Radial Bargmann representation for the Fock space of type B

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Abstract

Let $\nu_{\alpha,q}$ be the probability and orthogonality measure for the q-Meixner-Pollaczek orthogonal polynomials, which has appeared in [BEH15] as the distribution of the (α, q)-Gaussian process (the Gaussian process of type B) over the (α, q)-Fock space (the Fock space of type B). The main purpose of this paper is to find the radial Bargmann representation of $\nu_{\alpha,q}$. Our main results cover not only the representation of q-Gaussian distribution by [LM95], but also of q^2-Gaussian and symmetric free Meixner distributions on $\mathbb{R}$. In addition, non-trivial commutation relations satisfied by (α, q)-operators are presented.

Keywords: Radial Bargmann representation, deformation, Fock spaces, q-orthogonal polynomials.

2010 Mathematics Subject Classification: 33D45, 46L53, 60E99.

1 Introduction

Bożejko-Ejsmont-Hasebe [BEH15] considered a deformation of the (algebraic) full Fock space with two parameters α and q, namely, the (α, q)-Fock space (or the Fock space of type B) $\mathcal{F}_{\alpha,q}(H)$ over a complex Hilbert space H. The deformation with $\alpha = 0$ is equivalent to the q-deformation by Bożejko-Speicher [BS91] and Bożejko-Kümmerer-Speicher [BKS97], and the corresponding q-Bargmann-Fock space has been constructed by van Leeuwen-Maassen [LM95].

For the construction of $\mathcal{F}_{\alpha,q}(H)$, their starting point is to replace the Coxeter group of type A, that is, symmetric group S_n for the q-Fock space by the Coxeter group of type B, $\Sigma_n := \mathbb{Z}_2^n \rtimes S_n$ in (A.1) of the Appendix A. This replacement provides us a more general symmetrization operator on $H^{\otimes n}$ than that of [BS91] to define the (α, q)-inner product $\langle \cdot, \cdot \rangle_{\alpha,q}$ in (A.3). One can define annihilation $B^-_{\alpha,q}(f)$ and creation $B^+_{\alpha,q}(f)$ operators acting on $\mathcal{F}_{\alpha,q}(H)$, and the (α, q)-Gaussian process (the Gaussian process of type B) $G_{\alpha,q}(f)$ for $f \in H$ as the sum of them, $G_{\alpha,q}(f) := B^-_{\alpha,q}(f) + B^+_{\alpha,q}(f)$. It is one of their main interests to find a probability distribution $\mu_{\alpha,q,f}$ on $\mathbb{R}$ of $G_{\alpha,q}(f)$, $\|f\|_H = 1$, with respect to the vacuum state $\langle \Omega, \cdot\, \Omega \rangle_{\alpha,q}$. $\mathcal{F}_{\alpha,q}(H)$ equipped with $\langle \cdot, \cdot \rangle_{\alpha,q}$, $B^-_{\alpha,q}(f)$, and $B^+_{\alpha,q}(f)$ is a typical example of interacting Fock spaces in the sense of Accardi-Bożejko [AB98]. It suggests that the theory of orthogonal polynomials plays intrinsic roles in all previous works mentioned above. In fact, the measure $\mu_{\alpha,q,f}$ given in [BEH15, Theorem 3.3] is derived essentially from the orthogonality measure $\nu_{\alpha,q}$ associated with the q-Meixner-Pollaczek orthogonal polynomials $\{P_n^{(\alpha,q)}(x)\}$ for $\alpha, q \in (-1, 1)$ given by the recurrence relation,

$$\begin{cases} P_0^{(\alpha,q)}(x) = 1, \ P_1^{(\alpha,q)}(x) = x, \\ xP_n^{(\alpha,q)}(x) = P_{n+1}^{(\alpha,q)}(x) + (1 + \alpha q^{n-1})[n]_q P_{n-1}^{(\alpha,q)}(x), \quad n \geq 1 \end{cases}$$

where $[n]_q = 1 + q + \cdots + q^{n-1}$ is the q number. However, the Bargmann representation (measure on $\mathbb{C}$) of $\nu_{\alpha,q}$ has not been obtained yet except the case of $\alpha = 0$ for $0 \leq q < 1$ [LM95], for $q = 1$ [Barg61][AKK03], for $q = 0$ [Bi97], and t-deformed cases of these [AKW16][KW14], and for $q > 1$ [Kr98].

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Therefore, the main purpose of this paper is to find the radial Bargmann representation of the probability measure $\nu_{\alpha,q}$ on $\mathbb{R}$. Our results cover the radial Bargmann representations of q-Gaussian, symmetric free Meixner (Kesten) and q^2-Gaussian distributions on $\mathbb{R}$.

The organization of this paper will be as follows. In Section 2, we shall explain how the (α, q)-Fock space is related to the notion of one-mode interacting Fock spaces and Bargmann representation. In Section 3, the radial Bargmann representation of $\nu_{\alpha,q}$ is constructed explicitly in Theorem 3.11. In Section 4, commutation relations satisfied by one-mode (α, q)-annihilation and creation operators will be treated. In the Appendix, we shall give a minimum reference on the Coxeter group of type B extracted from [BEH15].

2 Key ideas and our purpose

Let us point out some of the keys to calculate the distribution of $G_{\alpha,q}(f)$ in [BEH15]. It is shown that a linear map, $\Phi : \mathrm{Span}\{f^{\otimes n} \mid f \in H,\ n \geq 0\} \to L^2(\mathbb{R}, \mu_{\alpha,q,f})$ given by $\Phi(f^{\otimes n}) = P_n^{(\alpha\langle f,\overline{f}\rangle_H,q)}(x)$, is an isometry and a relation under $\|f\|_H = 1$,

$$\begin{aligned} G_{\alpha,q}(f)f^{\otimes n} &= (B^+_{\alpha,q}(f) + B^-_{\alpha,q}(f))(f^{\otimes n}) \\ &= f^{\otimes(n+1)} + (1 + \alpha\langle f, \overline{f}\rangle_H)q^{n-1}[n]_q f^{\otimes(n-1)}, \end{aligned}$$

is satisfied where $\overline{f}$ denotes a self-adjoint involution of $f \in H$ in (A.2). This corresponds to the three terms recursion relation satisfied by $P_n^{(\alpha\langle f,\overline{f}\rangle_H,q)}(x)$ through Φ. Then, it is proved that $\mu_{\alpha,q,f} = \nu_{\alpha\langle f,\overline{f}\rangle_H,q}$ (see $\nu_{\alpha,q}$ in (3.3)) in the sense of

$$\langle \Omega, G_{\alpha,q}(f)^n\Omega\rangle_{\alpha,q} = \int x^n \mu_{\alpha,q,f}(dx) \tag{2.1}$$

where Ω denotes the vacuum vector. Therefore, in order to get the Bargmann representation of $\nu_{\alpha\langle f,\overline{f}\rangle_H,q}$, it is enough to consider that of $\nu_{\alpha,q}$ in the sense of Definition 2.2 given later.

Since the structure mentioned above can be reduced to the one-mode analogue of (α, q)-Fock spaces, let us recall fundamental relationships between one-mode interacting Bargmann-Fock spaces and the theory of orthogonal polynomials of one variable.

Definition 2.1. Let $\{\omega_n\}_{n=0}^\infty$ with $\omega_0 := 1$ be an infinite sequence of positive real numbers and $\{\alpha_n\}_{n=0}^\infty$ be of real numbers. A one-mode interacting Bargmann-Fock space $\mathcal{B}$ is defined as $\bigoplus_{n=0}^\infty \mathbb{C}\Phi_n$ equipped with $\Phi_n := z^n/[\omega_n]!$, $[\omega_n]! := \prod_{k=0}^n \omega_k$, the inner product $\langle \Phi_m, \Phi_n\rangle_{\mathcal{B}} = \delta_{m,n}$ for all $m, n \in \mathbb{N} \cup \{0\}$, operators of creation a^+, annihilation a^-, and conservation a° defined by

$$\begin{cases} a^+\Phi_n := \sqrt{\omega_{n+1}}\Phi_{n+1}, & n \geq 0, \\ a^-\Phi_0 = 0,\ a^-\Phi_n := \sqrt{\omega_n}\Phi_{n-1}, & n \geq 1, \\ a^\circ\Phi_n := \alpha_n\Phi_n, & n \geq 0. \end{cases} \tag{2.2}$$

Let $(\{\omega_n\}_{n=0}^\infty, \{\alpha_n\}_{n=0}^\infty)$ be a pair of sequences in Definition 2.1 and define a sequence of monic polynomials $\{P_n(x)\}$ recurrently by

$$\begin{cases} P_0(x) = 1,\ P_1(x) = x - \alpha_0, \\ xP_n(x) = P_{n+1}(x) + \omega_n P_{n-1} + \alpha_n P_n(x), \quad n \geq 1. \end{cases} \tag{2.3}$$

Then there exists a probability measure μ on $\mathbb{R}$ with finite moments of all orders such that $\{P_n(x)\}$ is the orthogonal polynomials with $\langle P_m(x), P_n(x)\rangle_{L^2(\mathbb{R},\mu)} = \delta_{m,n}[\omega_n]!$ for all $m, n \in \mathbb{N}\cup\{0\}$. (See [Chi78][HO07], for example.)

It is easy to see that a linear map

$$U : \mathcal{B} = \bigoplus_{n=0}^{\infty} \mathbb{C}\Phi_n \to L^2(\mathbb{R}, \mu)$$

defined by $U\left(\sqrt{[\omega_n]!}\Phi_n\right) = P_n(x)$ is an isometry and in addition $a^+ + a^- + a^\circ = U^*XU$ is satisfied due to (2.2) and (2.3), where X represents the multiplication operator by x in $L^2(\mathbb{R}, \mu)$. This intertwining relation provides a notion of the quantum decomposition of a classical random variable X and

$$\langle \Phi_0, (a^+ + a^- + a^\circ)^n \Phi_0 \rangle_{\mathcal{B}} = \int x^n \mu(dx). \tag{2.4}$$

Therefore, if $\omega_n = (1 + \alpha q^{n-1})[n]_q,\ \alpha_n = 0$, the equality in (2.4) is a one-mode analogue of (2.1).

Now it is interesting to consider the following moment problem to realize the inner product by the integral:

Problem 1. For a given $\{\omega_n\}$ of μ, find a probability measure γ_μ satisfying the equality,

$$\int_{\mathbb{C}} \overline{z}^m z^n \gamma_\mu(d^2 z) = \delta_{m,n}[\omega_n]! \tag{2.5}$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

Definition 2.2. A measure γ_μ satisfying the equality (2.5) is called a Bargmann representation (measure on $\mathbb{C}$) of a measure μ on $\mathbb{R}$.

It was proved in [Sz07] (see also [AKW16][KW14]) that if a measure μ admits any Bargmann representation, then it also admits a radial (rotation invariant) Bargmann representation

$$\gamma_\mu(d^2 z) = \frac{1}{2\pi}\lambda_{[0,2\pi)}(d\theta)\rho_\mu(dr),\ z = re^{i\theta},\ r \geq 0,\ \theta \in [0, 2\pi),$$

where $\lambda_{[0,2\pi)}$ is the Lebesgue measure on $[0, 2\pi)$. It says that the angular part takes care of orthogonality of (2.5). Therefore, Problem 1 can be transformed into the following Problem 2:

Problem 2. Find a positive radial measure ρ_μ satisfying

$$\int_0^\infty r^{2n}\rho_\mu(dr) = [\omega_n]!$$

for all $m, n \in \mathbb{N} \cup \{0\}$.

Main Purpose: We shall consider Problem 2 associated with $\omega_n = (1 + \alpha q^{n-1})[n]_q,\ \alpha_n = 0$ of $\nu_{\alpha.q}$ in Section 3. Furthermore, commutation relations satisfied by a^+, a^- acting on $\mathcal{B}$ associated with $\omega_n = (1 + \alpha q^{n-1})[n]_q$ will be presented in Section 4.

Remark 2.3. (1) One can notice that γ_μ and ρ_μ are determined only by $[\omega_n]!$. Therefore, it is enough in general for the Bargmann representation in the above sense to consider the symmetric measure μ with $\alpha_n = 0$ for all n, which implies that a° is a zero operator.

(2) If μ is symmetric, then $\alpha_n = 0$ for all n is implied. The converse statement is true if μ is determined by its moments.

3 (α, q)-Bargmann representation

3.1 q-Meixner-Pollaczek polynomials

Let us recall standard notations from q-calculus, which can be found in [GR04][KLS10] for example. The q-shifted factorials are defined by

$$(a; q)_0 := 1, \quad (a; q)_k := \prod_{\ell=1}^{k}(1 - aq^{\ell-1}),\ k = 1, 2, \ldots, \infty,$$

and the product of q-shifted factorials is defined by

$$(a_1, a_2; q)_k := (a_1; q)_k(a_2; q)_k, \qquad k = 1, 2, \ldots, \infty.$$

Remark 3.1. The q-shifted factorials are a natural extension of the Pochhammer symbol $(\cdot)_n$ because one can see that $\lim_{q\to 1}[k]_q = k$ implies

$$\lim_{q\to 1} \frac{(q^k;q)_n}{(1-q)^n} = (k)_n, \tag{3.1}$$

where $(k)_0 := 1,\ (k)_n := k(k+1)\cdots(k+n-1),\ n \geq 1$.

As we have mentioned, $\{P_n^{(\alpha,q)}(x)\}$ for $\alpha, q \in (-1,1)$ is the q-Meixner-Pollaczek polynomials satisfying the recurrence relation,

$$\begin{cases} P_0^{(\alpha,q)}(x) = 1,\ P_1^{(\alpha,q)}(x) = x, \\ xP_n^{(\alpha,q)}(x) = P_{n+1}^{(\alpha,q)}(x) + (1+\alpha q^{n-1})[n]_q P_{n-1}^{(\alpha,q)}(x), \quad n \geq 1. \end{cases} \tag{3.2}$$

It is known in [KLS10, 14.9.2] and [BEH15, page 1781] that the orthogonality measure $\nu_{\alpha.q}$ for such polynomials has the density of the form,

$$\frac{(q,\gamma^2;q)_\infty}{2\pi}\sqrt{\frac{1-q}{4-(1-q)x^2}}\left(\frac{g(x,1;q)g(x,-1;q)g(x,\sqrt{q};q)g(x,-\sqrt{q};q)}{g(x,i\gamma;q)g(x,-i\gamma;q)}\right), \tag{3.3}$$

supported on the interval $(-2/\sqrt{1-q}, 2/\sqrt{1-q})$ where

$$g(x,b;q) = \prod_{k=0}^{\infty}(1-4bx(1-q)^{-1/2}q^k + b^2q^{2k}),$$

and

$$\gamma = \begin{cases} \sqrt{-\alpha}, & \alpha < 0, \\ i\sqrt{\alpha}, & \alpha \geq 0. \end{cases}$$

Example 3.2. (1) If $\alpha = 0$, then q-Meixner-Pollaczek polynomials get back to the q-Hermite polynomials $H_n^{(q)}(x)$ whose orthogonality measure is the standard q-Gaussian measure on $(-2/\sqrt{1-q}, 2/\sqrt{1-q})$ given by

$$\nu_q(dx) := \frac{\sqrt{1-q}}{\pi}\sin\theta\prod_{n=1}^{\infty}(1-q^n)|1-q^ne^{2i\theta}|^2dx,$$

where $x\sqrt{1-q} = 2\cos\theta,\ \theta \in [0,\pi]$. Furthermore, one can get the standard Gaussian law as $q \to 1$, the Bernoulli law as $q \to -1$, and the standard Wigner's semi-circle law if $q = 0$. See [BKS97][BS91].

(2) The measure $\nu_{\alpha,0}$ is the symmetric free Meixner law [An03][BB06][SY01].

(3) The measure $\nu_{q,q}$ is the q^2-Gaussian law scaled by $\sqrt{1+q}$.

(4) If $\alpha = -q^{2\beta}$ as suggested in Remark 3.1, then the measure $\nu_{-q^{2\beta},q}$ under a certain scaling converges to the classical symmetric Meixner law as $q \uparrow 1$,

$$\frac{2^{2\beta}}{2\pi\Gamma(2\beta)}|\Gamma(\beta+ix)|^2dx, \quad x \in \mathbb{R}. \tag{3.4}$$

See also [KLS10, 14.9.15].

3.2 Problem

For $\alpha, q \in (-1,1)$, we would like to know when there exists a radial measure $\rho_{\nu_{\alpha,q}}$ satisfying

$$\int_0^\infty r^{2k}\,\rho_{\nu_{\alpha,q}}(dr) = (-\alpha;q)_k[k]_q!, \qquad k \in \mathbb{N}\cup\{0\}. \tag{3.5}$$

Here $[k]_q!$ denotes the q-factorials defined by

$$[0]_q! := 1, \quad [k]_q! := \prod_{\ell=1}^{k}[\ell]_q = \frac{(q;q)_k}{(1-q)^k},\ k \geq 1.$$

It is easy to get the inequality for $\alpha, q \in (-1,1)$,

$$|(-\alpha;q)_k[k]_q!| \leq \left(\frac{4}{1-|q|}\right)^k, \ k \in \mathbb{N}\cup\{0\}. \tag{3.6}$$

Due to Carleman criterion for the moment problem, this inequality implies that a radial measure $\rho_{\nu_{\alpha,q}}$ is determined uniquely by the sequence $\{(-\alpha;q)_k[k]_q!\}$.

We shall follow the procedure below to construct $\rho_{\nu_{\alpha,q}}$ in (3.5).

(1) Recall the radial part of the q-Gaussian measure on $\mathbb{C}$ (q-Bargmann measure), $\rho_{\nu_q} = \rho_{\nu_{0,q}}$, obtained in [LM95],

$$\int_0^\infty r^{2k}\, \rho_{\nu_q}(dr) = [k]_q!. \tag{3.7}$$

(2) Find a radial (possibly signed) measure $\rho_{\alpha,q}$ having the moment $(-\alpha;q)_k$.

(3) Compute the multiplicative (Mellin) convolution $\rho_{\nu_q} \circledast \rho_{\alpha,q}$ to get $\rho_{\nu_{\alpha,q}}$.

Remark 3.3. It is known [LM95] that a radial measure ρ_{ν_q} in (3.7) does not exist for $q<0$. However, one can see that the positivity assumption on q can be relaxed for $\rho_{\nu_{\alpha,q}}$ if $\alpha = q$. It will be discussed right after the proof of Proposition 3.6 and in Proposition 3.7.

3.3 Construction of (α, q)-radial measures

Lemma 3.4. *Suppose that $\alpha \in (-1,1)$ and $q \in [0,1)$. Let*

$$\rho_{\alpha,q} := (-\alpha;q)_\infty \sum_{n=0}^\infty \frac{(-\alpha)^n}{(q;q)_n}\delta_{q^{n/2}},$$

which is a signed measure. Then we have

$$\int_0^\infty r^{2k}\, \rho_{\alpha,q}(dr) = (-\alpha;q)_k, \qquad k \in \mathbb{N}\cup\{0\}.$$

In particular, if taking $\alpha=-q$, then one can see $\rho_{\nu_q} = D_{(1-q)^{-1/2}}(\rho_{-q,q})$, namely,

$$\int_0^\infty r^{2k}\, D_{(1-q)^{-1/2}}(\rho_{-q,q})(dr) = \frac{(q;q)_k}{(1-q)^k} = [k]_q!,$$

where $D_t(\lambda)$ is the push-forward of λ by the map $x \mapsto tx$ for a measure λ on $\mathbb{R}$.

Proof. Firstly, we have

$$\int_0^\infty r^{2k}\, \rho_{\alpha,q}(dr) = (-\alpha;q)_\infty \sum_{n=0}^\infty \frac{(-\alpha q^k)^n}{(q;q)_n}.$$

Since Euler's formula (see [GR04, 1.3.15]),

$$\frac{1}{(a;q)_\infty} = \sum_{n=0}^\infty \frac{a^n}{(q;q)_n}, \tag{3.8}$$

is known, we replace a by $-\alpha q^k$ in (3.8) to obtain

$$\begin{aligned}\int_0^\infty r^{2k}\, \rho_{\alpha,q}(dr) &= \frac{(-\alpha;q)_\infty}{(-\alpha q^k;q)_\infty} \\ &= (-\alpha;q)_k.\end{aligned}$$

The proof is complete. ∎

Remark 3.5. (1) The last equality in proof is due to the formula

$$(a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty}.$$

See [GR04, 1.2.30], for example.

(2) Euler's formula is considered as the q-analogue of exponential function e^a due to

$$\lim_{q\to 1} \frac{1}{((1-q)a;q)_n} = e^a.$$

Let

$$\begin{bmatrix} n \\ \ell \end{bmatrix}_q := \frac{[n]_q!}{[\ell]_q![n-\ell]_q!} = \frac{(q;q)_n}{(q;q)_\ell (q;q)_{n-\ell}}$$

be the q-binomial coefficients and $h_n(z \mid q)$ be the Rogers-Szegö polynomials [GR04][S05] defined by

$$h_n(z \mid q) = \sum_{\ell=0}^{n} \begin{bmatrix} n \\ \ell \end{bmatrix}_q z^\ell.$$

Proposition 3.6. *Suppose that* $\alpha \in (-1,1)$ *and* $q \in [0,1)$*. Let*

$$\rho_{\nu_{\alpha,q}} := \begin{cases} (-\alpha, q; q)_\infty \displaystyle\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} \mid q) \delta_{(1-q)^{-1/2} q^{n/2}}, & q > 0, \\ -\alpha \delta_0 + (1+\alpha)\delta_1, & q = 0, \end{cases} \tag{3.9}$$

which is a signed measure in general. Then we have

$$\int_0^\infty r^{2k} \, \rho_{\nu_{\alpha,q}}(dr) = \frac{(-\alpha, q; q)_k}{(1-q)^k} = (-\alpha; q)_k [k]_q!, \qquad k \in \mathbb{N} \cup \{0\}. \tag{3.10}$$

Proof. First of all, it is easy to show (3.10) for the case $q = 0$. Therefore, let us assume $q > 0$.

One can compute the multiplicative (Mellin) convolution $\circledast$ to get $\rho_{\nu_{\alpha,q}}$ as follows:

$$\begin{aligned} \rho_{\nu_{\alpha,q}} &= \rho_{\alpha,q} \circledast D_{(1-q)^{-1/2}}(\rho_{-q,q}) \\ &= (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} \frac{(-\alpha)^\ell q^{n-\ell}}{(q;q)_\ell (q;q)_{n-\ell}} \right) \delta_{(1-q)^{-1/2} q^{n/2}} \\ &= (-\alpha, q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} \mid q) \delta_{(1-q)^{-1/2} q^{n/2}}. \end{aligned}$$

On the other hand, by Lemma 3.4, we have

$$\int_0^\infty r^{2k} \, D_{(1-q)^{-1/2}}(\rho_{-q,q})(dr) = \frac{(q;q)_k}{(1-q)^k} = [k]_q!.$$

Therefore, one can get

$$\begin{aligned} \int_0^\infty r^{2k} \, \rho_{\nu_{\alpha,q}}(dr) &= \int_0^\infty r^{2k} \, \rho_{\alpha,q}(dr) \int_0^\infty r^{2k} \, D_{(1-q)^{-1/2}}(\rho_{-q,q})(dr) \\ &= (-\alpha; q)_k [k]_q!, \qquad k \in \mathbb{N} \cup \{0\}. \end{aligned}$$

□

In Proposition 3.6, we have obtained $\rho_{\nu_{\alpha,q}}$ for $\alpha \in (-1,1)$ and $q \in (0,1)$. Due to the term

$$\delta_{(1-q)^{-1/2} q^{n/2}} \ \text{ in } \ \rho_{\nu_{\alpha,q}},$$

it seems impossible for $q \in (-1,0)$ to define $\rho_{\nu_{\alpha,q}}$. However, if $-1 < \alpha = q < 0$ then $\nu_{q,q}$ coincides with a scaled q^2-Gaussian measure, and hence the Bargmann measure exists.

Proposition 3.7. *Suppose $-1 < \alpha = q < 0$. We define*

$$\begin{aligned}\rho_{\nu_{q,q}} &:= D_{(1+q)^{1/2}}(\rho_{\nu_{q^2}}) \\ &= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_n} \delta_{(1-q)^{-1/2}(-q)^n}.\end{aligned} \tag{3.11}$$

Then one can see

$$\int_0^\infty r^{2k}\, \rho_{\nu_{q,q}}(dr) = (1+q)^k [k]_{q^2}! = (-q;q)_k [k]_q!.$$

Proof. One can see by direct computations

$$\begin{aligned}(-q;q)_k[k]_q! &= \left\{\prod_{\ell=1}^{k}(1-(-q)q^{\ell-1})\right\}\left\{\prod_{\ell=1}^{k}\frac{1-q^\ell}{1-q}\right\} \\ &= (1+q)^k \prod_{\ell=1}^{k}\frac{1-q^{2\ell}}{1-q^2} \\ &= (1+q)^k [k]_{q^2}!.\end{aligned}$$

Thus $\rho_{\nu_{q,q}}$ can be defined as the radial measure for q^2-Gaussian measure on $\mathbb{C}$ scaled by $(1+q)^{1/2}$, namely, $\rho_{\nu_{q,q}} = D_{(1+q)^{1/2}}(\rho_{\nu_{q^2}})$. □

Remark 3.8. If we use the fact that $h_n(-1 \mid q) = 0$ for odd $n \geq 1$ (see proof of Lemma 3.9 below), we can extend the definition (3.9) to the case $-1 < \alpha = q < 0$. This will give an alternative way to define $\rho_{\nu_{q,q}}$ for $-1 < q < 0$, but both definitions give the same measure.

We need some properties of the Rogers-Szegö polynomials to know when the measure $\rho_{\nu_{\alpha,q}}$ becomes positive.

Lemma 3.9 ([MGH90])**.** *Suppose that $q \in (-1,1)$.*

(1) *If $n \geq 0$ is odd, then $h_n(x \mid q) \geq 0$ if and only if $x \geq -1$. Moreover, the point $x = -1$ is the unique zero of $h_n(x \mid q)$ on $\mathbb{R}$.*

(2) *If $n \geq 0$ is even, then $h_n(x \mid q) > 0$ for all $x \in \mathbb{R}$.*

Proof. It is known that all the zeros of $h_n(z \mid q)$ lie on the unit circle $|z| = 1$. See [MGH90] or [S05, Theorem 1.6.11]. Note that the result was obtained for $q \in [0,1)$, but the proof can be extended to $q \in (-1,1)$ without any modifications.

By definition, one can see

$$\begin{bmatrix} n \\ \ell \end{bmatrix}_q = \frac{(1-q^{n-\ell+1})(1-q^{n-\ell+2})\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^\ell)} > 0,$$

and hence $h_n(1 \mid q) > 0$ for all $n \geq 0$. Thus, $h_n(x \mid q) \neq 0$ for $x \in \mathbb{R} \setminus \{-1\}$. It then suffices to show that $h_n(-1 \mid q) > 0$ for all even $n \geq 0$ and $h_n(-1 \mid q) = 0$ for all odd $n \geq 1$. The recurrence relation for the Rogers-Szegö polynomials is known to be

$$h_{n+1}(z \mid q) = (z+1)h_n(z \mid q) - (1-q^n)z h_{n-1}(z \mid q), \qquad n \geq 1. \tag{3.12}$$

See [S05, 1.6.76] (note that formula (1.6.76) has an error of a sign). It is easy to see that $h_0(-1 \mid q) = 1 > 0, h_1(-1 \mid q) = 0$, so by induction and (3.12) one can show $h_n(-1 \mid q) > 0$ for all even $n \geq 0$ and $h_n(-1 \mid q) = 0$ for all odd $n \geq 1$. □

We need the following lemma in proof of Theorem 3.11 for the non-existence part of a radial Bargmann measure.

Lemma 3.10. *Let μ be a signed measure on $\mathbb{R}$ with compact support and let ν be a nonnegative measure on $\mathbb{R}$. If μ and ν have the same finite moments of all orders, then $\mu = \nu$.*

Proof. We denote by m_n the moments of μ (and ν by assumption). Since μ is compactly supported, say on $[-R,R]$,

$$|m_n| = \left|\int_{[-R,R]} x^n\,\mu(dx)\right| \le \|\mu\| R^n, \qquad n \in \mathbb{N}\cup\{0\},$$

where $\|\mu\|$ denotes the total variation of μ. Therefore, ν is also supported on $[-R,R]$. By Weierstrass' approximation, we have

$$\int_{[-R,R]} f(x)\,\mu(dx) = \int_{[-R,R]} f(x)\,\nu(dx) \tag{3.13}$$

for all $f \in C([-R,R])$. This implies that $\mu = \nu$ since, if we use the Hahn decomposition $\mu = \mu_+ - \mu_-$, then (3.13) implies

$$\int_{[-R,R]} f(x)\,\mu_+(dx) = \int_{[-R,R]} f(x)\,(\nu+\mu_-)(dx),$$

and hence $\mu_+ = \nu + \mu_-$ as nonnegative finite measures. □

In summary, the complete answer to the unique existence of a radial Bargmann representation of $\nu_{\alpha,q}$ is stated as follows:

Theorem 3.11. *Suppose that $\alpha, q \in (-1,1)$. The probability measure $\nu_{\alpha,q}$ has a radial Bargmann representation if and only if either (i) $q \ge 0$ and $\alpha \le q$ or (ii) $\alpha = q \ne 0$.*

In fact, the radial measure is given uniquely by

$$\rho_{\nu_{\alpha,q}} = \begin{cases} -\alpha\delta_0 + (1+\alpha)\delta_1 & (\alpha \le q = 0), \\ (-\alpha,q;q)_\infty \displaystyle\sum_{n=0}^\infty \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} \mid q)\delta_{(1-q)^{-1/2}q^{n/2}} & (q>0,\ \alpha<q), \\ (q^2;q^2)_\infty \displaystyle\sum_{n=0}^\infty \frac{q^{2n}}{(q^2;q^2)_n}\delta_{(1-q)^{-1/2}|q|^n} & (\alpha = q \ne 0). \end{cases}$$

Proof. **1. Existence and uniqueness.** If $q \in [0,1)$ and $\alpha \le q$, then by Proposition 3.6 and Lemma 3.9, the signed measure $\rho_{\nu_{\alpha,q}}$ is in fact a nonnegative measure and becomes the radial part of a Bargmann measure. The case $\alpha = q < 0$ was discussed in Proposition 3.7. Due to Carleman criterion for the moment problem, the inequality given in (3.6) guarantees the uniqueness of $\rho_{\nu_{\alpha,q}}$ for these cases.
2. Non-existence. (1) If $q \in (0,1)$ and $\alpha > q$, then $\rho_{\nu_{\alpha,q}}$ is not a nonnegative measure and is really a signed measure since $h_n(-\alpha/q \mid q) < 0$ for odd integers $n \ge 0$ and $q > 0$ from Lemma 3.9. By Lemma 3.10, if a radial Bargmann measure exists, then it must be equal to the signed measure $\nu_{\alpha,q}$. This is a contradiction. Thus, a radial Bargmann measure does not exist.
(2) If $q = 0$ and $\alpha > q = 0$ then by (3.9) $\nu_{\alpha,0}$ is really a signed measure, and hence by the same argument as above, a radial Bargmann measure does not exist.
(3) Let

$$\beta_k(\alpha,q) := (-\alpha;q)_k [k]_q!, \qquad k \ge 0, \alpha, q \in (-1,1).$$

Given $q < 0$ and $\alpha \ne q$, suppose that there exists a radial part of a Bargmann measure, ρ. Let ρ^2 be the push-forward of ρ by the map $x \mapsto x^2$. Then,

$$\beta_k(\alpha,q) = \int_0^\infty x^k\,\rho^2(dx) = \int_0^\infty x^{2k}\,\rho(dx). \tag{3.14}$$

By the way, by Proposition 3.6 it holds that $\beta_k(\alpha,q') = \int_0^\infty x^{2k}\,\rho_{\nu_{\alpha,q'}}(dx)$ for any $q' > 0$, that is,

$$\beta_k(\alpha,q') = (-\alpha,q';q')_\infty \sum_{n=0}^\infty \frac{(q')^n}{(q';q')_n} h_n(-\alpha(q')^{-1} \mid q') \frac{(q')^{kn}}{(1-q')^k}, \qquad q' > 0, \tag{3.15}$$

which is true even for $q' = q$ by analytic continuation.

Now let us consider the signed measure

$$\mu := (-\alpha,q;q)_\infty \sum_{n=0}^\infty \frac{q^n}{(q;q)_n} h_n(-\alpha q^{-1} \mid q)\delta_{(1-q)^{-1}q^n}, \qquad \alpha \ne q < 0,$$

supported on the points $\frac{q^n}{1-q}$ for $n = 0, 1, 2, 3, \ldots$. Then by (3.15) for $q' = q$ and by (3.14),

$$\int_{\mathbb{R}} x^k \, \mu(dx) = \beta_k(\alpha, q) = \int_0^\infty x^k \, \rho^2(dx), \qquad k \in \mathbb{N} \cup \{0\}.$$

By Lemma 3.10, the signed measure μ and the probability measure ρ^2 should be equal. However, the support of μ is not contained in $[0, \infty)$, and hence μ cannot be equal to ρ^2. This is a contradiction. □

Example 3.12. (1) The radial measure $\rho_{\nu_{0,q}}$ for $q \in [0, 1)$ is of the q-Bargmann [LM95].

(2) The radial measure $\rho_{\nu_{q,q}}$ for $q \in (-1, 1)$ is of the q^2-Bargmann.

(3) $\lim_{q \uparrow 1} \rho_{\nu_{\alpha,q}}$ is of the classical Bargmann [Barg61][AKK03].

(4) Consider $\alpha = -q^{2\beta}, \beta > 0$. This choice of α is suggested by (3.1) in Remark 3.1. In fact, one can see

$$\lim_{q \uparrow 1} \frac{(1 - q^{2\beta+n-1})[n]_q}{4(1-q)} = \frac{1}{4}(n + 2\beta - 1)n.$$

This limit sequence is the Jacobi sequence of the symmetric Meixner distribution in (3.4), so that $\rho_{\nu_{-q^{2\beta},q}}$ under suitable scaling converges weakly as $q \uparrow 1$ to the radial measure with the density,

$$\frac{2\pi r}{\Gamma(2\beta)} \int_0^\infty h(r, t/4) e^{-t} t^{2\beta-1} dt$$

where

$$h(r,t) = \frac{1}{\pi t} \exp\left(-\frac{r^2}{t}\right), \; r \in \mathbb{R}, \; t > 0.$$

This is an integral representation of the radial density for the Bessel kernel measure, which can be also represented by the modified Bessel function [As05][As09].

(5) $\rho_{\nu_{\alpha,0}}$ for $\alpha \in (-1, 0]$ is the radial measure for the symmetric free Meixner distribution. See Remark 3.13 below.

Remark 3.13. Let μ_t be a t-deformed probability measure of a probability measure μ on $\mathbb{R}$ defined through the Cauchy transform G_μ of μ,

$$\frac{1}{G_{\mu_t}(z)} := \frac{t}{G_\mu(z)} + (1-t)z, \quad t \geq 0,$$

examined by Bożejko-Wysoczański [BW98, BW01]. Krystek-Wojakowski [KW14] discussed the radial Bargmann representation of a t-deformed probability measure μ_t, t-Bargmann representation for short, and obtained necessary and sufficient condition for the admissibility of the representation. The t-Bargmann representation of the Kesten measure κ_t has the form,

$$\rho_{\kappa_t} = \left(1 - \frac{1}{t}\right)\delta_0 + \frac{1}{t}\delta_{\sqrt{t}}, \quad t \geq 1.$$

In [AKW16], the t-Bargmann representation of a symmetric free Meixner law $\varphi_{s,t}$ with two positive parameters s, t is treated and is admitted if and only if $t \geq 1$. In fact, one can see $\rho_{\varphi_{s,t}} = D_s(\rho_{\kappa_t})$ and hence

$$\rho_{\nu_{(1-t)/t,0}} = \rho_{\varphi_{1/\sqrt{t},t}} = D_{1/\sqrt{t}}(\rho_{\kappa_t}), \quad t \geq 1.$$

Therefore, the case (5) in Example 3.12 can be viewed as a t-Bargmann representation, too.

Furthermore, let us state the t-deformed version of Theorem 3.11 for $q \neq 0$ without proof:

Proposition 3.14. *The t-deformed version of $\rho_{\nu_{\alpha,q}}$ for $q \neq 0$ is given by*

$$\left(1 - \frac{1}{t}\right)\delta_0 + \frac{1}{t}\rho_{\nu_{\alpha,q}}, \quad t \geq 1.$$

Remark 3.15. The t-Bargmann representation of ν_q is treated in [KW14] for $q = 1$ and [AKW16] for $0 \le q < 1$.

Before closing this section, let us give a short remark about relations with the free infinite divisibility. Many of particular examples have so far suggested that the free infinite divisibility of a probability measure implies the existence of a radial Bargmann representation. The converse is not true in general because the Askey-Wimp-Kerov distribution $\mu_{9/10}$ for instance, discussed in [BBLS11], is not freely infinitely divisible, but it has a Bargmann representation with a gamma distribution as its radial measure. However, not many counterexamples have been found.

Therefore, we conjecture that the free infinite divisibility of our (α, q)-Gaussian distribution is equivalent to the existence of its radial Bargmann measure:

Conjecture. Suppose that $\alpha, q \in (-1, 1)$. The probability measure $\nu_{\alpha,q}$ is freely infinitely divisible if and only if if and only if $\alpha = q$ or $\alpha < q \ge 0$.

This conjecture is guaranteed to be true in the restricted subfamilies $\{\nu_{\alpha,0} \mid \alpha \in (-1, 1)\}$ ([SY01, Theorem 3.2]), $\{\nu_{0,q} \mid -1 < q < 1\}$ ([ABBL10] and [AH13, Example 3.11] for the free infinite divisibility), and $\{\nu_{q,q} \mid q \in (-1, 1)\}$ (all measures in this family are freely infinitely divisible since they are q^2-Gaussians).

4 Commutation relations among one-mode (α, q)-operators

Definition 4.1. Suppose that $\alpha, q \in (-1, 1)$ and f is analytic on $\mathbb{C}$.

(1) Let Z be the multiplication operator defined by

$$(Zf)(z) := zf(z).$$

(2) Let D_q be the Jackson derivative given by

$$(D_q f)(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

(3) The α-deformed Jackson derivative is given as

$$D_{\alpha,q} := \begin{cases} D_q + \alpha q^{2N} D_{1/q}, & q \neq 0, \\ D_0 + \alpha \frac{d}{dz}\big|_0, & q = 0, \end{cases}$$

where N is the number operator. For $q \neq 0$, we can also write

$$D_{\alpha,q} = D_q + \frac{\alpha}{q^2} D_{1/q} q^{2N}.$$

Remark 4.2. It is easy to check that the α-deformed Jackson derivative is equivalently defined as

$$(D_{\alpha,q} f)(z) = (D_q f)(z) + \alpha (D_{1/q} f)(q^2 z), \quad q \neq 0.$$

For example, if $f(z) = z^n$, $(D_{\alpha,q} f)(z) = (1 + \alpha q^{n-1})[n]_q z^{n-1}$ holds. In fact, the α-deformed Jackson derivative is an analogue of the operator in [BEH15, Theorem2.5].

Then, one can realize one-mode analogue of (α, q)-operators on an appropriate domain of the one-mode interacting Bargmann-Fock space $\mathcal{B}$ with $\omega_n = (1 + \alpha q^{n-1})[n]_q$ and $\alpha_n = 0$ by

$$a^+ := Z, \; a^- := D_{\alpha,q}, \text{ and } \Phi_n := \frac{z^n}{\sqrt{[\omega_n]!}}.$$

In fact, it is easy to check that

$$\begin{cases} a^+ \Phi_n = \sqrt{\omega_{n+1}} \Phi_{n+1}, \\ a^- \Phi_n = \sqrt{\omega_n} \Phi_{n-1}, \end{cases}$$

hold and the q-commutation relation, one-mode analogue of (A.4),

$$\begin{aligned}[a^-, a^+]_q \Phi_n &:= (a^- a^+ - q a^+ a^-)\Phi_n \\ &= (I + \alpha q^{2N})\Phi_n,\end{aligned}$$

is satisfied. Let us put $M_{\alpha,q} = I + \alpha q^{2N}$ and then one can get the expression,

$$M_{\alpha,q} = (1+\alpha)I - \alpha(1-q^2) Z D_{q^2},$$

due to $(ZD_{q^2})\Phi_n = [n]_{q^2}\Phi_n$.

Therefore one can obtain the following

Theorem 4.3. *Suppose $\alpha \in (-1,1)$ and $q \in (-1,1)$. Then the following are satisfied.*

(1) $[a^-, a^+]_q = M_{\alpha,q}$, $[a^-, M_{\alpha,q}]_{q^2} = (1-q^2)a^-$, $[M_{\alpha,q}, a^+]_{q^2} = (1-q^2)a^+$.

(2) $M_{\alpha,q} = (1+\alpha)I - \alpha(1-q^2)ZD_{q^2}$.

(3) *In particular, if $\alpha = q$, then one can obtain a more refined relation,* $[a^-, a^+]_{q^2} = (1+q)I$.

Example 4.4. (1) $\alpha = 0$ implies $[a^-, a^+]_q = I$. Hence $M_{0,q} = I$ commutes with both a^+ and a^-,

$$[a^-, M_{0,q}]_1 = [M_{0,q}, a^+]_1 = 0.$$

Therefore, the case $\alpha \neq 0$ provides non-trivial commutation relations.

(2) If $\alpha = -q^{2\beta}$ for $\beta > 0$, then the limiting case of the scaled operator is obtained as

$$\lim_{q\uparrow 1} \frac{M_{-q^{2\beta},q}}{1-q^2} = \lim_{q\uparrow 1} \frac{I - q^{2\beta}q^{2N}}{1-q^2} = N + \beta.$$

Moreover, let us consider the scaled operators,

$$A^{\pm} := \lim_{q\uparrow 1} \frac{a^{\pm}}{\sqrt{1-q^2}}.$$

Then one can get

$$[A^-, A^+]_1 = N + \beta$$

and hence

$$[A^-, N]_1 = A^-, \ [N, A^+]_1 = A^+.$$

It should be noted that these are the commutation relations for the classical Meixner-Pollaczek polynomials with respect to the symmetric Meixner distribution in (3.4). See [As08].

Acknowledgments. N. Asai was partially supported by Grant-in-Aid for Scientific Research (C) 23540131, JSPS. M. Bożejko was partially supported by the MAESTRO grant DEC-2011/02/A/ ST1/00119 and OPUS grant DEC-2012/05/B/ST1/00626 of National Center of Science. T. Hasebe was supported by Grant-in-Aid for Young Scientists (B) 15K17549, JSPS.

A Appendix

Let Σ_n be the set of bijections σ of the $2n$ points $\{\pm 1, \pm 2, \cdots, \pm n\}$ with $\sigma(-k) = -\sigma(k)$. Equipped with the composition operation as a product, Σ_n becomes what is called a Coxeter group of type B. It is generated by $\pi_0 := (1, -1)$ and $\pi_i := (i, i+1)$, $1 \le i \le n-1$, which satisfy the generalized braid relations

$$\begin{cases} \pi_i^2 = e, & 0 \le i \le n-1, \\ (\pi_0\pi_1)^4 = (\pi_i \pi_{i+1})^3 = e, & 1 \le i \le n-1, \\ (\pi_i\pi_j)^2 = e, & |i-j| \ge 2, \ 0 \le i,j \le n-1. \end{cases} \tag{A.1}$$

An element $\sigma \in \Sigma_n$ expresses an irreducible form,

$$\sigma = \pi_{i_1} \cdots \pi_{i_k}, \quad 0 \le i_1, \ldots, i_k \le n-1,$$

and in this case

$$\begin{aligned} &\ell_1(\sigma) := \text{the number of } \pi_0 \text{ in } \sigma, \\ &\ell_2(\sigma) := \text{the number of } \pi_i,\ 1 \le i \le n-1, \text{ in } \sigma \end{aligned}$$

are well defined. Let H be a separable Hilbert space. For a given self-adjoint involution $f \mapsto \overline{f}$ for $f \in H$, an action of Σ_n on $H^{\otimes n}$ is defined by

$$\begin{cases} \pi_0(f_1 \otimes \cdots \otimes f_n) = \overline{f_1} \otimes f_2 \otimes \cdots \otimes f_n, & n \ge 1, \\ \pi_i(f_1 \otimes \cdots \otimes f_n) = f_1 \otimes \cdots \otimes f_{i-1} \otimes f_{i+1} \otimes f_i \otimes f_{i+2} \otimes \cdots \otimes f_n, & n \ge 2,\ 1 \le i \le n-1. \end{cases} \tag{A.2}$$

The (α, q)-inner product on the full Fock space $\mathcal{F}(H)$ is defined by

$$\langle f_1 \otimes \cdots \otimes f_m, g_1 \otimes \cdots \otimes g_n \rangle_{\alpha,q} := \delta_{m,n} \sum_{\sigma \in \Sigma_n} \alpha^{\ell_1(\sigma)} q^{\ell_2(\sigma)} \prod_{j=1}^{n} \langle f_j, g_{\sigma(j)} \rangle_H,\ \alpha, q \in (-1,1) \tag{A.3}$$

with conventions $0^0 = 1$ and $g_{-k} = \overline{g_k},\ k = 1, 2, \ldots, n$. For example, if one may define the involution as $\overline{f} := -f$, then $g_{-k} = -g_k$. Equipped with this inner product the full Fock space $\mathcal{F}(H)$ is denoted by $\mathcal{F}_{\alpha,q}(H)$ to emphasize on the dependence of the inner product on α, q.

The (α, q)-creation operator $B^+_{\alpha,q}(f)$ is the usual left creation operator on the full Fock space, and the (α, q)-annihilation operator $B^-_{\alpha,q}(f)$ is its adjoint with respect to the inner product $\langle \cdot, \cdot \rangle_{\alpha,q}$. They satisfy the commutation relation

$$B^-_{\alpha,q}(f) B^+_{\alpha,q}(g) - q B^+_{\alpha,q}(g) B^-_{\alpha,q}(f) = \langle f, g \rangle_H I + \alpha \langle \overline{f}, g \rangle_H q^{2N}, \quad f, g \in H. \tag{A.4}$$

The readers can consult [BEH15] for details.

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