it is a $c/2$-hyperbolic time for x with respect to every g sufficiently close to f. Consequently

$$K(c/2,\delta)^{-2}e^{-|P_f-P_g|k} \leqslant \frac{\nu_g(B(x,k,\delta))}{\nu_f(B(x,k,\delta))} \leqslant K(c/2,\delta)^2 e^{|P_f-P_g|k},$$

which proves that the conformal measures ν_f and ν_g are comparable at hyperbolic times and that $J_\nu g = d(g_*^{-1}\nu)/d\nu$ is a well defined object in the domain of each inverse branch g^{-1}. So, in general, the relation above indicates that the continuity of the topological pressure should play a crucial role to obtain the continuity of the Jacobian $\mathcal{F} \ni g \to J_\nu g$.

Given $n \geqslant 1$ define $f_x^n : \mathcal{F}^{\mathbb{N}} \to M$ given by $f_x^n(\underline{g}) := \underline{g}^n(x)$. Since f is non-uniformly expanding and non-uniformly expanding along random orbits then there are finitely many ergodic stationary measures absolutely continuous with respect to ν. More precisely,

Theorem 7.3. *Let $(\theta_\varepsilon)_\varepsilon$ be a non-degenerate random perturbation of $f \in \mathcal{F}$. Given $\varepsilon > 0$ there are finitely many ergodic stationary measures $\mu_1^\varepsilon, \mu_2^\varepsilon, \dots, \mu_l^\varepsilon$ that are absolutely continuous with respect to the conformal measure ν and*

$$\mu_i^\varepsilon = \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\int \underline{f}_*^j\big(\nu \mid B\big(\mu_i^\varepsilon\big)\big)\, d\theta_\varepsilon^{\mathbb{N}}(\underline{f}), \tag{7.1}$$

for every $1 \leqslant i \leqslant l$. In addition, $l \geqslant 1$ can be taken constant for every sufficiently small ε.

Proof. This proof follows closely the one of Theorem C in [2]. For that reason we give a brief sketch of the proof and refer the reader to [2] for details. It is easy to check that any accumulation point μ^ε of the sequence of probability measures

$$\frac{1}{n}\sum_{j=0}^{n-1}\big(f_x^j\big)_*\theta_\varepsilon^{\mathbb{N}} \tag{7.2}$$

on M is a stationary measure. Moreover, any stationary measure μ^ε is absolutely continuous with respect to ν because of the non-degeneracy of the random perturbation and

$$\mu^\varepsilon(E) = \int \mu^\varepsilon\big(g^{-1}(E)\big)\, d\theta_\varepsilon(g) = \int 1_E\big(g(x)\big)\, d\theta_\varepsilon(g)\, d\mu^\varepsilon(x) = \int \big((f_x)_*\theta_\varepsilon^{\mathbb{N}}\big)(E)\, d\mu^\varepsilon$$

for every measurable set E.

On the other hand, by the ergodic decomposition of the F-invariant probability measure $\theta_\varepsilon^{\mathbb{N}} \times \mu^\varepsilon$ there are ergodic stationary measures. We prove that there can be at most finitely many of them. Indeed, a point x belongs to the basin of attraction $B(\mu^\varepsilon)$ of an ergodic stationary measure μ^ε if and only if

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi\big(\underline{f}^j(x)\big) \to \int \psi \, d\mu^\varepsilon \tag{7.3}$$

for every $\psi \in C(M)$ and $\theta_\varepsilon^{\mathbb{N}}$-almost every $\underline{f} \in \mathcal{F}^{\mathbb{N}}$. In addition, if $x \in B(\mu^\varepsilon)$ then $g(x) \in B(\mu^\varepsilon)$ for every $g \in \operatorname{supp}(\theta_\varepsilon)$. Furthermore, the non-degeneracy of the random perturbation implies that $B(\mu^\varepsilon)$ contains the ball of radius r_ε centered at $f(x)$. Then, the compactness of M implies that there are finitely many ergodic absolutely continuous stationary measures $\mu_1^\varepsilon, \ldots, \mu_l^\varepsilon$, with $1 \leqslant l \leqslant l(\varepsilon)$. Since $\nu(B(\mu_i^\varepsilon)) > 0$, integrating (7.3) with respect to ν and using the dominated convergence theorem one obtains

$$\int \psi \, d\mu_i^\varepsilon = \lim_n \frac{1}{n}\sum_{j=0}^{n-1} \int_{B(\mu_i^\varepsilon)} \psi \circ \underline{f}^j \, d\nu = \lim_n \frac{1}{n}\sum_{j=0}^{n-1} \int \psi \, d\underline{f}^j_*\big(\nu \mid B\big(\mu_i^\varepsilon\big)\big)$$

for every $\psi \in C(M)$ and $\theta_\varepsilon^{\mathbb{N}}$-almost every $\underline{f} \in \mathcal{F}$. This proves the first statement of the theorem.

It remains to show that $l = l(\varepsilon)$ can be chosen constant for every sufficiently small ε. The support of each stationary measure μ_i^ε is an invariant set with non-empty interior (see [2]). Since f is non-uniformly expanding then $\operatorname{supp}(\mu_i^\varepsilon)$ contains some hyperbolic pre-ball $V_n(x)$ associated to f and, by invariance, a ball of radius δ. This proves that $l(\varepsilon) \leqslant l_0$ for every small $\varepsilon > 0$. On the other direction, since the set $\operatorname{supp}(\mu_i^\varepsilon)$ has positive ν-measure and is forward invariant by f it must be contained in the support of some ergodic stationary measure $\mu_i^{\varepsilon'}$ for every ε' smaller than ε. This proves the l can be taken constant for small ε and completes sketch of the proof of the theorem. □

Now we are in a position to prove that the equilibrium states constructed in Theorem A are stochastically stable.

Proof of Theorem E. Let $(\mu^\varepsilon)_{\varepsilon>0}$ be a sequence of stationary measures absolutely continuous with respect to ν and let η be any weak* accumulation point. Theorem 7.3 implies that there is $l \geqslant 1$ such that there are exactly l ergodic stationary measures $\mu_1^\varepsilon, \ldots, \mu_l^\varepsilon$ that are absolutely continuous with respect to ν, for every sufficiently small ε. Furthermore,

$$\mu_i^\varepsilon = \lim_{n\to\infty} \nu_{n,i}^\varepsilon \quad \text{where} \quad \nu_{n,i}^\varepsilon = \frac{1}{n}\sum_{j=0}^{n-1} \int \underline{f}^j_*\big(\nu \mid B(\mu_i^\varepsilon)\big)\, d\theta_\varepsilon^{\mathbb{N}}(\underline{f}).$$

Proceed as in the beginning of Subsection 5.1 and write $\nu_n^\varepsilon \leqslant \xi_n^\varepsilon + \frac{1}{n}\sum_{j=0}^{n-1} \eta_j^\varepsilon$ with

$$\xi_{n,i}^\varepsilon = \frac{1}{n}\sum_{j=0}^{n-1} \int_{B(\mu_i^\varepsilon)} \underline{f}^j_*\big(\nu \mid H_j(\underline{f})\big)\, d\theta_\varepsilon^{\mathbb{N}}(\underline{f})$$

and

$$\eta_{n,j}^\varepsilon = \sum_{k>0} \int_{B(\mu_i^\varepsilon)} \underline{f}^k_*\big(\big[\underline{f}^j_*\big(\nu \mid H_j(\underline{f})\big)\big] \bigm| \big\{n_1^\varepsilon\big(\cdot, \sigma^j(\underline{f})\big) > k\big\}\big)\, d\theta_\varepsilon^{\mathbb{N}}(\underline{f}).$$

The arguments from Section 5 and the uniform integrability of $\varepsilon \mapsto n_1^\varepsilon \in L^1(\theta_\varepsilon^{\mathbb{N}} \times \nu)$ yield that each measure $\nu_{n,i}^\varepsilon$ is absolutely continuous with respect to ν with density bounded from above by a constant depending only on ε. By weak* convergence it follows that η is also absolutely continuous with respect to ν and, consequently, η belongs to the convex hull of finitely many ergodic equilibrium states $\mu_1, \ldots, \mu_k$ for f with respect to ϕ. This completes the proof of the theorem. □

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