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PARTIAL TOPOLOGICAL PRODUCTS IN $\mathcal{MAP}$

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Abstract. In this paper we continue with the study of the category $\mathcal{MAP}$ of continuous maps and their morphisms, introduced in [3]. This category is an extension of both the category $\mathcal{TOP}_Y$ (of continuous maps into a fixed space Y and their morphisms) and $\mathcal{TOP}$ (of topological spaces and continuous maps as morphisms). Partial products are used to obtain universal type theorems for T_0, Tychonoff and zero-dimensional maps. Finally we introduce zero-dimensional and strongly zero-dimensional maps and generalize some well known results in the category $\mathcal{TOP}$ concerning zero-dimensional and strongly zero-dimensional spaces to the category $\mathcal{MAP}$.

1. Introduction

The study of General Topology is usually concerned with the category $\mathcal{TOP}$ of topological spaces as objects, and continuous maps as morphisms. One of the most important operations on objects in $\mathcal{TOP}$ is the Tychonoff product which gives rise to many interesting results and examples. The Tychonoff product of an arbitrary number of topological spaces was defined by A.Tychonoff in 1930 [15].

The concepts of *space* and *map* are equally important and one can even look at a space as a map from this space onto a singleton space and in this manner identify these two concepts. With this in mind, a branch of General Topology which has become known as General Topology of Continuous Maps, or Fibrewise General Topology, was initiated. This field of research is concerned most of all in extending the main notions and results concerning topological spaces to that of continuous maps. For an arbitrary topological space Y one considers the category $\mathcal{TOP}_Y$, the objects of which are continuous maps into the space Y, and for the objects $f : X \to Y$ and $g : Z \to Y$, a morphism from f into g is a continuous map $\lambda : X \to Z$ with the property $f = g \circ \lambda$. This situation is a generalization of the category $\mathcal{TOP}$, since the category $\mathcal{TOP}$ is isomorphic to the particular case

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of $\mathcal{TOP}_Y$ in which the space Y is a singleton space. As is the case in $\mathcal{TOP}$, one of the most important operations on objects in the category $\mathcal{TOP}_Y$ is the fibrewise product of maps defined by B.A.Pasynkov [10, 11, 12].

As mentioned above, the Tychonoff product gives rise to many interesting results. In particular, results concerning universal spaces. Recall that a space X is said to be *universal* for all spaces having a topological property $\mathcal{P}$ if the space X has property $\mathcal{P}$ and every space having property $\mathcal{P}$ is homeomorphically embeddable in X. Universal spaces are very useful since they reduce the study of a class of spaces having some topological property $\mathcal{P}$ to the study of subspaces of a fixed space. We will be interested in the following three results obtained respectively by A.Tychonoff [15], P.S.Alexandroff [1] and N.Vedenissoff [16].

Theorem 1.1. *The Tychonoff cube $I^{\mathfrak{m}}$ is universal for all Tychonoff spaces of weight $\mathfrak{m} \geqslant \aleph_0$.*

Theorem 1.2. *The Alexandroff cube $F^{\mathfrak{m}}$ is universal for all T_0-spaces of weight $\mathfrak{m} \geqslant \aleph_0$.*

Theorem 1.3. *The Cantor cube $D^{\mathfrak{m}}$ is universal for all zero-dimensional spaces of weight $\mathfrak{m} \geqslant \aleph_0$.*

Completely regular and Tychonoff maps were defined by B.A.Pasynkov in 1984. These definitions made it possible to generalize and obtain an analogue to Theorem 1.1 in the category $\mathcal{TOP}_Y$ [12].

Theorem 1.4. *A Tychonoff map $f : X \to Y$ has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$) if and only if, the map f is homeomorphically embeddable into the projection p of a partial topological product $P = P(Y, \{Z_\alpha\}, \{O_\alpha\} : \alpha \in \mathcal{A})$, where $Z_\alpha = I$ for every $\alpha \in \mathcal{A}$ and $|\mathcal{A}| \leqslant \mathfrak{m}$.*

The following result was also given as a corollary to Theorem 1.4 in [12].

Corollary 1.5. *A continuous map is Tychonoff if and only if it is homeomorphically embeddable into the projection of a partial topological product, all the fibres of which are segments.*

B.A.Pasynkov also generalized and obtained an analogue to the result in $\mathcal{TOP}$ of the existence of a compactification for a Tychonoff space having the same weight and also constructed a maximal Tychonoff compactification for a Tychonoff map (i.e. an analogue to the Stone-Čech compactification) [12].

In [3], a category of maps $\mathcal{MAP}$ in which one does not restrain oneself with a fixed base space Y was introduced. The aim of this paper is to generalize and obtain an analogue to Theorems 1.1, 1.2 and 1.3 in the category $\mathcal{MAP}$. For more details and undefined terms on the General Topology of Continuous Maps one can consult [2, 3, 4, 5, 7, 8, 9, 12, 13].

2. Preliminary notions on the category $\mathcal{MAP}$

The objects of $\mathcal{MAP}$ are continuous maps from any topological space into any topological space. For two objects $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, a morphism from f_1 into f_2 is a pair of continuous maps $\{\lambda_T, \lambda_B\}$, where $\lambda_T : X_1 \to X_2$ and $\lambda_B : Y_1 \to Y_2$, such that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{\lambda_T} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y_1 & \xrightarrow[\lambda_B]{} & Y_2 \end{array}$$

is commutative. It is not difficult to see that this definition of a morphism in $\mathcal{MAP}$ satisfies the necessary axioms that morphisms should satisfy in any category (see, for example, [14]).

Let $\mathcal{P}_T$ and $\mathcal{P}_B$ be two topological/set theoretic properties of maps (for example: closed, open, 1-1, onto, etc.). If λ_T has property $\mathcal{P}_T$ and λ_B has property $\mathcal{P}_B$ then we say that $\{\lambda_T, \lambda_B\}$ is a $\{\mathcal{P}_T, \mathcal{P}_B\}$*-morphism*. If $\mathcal{P}_T$ is the *continuous* property, then we say that $\{\lambda_T, \lambda_B\}$ is a $\{*, \mathcal{P}_B\}$*-morphism*, similarly for $\mathcal{P}_B$. Therefore, a $\{*, *\}$-morphism is just a morphism. Also, if $\mathcal{P}_T = \mathcal{P}_B = \mathcal{P}$ then a $\{\mathcal{P}_T, \mathcal{P}_B\}$-morphism is called a $\mathcal{P}$*-morphism*.

As noted in the introduction, separation axioms for maps have already been defined in the category $\mathcal{TOP}_Y$ and since these axioms involve only one map, they have also been defined for the category $\mathcal{MAP}$. We only give the definitions of functionally Hausdorff and Tychonoff maps, for the other separation axioms one can consult for example [12, 13, 2, 4, 5, 3].

Definition 2.1. The subsets A and B of the space X are said to be *functionally separated in* $U \subset X$, if the sets $A \cap U$ and $B \cap U$ are functionally separated in U (that is, there exists a continuous function $\phi : U \to [0, 1]$ such that $A \cap U \subset \phi^{-1}(0)$ and $B \cap U \subset \phi^{-1}(1)$).

Definition 2.2. A continuous map $f : X \to Y$ is said to be *functionally Hausdorff or* $T_{2\frac{1}{2}}$, if for every two distinct points x and x' in X lying in the same fibre, there exists a neighborhood O of the point $f(x)$, such that the sets $\{x\}$ and $\{x'\}$ are functionally separated in $f^{-1}O$.

Definition 2.3. A continuous map $f : X \to Y$ is said to be *completely regular*, if for every point $x \in X$ and every closed set F in X, not containing the point x, there exists a neighborhood O of the point $f(x)$, such that the sets $\{x\}$ and F are functionally separated in $f^{-1}O$.

Definition 2.4. A completely regular T_0-map is called a *Tychonoff* (or $T_{3\frac{1}{2}}$-) map, where a map $f : X \to Y$ is said to be a T_0-map if for every two distinct points $x, x' \in X$ lying in the same fibre, at least one of the points x, x' has a neighborhood in X which does not contain the other point.

It can be easily verified that every Tychonoff map is functionally Hausdorff.

We now give the definition of a submap as an analogue of subspace. Since we do not restrict ourselves to a fixed base space Y our definition slightly differs from that given in the category $\mathcal{TOP}_Y$ [12]. This definition was introduced in [3].

Definition 2.5. The map $g : A \to B$ is said to be a (closed, open, everywhere dense, etc.) submap of the map $f : X \to Y$, if g is the restriction of the map f on the (closed, open, everywhere dense, etc.) subset A of the space X and $g(A) = f(A) \subset B \subset Y$.

The following result is known [3].

Proposition 2.1. *Any submap of a T_i-map is a T_i-map for $i \leqslant 3\frac{1}{2}$. Prenormality, functional prenormality, normality, functional normality, collectionwise prenormality and collectionwise normality are hereditary with respect to closed submaps.*

The proof of the following proposition for the case $B = Y$ can be found in [13]. For the situation given below the proof is analogous and so is omitted. Remember that in $\mathcal{TOP}_Y$ (and also in $\mathcal{MAP}$), by a *compact map* we mean a *perfect map*, namely, a closed map with compact fibres. It is evident that a closed submap of a compact map is compact.

Proposition 2.2. *Let the compact map $g : A \to B$ be a submap of a T_2-map $f : X \to Y$ and let B be a closed subset of Y, the g is a closed submap of f.*

Finally, we give the definitions of base and weight for a continuous map, both given by B.A.Pasynkov [10, 12].

Definition 2.6. Let $f : X \to Y$ be a map of topological spaces. A set $U \subset X$ is said to be *f-functionally open*, if there exists an open subset O of Y such that $U \subset f^{-1}O$ and U is functionally open in $f^{-1}O$.

Definition 2.7. Let $f : X \to Y$ be a map of topological spaces. A collection $\mathfrak{B}_f$ of open (resp. f-functionally open, functionally open) subsets of X is called a *base* (resp. *f-functionally open base, functionally open base*), *for the map f* if for every point $x \in X$ and every neighborhood U_x of x in X there exists a neighborhood O_y of the point $y = f(x)$ in Y and an element $V \in \mathfrak{B}_f$ such that $x \in f^{-1}O_y \cap V \subset U_x$.

Definition 2.8. A collection $\mathfrak{S}_f$ of open (resp. f-functionally open, functionally open) subsets of X is called a *subbase* (resp. *f-functionally open subbase, functionally open subbase*), *for the map f* if the intersection of finite subcollections of the collection $\mathfrak{S}_f$ constitute a base for the map f.

Definition 2.9. The minimal cardinal number of the form $|\mathfrak{B}_f|$, where $\mathfrak{B}_f$ is a base (resp. f-functionally open base, functionally open base) for the map f (if such bases exist), is called the *weight* (resp. *f-functional weight, functional weight*) *of the continuous map f* and is denoted by $\mathfrak{w}(f)$ (resp. $\mathfrak{W}(f), \mathfrak{W}'(f)$).

A proof for the following two propostions can be found in [13].

Proposition 2.3. *For a continuous map $f : X \to Y$ the following hold:*

1. *If the respective bases are defined, then*

$$\mathfrak{w}(f) \leqslant \mathfrak{W}(f) \text{ and } \mathfrak{W}(f) \leqslant \mathfrak{W}'(f);$$

2. *If $g : A \to B$ is a submap of the map f, then every (resp. every functionally open, every f-functionally open) base of f induces a base (resp. a functionally open base, a g-functionally open base) of g and*

$$\mathfrak{w}(g) \leqslant \mathfrak{w}(f),\ \mathfrak{W}(g) \leqslant \mathfrak{W}(f) \text{ and } \mathfrak{W}'(g) \leqslant \mathfrak{W}'(f),$$

when the respective bases are defined.

Proposition 2.4. *The map $f : X \to Y$ is completely regular if and only if there exists an f-functionally open base of f.*

The above proposition shows in particular that for a Tychonoff map f, the weight $\mathfrak{W}(f)$ is defined.

3. Elementary partial topological products

Elementary partial topological products were defined by B.A.Pasynkov in 1964 [10, 11]. By taking fan products of elementary partial topological products, which are called partial topological products, he proved Theorem 1.4, the analogue of Theorem 1.1 in the category $\mathcal{TOP}_Y$. In this section we give the definition of elementary partial topological products, as given by B.A.Pasynkov, and in the following sections we go on to define partial topological products for both the Tychonoff product of maps and fan product relative to an inverse system, the two types of products in the category $\mathcal{MAP}$ introduced in [3]. In the following sections, using the same approach of B.A.Pasynkov in proving Theorem 1.4, we use these definitions to obtain analogues of Theorems 1.1, 1.2 and 1.3 (and so also Theorem 1.4) in the category $\mathcal{MAP}$.

Definition 3.1. Let Y and Z be topological spaces and let O be an open subset of Y. Consider the disjoint union D of the sets $Y \setminus O$ and $O \times Z$ and define a map $p : D \to Y$ by letting $p(y) = y$ if $y \in Y \setminus O$ and $p(y, z) = y$ if $(y, z) \in O \times Z$. Let Ω_Y and $\Omega_{O\times Z}$ be the topologies of Y and $O \times Z$ respectively. The elementary partial topological product ($\equiv$ EPTP) with base space Y, fibre Z and open set O is the set D endowed with the topology generated by the base $p^{-1}\Omega_Y \cup \Omega_{O\times Z}$ and is denoted by $P(Y, Z, O)$. The continuous map $p : P(Y, Z, O) \to Y$ is called the *projection of the EPTP $P(Y, Z, O)$*. The projection q of the product $O \times Z \subset P(Y, Z, O)$ onto the factor Z is called the *side projection of the EPTP $P(Y, Z, O)$*.

Thus, the EPTP $P(Y, Z, O)$ induces on $O \times Z$ the topology of the topological product $O \times Z$, and on $Y \setminus O$, the subspace topology as a subspace of Y. Also, the projection p is continuous, open and its restriction on $Y \setminus O$ is a homeomorphic embedding. The following result can be found in [13].

Proposition 3.1. *The projection $p : P \to Y$ of the EPTP $P = P(Y, Z, O)$ satisfies the inequality $\mathfrak{w}(p) \leqslant \mathfrak{w}(Z) + 1$. If the fibre Z is a T_i-space, then the projection p is a T_i-map, for $i \leqslant 3$. If the fibre Z is completely regular, then the projection p is completely regular and $\mathfrak{W}(p) = \mathfrak{w}(Z) + 1$. If moreover, the set $O \subset Y$ is functionally open, then the weight $\mathfrak{W}'(p)$ is defined and $\mathfrak{W}'(p) = \mathfrak{W}(p)$.*

The following definition will be of importance later.

Definition 3.2. Let there be given an EPTP $P = P(Y, Z, O)$, topological spaces X and B, and continuous maps $f : X \to B$, $\lambda^B : B \to Y$ and $g : f^{-1}(\lambda^B)^{-1}O \to Z$. The map $\triangle(f, g; \lambda^B) : X \to P$, mapping a point $x \in X \backslash f^{-1}(\lambda^B)^{-1}O$ onto the point $\lambda^B \circ f(x) \in Y \backslash O \subset P$ and a point $x \in f^{-1}(\lambda^B)^{-1}O$ onto the point $(\lambda^B \circ f(x), g(x)) \in O \times Z \subset P$, will be called the *diagonal product of the maps f and g over the map λ^B*. If $B = Y$ and $\lambda^B = \mathrm{id}_Y$ then $\triangle(f, g; \lambda^B) \equiv \triangle(f, g; \mathrm{id}_Y) \equiv \triangle(f, g)$ and is called simply the *diagonal product of the maps f and g*.

It is not difficult to see that the map $\triangle(f, g; \lambda^B)$ is a continuous map and that the projection $p : P(Y, Z, O) \to Y$ and side projection $q : O \times Z \subset P(Y, Z, O) \to Z$ satisfy the following relations:

$$p \circ \triangle(f, g; \lambda^B) = \lambda^B \circ f;$$
$$q \circ \triangle(f, g; \lambda^B)|_{f^{-1}(\lambda^B)^{-1}O} = g.$$

A proof of the following result can be found in [13].

Proposition 3.2. *If $P = P(Y, Z, O)$ is an EPTP, $p : P \to Y$ is its projection and $pr : Y \times Z \to Y$ is the projection of the topological product $Y \times Z$ onto the factor Y, then there exists a continuous onto map $\psi : Y \times Z \to P$ such that $pr = p \circ \psi$.*

As a corollary to the above proposition we have:

Corollary 3.3. *The projection $p : P \to Y$ of an EPTP $P = P(Y, Z, O)$ with compact fibre Z is compact.*

Proof. Since the space Z is compact, it follows that the map $pr : Y \times Z \to Y$ is compact. Since the map $\psi : Y \times Z \to P$ is onto, the result follows from the relation $pr = p \circ \psi$. □

4. Tychonoff products

Tychonoff products of maps is taken to be the Tychonoff product of objects in the category $\mathcal{MAP}$ [3]. We recall the definition.

Definition 4.1. Let $\{f_\alpha : \alpha \in \mathcal{A}\}$ be a collection of continuous maps, where $f_\alpha : X_\alpha \to Y_\alpha$. The Tychonoff product of the maps $\{f_\alpha : \alpha \in \mathcal{A}\}$, which is denoted by $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$, is the continuous map which assigns to the point $x = \{x_\alpha\} \in \prod\{X_\alpha : \alpha \in \mathcal{A}\}$ the point $\{f_\alpha(x_\alpha)\} \in \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$.

If $pr^\alpha_T : \prod\{X_\alpha : \alpha \in \mathcal{A}\} \to X_\alpha$ and $pr^\alpha_B : \prod\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ are the projections, then the diagram

$$\begin{array}{ccc} \prod\{X_\alpha : \alpha \in \mathcal{A}\} & \xrightarrow{pr^\alpha_T} & X_\alpha \\ {\scriptstyle \prod\{f_\alpha:\alpha\in\mathcal{A}\}}\downarrow & & \downarrow{\scriptstyle f_\alpha} \\ \prod\{Y_\alpha : \alpha \in \mathcal{A}\} & \xrightarrow[pr^\alpha_B]{} & Y_\alpha \end{array}$$

is commutative. Therefore, the pair $\{pr^\alpha_T, pr^\alpha_B\}$ is a $\{\text{onto}, \text{onto}\}$-morphism of $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$ into f_α.

The following two results were proved in [3].

Proposition 4.1. *Let $f = \prod\{f_\alpha : \alpha \in \mathcal{A}\} : X = \prod\{X_\alpha : \alpha \in \mathcal{A}\} \to Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ be the Tychonoff product of the maps $f_\alpha : X_\alpha \to Y_\alpha$. If $\mathfrak{B}_{f_\alpha}$ is a base for the map f_α for every $\alpha \in \mathcal{A}$, then the collection $\mathfrak{S}_f = \bigcup\{(pr_T^\alpha)^{-1}\mathfrak{B}_{f_\alpha} : \alpha \in \mathcal{A}\}$ is a subbase for the map f and the weight $\mathfrak{w}(f) \leqslant \max\{|\mathcal{A}|, \sup\{\mathfrak{w}(f_\alpha) : \alpha \in \mathcal{A}\}, \aleph_0\}$.*

Remark 4.1. It is not difficult to show that in the above context, if the map f_α is completely regular for every $\alpha \in \mathcal{A}$, then $\mathfrak{W}(f) \leqslant \max\{|\mathcal{A}|, \sup\{\mathfrak{W}(f_\alpha) : \alpha \in \mathcal{A}\}, \aleph_0\}$.

Proposition 4.2. *The Tychonoff product $\prod\{f_\alpha : \alpha \in \mathcal{A}\}$ of T_i-maps f_α is a T_i-map for $i \leqslant 3\frac{1}{2}$.*

Let there be given a continuous map $f : X \to Y$, a collection of continuous maps $f_\alpha : X_\alpha \to Y_\alpha, \alpha \in \mathcal{A}$, and a collection of morphisms $\{\lambda_\alpha^T, \lambda_\alpha^B\} : f \to f_\alpha, \alpha \in \mathcal{A}$. Consider the standard diagonal products $\triangle\lambda_\alpha^T \equiv \triangle\{\lambda_\alpha^T : \alpha \in \mathcal{A}\} : X \to \prod\{X_\alpha : \alpha \in \mathcal{A}\}$ and $\triangle\lambda_\alpha^B \equiv \triangle\{\lambda_\alpha^B : \alpha \in \mathcal{A}\} : Y \to \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$. It is not difficult to see that $\{\triangle\lambda_\alpha^T, \triangle\lambda_\alpha^B\}$ is a morphism of the map f into the Tychonoff product $\prod f_\alpha \equiv \prod\{f_\alpha : \alpha \in \mathcal{A}\}$. We therefore have

$$\prod f_\alpha \circ \triangle\lambda_\alpha^T = \triangle\lambda_\alpha^B \circ f.$$

Proposition 4.3. *If under the above conditions we have:*

1. *the collection $\{\lambda_\alpha^B : \alpha \in \mathcal{A}\}$ separates points and for every point $y \in Y$ and every two distinct points x and x' in the fibre $f^{-1}y$ there exists some $\alpha \in \mathcal{A}$, such that $\lambda_\alpha^T(x) \neq \lambda_\alpha^T(x')$, and*
2. *for every closed set F in X and every point $x \in X \setminus F$, there exists some $\alpha \in \mathcal{A}$ and an open set U in X_α such that $x \in (\lambda_\alpha^T)^{-1}U \subset (X \setminus F)$,*

then the morphism $\{\triangle\lambda_\alpha^T, \triangle\lambda_\alpha^B\}$ is a {homeomorphic embedding, 1-1}-morphism of f into the Tychonoff product $\prod f_\alpha$.

Proof. Take any two distinct points x and x' in X. If $f(x) \neq f(x')$, then by condition (1), there exists some $\alpha \in \mathcal{A}$ such that $\lambda_\alpha^B \circ f(x) \neq \lambda_\alpha^B \circ f(x')$. Therefore, from the relation $\lambda_\alpha^B \circ f = f_\alpha \circ \lambda_\alpha^T$ we get that $\lambda_\alpha^T(x) \neq \lambda_\alpha^T(x')$. If $f(x) = f(x')$, then again by condition (1), there exists some $\alpha \in \mathcal{A}$ such that $\lambda_\alpha^T(x) \neq \lambda_\alpha^T(x')$. In both cases we have the inequality $\triangle\lambda_\alpha^T(x) \neq \triangle\lambda_\alpha^T(x')$. We have thus shown that the continuous map $\triangle\lambda_\alpha^T$ is 1-1. That $\triangle\lambda_\alpha^B$ is a 1-1 map follows from the fact that the collection $\{\lambda_\alpha^B : \alpha \in \mathcal{A}\}$ separates points.

We now show that the corestriction of $\triangle\lambda_\alpha^T$ onto its image, that is $\triangle\lambda_\alpha^T$ as a map from X onto $\triangle\lambda_\alpha^T(X)$, is an open map. Take any open set V in X. By condition (2), for every $x \in V$ there exists some $\alpha(x) \in \mathcal{A}$ and an open set $U(x)$ in $X_{\alpha(x)}$, such that $x \in (\lambda_{\alpha(x)}^T)^{-1}U(x) \subset V$. From this follows that $x \in (\triangle\lambda_\alpha^T)^{-1}(pr_T^{\alpha(x)})^{-1}U(x) \subset V$. The set $U = \bigcup\{(pr_T^{\alpha(x)})^{-1}U(x) : x \in V\}$ is open in $\prod X_\alpha$ and $V = (\triangle\lambda_\alpha^T)^{-1}U$. Hence $\triangle\lambda_\alpha^T(V) = U \cap \triangle\lambda_\alpha^T(X)$ and so the map $\triangle\lambda_\alpha^T : X \to \prod X_\alpha$ is open, from which we conclude that $\triangle\lambda_\alpha^T$ is a homeomorphic embedding. □

We now introduce and define Tychonoff partial topological products.

Definition 4.2. . Let $P_\alpha = P(Y_\alpha, Z_\alpha, O_\alpha)$ be an EPTP with base space Y_α, fibre Z_α and open set O_α for every α in some indexing set $\mathcal{A}$ and let $p_\alpha : P_\alpha \to Y_\alpha$ be the corresponding projection of the EPTP P_α. The Tychonoff product $\prod P_\alpha \equiv \prod\{P_\alpha : \alpha \in \mathcal{A}\}$ is called the *Tychonoff partial topological product (≡ TPTP) of the EPTPs* $P_\alpha, \alpha \in \mathcal{A}$. The Tychonoff product $\prod p_\alpha \equiv \prod\{p_\alpha : \alpha \in \mathcal{A}\}$ of the projections p_α is called the *projection of the TPTP* $\prod P_\alpha$ *onto its base*. The projection of the TPTP $\prod P_\alpha$ onto the EPTP P_α is denoted by pr_α.

We now formulate our main theorem of this section, an analogue of Theorem 1.1 in the category $\mathcal{MAP}$ with respect to Tychonoff products. Below, by I we denote the *unit interval* $[0,1] \subset \mathbb{R}$.

Theorem 4.4. *For a Tychonoff map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, I, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, I, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$.*

We can write down the following corollaries to the above theorem. Since a $T_{2\frac{1}{2}}$ compact map is Tychonoff, we have:

Corollary 4.5. *For a $T_{2\frac{1}{2}}$ compact map $f : X \to Y$ into a Hausdorff space Y the following are equivalent:*

1. *The map f has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a {closed homeomorphic embedding,homeomorphic embedding}-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, I, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$;*
3. *There exists a {closed homeomorphic embedding,homeomorphic embedding}-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, I, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$.*

Corollary 4.6. *For a continuous map $f : X \to Y$ the following are equivalent:*

1. *The map f is Tychonoff;*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, I, O_\alpha)$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, I, O_\alpha)$.*

Corollary 4.7. *For a continuous map $f : X \to Y$ into a Hausdorff space Y the following are equivalent:*

1. *The map f is $T_{2\frac{1}{2}}$ and compact;*
2. *There exists a {closed homeomorphic embedding,homeomorphic embedding}-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, I, O_\alpha)$;*
3. *There exists a {closed homeomorphic embedding,homeomorphic embedding}-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, I, O_\alpha)$.*

We now give some results to help us prove the above theorem and corollaries.

Let $\prod P_\alpha \equiv \prod\{P_\alpha : \alpha \in \mathcal{A}\}$ be the TPTP of the EPTPs $P_\alpha = P(Y_\alpha, Z_\alpha, O_\alpha)$ and let there be given topological spaces X and Y, and continuous maps $f : X \to Y$, $\lambda^B_\alpha : Y \to Y_\alpha$ and $g_\alpha : f^{-1}(\lambda^B_\alpha)^{-1}O_\alpha \to Z_\alpha$ for every $\alpha \in \mathcal{A}$. Then, the diagonal product of the maps f and g_α over the map λ^B_α, $\triangle_\alpha \equiv \triangle(f, g_\alpha; \lambda^B_\alpha) : X \to P_\alpha$, is defined and we have that $\{\triangle_\alpha, \lambda^B_\alpha\} : f \to p_\alpha$, where $p_\alpha : P_\alpha \to Y_\alpha$ is the projection of the EPTP P_α. Therefore, by taking the standard diagonal products, we get a morphism $\{\triangle(\triangle_\alpha), \triangle\lambda^B_\alpha\}$ of the map f into the projection $\prod p_\alpha$ of the TPTP $\prod P_\alpha$, and so

$$\prod p_\alpha \circ \triangle(\triangle_\alpha) = \triangle\lambda^B_\alpha \circ f.$$

Proposition 4.8. *If under the above conditions we have:*

1. *the collection $\{\lambda^B_\alpha : \alpha \in \mathcal{A}\}$ separates points and for every point $y \in Y$ and every two distinct points x and x' in the fibre $f^{-1}y$ there exists some $\alpha \in \mathcal{A}$, such that $y \in (\lambda^B_\alpha)^{-1}O_\alpha$ and $g_\alpha(x) \neq g_\alpha(x')$, and*
2. *the collection $\mathfrak{B}_f = \bigcup\{\{g_\alpha^{-1}W : W$ open in $Z_\alpha\} : \alpha \in \mathcal{A}\}$ is a base for the map f and $\mu^B_\alpha \equiv \lambda^B_\alpha|_{(\lambda^B_\alpha)^{-1}O_\alpha} : (\lambda^B_\alpha)^{-1}O_\alpha \to O_\alpha \cap \lambda^B_\alpha(Y)$ is an open map,*

then the morphism $\{\triangle(\triangle_\alpha), \triangle\lambda^B_\alpha\}$ is a {homeomorphic embedding, 1-1 local homeomorphic embedding}-morphism of f into the projection $\prod p_\alpha$ of the TPTP $\prod P_\alpha$.

Proof. If two distinct points x and x' in the fibre $f^{-1}y$, and some index $\alpha \in \mathcal{A}$ satisfy $g_\alpha x \neq g_\alpha x'$, then we have $\triangle_\alpha x \neq \triangle_\alpha x'$.

Now let F be a closed subset of X and let $x \in X \setminus F$. By the hypothesis, there exists an index $\alpha \in \mathcal{A}$, an open set W in Z_α and an open set O in Y, such that $x \in f^{-1}O \cap g_\alpha^{-1}W \subset X \setminus F$. Let $H = \mu^B_\alpha\left(O \cap (\lambda^B_\alpha)^{-1}O_\alpha\right)$ and let $\hat{H}$ be an open subset of Y_α satisfying $H = \hat{H} \cap \lambda^B_\alpha(Y)$. We then have $x \in f^{-1}O \cap g_\alpha^{-1}W = (\triangle_\alpha)^{-1}\Big((\hat{H} \cap O_\alpha) \times W\Big) \subset X \setminus F$ and the set $(\hat{H} \cap O_\alpha) \times W$ is open in P_α.

The result now follows from Proposition 4.3. □

Corollary 4.9. *If under the above conditions we have:*

1. *the collection $\{\lambda^B_\alpha : \alpha \in \mathcal{A}\}$ separates points and the map f is a T_0-map, and*
2. *the collection $\mathfrak{B}_f = \bigcup\{\{g_\alpha^{-1}W : W$ open in $Z_\alpha\} : \alpha \in \mathcal{A}\}$ is a base for the map f and $\mu^B_\alpha \equiv \lambda^B_\alpha|_{(\lambda^B_\alpha)^{-1}O_\alpha} : (\lambda^B_\alpha)^{-1}O_\alpha \to O_\alpha \cap \lambda^B_\alpha(Y)$ is an open map,*

then the morphism $\{\triangle(\triangle_\alpha), \triangle\lambda^B_\alpha\}$ is a {homeomorphic embedding, 1-1 local homeomorphic embedding}-morphism of f into the projection $\prod p_\alpha$ of the TPTP $\prod P_\alpha$.

Proof. Take any point $y \in Y$ and any two distinct points x and x' in the fibre $f^{-1}y$. Since f is a T_0-map, one can assume without loss of generality, that the point x has a neighborhood U satisfying $x' \notin U$. Thus, by property 2., there exists some $\alpha \in \mathcal{A}$, an open subset W of Z_α and an open subset O of Y satisfying $x \in f^{-1}O \cap W$. Therefore, we can conclude that $g_\alpha(x) \in W$, while $g_\alpha(x') \notin W$, since $x \in f^{-1}y \cap W$. Hence, property 1. of Proposition 4.8 is satisfied. □

Corollary 4.10. *If under the above conditions we have:*

1. the space Y is a T_0-space and the map f is a T_0-map, and
2. the collection $\mathfrak{B}_f = \bigcup\{\{g_\alpha^{-1}W : W$ open in $Z_\alpha\} : \alpha \in \mathcal{A}\}$ is a base for the map f and $\mu_\alpha^B \equiv \lambda_\alpha^B|_{(\lambda_\alpha^B)^{-1}O_\alpha} : (\lambda_\alpha^B)^{-1}O_\alpha \to O_\alpha \cap \lambda_\alpha^B(Y)$ is an open map,

then the morphism $\{\triangle(\triangle_\alpha), \triangle\lambda_\alpha^B\}$ is a {homeomorphic embedding, 1-1 local homeomorphic embedding}-morphism of f into the projection $\prod p_\alpha$ of the TPTP $\prod P_\alpha$.

Proof. Take any two distinct points y and y' in the T_0-space Y. Without loss of generality, one can assume that the point y has a neighborhood O satisfying $y' \notin O$. There exists some $\alpha \in \mathcal{A}$ such that $y \in (\lambda_\alpha^B)^{-1}O_\alpha$. Then, by property 2., it is not difficult to see that $\mu_\alpha^B(y) \neq \mu_\alpha^B(y')$. The result now follows from Corollary 4.9. □

Finally we need the following results for the case of compact fibres.

Proposition 4.11. *Let $\prod P_\alpha \equiv \prod\{P_\alpha : \alpha \in \mathcal{A}\}$ be the TPTP of the EPTPs $P_\alpha = P(Y_\alpha, Z_\alpha, O_\alpha)$, where the fibres Z_α are compact for every $\alpha \in \mathcal{A}$. Then, the projection $\prod p_\alpha \equiv \prod\{p_\alpha : \alpha \in \mathcal{A}\}$ of the TPTP $\prod P_\alpha$ onto its base is a compact map.*

Proof. By Corollary 3.3 we have that the projections $p_\alpha : P_\alpha \to Y_\alpha$ are compact. Therefore, we can conclude that the projection $\prod p_\alpha$ is also compact, as a Tychonoff product of compact maps (see [6, 3]). □

Corollary 4.12. *Let $\prod P_\alpha$ be the TPTP of the EPTPs $P_\alpha = P(Y_\alpha, Z_\alpha, O_\alpha)$, where the fibres Z_α are compact and metrizable for every $\alpha \in \mathcal{A}$. Then, the projection $\prod p_\alpha$ of the TPTP $\prod P_\alpha$ onto its base is a compact Tychonoff map with weight $\mathfrak{W}(\prod p_\alpha) \leqslant \max(|\mathcal{A}|, \aleph_0)$.*

Proof. The fact that the map $\prod p_\alpha$ is compact follows from Proposition 4.11 and the fact that it is Tychonoff, together with the inequality for the weight $\mathfrak{W}(\prod p_\alpha)$, follows from Propositions 3.1 and 4.2 and Remark 4.1. □

Proof of Theorem 4.4. We begin by showing that (1) implies (2). Let $f : X \to Y$ be a Tychonoff map with weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$, where $\mathfrak{m} \geqslant \aleph_0$. Let $\mathfrak{B}_f = \{U_\alpha : \alpha \in \mathcal{A}\}$ be an f-functionally open base for the map f with $|\mathcal{A}| \leqslant \mathfrak{m}$. For every $\alpha \in \mathcal{A}$, take an open subset O_α of Y and a continuous map $g_\alpha : f^{-1}O_\alpha \to I$ satisfying $U_\alpha = g_\alpha^{-1}(]0, 1])$. Let $P_\alpha = P(Y, I, O_\alpha)$ be an EPTP for every $\alpha \in \mathcal{A}$ and let $\mathrm{id} : Y \to Y$ be the identity map. By Corollary 4.9 we can conclude that the morphism $\{\triangle(\triangle_\alpha), \triangle\mathrm{id}_\alpha\}$, where $\triangle_\alpha = \triangle(f, g_\alpha; \mathrm{id})$ and $\mathrm{id}_\alpha = \mathrm{id}$, is a homeomorphic embedding-morphism of f into the projection $\prod p_\alpha$ of the TPTP $\prod P_\alpha$.

That (3) follows from (2) is evident. We are left to show that (3) implies (1). If there exists a homeomorphic embedding-morphism of the map f into the projection $\prod p_\alpha$ of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, I, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$, then by Proposition 2.3 and Corollary 4.12 we have

$$\mathfrak{W}(f) \leqslant \mathfrak{W}(\prod p_\alpha) \leqslant |\mathcal{A}| \leqslant \mathfrak{m}.$$

□

Proof of Corollary 4.5. We only need to show that (1) implies (2). Since the space Y is Hausdorff, the diagonal $\triangle\mathrm{id}_\alpha(Y)$, where $\mathrm{id}_\alpha \equiv \mathrm{id} : Y \to Y$, is a closed subset of the Tychonoff product $Y^{|\mathcal{A}|}$. Therefore, the result follows from Proposition 2.2. □

Proof of Corollary 4.6. We only need to show that (3) implies (1) and this follows from Corollary 4.12 and Proposition 2.1. □

Proof of Corollary 4.7. The fact that (1) implies (2) follows from Corollary 4.5. That (2) implies (3) is evident and the proof that (3) implies (1) follows from Corollary 4.12 and the fact, as was already noted, that a closed submap of a compact map is compact. □

We end this section by a universal type theorem for T_0-maps in $\mathcal{MAP}$, an analogue to Theorem 1.2 in $\mathcal{TOP}$. Below, by the space F we denote the two point set $\{0, 1\}$ with the topology consisting of the empty set, the set $\{0\}$ and the whole space. Clearly, the space F is T_0.

Theorem 4.13. *For a T_0-map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{w}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, F, Y)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, F, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$.*

Proof. We begin by showing that (1) implies (2). Let $f : X \to Y$ be a T_0-map with weight $\mathfrak{w}(f) \leqslant \mathfrak{m}$, where $\mathfrak{m} \geqslant \aleph_0$. Let $\mathfrak{B}_f = \{U_\alpha : \alpha \in \mathcal{A}\}$ be a base for the map f with $|\mathcal{A}| \leqslant \mathfrak{m}$. For every $\alpha \in \mathcal{A}$, take a continuous map $g_\alpha : X \to F$ satisfying $U_\alpha = g_\alpha^{-1}\{0\}$. Let $P_\alpha = P(Y, F, Y)$ be an EPTP for every $\alpha \in \mathcal{A}$ and let $\mathrm{id} : Y \to Y$ be the identity map. By Corollary 4.9 we can conclude that the morphism $\{\triangle(\triangle_\alpha), \triangle\mathrm{id}_\alpha\}$, where $\triangle_\alpha = \triangle(f, g_\alpha; \mathrm{id})$ and $\mathrm{id}_\alpha = \mathrm{id}$, is a homeomorphic embedding-morphism of f into the projection $\prod p_\alpha$ of the TPTP $\prod P_\alpha$.

That (3) follows from (2) is evident. We are left to show that (3) implies (1). If there exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, F, O_\alpha)$ and $|\mathcal{A}| \leqslant \mathfrak{m}$, then by Propositions 2.3 and 3.1 we have

$$\mathfrak{w}(f) \leqslant \mathfrak{w}(\prod p_\alpha) \leqslant |\mathcal{A}| \leqslant \mathfrak{m}.$$

□

Corollary 4.14. *For a continuous map $f : X \to Y$ the following are equivalent:*

1. *The map f is T_0;*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y, F, Y)$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a TPTP $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$, where the EPTP $P_\alpha = P(Y_\alpha, F, O_\alpha)$.*

5. Fan products

We recall the definition of fan product with respect to a collection of maps and an inverse system, introduced in [3]. For undefined terms with respect to inverse systems one can consult [6].

Suppose we are given a collection of maps $f_\sigma : X_\sigma \to Y_\sigma$ for every $\sigma \in \Sigma$, where the indexing set Σ is directed by the relation $\leqslant$. We further suppose that we are given an inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. We denote by P, the subspace of the Tychonoff product $\prod\{X_\sigma : \sigma \in \Sigma\}$ given by

$$\left\{\{x_\sigma\} : \lambda^\sigma_\rho(f_\sigma x_\sigma) = f_\rho x_\rho \text{ for every } \sigma, \rho \in \Sigma \text{ satisfying } \rho \leqslant \sigma\right\}.$$

We call this space, the *fan product of the spaces X_σ with respect to the maps f_σ and the inverse system* $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. The space P is denoted by $\prod\left\{X_\sigma, f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$.

For every $\sigma \in \Sigma$, the restriction of the projection $pr_\sigma : \prod\{X_\sigma : \sigma \in \Sigma\} \to X_\sigma$ on the subspace P will be denoted by π_σ and is called the *projection of the fan product P to X_σ*. From the definition of fan product we have $\lambda^\sigma_\rho \circ f_\sigma \circ \pi_\sigma = f_\rho \circ \pi_\rho$ for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$. In this way one can define a map $p : P \to \varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$, called the *projection of the fan product P to the limit of the inverse system* $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$, by

$$p = \prod\{f_\sigma \circ \pi_\sigma : \sigma \in \Sigma\}.$$

It is evident that the projections p and $\pi_\sigma, \sigma \in \Sigma$ are continuous maps. The projection p is called the *fibrewise product of the maps f_σ with respect to the inverse system* $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ and is denoted by $\prod\left\{f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$. It is not difficult to see that for every point $y = \{y_\sigma\} \in \varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$, the preimage $p^{-1}y$ is homeomorphic to the Tychonoff product of the fibres $f_\sigma^{-1}y_\sigma$, that is $\prod\{f_\sigma^{-1}y_\sigma : \sigma \in \Sigma\}$.

The folowing two results were proved in [3].

Proposition 5.1. *Let $p : P \to \varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ be the fibrewise product of the maps f_σ with respect to the inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. If $\mathfrak{B}_{f_\sigma}$ is a base for the map f_σ for every $\sigma \in \Sigma$, then the collection $\mathfrak{S}_p = \bigcup\{\pi_\sigma^{-1}\mathfrak{B}_{f_\sigma} : \sigma \in \Sigma\}$ is a subbase for the map p and the weight $\mathfrak{w}(p) \leqslant \max\{|\Sigma|, \sup\{\mathfrak{w}(f_\sigma) : \sigma \in \Sigma\}, \aleph_0\}$.*

Remark 5.1. It is not difficult to show that in the above context, if the map f_σ is completely regular for every $\sigma \in \Sigma$, then $\mathfrak{W}(p) \leqslant \max\{|\Sigma|, \sup\{\mathfrak{W}(f_\alpha) : \sigma \in \Sigma\}, \aleph_0\}$.

Proposition 5.2. *The fibrewise product $p = \prod\left\{f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$ of T_i-maps f_σ with respect to the inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ is a T_i-map for $i \leqslant 3\frac{1}{2}$.*

Let there be given a continuous map $f : X \to Y$, a collection of continuous maps $f_\sigma : X_\sigma \to Y_\sigma, \sigma \in \Sigma$, where Σ is a non-empty directed set, an inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ and a collection of morphisms $\{\mu^T_\sigma, \mu^B_\sigma\} : f \to f_\sigma, \sigma \in \Sigma$, where $\lambda^\sigma_\rho \circ \mu^B_\sigma = \mu^B_\rho$ for any $\rho, \sigma \in \Sigma$ satisfying $\rho \leqslant \sigma$. It is not difficult to see that under the above conditions, the standard diagonal product $\triangle\mu^T_\sigma \equiv \triangle\{\mu^T_\sigma : \sigma \in \Sigma\} : X \to \prod\{X_\sigma : \sigma \in \Sigma\}$ has its image in $P = \prod\left\{X_\sigma, f_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$ and the standard diagonal product $\triangle\mu^B_\sigma \equiv \triangle\{\mu^B_\sigma : \sigma \in \Sigma\} : Y \to \prod\{Y_\sigma : \sigma \in \Sigma\}$ has its image in $\varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. One can also see that the diagonal product $\{\triangle\mu^T_\alpha, \triangle\mu^B_\alpha\}$ is a morphism of the map f into the projection $p : P \to \varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. We therefore have

$$p \circ \triangle\mu^T_\sigma = \triangle\mu^B_\sigma \circ f.$$

As a corollary to Proposition 4.3 we have the following result.

Corollary 5.3. *If under the above conditions we have:*

1. *the collection $\{\mu^B_\sigma : \sigma \in \Sigma\}$ separates points and for every point $y \in Y$ and every two distinct points x and x' in the fibre $f^{-1}y$ there exists some $\sigma \in \Sigma$, such that $\mu^T_\sigma(x) \neq \mu^T_\sigma(x')$, and*
2. *for every closed set F in X and every point $x \in X \setminus F$, there exists some $\sigma \in \Sigma$ and an open set U in X_σ such that $x \in (\mu^T_\sigma)^{-1}U \subset (X \setminus F)$,*

then the morphism $\{\triangle\mu^T_\sigma, \triangle\mu^B_\sigma\}$ is a $\{$homeomorphic embedding, 1-1$\}$-morphism of f into the projection p.

We now introduce and define Fan partial topological products.

Definition 5.1. . Let $P_\sigma = P(Y_\sigma, Z_\sigma, O_\sigma)$ be an EPTP with base space Y_σ, fibre Z_σ and open set O_σ for every σ in some directed set Σ and let $p_\sigma : P_\sigma \to Y_\sigma$ be the corresponding projection of the EPTP P_σ. Also, let there be given an inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$. The fan product $P = \prod\left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$ is called the *Fan partial topological product ($\equiv$ FPTP) of the EPTPs $P_\sigma, \sigma \in \Sigma$, with respect to the inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$*. The fibrewise product $p = \prod\left\{p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$ of the projections p_σ with respect to the inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ is called the *projection of the FPTP P onto its base*. The projection of the FPTP P onto the EPTP P_σ is denoted by π_σ.

We now formulate our main theorem of this section, an analogue of Theorem 1.1 in the category $\mathcal{MAP}$ with respect to Fan products. Remember that in the above context, if Y_0 is a topological space and $Y_\sigma = Y_0$ for every $\sigma \in \Sigma$, and we further have the binding maps $\lambda^\sigma_\rho = \mathrm{id}_{Y_0}$ for every $\sigma, \rho \in \Sigma$ satisfying $\rho \leqslant \sigma$, then the inverse system $\boldsymbol{S}(Y_0, \Sigma) = \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ is called the *constant inverse system* of the space Y_0 on the set Σ and we have that the limit $\varprojlim\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ is homeomorphic to Y_0.

Theorem 5.4. *For a Tychonoff map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\{P_\sigma, p_\sigma, \boldsymbol{S}(Y, \Sigma)\}$, where the EPTP $P_\sigma = P(Y, I, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod\left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$, where the EPTP $P_\sigma = P(Y_\sigma, I, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$.*

We can write down the following corollaries to the above theorem. Since a $T_{2\frac{1}{2}}$ compact map is Tychonoff, we have:

Corollary 5.5. *For a $T_{2\frac{1}{2}}$ compact map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{W}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a {closed homeomorphic embedding,homeomorphic embedding}-morphism of the map f into the projection of a FPTP $P = \prod \{P_\sigma, p_\sigma, \boldsymbol{S}(Y, \Sigma)\}$, where the EPTP $P_\sigma = P(Y, I, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$;*
3. *There exists a {closed homeomorphic embedding,homeomorphic embedding}-morphism of the map f into the projection of a FPTP $P = \prod \left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$, where the EPTP $P_\sigma = P(Y_\sigma, I, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$.*

Corollary 5.6. *For a continuous map $f : X \to Y$ the following are equivalent:*

1. *The map f is Tychonoff;*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod \{P_\sigma, p_\sigma, \boldsymbol{S}(Y, \Sigma)\}$, where the EPTP $P_\sigma = P(Y, I, O_\sigma)$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod \left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$, where the EPTP $P_\sigma = P(Y_\sigma, I, O_\sigma)$.*

Corollary 5.7. *For a continuous map $f : X \to Y$ the following are equivalent:*

1. *The map f is $T_{2\frac{1}{2}}$ and compact;*
2. *There exists a {closed homeomorphic embedding,homeomorphic embedding}-morphism of the map f into the projection of a FPTP $P = \prod \{P_\sigma, p_\sigma, \boldsymbol{S}(Y_0, \Sigma)\}$, where the EPTP $P_\sigma = P(Y, I, O_\sigma)$;*
3. *There exists a {closed homeomorphic embedding,homeomorphic embedding}-morphism of the map f into the projection of a FPTP $P = \prod \left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$, where the EPTP $P_\sigma = P(Y_\sigma, I, O_\sigma)$.*

Remark 5.2. One can note that contrary to Corollaries 4.5 and 4.7, in Corollaries 5.5 and 5.7 the Hausdorffness of the space Y is not necessary to ensure closeness of the *top* homeomorphic embedding.

We now give some results to help us prove the above theorem and corollaries.

Let $P = \prod \left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$ be the FPTP of the EPTPs $P_\sigma, \sigma \in \Sigma$, with respect to the inverse system $\{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}$ and let there be given topological spaces X and Y, and continuous maps $f : X \to Y$, $\mu^B_\sigma : Y \to Y_\sigma$ with $\lambda^\sigma_\rho \circ \mu^B_\sigma = \mu^B_\rho$ for any $\rho, \sigma \in \Sigma$ satisfying $\rho \leqslant \sigma$, and $g_\sigma : f^{-1}(\mu^B_\sigma)^{-1}O_\sigma \to Z_\sigma$ for every $\sigma \in \Sigma$. Then, the diagonal product of the maps f and g_σ over the map μ^B_σ, $\triangle_\sigma \equiv \triangle(f, g_\sigma; \mu^B_\sigma) : X \to P_\sigma$, is defined and we have that $\{\triangle_\sigma, \mu^B_\sigma\} : f \to p_\sigma$, where $p_\sigma : P_\sigma \to Y_\sigma$ is the projection of the EPTP P_σ. It is not difficult to see that by taking the standard diagonal products, we get a morphism $\{\triangle(\triangle_\sigma), \triangle\mu^B_\sigma\}$ of the map f into the projection p of the FPTP P, and so

$$p \circ \triangle(\triangle_\sigma) = \triangle\mu^B_\sigma \circ f.$$

As corollaries to Proposition 4.8 and Corollaries 4.9 and 4.10 we have the following results.

Corollary 5.8. *If under the above conditions we have:*

1. *the collection* $\{\mu_\sigma^B : \sigma \in \Sigma\}$ *separates points and for every point* $y \in Y$ *and every two distinct points* x *and* x' *in the fibre* $f^{-1}y$ *there exists some* $\sigma \in \Sigma$, *such that* $y \in (\mu_\sigma^B)^{-1}O_\sigma$ *and* $g_\sigma(x) \neq g_\sigma(x')$, *and*
2. *the collection* $\mathfrak{B}_f = \bigcup\{\{g_\sigma^{-1}W : W$ *open in* $Z_\sigma\} : \sigma \in \Sigma\}$ *is a base for the map* f *and* $\eta_\sigma^B \equiv \mu_\sigma^B|_{(\mu_\sigma^B)^{-1}O_\sigma} : (\mu_\sigma^B)^{-1}O_\sigma \to O_\sigma \cap \mu_\sigma^B(Y)$ *is an open map,*

then the morphism $\{\triangle(\triangle_\sigma), \triangle\mu_\sigma^B\}$ *is a* {*homeomorphic embedding, 1-1 local homeomorphic embedding*}*-morphism of* f *into the projection* p *of the FPTP* P.

Corollary 5.9. *If under the above conditions we have:*

1. *the collection* $\{\mu_\sigma^B : \sigma \in \Sigma\}$ *separates points and the map* f *is a* T_0*-map, and*
2. *the collection* $\mathfrak{B}_f = \bigcup\{\{g_\sigma^{-1}W : W$ *open in* $Z_\sigma\} : \sigma \in \Sigma\}$ *is a base for the map* f *and* $\eta_\sigma^B \equiv \mu_\sigma^B|_{(\mu_\sigma^B)^{-1}O_\sigma} : (\mu_\sigma^B)^{-1}O_\sigma \to O_\sigma \cap \mu_\sigma^B(Y)$ *is an open map,*

then the morphism $\{\triangle(\triangle_\sigma), \triangle\mu_\sigma^B\}$ *is a* {*homeomorphic embedding, 1-1 local homeomorphic embedding*}*-morphism of* f *into the projection* p *of the FPTP* P.

Corollary 5.10. *If under the above conditions we have:*

1. *the space* Y *is a* T_0*-space and the map* f *is a* T_0*-map, and*
2. *the collection* $\mathfrak{B}_f = \bigcup\{\{g_\sigma^{-1}W : W$ *open in* $Z_\sigma\} : \sigma \in \Sigma\}$ *is a base for the map* f *and* $\eta_\sigma^B \equiv \mu_\sigma^B|_{(\mu_\sigma^B)^{-1}O_\sigma} : (\mu_\sigma^B)^{-1}O_\sigma \to O_\sigma \cap \mu_\sigma^B(Y)$ *is an open map,*

then the morphism $\{\triangle(\triangle_\sigma), \triangle\mu_\sigma^B\}$ *is a* {*homeomorphic embedding, 1-1 local homeomorphic embedding*}*-morphism of* f *into the projection* p *of the FPTP* P.

Finally we need the following results for the case of compact fibres, the first of which was proved in [3].

Proposition 5.11. *The fibrewise product* $p = \prod\left\{f_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$ *of compact maps* f_σ *with respect to the inverse system* $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$ *is a compact map.*

Proposition 5.12. *Let* $P = \prod\left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$ *be the FPTP of the EPTPs* $P_\sigma, \sigma \in \Sigma$, *with respect to the inverse system* $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$, *where the fibres* Z_σ *are compact for every* $\sigma \in \Sigma$. *Then, the projection* p *of the FPTP* P *onto its base is a compact map.*

Proof. By Corollary 3.3 we have that the projections $p_\sigma : P_\sigma \to Y_\sigma$ are compact. Therefore, we can conclude by Proposition 5.11, that the projection p is also compact as a fibrewise product of compact maps. □

Corollary 5.13. *Let* $P = \prod\left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$ *be the FPTP of the EPTPs* $P_\sigma, \sigma \in \Sigma$, *with respect to the inverse system* $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$, *where the fibres* Z_σ *are compact and metrizable for every* $\sigma \in \Sigma$. *Then, the projection* p *of the FPTP* P *onto its base is a compact Tychonoff map with weight* $\mathfrak{W}(p) \leqslant \max(|\Sigma|, \aleph_0)$.

Proof. The fact that the map p is compact follows from Proposition 5.12 and the fact that it is Tychonoff, together with the inequality for the weight $\mathfrak{W}(p)$, follows from Propositions 3.1 and 5.2 and Remark 5.1. □

Proof of Theorem 5.4. The proof of (1) implies (2) follows on the same lines as that of (1) implies (2) in Theorem 4.4, using the corresponding results of this section. That (3) follows from (2) is evident. We are left to show that (3) implies (1). If there exists a homeomorphic embedding-morphism of the map f into the projection p of a FPTP $P = \prod \left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$, where the EPTP $P_\sigma = P(Y_\sigma, I, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$, then by Proposition 2.3 and Corollary 5.13 we have

$$\mathfrak{W}(f) \leqslant \mathfrak{W}(p) \leqslant |\Sigma| \leqslant \mathfrak{m}.$$

□

Proof of Corollary 5.5. We only need to show that (1) implies (2). Since the diagonal $\triangle\mathrm{id}_\sigma(Y)$, where $\mathrm{id}_\sigma \equiv \mathrm{id} : Y \to Y$, is homeomorphic to the limit of the constant inverse system $\boldsymbol{S}(Y, \Sigma)$, the result follows from Proposition 2.2. □

Proof of Corollary 5.6. We only need to show that (3) implies (1) and this follows from Corollary 5.13 and Proposition 2.1. □

Proof of Corollary 5.7. The fact that (1) implies (2) follows from Corollary 5.5. That (2) implies (3) is evident and the proof that (3) implies (1) follows from Corollary 5.13 and the fact that a closed submap of a compact map is compact. □

Finally, we end this section by a universal type theorem for T_0-maps in $\mathcal{MAP}$ for Fan poducts corresponding to Theorem 4.13. This is an analogue of Theorem 1.2 in the category $\mathcal{MAP}$ with respect to Fan products. The proof is omitted as it is analogous, using the corresponding results of this section, to that of Theorem 4.13.

Theorem 5.14. *For a T_0-map $f : X \to Y$ the following are equivalent:*

1. *The map f has weight $\mathfrak{w}(f) \leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$);*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod \{P_\sigma, p_\sigma, \boldsymbol{S}(Y, \Sigma)\}$, where the EPTP $P_\sigma = P(Y, F, Y)$ and $|\Sigma| \leqslant \mathfrak{m}$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod \left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$, where the EPTP $P_\sigma = P(Y_\sigma, F, O_\sigma)$ and $|\Sigma| \leqslant \mathfrak{m}$.*

Corollary 5.15. *For a continuous map $f : X \to Y$ the following are equivalent:*

1. *The map f is T_0;*
2. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod \{P_\sigma, p_\sigma, \boldsymbol{S}(Y, \Sigma)\}$, where the EPTP $P_\sigma = P(Y, F, Y)$;*
3. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod \left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\}$, where the EPTP $P_\sigma = P(Y_\sigma, F, O_\sigma)$.*

6. Zero-dimensional and strongly zero-dimensional maps

In this section we define zero-dimensional and strongly zero-dimensional maps. We note that our definition of zero-dimensional maps differ from that given in [6].

Using our definition one can see that many properties of zero-dimensional spaces can be generalized from the category $\mathcal{TOP}$ to the category $\mathcal{MAP}$.

Definition 6.1. Let there be given a continuous map $f : X \to Y$. A set $U \subset X$ is said to be *f-closed-open (f-clopen)*, if there exists an open subset O of Y such that $U \subset f^{-1}O$ and U is clopen in $f^{-1}O$.

Definition 6.2. Let there be given a continuous map $f : X \to Y$, where $X \neq \emptyset$. The map f is called *zero-dimensional* if it is a T_1-map and has a base $\mathfrak{B}_f$ consisting of f-clopen sets, where a map $f : X \to Y$ is said to be a T_1-map if for every two distinct points $x, x' \in X$ lying in the same fibre, each of the points x, x' has a neighborhood in X which does not contain the other point.

Note that if the set U is f-clopen then it is also open in X but is not necessarily closed in X. It is not difficult to see that every zero-dimensional map is Tychonoff. Remember that for a continuous map $f : X \to Y$, two subsets A and B of the space X are said to be *f-functionally separated* if for every point $y \in Y$ there exists a neighborhood O of y in Y, such that the sets A and B are functionally separated in $f^{-1}O$.

Definition 6.3. Let there be given a Tychonoff map $f : X \to Y$, where $X \neq \emptyset$. The map f is called *strongly prezero-dimensional* if for every pair A, B of functionally separated subsets of the space X and for every $y \in Y$, there exists a neighborhood O of y in Y and a clopen (in $f^{-1}O$) set $U \subset f^{-1}O$, such that $A \cap f^{-1}O \subset U \subset f^{-1}O \setminus B$. The map f is called *strongly zero-dimensional* if for every open set $O \subset Y$, the map $f|_{f^{-1}O} : f^{-1}O \to O$ is strongly prezero-dimensional.

One can note that if the map $f : X \to Y$ is strongly zero-dimensional, then for every pair A, B of f-functionally separated subsets of the space X and for every $y \in Y$, there exists a neighborhood O of y in Y and a clopen (in $f^{-1}O$) set $U \subset f^{-1}O$, such that $A \cap f^{-1}O \subset U \subset f^{-1}O \setminus B$.

Proposition 6.1. *Every strongly zero-dimensional map is zero-dimensional.*

Proof. Say $f : X \to Y$ is a strongly zero-dimensional map. Take any closed set F in X and any point $x \in X \setminus F = W$. Also, let $y = f(x)$. Since f is Tychonoff, there exists a neighborhood O of y in Y and a function $\phi : f^{-1}O \to [0, 1]$ such that $x \in \phi^{-1}(0)$ and $F \cap f^{-1}O \subset \phi^{-1}(1)$. Furthermore, there exists a neighborhood O' of y in Y such that $O' \subset O$, and a clopen (in $f^{-1}O'$) set U in $f^{-1}O'$ such that $x \in U \subset f^{-1}O' \setminus F = f^{-1}O' \cap W$. Therefore, f is a zero-dimensional map as required to prove. □

Theorem 6.2. *A Tychonoff map $f : X \to Y$, where $X \neq \emptyset$, is strongly zero-dimensional if and only if, for every $y \in Y$, every neighborhood O of y in Y and every finite f-functionally open cover $\mathcal{U} = \{U_i : i = 1, \ldots, k\}$ of $f^{-1}O$ there exists a neighborhood $O' \subset O$ of y and a finite refinement $\mathcal{V} = \{V_i : i = 1, \ldots, k\}$ of $\mathcal{U} \wedge f^{-1}O'$ such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Note that by the hypothesis, $\bigcup \mathcal{V} = \bigcup\{V_i : i = 1, \ldots, k\} = f^{-1}O'$.*

Proof. Let $f : X \to Y$ be a strongly zero-dimensional map and take any point y in Y. Let O be any neighborhood of y in Y and consider a finite f-functionally open cover $\mathcal{U} = \{U_i : i = 1, \dots, k\}$ of $f^{-1}O$. We apply induction with respect to k. For $k = 1$, the existence of the refinement $\mathcal{V}$ is obvious. Assume that the existence of $\mathcal{V}$ is true for every $k < m$ and consider the case of $k = m$. By the inductive hypothesis, there exists a neighborhood $O_1 \subset O$ of y in Y and a cover $\{W_1, \dots, W_{m-1}\}$ of $f^{-1}O_1$ consisting of pairwise disjoint open sets (and so are f-clopen), satisfying

$$W_i \subset U_i \text{ for } i < m-1, \quad \text{and } W_{m-1} \subset U_{m-1} \cup U_m.$$

Consider the sets $W_{m-1} \setminus U_{m-1}$ and $W_{m-1} \setminus U_m$, which are disjoint and f-functionally closed. Thus, there exist neighborhoods O_2 and O_3 of y in Y and functions $\psi, \phi : f^{-1}(O_2 \cap O_3) = f^{-1}O_4 \to [0,1]$ satisfying

$$W_{m-1} \setminus U_{m-1} = \psi^{-1}(0)$$
$$W_{m-1} \setminus U_m = \phi^{-1}(0)$$

Let $\eta : f^{-1}O_4 \to [0,1]$ be defined by $\eta(x) = \frac{\psi(x)}{\psi(x)+\phi(x)}$. Then $W_{m-1} \setminus U_{m-1} = \eta^{-1}(0)$ and $W_{m-1} \setminus U_m = \eta^{-1}(1)$ and therefore the sets $W_{m-1} \setminus U_{m-1}$ and $W_{m-1} \setminus U_m$ are functionally separated in $f^{-1}O_4$. There exists a neighborhood $O_5 \subset O_4$ of y in Y and a clopen (in $f^{-1}O_5$) set $U \subset f^{-1}O_5$ such that

$$(W_{m-1} \setminus U_{m-1}) \cap f^{-1}O_5 \subset U \subset f^{-1}O_5 \setminus (W_{m-1} \setminus U_m).$$

The latter inclusion implies that

$$U \subset (f^{-1}O_5 \setminus W_{m-1}) \cup (f^{-1}O_5 \cap U_m),$$

and hence

$$(W_{m-1} \setminus U) \cap f^{-1}O_5 \subset U_{m-1} \text{ and } U \cap W_{m-1} \subset U_m.$$

Consequently, one can easily see that the collection $\mathcal{V} = \{V_i : i = 1, \dots, m\}$, where

$$V_i = W_i \cap f^{-1}O_5 \text{ for } i < m-1,$$
$$V_{m-1} = (W_{m-1} \setminus U) \cap f^{-1}O_5 \quad \text{and } V_m = U \cap W_{m-1},$$

is the desired refinement of $\mathcal{U} \wedge f^{-1}O_5$.

Conversely, take any pair A, B of functionally separated subsets of the space X and take any point y in Y. There exists a function $\phi : X \to [0,1]$ such that $A \subset \phi^{-1}(0)$ and $B \subset \phi^{-1}(1)$. The sets $U_1 = \phi^{-1}(]0,1])$ and $U_2 = \phi^{-1}([0,1[)$ form a functionally open (and so an f-functionally open) cover of X. By the hypothesis, there exists a neighborhood O of y in Y and a disjoint open refinement $\mathcal{V} = \{V_1, V_2\}$ of $(\mathcal{U} = \{U_1, U_2\}) \wedge f^{-1}O$. Therefore, the f-clopen set V_2 satisfies

$$A \cap f^{-1}O \subset V_2 \subset f^{-1}O \setminus B,$$

which proves that the map f is strongly prezero-dimensional. In an analogous manner one can show that f is strongly zero-dimensional. □

Recall that a map $f : X \to Y$ is called *functionally prenormal* if every two disjoint closed sets in X are f-functionally separated. The map f is called *functionally normal* if for every open set O in Y the map $f : f^{-1}O \to O$ is functionally prenormal. A functionally normal $T_{3\frac{1}{2}}$-map ia called a $T_{4\frac{1}{2}}$-map. Functionally normal (as well as normal) maps were defined by B.A.Pasynkov [12]. With slight modifications in the proof of Theorem 6.2 one can get the following result.

Theorem 6.3. *A $T_{4\frac{1}{2}}$-map $f : X \to Y$, where $X \neq \emptyset$, is strongly zero-dimensional if and only if, for every $y \in Y$, every neighborhood O of y in Y and every finite open cover $\mathcal{U} = \{U_i : i = 1, \dots, k\}$ of $f^{-1}O$ there exists a neighborhood $O' \subset O$ of y and a finite refinement $\mathcal{V} = \{V_i : i = 1, \dots, k\}$ of $\mathcal{U} \wedge f^{-1}O'$ such that $V_i \cap V_j = \emptyset$ whenever $i \neq j$. Note that by the hypothesis, $\bigcup \mathcal{V} = \bigcup \{V_i : i = 1, \dots, k\} = f^{-1}O'$.*

Remember that a map $f : X \to Y$ is called *finally compact* if f is closed and for every $y \in Y$ the fibre $f^{-1}y$ is finally compact, that is every open cover of $f^{-1}y$ has a countable subcover. A finally compact T_3-map is called a *Lindelöf* map. Thus, every compact (Hausdorff) map is finally compact (Lindelöf) and every Lindelöf map is a T_4 paracompact map [2].

Theorem 6.4. *Every zero-dimensional Lindelöf map is strongly zero-dimensional.*

Proof. Let A, B be a pair of functionally separated subsets of the space X and take any point $y \in Y$. There exists a function $\phi : X \to [0, 1]$ such that

$$A \subset \phi^{-1}(0) = A' \text{ and } B \subset \phi^{-1}(1) = B'.$$

Then the sets A' and B' are closed and disjoint in X.

For every $x \in f^{-1}y$ one can find an f-clopen set $U(x)$ and a neighborhood $O_{U(x)}$ of y in Y such that

$$(U(x) \cap f^{-1}O_{U(x)}) \cap A' = \emptyset, \quad \text{or } (U(x) \cap f^{-1}O_{U(x)}) \cap B' = \emptyset.$$

Let $U'(x) = U(x) \cap f^{-1}O_{U(x)}$, where one can assume that $U'(x)$ is clopen in $f^{-1}O_{U(x)}$. Since the map f is Lindelöf, there exists a countable subcollection $\{U'(x_i) : i < \omega\}$ of $\{U'(x) : x \in f^{-1}y\}$ covering $f^{-1}y$. Let

$$W(x_i) = \left[U'(x_i) \setminus \bigcup_{j<i} U'(x_j) \right] \bigcap f^{-1} \left(\bigcap_{j \leqslant i} O_{U(x_j)} \right) \text{ for } i < \omega.$$

The collection $\mathcal{W} = \{W(x_i) : i < \omega\}$ consists of f-clopen and pairwise disjoint sets, and covers $f^{-1}y$. By the closedness of f, there exists a neighborhood O of y in Y such that $\mathcal{W}$ covers $f^{-1}O$. Let $\mathcal{W}' = \mathcal{W} \wedge f^{-1}O$. The set $U = \bigcup \{W_i : A \cap W_i \neq \emptyset, W_i \in \mathcal{W}'\}$ has the following properties:

$$U \text{ is clopen in } f^{-1}O \text{ and } A \cap f^{-1}O \subset U \subset f^{-1}O \setminus B.$$

Thus f is strongly prezero-dimensional. Analogously one can prove that the map f is strongly zero-dimensional. □

Corollary 6.5. *Zero-dimensionality and strong zero-dimensionality are equivalent in the realm of compact maps $f : X \to Y$, where $X \neq \emptyset$.*

Theorem 6.6. *If $f : X \to Y$ is a zero-dimensional map, then so is any submap $g : A \to B$, where $A \neq \emptyset$.*

If $f : X \to Y$ is a strongly zero-dimensional map and $g : A \to B$ is a submap of f, where $A \neq \emptyset$, B is closed in Y and for every open in Y set O, every continuous function $\phi : g^{-1}(O \cap B) \to [0, 1]$ is continuously extendable to a function $\psi : f^{-1}O \to [0, 1]$, then g is also strongly zero-dimensional.

Proof. The first part of the theorem follows from the definitions. The second part follows from Definition 6.3, since under the assumptions on the map g, any two $g|_{g^{-1}(O\cap B)}$-functionally separated subsets of the space $g^{-1}(O \cap B)$ are $f|_{f^{-1}O}$-functionally separated in $f^{-1}O$ for any open set O in Y. □

We now prove the following result concerning the maximal Tychonoff compactification $\beta f : \beta_f X \to Y$ of a Tychonoff map $f : X \to Y$. A compact map $bf : b_f X \to Y$ is said to be a compactification of $f : X \to Y$ if there exists a {dense homeomorphic embedding}-morphism $\{\lambda, \mathrm{id}_Y\} : f \to bf$ [17, 18]. In this situation we usually identify X with $\lambda(X)$ and so $b_f X = [X]_{b_f X}$ and $f = bf|_X$, where by $[X]_{b_f X}$ we mean the closure of X in $b_f X$. For details concerning compactifications of Tychonoff maps, in particular the construction of βf, one can consult [12, 13, 9].

Theorem 6.7. *The compactification $\beta f : \beta_f X \to Y$ of a Tychonoff map $f : X \to Y$ is strongly zero-dimensional if and only if the map f is strongly zero-dimensional.*

For the proof of the above theorem we need some preliminary lemmas.

Lemma 6.8 (Pasynkov [12])**.** *For a Tychonoff compactification $bf : b_f X \to Y$ of a Tychonoff map $f : X \to Y$, the following properties are equivalent:*

1. *$bf \equiv \beta f : \beta_f X \to Y$, where by equivalence we understand that bf and βf are canonically isomorphic, i.e. there exists a homeomorphism $\lambda : \beta_f X \to b_f X$ equal to the identity on X such that $\beta f = bf \circ \lambda$;*
2. *For any open set O in Y and any continuous bounded function $\phi : f^{-1}O \to [a, b]$, there exists a continuous extension of ϕ on $(bf)^{-1}O$;*
3. *For any open set O in Y and any two functionally separated in $f^{-1}O$ subsets F and H we have*

$$[F]_{b_f X} \cap [H]_{b_f X} \cap (bf)^{-1}O = \emptyset.$$

Below, by the space D we understand the two point set $\{0, 1\}$ with the discrete topology.

Lemma 6.9. *Let $\beta f : \beta_f X \to Y$ be the maximal Tychonoff compactification of a Tychonoff map $f : X \to Y$. Let $U \subset X$ be an f-clopen subset, i.e. there exists an open set O in Y such that $U \subset f^{-1}O$ and U is clopen in $f^{-1}O$. Then $[U]_{(\beta f)^{-1}O}$ is clopen in $(\beta f)^{-1}O$ and so is a βf-clopen set.*

Proof. There exists a function $\phi : f^{-1}O \to D \subset [0, 1]$ such that $U = \phi^{-1}(0)$ and $f^{-1}O \backslash U = \phi^{-1}(1)$. Therefore, U and $f^{-1}O \backslash U$ are functionally separated in $f^{-1}O$. Consequently, by Lemma 6.8, $[U]_{(\beta f)^{-1}O}$ is clopen in $(\beta f)^{-1}O$. □

Proof of Theorem 6.7. If the map βf is strongly zero-dimensional then so is the map f by Theorem 6.6 and Lemma 6.8.

We need to show that if the map f is strongly zero-dimensional then so is the map βf. Let A, B be a pair of functionally separated subsets in $(\beta f)^{-1}O$, where O is an open set in Y. There exists a function $\phi : (\beta f)^{-1}O \to [0,1]$ such that $A \subset \phi^{-1}(0)$ and $B \subset \phi^{-1}(1)$.

Consider the sets $A_1 = X \cap \phi^{-1}([0, \frac{1}{3}[)$ and $B_1 = X \cap \phi^{-1}(]\frac{2}{3}, 1])$ which are functionally separated in $f^{-1}O$. Since the map f is strongly zero-dimensional, for any point $y \in O$, there exists a neighborhood $O'(y) \subset O$ of y in Y and a clopen in $f^{-1}O'(y)$ set $U \subset f^{-1}O'(y)$ such that $A_1 \cap f^{-1}O'(y) \subset U \subset f^{-1}O'(y) \setminus B_1$. By Lemma 6.9 we have that $[U]_{(\beta f)^{-1}O'(y)}$ is clopen in $(\beta f)^{-1}O'(y)$.

Since X is dense in $\beta_f X$ we have $A \subset [A_1]_{(\beta f)^{-1}O}$ and $B \subset [B_1]_{(\beta f)^{-1}O}$. From Lemma 6.8 we also have that $[B_1]_{(\beta f)^{-1}O'(y)} \cap [U]_{(\beta f)^{-1}O'(y)} = \emptyset$ and consequently

$$A \cap (\beta f)^{-1}O'(y) \subset [A_1]_{(\beta f)^{-1}O'(y)} = [A_1 \cap f^{-1}O'(y)]_{(\beta f)^{-1}O'(y)} \subset$$
$$[U]_{(\beta f)^{-1}O'(y)} \subset (\beta f)^{-1}O'(y) \setminus [B_1]_{(\beta f)^{-1}O'(y)} \subset$$
$$(\beta f)^{-1}O'(y) \setminus \left(B \cap (\beta f)^{-1}O'(y)\right).$$

□

Proposition 6.10. *The Tychonoff product $f = \prod\{f_\alpha : \alpha \in \mathcal{A}\} : X = \prod\{X_\alpha : \alpha \in \mathcal{A}\} \to Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$, where $\mathcal{A} \neq \emptyset$ and $X_\alpha \neq \emptyset$ for every $\alpha \in \mathcal{A}$, is zero-dimensional if and only if all the maps f_α are zero-dimensional.*

Proof. The fact that f_α is zero-dimensional if the Tychonoff product f is zero-dimensional follows from Theorem 6.6.

Conversely, say f_α is zero-dimensional for every $\alpha \in \mathcal{A}$. Then f is a T_1-map by Proposition 4.2. Let $\mathfrak{B}_{f_\alpha}$ be an f_α-clopen base for the map f_α, for every $\alpha \in \mathcal{A}$. Then by Proposition 4.1, the collection $\mathfrak{S}_f = \cup\{(pr_T^\alpha)^{-1}\mathfrak{B}_{f_\alpha} : \alpha \in \mathcal{A}\}$ is an f-clopen subbase for the map f and thus it is not difficult to see that f has an f-clopen base. □

In a similar fashion one can prove the following results.

Proposition 6.11. *Let $p : P = \prod \left\{X_\sigma, f_\sigma, \{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}\right\} \to \varprojlim\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$ be the fibrewise product of the maps f_σ with respect to the inverse system $\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$, where $\varprojlim\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\} \neq \emptyset$. If all the maps f_σ are zero-dimensional then the map p is also zero-dimensional.*

Proposition 6.12. *Let $\varprojlim \boldsymbol{S}_f = \varprojlim \left\{f_\sigma, \{\pi_\rho^\sigma, \lambda_\rho^\sigma\}, \Sigma\right\} : \varprojlim \boldsymbol{S}_T = \varprojlim\{X_\sigma, \pi_\rho^\sigma, \Sigma\} \to \varprojlim \boldsymbol{S}_B = \varprojlim\{Y_\sigma, \lambda_\rho^\sigma, \Sigma\}$ be the limit map of the inverse system $\boldsymbol{S}_f = \left\{f_\sigma, \{\pi_\rho^\sigma, \lambda_\rho^\sigma\}, \Sigma\right\}$. If all the maps f_σ are zero-dimensional and $\varprojlim \boldsymbol{S}_T \neq \emptyset$ then the map $\varprojlim \boldsymbol{S}_f$ is also zero-dimensional.*

With respect to sums we have the following result. The proof is not difficult and so is omitted.

Proposition 6.13. *The sum* $f = \bigoplus\{f_\alpha : \alpha \in \mathcal{A}\} : X = \bigoplus\{X_\alpha : \alpha \in \mathcal{A}\} \to Y = \bigoplus\{Y_\alpha : \alpha \in \mathcal{A}\}$*, where* $\mathcal{A} \neq \emptyset$ *and* $X_\alpha \neq \emptyset$ *for every* $\alpha \in \mathcal{A}$*, is zero-dimensional (strongly zero-dimensional) if and only if all the maps* f_α *are zero-dimensional (strongly zero-dimensional).*

We next prove a lemma concerning EPTPs having zero-dimensional fibre.

Lemma 6.14. *The projection* $p : P \to Y$*, where* $Y \neq \emptyset$*, of an EPTP* $P = P(Y, Z, O)$ *with zero-dimensional fibre* Z *is zero-dimensional and* $\mathfrak{W}(p) = \mathfrak{w}(Z) + 1$*.*

Proof. Let p be the projection of an EPTP with zero-dimensional fibre Z. That the map p is a T_1-map and that $\mathfrak{W}(p) = \mathfrak{w}(Z) + 1$ follows from Proposition 3.1. Let $\mathfrak{B}$ be a base for the space Z consisting of clopen sets and of cardinality $\mathfrak{w}(Z)$. The collection $\mathfrak{B}_p$ consisting of the sets $\{P\}$ and $\{O \times V : V \in \mathfrak{B}\}$ is a base for the map p. It is not difficult to see that $\mathfrak{B}_p$ consists of p-clopen sets and therefore the map p is zero-dimensional. □

Next we have the following universal type theorem for zero-dimensional maps, the proof of which is analogous to that of Theorems 4.4 and 5.4 and so is omitted. This is an analogue of Theorem 1.3 in the category $\mathcal{MAP}$.

Theorem 6.15. *For a zero-dimensional map* $f : X \to Y$ *the following are equivalent:*

1. *The map* f *has weight* $\mathfrak{W}(f) \leqslant \mathfrak{m}$ *(*$\mathfrak{m} \geqslant \aleph_0$*);*
2. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a TPTP* $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$*, where the EPTP* $P_\alpha = P(Y, D, O_\alpha)$ *and* $|\mathcal{A}| \leqslant \mathfrak{m}$*;*
3. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a TPTP* $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$*, where the EPTP* $P_\alpha = P(Y_\alpha, D, O_\alpha)$ *and* $|\mathcal{A}| \leqslant \mathfrak{m}$*;*
4. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a FPTP* $P = \prod\{P_\sigma, p_\sigma, \boldsymbol{S}(Y, \Sigma)\}$*, where the EPTP* $P_\sigma = P(Y, D, O_\sigma)$ *and* $|\Sigma| \leqslant \mathfrak{m}$*;*
5. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a FPTP* $P = \prod\left\{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\right\}$*, where the EPTP* $P_\sigma = P(Y_\sigma, D, O_\sigma)$ *and* $|\Sigma| \leqslant \mathfrak{m}$*.*

We can write down the following corollary to the above theorem. We omit the proof as it is analogous to that of Corollaries 4.6 and 5.6.

Corollary 6.16. *For a continuous map* $f : X \to Y$*, where* $X \neq \emptyset$*, the following are equivalent:*

1. *The map* f *is zero-dimensional;*
2. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a TPTP* $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$*, where the EPTP* $P_\alpha = P(Y, D, O_\alpha)$*;*
3. *There exists a homeomorphic embedding-morphism of the map* f *into the projection of a TPTP* $\prod\{P_\alpha : \alpha \in \mathcal{A}\}$*, where the EPTP* $P_\alpha = P(Y_\alpha, D, O_\alpha)$*;*

4. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod \{P_\sigma, p_\sigma, \boldsymbol{S}(Y,\Sigma)\}$, where the EPTP $P_\sigma = P(Y, D, O_\sigma)$;*
5. *There exists a homeomorphic embedding-morphism of the map f into the projection of a FPTP $P = \prod \{P_\sigma, p_\sigma, \{Y_\sigma, \lambda^\sigma_\rho, \Sigma\}\}$, where the EPTP $P_\sigma = P(Y_\sigma, D, O_\sigma)$.*

Finally we have the following result concerning Tychonoff compactifications.

Theorem 6.17. *Every zero-dimensional map $f : X \to Y$ of weight $\mathfrak{W}(f) = \mathfrak{m} \geqslant \aleph_0$ has a zero-dimensional compactification $bf : b_f X \to Y$ of weight $\mathfrak{W}(bf) = \mathfrak{m}$.*

Proof. By Theorem 6.15 (4), the map f can be identified with a submap of the projection $p : P \to \varprojlim \boldsymbol{S}(Y,\Sigma) \cong Y$ of a FPTP $P = \prod \{P_\sigma, p_\sigma, \boldsymbol{S}(Y,\Sigma)\}$, where the EPTP $P_\sigma = P(Y, D, O_\sigma)$ and $|\Sigma| = \mathfrak{m}$. Let $b_f X$ be the closure of X in P and let $bf = p|_{b_f X}$. By Proposition 5.12 the map p is compact, and therefore so is bf as a closed submap of a compact map. This implies that bf is a compactification of the map f and by Lemma 6.14 and Proposition 6.11, we have that bf is zero-dimensional. Finally we have

$$\mathfrak{m} = \mathfrak{W}(f) \leqslant \mathfrak{W}(bf) \leqslant \mathfrak{W}(p) = \mathfrak{m},$$

from which follows the equality $\mathfrak{W}(bf) = \mathfrak{m}$. □

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