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On metrizable type (*MT*-) maps and spaces

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Abstract

In this paper we define and study *MT*-maps, which are the fibrewise topological analogue of metrizable spaces, i.e., the extension of metrizability from the category *Top* to the category Top_Y. Several characterizations and properties of *MT*-maps are proved. The notion of an *MT*-space as an *MT*-map preimage of a metrizable space is introduced. Examples of *MT*-spaces and their relation with *M*-spaces are given. Finally it is deduced that an *MT*-space with a G_δ-diagonal is metrizable.

Keywords: *MT*-map; *MT*-space; Fibrewise Topology

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1. Preliminaries

Fibrewise General Topology or General Topology of Continuous Maps is concerned most of all in extending the main notions and results concerning spaces to continuous maps. In this way one can see some well-known results in a new and clearer light and one can also be led to further developments which otherwise would not have suggested themselves. This is usually done in the following way.

For an arbitrary topological space Y one considers the category Top_Y, the objects of which are continuous maps into the space Y, and for the objects $f: X \to Y$ and $g: Z \to Y$, a morphism from f into g is a continuous map $\lambda: X \to Z$ with the property $f = g \circ \lambda$. This is denoted by $\lambda: f \to g$. We note that this situation is a generalization of the category

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Top (of topological spaces and continuous maps as morphisms), since the category *Top* is isomorphic to the particular case of Top_Y in which the space Y is a singleton set.

In defining properties of a continuous map $f : X \to Y$ one does not directly involve any properties on the spaces X and Y (except the existence of a topology). Such were the definitions given in [3,4,14,16] for the separation axioms, compactness, paracompactness, weight and others. In most cases there is some choice in defining these properties and one usually prefers the simplest and the one that gives the most complete generalization of the corresponding results in the category *Top*. It would be beneficial to have a more systematic way of extending definitions and results from the category *Top* to the category Top_Y and some hope is provided by the link between fibrewise topology and topos theory [7,8,10, 11]. Unfortunately, as was noted in [6], this approach has several drawbacks.

Research in the general topology of continuous maps showed a strong analogy in the behaviour of spaces and maps and it was possible to extend the main notions and results of spaces to that of maps. Most of the results obtained so far in this field can be found in [3,4,6,9,14,15], where one can also find an extensive bibliography on the subject.

Unless otherwise stated, Y is a fixed topological space with topology τ. The collection of all neighborhoods (nbd(s)) of a point $y \in Y$ is denoted by $N(y)$. A morphism $\lambda : f \to g$ is called surjective, closed, perfect, etc., if, respectively, such is the map $\lambda : X \to Z$. If $[\lambda X] = Z$ then the morphism λ is said to be dense and if $\lambda : X \to Z$ is a homeomorphism then the morphism λ is said to be an isomorphism. Here by $[\,\cdot\,]$ or $[\,\cdot\,]_X$ we mean the closure operator in the respective space.

We now give some definitions and results concerning maps. For more details one can consult [6] and [15].

Definition 1.1. A continuous map $f : X \to Y$ is called a T_i-map, $i = 0, 1, 2$, if for all $x, x' \in X$ such that $x \neq x'$, $fx = fx'$ the following condition is, respectively satisfied:

$i = 0$: at least one of the points x, x' has a nbd in X not containing the other point;

$i = 1$: each of the points x, x' has a nbd in X not containing the other point;

$i = 2$: the points x and x' have disjoint nbds in X.

A T_2-map is also called Hausdorff. We note that for $i = 0, 1$ the property for a map $f : X \to Y$ to be a T_i-map, is equivalent to the property that all the fibers $f^{-1}y$, $y \in Y$, are T_i-spaces. This is not the case for T_2-maps.

Definition 1.2. The subsets A and B of the space X are said to be, respectively:
(a) nbd separated in $U \subset X$,
(b) functionally separated in $U \subset X$,
if the sets $A \cap U$ and $B \cap U$
(a) have disjoint nbds in U,
(b) are functionally separated in U (i.e., there exists a continuous function $\phi : U \to [0, 1]$ such that $A \cap U \subset \phi^{-1}(0)$ and $B \cap U \subset \phi^{-1}(1)$).

Definition 1.3. A continuous map $f : X \to Y$ is called completely regular (regular), if for every point $x \in X$ and every closed set F in X, $x \notin F$, there exists a nbd $O \in N(fx)$, such that the sets $\{x\}$ and F are functionally separated (nbd separated) in $f^{-1}O$. A completely regular (regular) T_0-map is called Tychonoff or $T_{3\frac{1}{2}}$- (T_3-) map.

It can be easily verified that every T_j-map is a T_i-map for $j, i = 0, 1, 2, 3, 3\frac{1}{2}$ and $i \leqslant j$.

Definition 1.4. A continuous map $f : X \to Y$ is called functionally prenormal (prenormal) if for every $y \in Y$ and every two disjoint, closed sets F and H in X, there exists $O \in N(y)$ such that F and H are functionally separated (nbd separated) in $f^{-1}O$. If for every $O \in \tau$, the map $f|_{f^{-1}O} : f^{-1}O \to O$ is functionally prenormal (prenormal) then f is called functionally normal (normal). A normal T_3-map is called a T_4-map.

Remark 1.5. In [6] a functionally prenormal (prenormal) map in the sense of Definition 1.4 is called a functionally normal (normal) map. We will use the terminology of Definition 1.4 as is [15].

The following results can be found in [15].

Proposition 1.6. *If a map $f : X \to Y$ is closed we have*:

(a) *If for every $y \in Y$, every $x \in f^{-1}y$ and every closed* (*in $f^{-1}y$*) *set A, such that $x \notin A$, the sets $\{x\}$ and A are nbd separated in X, then f is regular.*
(b) *If for every $y \in Y$, every two disjoint, closed* (*in $f^{-1}y$*) *sets are nbd separated in X, then f is normal.*

Remark 1.7. The above proposition shows that a closed normal T_1-map is T_3 and thus T_4.

Proposition 1.8. *If a space X is* (a) *a T_i-space, $i = 0, 1, 2$,* (b) *regular,* (c) *completely regular, then a continuous map $f : X \to Y$ is, respectively* (a) *a T_i-map, $i = 0, 1, 2$,* (b) *regular,* (c) *completely regular.*

Proposition 1.9. *A continuous map of a* (a) *normal space is functionally prenormal,* (b) *hereditary normal space is functionally normal.*

Proposition 1.10. *If a space Y and the map $f : X \to Y$ are*: (a) *a T_i-space and a T_i-map, respectively $i = 0, 1, 2$,* (b) *regular,* (c) *completely regular, then the space X will be, respectively* (a) *a T_i-space, $i = 0, 1, 2$,* (b) *regular,* (c) *completely regular.*

Finally we give the definition of submaps and compact maps [16].

Definition 1.11. The restriction of the map $f : X \to Y$ on a (closed, etc.) subset of the space X is called a (closed, etc.) submap of the map f.

Definition 1.12. By a compact map is meant a perfect (i.e., continuous, closed and fibrewise compact) map.

Note that a closed submap of a compact map is compact.

2. *MT*-maps: definition, characterizations and invariance

We first give some results and definitions with respect to paracompact maps which one can find in [3].

Let $f: X \to Y$ be a continuous map of a topological space X into a topological space (Y, τ). For $y \in Y$, a collection of subsets of X is said to be y-locally finite if for every $x \in f^{-1}y$, there exists a nbd O_x of x in X, such that O_x meets finitely many elements of the collection. If the collection $\mathcal{U} = \{U_\alpha\colon \alpha \in \mathcal{A}\}$ is a y-locally finite open in X collection, then $\mathcal{U}$ is locally finite in $\bigcup_{x \in f^{-1}y} O_x$, i.e., for every $z \in \bigcup_{x \in f^{-1}y} O_x$, z has a nbd in X which meets finitely many elements of $\mathcal{U}$. In particular, if f is closed and $\mathcal{U}$ covers $f^{-1}y$, then there exists a nbd $O_y \in N(y)$ such that $\mathcal{U}$ is a cover of $f^{-1}O_y$ and is locally finite in $f^{-1}O_y$, that is for every $z \in f^{-1}O_y$, z has a nbd in $f^{-1}O_y$ (and so in X) such that it intersects finitely many elements of $\mathcal{U}$.

Definition 2.1. A continuous map $f: X \to Y$ is said to be paracompact if for every point $y \in Y$ and every open (in X) cover $\mathcal{U} = \{U_\alpha\colon \alpha \in \mathcal{A}\}$ of the fibre $f^{-1}y$ (i.e., $f^{-1}y \subset \bigcup\{U_\alpha\colon \alpha \in \mathcal{A}\}$), there exists a nbd O_y of y such that $f^{-1}O_y$ is covered by $\mathcal{U}$ and $(f^{-1}O_y \wedge \mathcal{U})$ has a y-locally finite open refinement.

Note that if f is paracompact then it is a closed map and is fibrewise paracompact, i.e., for every $y \in Y$, $f^{-1}y$ is paracompact. The converse of this statement is not true even for Tychonoff maps, that is there is a closed Tychonoff map with paracompact fibers which is not paracompact (Example 2.10). Also every compact map is paracompact, and every closed submap of a paracompact map is paracompact.

Proposition 2.2. *A paracompact T_2-map is regular and normal* (*and so is a T_4-map*).

Definition 2.3. Let $f: X \to Y$ be a continuous map and $y \in Y$. Let $\mathcal{U}$ be an open (in X) cover of $f^{-1}y$. The collection $\mathcal{V}$ of subsets of X is said to be a y-star refinement of $\mathcal{U}$ if $V \cap f^{-1}y \neq \emptyset$ for every $V \in \mathcal{V}$ and there exists a nbd $O_y \in N(y)$ such that $\bigcup \mathcal{V} = f^{-1}O_y$, $\mathcal{U}$ covers $f^{-1}O_y$ and $\{\mathrm{St}(V, \mathcal{V})\colon V \in \mathcal{V}\} < \mathcal{U} \wedge f^{-1}O_y$.

We finally give some characterizations of paracompact maps obtained in [3] which we will need below.

Theorem 2.4. *For a T_1-map $f: X \to Y$ the following are equivalent*:

(i) *The map f is paracompact T_2.*

(ii) *For every $y \in Y$ and every open* (*in X*) *cover $\mathcal{U}$ of the fibre $f^{-1}y$, there exists an open y-star refinement $\mathcal{V}$.*

(iii) *The map f is regular and for every $y \in Y$ and every open* (*in* X) *cover* $\mathcal{U}$ *of the fibre* $f^{-1}y$, *there exists a nbd* $O_y \in N(y)$ *such that* $f^{-1}O_y$ *is covered by* $\mathcal{U}$ *and* $(f^{-1}O_y \wedge \mathcal{U})$ *has a* y-σ-*discrete open refinement.*

We now define collectionwise normality for maps.

Definition 2.5. A T_1-map f is said to be *collectionwise prenormal* if for every discrete collection $\{F_s\colon s \in \mathcal{S}\}$ of closed subsets of X and for every $y \in Y$, there exist $O_y \in N(y)$ and a collection of open subsets $\{U_s\colon s \in \mathcal{S}\}$, such that $F_s \cap f^{-1}O_y \subset U_s$ and U_s are discrete in $f^{-1}O_y$. The map f is said to be *collectionwise normal* if for every $O \in \tau$, the map $f|_{f^{-1}O} : f^{-1}O \to O$ is collectionwise prenormal.

Proposition 2.6. *A T_1-map f is collectionwise normal if and only if for every $O \in \tau$, every closed discrete* (*in* $f^{-1}O$) *collection* $\{F_s\colon s \in \mathcal{S}\}$ *and every* $y \in O$, *there exists* $O_y \in N(y)$, $O_y \subset O$ *such that* $\{F_s \cap f^{-1}O_y\colon s \in \mathcal{S}\}$ *are nbd separated.*

Proof. Let $O \in \tau$ and $\{F_s\colon s \in \mathcal{S}\}$ a discrete collection of closed subsets of $f^{-1}O$. Let $y \in O$. There exists an open set $O_y \subset O$ such that $\{F_s \cap f^{-1}O_y\}$ are nbd separated, say by $\{U_s\colon s \in \mathcal{S}\}$. Let

$$A = \bigcup_{s \in \mathcal{S}} (F_s \cap f^{-1}O_y) \quad \text{and} \quad B = f^{-1}O_y \setminus \bigcup_{s \in \mathcal{S}} U_s.$$

These two sets are closed and disjoint in $f^{-1}O_y$ and so, since f is normal, there exists open sets U and V in X, and an open set O'_y in Y such that, $A' \subset U \subset f^{-1}O'_y$, $B' \subset V \subset f^{-1}O'_y$ and $V \cap U = \emptyset$, where $A' = A \cap f^{-1}O'_y$ and $B' = B \cap f^{-1}O'_y$. Now consider $V_s = U_s \cap U$. We have that $F_s \cap f^{-1}O'_y \subset V_s$ and the collection $\{V_s\colon s \in \mathcal{S}\}$ is discrete in $f^{-1}O'_y$. □

Proposition 2.7. *Every paracompact T_1-map is collectionwise normal.*

Proof. Let $f : X \to Y$ be a paracompact map and let $O \in \tau$. Consider the restriction $f|_{f^{-1}O} : f^{-1}O \to O$ and let $\{F_s\colon s \in \mathcal{S}\}$ be a discrete in $f^{-1}O$ collection of closed (in $f^{-1}O$) subsets of $f^{-1}O$. Take an arbitrary point $y \in O$. For every $x \in f^{-1}y$ choose a nbd $H_x \subset f^{-1}O$ of x which meets at most one set of the collection $\{F_s\colon s \in \mathcal{S}\}$. Let $\mathcal{H} = \{H_x\colon x \in f^{-1}y\}$, then $\mathcal{H}$ is an open (in $f^{-1}O$ and so in X) cover of $f^{-1}y$. Let $\mathcal{V}$ be an open y-star refinement of $\mathcal{H}$. One can assume that $\mathcal{V}$ consists of open subsets of $f^{-1}O$, say $\mathcal{V} = \{V_t\colon t \in \mathcal{T}\}$. Thus, there exists $O_y \subset O$, $O_y \in N(y)$ such that $\{\mathrm{St}(V_t, \mathcal{V})\colon t \in \mathcal{T}\} < \mathcal{H} \wedge f^{-1}O_y$. We show that every element of $\mathcal{V}$ meets at most one element of the collection $\{G_s\colon s \in \mathcal{S}\}$, where $G_s = \mathrm{St}(F_s, \mathcal{V})$. For every $t \in \mathcal{T}$, there exists an $x \in f^{-1}y$ such that $\mathrm{St}(V_t, \mathcal{V}) \subset H_x \cap f^{-1}O_y$ and so if $V_t \cap G_s \neq \emptyset$ then $H_x \cap F_s \neq \emptyset$. □

Definition 2.8. The sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open (in X) covers of $f^{-1}y$, $y \in Y$, is said to be a *y-development* if for every $x \in f^{-1}y$ and every nbd $U(x)$ of x in X, there exist

$i < \omega$ and $O \in N(y)$ such that $x \in \mathrm{St}(x, \mathcal{W}_i) \cap f^{-1}O \subset U(x)$. One can assume that $\mathcal{W}_i = \{W_{i\alpha}\colon \alpha \in \mathcal{A}_i\}$, where $W_{i\alpha} \cap f^{-1}y \neq \emptyset$, for every $i < \omega$ and for every $\alpha \in \mathcal{A}_i$. The map f is said to have an *f-development* if it has a y-development for every $y \in Y$.

We now give our definition of a metrizable type map.

Definition 2.9. A closed map $f : X \to Y$ is said to be an *MT-map* if it is collectionwise normal and has an f-development.

We thus see that an *MT*-map f is closed, T_4 and has metrizable fibers. We now give an example of a $T_{3\frac{1}{2}}$ closed map with metrizable fibers which is not an *MT*-map.

Example 2.10. Let L be the Niemytzki plane and let $L_1 \subset L$ be the line $y = 0$. Then L_1 is closed in L and so the quotient map $q : L \to L/L_1$ is a closed map. The map q is Tychonoff since L is Tychonoff and every fibre of q is metrizable. Since there exist closed in L subsets $A \subset L_1$ and $B \subset L_1$ which are not nbd separated, we have that q is not a prenormal map, and so cannot be an *MT*-map.

Definition 2.11. A collection $\mathcal{B}_y$ of open sets of X is said to be a *base at y for the map f*, $y \in Y$, if for every $x \in f^{-1}y$ and every open nbd $U(x)$ of x there exist $O \in N(y)$ and $B \in \mathcal{B}_y$ such that $x \in B \cap f^{-1}O \subset U(x)$. One can assume that for every $B \in \mathcal{B}_y$ we have $B \cap f^{-1}y \neq \emptyset$.

Thus $\mathcal{B}_f = \{\mathcal{B}_y\colon y \in Y\}$, where $\mathcal{B}_y$ is a base at y for f, will give a base for the map f. Conversely, if $\mathcal{B}_f$ is a base for f, by taking $\mathcal{B}_f(y) = \{B \in \mathcal{B}_f\colon B \cap f^{-1}y \neq \emptyset\}$ one gets a base at $y \in Y$ for the map f.

Theorem 2.12. *For a continuous map $f : X \to Y$ the following are equivalent*:

(1) *f is an MT-map*;

(2) *f is a closed T_3-map with a y-σ-discrete y-base for every $y \in Y$*;

(3) *f is a closed T_3-map with a y-σ-locally finite y-base for every $y \in Y$.*

Proof. We prove only implication (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) follows on the same footsteps as the proof of the analogous result in the category *Top*.

We need to show that every *MT*-map f has a y-σ-discrete y-base for every $y \in Y$. We first show that every *MT*-map is paracompact. Let y be an arbitrary point of Y and $\{U_s\colon s \in \mathcal{S}\}$ be an open cover of $f^{-1}y$. Take a well-ordering relation $\prec$ on the set $\mathcal{S}$ and let

$$F_{s,i} = X \setminus \left\{ \mathrm{St}(X \setminus U_s, \mathcal{W}_i) \cup \bigcup_{s' \prec s} U_{s'} \right\},$$

where $\mathcal{W}_i$, $i < \omega$ is a y-development. The sets $F_{s,i}$ are closed in X. Consider

$$F'_{s,i} = F_{s,i} \cap \left(\bigcup \mathcal{W}_i \right) \subset U_s \cap \left(\bigcup \mathcal{W}_i \right).$$

Fix an $i < \omega$. There exists an open (in Y) set $O_y(i) \in N(y)$ such that $f^{-1}O_y(i) \subset \bigcup \mathcal{W}_i$. Then $\mathcal{F}'_i = \{F'_{s,i} \cap f^{-1}O_y(i)\colon\ s \in \mathcal{S}\}$ is a collection of closed (in $f^{-1}O_y(i)$) sets. We now show that it is a y-discrete collection. Let $x \in f^{-1}y$ and denote by $s(x)$ the smallest element in $\mathcal{S}$ such that $x \in U_{s(x)}$. Consider the nbd of x, $U_{s(x)} \cap \mathrm{St}(x, \mathcal{W}_i)$. This nbd meets only one element of the collection $\mathcal{F}'_i$, namely the set $F'_{s(x),i} \cap f^{-1}O_y(i)$. Thus $\mathcal{F}'_i$ is y-discrete.

Now take an $O'_y(i) \in N(y)$ such that $f^{-1}O'_y(i) \subset \bigcup\{U_{s(x)} \cap \mathrm{St}(x, \mathcal{W}_i)\colon\ x \in f^{-1}y\}$ with $O'_y(i) \subset O_y(i)$. Then $\mathcal{F}''_i = \mathcal{F}'_i \cap f^{-1}O'_y(i)$ is a discrete and closed (in $f^{-1}O'_y(i)$) collection and so, by collectionwise normality, there exist open sets $U_{s,i}$ such that $F'_{s,i} \cap f^{-1}O'_y(i) \subset U_{s,i} \subset U_s \cap f^{-1}O'_y(i)$, for $s \in \mathcal{S}$ and $i < \omega$, and the collection $\{U_{s,i}\colon\ s \in \mathcal{S}\}$ is y-discrete for every $i < \omega$. Since $\{F'_{s,i}\colon\ s \in \mathcal{S},\ i < \omega\}$ forms a cover of the subset $f^{-1}y$, we get that $\{U_{s,i}\colon\ s \in \mathcal{S},\ i < \omega\}$ is a y-σ-discrete open refinement of $\{U_s\colon\ s \in \mathcal{S}\}$. Therefore, f is a paracompact map.

Finally, by taking a y-σ-discrete open refinement of $\mathcal{W}_i$ for every $i < \omega$, one gets a y-σ-discrete y-base. □

Note that in the proof of Theorem 2.12 we proved that an *MT*-map is paracompact. We now turn to other characterizations of *MT*-maps analogous to characterizations of metrizable spaces in terms of strong and normal developments. We first give some definitions.

Definition 2.13. The sequence $\mathcal{W}_1, \mathcal{W}_2, \ldots$ of open (in X) covers of $f^{-1}y$, $y \in Y$, is said to be a *strong y-development* if for every $x \in f^{-1}y$ and every nbd $U(x)$ of x in X, there exist a nbd $V(x)$ of x in X, $i < \omega$ and $O \in N(y)$ such that $x \in \mathrm{St}(V(x), \mathcal{W}_i) \cap f^{-1}O \subset U(x)$. One can assume that $\mathcal{W}_i = \{W_{i\alpha}\colon\ \alpha \in \mathcal{A}_i\}$, where $W_{i\alpha} \cap f^{-1}y \neq \emptyset$, for every $i < \omega$ and for every $\alpha \in \mathcal{A}_i$. The map f is said to have a *strong f-development* if it has a strong y-development for every $y \in Y$.

Definition 2.14. Let $\mathcal{W}_1, \mathcal{W}_2, \ldots$ be a y-development for $y \in Y$. If $\mathcal{W}_{i+1}$ is a y-star refinement of $\mathcal{W}_i$ for every $i < \omega$, then the y-development is said to be a *normal y-development*. The map f is said to have a *normal f-development* if it has a normal y-development for every $y \in Y$.

Theorem 2.15. *For a continuous map $f : X \to Y$ the following are equivalent*:

(1) *f is an MT-map*;

(2) *f is a closed T_0-map with a strong f-development*;

(3) *f is a closed T_0-map with a normal f-development.*

Proof. (1) ⇒ (3) We have already proved that an *MT*-map is paracompact and so the f-development can be arranged into a normal f-development.

(3) ⇒ (2) Take an arbitrary $y \in Y$ and let $\mathcal{W}_i$, $i < \omega$, be a normal y-development. Then for every $x \in f^{-1}y$ and every nbd $U(x)$ of x, there exists an $O \in N(y)$ and $i < \omega$ such that $x \in \mathrm{St}(x, \mathcal{W}_i) \cap f^{-1}O \subset U(x)$. We also have that $\{\mathrm{St}(W, \mathcal{W}_{i+1})\colon\ W \in \mathcal{W}_{i+1}\} <$

$\mathcal{W}_i \wedge f^{-1}O'$, for some $O' \in \tau$. Consider $O'' = O \cap O'$ and any $V \in \mathcal{W}_{i+1}$, $x \in V$. Then we get that $x \in \mathrm{St}(V, \mathcal{W}_{i+1}) \cap f^{-1}O'' \subset W \cap f^{-1}O'' \subset U(x)$, for some $W \in \mathcal{W}_i$.

(2) $\Rightarrow$ (1) Let $f : X \to Y$ be a closed T_0-map with a strong f-development. We show that in this case f is collectionwise normal. We first note that since each fibre is metrizable we have that f is a T_1-map. Now let $O \in \tau$ and $\mathcal{F} = \{F_s\colon s \in \mathcal{S}\}$ a closed discrete (in $f^{-1}O$) collection. Take any $y \in O \subset Y$. We may assume that $\mathcal{W}_{i+1}$ refines $\mathcal{W}_i \wedge f^{-1}O(i)$ for every $i < \omega$, where $O(i) \in N(y)$ and $O(i+1) \subset O(i)$. Also, since $\{\mathcal{W}_i\}$ is a strong y-development, we have the following condition: For every $x \in f^{-1}y$ and every nbd $U(x)$ of x, there exist an $i(x) < \omega$ and $O' \subset O(i(x))$ such that

$$\mathrm{St}^2(x, \mathcal{W}_{i(x)}) \cap f^{-1}O' \subset U(x),$$

where $\mathrm{St}^2(x, \mathcal{W}_i) = \mathrm{St}(\mathrm{St}(x, \mathcal{W}_i), \mathcal{W}_i)$.

Now for $F_s \in \mathcal{F}$ and $x \in F_s \cap f^{-1}y$, let $i(x)$ be such that $\mathrm{St}^2(x, \mathcal{W}_{i(x)}) \cap f^{-1}O(x)$, $O(x) \subset O(i(x))$, does not meet any $F_{s'} \in \mathcal{F}$, $s \neq s'$. Let

$$V(x) = \mathrm{St}(x, \mathcal{W}_{i(x)}) \cap f^{-1}O(x).$$

Then if $x \in F_s$ and $x' \in F_{s'} \neq F_s$, we get that $V(x) \cap V(x') = \emptyset$. Since if $V(x') = \mathrm{St}(x', \mathcal{W}_{i(x')}) \cap f^{-1}O(x')$ and say, without loss of generality, that $i(x') \geqslant i(x)$, we get that $O(x') \subset O(i(x')) \subset O(i(x))$. Thus, if say $z \in V(x) \cap V(x')$, we have $z \in f^{-1}O(i(x))$ and $z \in W' \in \mathcal{W}_{i(x')}$, $x' \in W'$. But $W' \subset W \cap f^{-1}O(i(x))$, for some $W \in \mathcal{W}_{i(x)}$, which implies that $z \in W \cap V(x) = W \cap \mathrm{St}(x, \mathcal{W}_{i(x)}) \cap f^{-1}O(x) \neq \emptyset$ and so $x' \in \mathrm{St}^2(x, \mathcal{W}_{i(x)}) \cap f^{-1}O(x)$, which is a contradiction. Now let

$$U(F_s) = \bigcup \{V(x)\colon x \in F_s \cap f^{-1}y\}$$

and for every $x \in f^{-1}y \setminus \bigcup \mathcal{F}$ let $U(x)$ be a nbd of x which does not meet any $F_s \in \mathcal{F}$. Since f is closed we conclude that there exists a nbd $O^* \in N(y)$ such that $O^* \subset O$ and $\{F_s\}$ are nbd separated in $f^{-1}O^*$. $\square$

The following four theorems follow easily from the corresponding results in the theory of general topological spaces, that is in the category *Top*.

Theorem 2.16. *If $f : X \to Y$ is an MT-map, then the following are equivalent*:
(1) *f has a countable y-base for every $y \in Y$*;
(2) *f is a Lindelöf map, that is a closed T_3-map with finally compact fibers* [3];
(3) *$f^{-1}y$ is separable for every $y \in Y$.*

Theorem 2.17. *If $f : X \to Y$ is an MT-map, then the following are equivalent*:
(1) *f is compact*;
(2) *$f^{-1}y$ is countably compact for every $y \in Y$*;
(3) *$f^{-1}y$ is sequentially compact for every $y \in Y$.*

Thus every compact *MT*-map has separable fibers.

Theorem 2.18. *A compact T_2-map is an MT-map if and only if it has a countable y-base for every $y \in Y$.*

Theorem 2.19. *A map f with a countable y-base for every $y \in Y$ is an MT-map if and only if it is closed and T_3.*

We now prove that the MT-property is invariant under perfect morphisms.

Proposition 2.20. *Let $f : X \to Y$ be an MT-map and $g : Z \to Y$ a continuous map. Then if $\lambda : f \to g$ is a perfect morphism of f onto g, g is also an MT-map.*

Proof. In [3] it is proved that under the above hypothesis the map g is paracompact T_2 (and so T_4 and closed). We now construct a y-σ-discrete base in $g^{-1}y$ for an arbitrary point $y \in Y$.

Let $\{\mathcal{G}_i\colon i < \omega\}$ be a normal y-development in $f^{-1}y$. For an arbitrary point $z \in g^{-1}y$ consider $U_i(z) = \mathrm{St}(\lambda^{-1}z, \mathcal{G}_i)$, $W_i(z) = Z \setminus \lambda(X \setminus U_i(z))$ and $V_i(z) = \lambda^{-1}(W_i(z)) \subset U_i(z)$. It follows from definition that $U_j(z) \subset U_i(z)$ if $j \geqslant i$. The collection $\mathcal{W}_i = \{W_i(z)\colon z \in g^{-1}y\}$ is an open (in Z) cover of $g^{-1}y$.

Let V be an open nbd of $z \in g^{-1}y$, then $\lambda^{-1}z \subset \lambda^{-1}V$. Since $\lambda^{-1}z$ is compact and the y-development is a normal sequence, we have that there exists an $i < \omega$ for which $\mathrm{St}(\lambda^{-1}z, \mathcal{G}_i) \cap f^{-1}O \subset \lambda^{-1}V$ for some nbd $O \in N(y)$. This implies that $W_i(z) \cap g^{-1}O \subset V$ and so $\{W_i(z)\colon i < \omega\}$ is a nbd f-base for each $z \in g^{-1}y$. We now show that for each $W_i(z)$ there exists a $j < \omega$ such that

$$\bigcup \bigl\{W_j(p)\colon z \in W_j(p)\bigr\} \subset W_i(z).$$

There exist an $O \in N(y)$ and a $j > i+1$ such that $U_j(z) \cap f^{-1}O \subset V_{i+1}(z) \subset U_{i+1}(z)$. Consider a point $p \in g^{-1}y$ such that $z \in W_j(p)$. We have that $\lambda^{-1}z \subset \lambda^{-1}W_j(p) = V_j(p) \subset U_j(p) = \mathrm{St}(\lambda^{-1}p, \mathcal{G}_j)$. Thus for every $x \in \lambda^{-1}z$, there exists a $G(x) \in \mathcal{G}_j$ with $x \in G(x)$ and $G(x) \cap \lambda^{-1}p \neq \emptyset$. This implies that $U_j(z) \cap \lambda^{-1}p \neq \emptyset$ and that $\lambda^{-1}p \subset V_{i+1}(z)$, since $V_{i+1}(z)$ contains $\lambda^{-1}\lambda x$ if it contains x.

Now let $q \in W_j(p)$. Since $\lambda^{-1}q \subset U_j(p)$ we have that for every $x \in \lambda^{-1}q$ there exists a $G(x) \in \mathcal{G}_j$, $x \in G(x)$, $G(x) \cap \lambda^{-1}p \neq \emptyset$. We have already showed that $\lambda^{-1}p \subset V_{i+1}(z) \subset U_{i+1}(z)$; and so there exists an $H \in \mathcal{G}_{i+1}$ with $G(x) \cap H \neq \emptyset$ and $H \cap \lambda^{-1}z \neq \emptyset$. Since $j > i + 1$ we have that $x \in U_i(z)$, which implies that $\lambda^{-1}q \subset U_i(z)$ and so $q \in W_i(z)$. $\square$

In some cases, if a certain subspace $Z \subset Y$ has a certain topological property and the map $f : X \to Y$ has the same property, then so does the subspace $f^{-1}Z \subset X$. For example, if Z is a T_i-space and f a T_i-map, for $i = 0, 1, 2, 3, 3\frac{1}{2}$, then so does the subspace $f^{-1}Z$ [15]. This is also true when Z and f are compact (paracompact), that is in this case the subspace $f^{-1}Z$ is compact [5] (paracompact [3]). The next example shows that this is not the case for an MT-map, that is the MT-map preimage of a metrizable space does not have to be metrizable.

Example 2.21. Consider the set $X = I \times I$, where $I = [0, 1]$, equipped with the lexicographic order and order topology. Then X is hereditary paracompact, compact LOTS which is not metrizable. Now let $f : X \to Y$, where $Y = I$ with standard metric topology, be the map $f(u, v) = u$, $(u, v) \in X$. Then the map f is continuous and closed. Also f is collectionwise normal (since it is paracompact and T_1) and one can check that it has an f-development. Thus f is an MT-map onto a metrizable space, while X is not metrizable.

With respect to Example 2.21, it will be interesting to see if there is an internal characterization of topological spaces that can be mapped by an MT-map onto a metrizable space. Such a characterization is given in Section 4 of this paper. To this end we now give an example of a closed map with metrizable fibers from a space X onto a metrizable space Y which is not an MT-map.

Example 2.22. Consider the set

$$D = \{(x, y)\colon\ (x, y) \in \mathbb{R}^2,\ y \geqslant 0\}$$

and let $D_1 \subset D$ be the line $y = 0$ and $D_2 = D \setminus D_1$. Let $U_n = \{(x, y)\colon\ 0 < y < \frac{1}{n}\}$ for every $n < \omega$, and for $x \in D_1$ put $V_n(x) = \{x\} \cup U_n$. Let $\mathcal{B}_1 = \{V_n(x)\colon\ x \in D_1, n < \omega\}$ and $\mathcal{B}_2$ the collection of all sets open in the usual topology of the plane and lying in D_2. It is not difficult to see that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ is a base for a topology on D and that D_1 is closed in D (with respect to this topology). Thus the quotient map $q : D \to D/D_1$ is a closed map with metrizable fibers. It is also not difficult to see that the space D/D_1 is metrizable and that q is not a prenormal map, for the same reason as that for Example 2.10. Thus q is not an MT-map. Note that the space D has a G_δ-diagonal but is not a T_2-space.

3. Fibrewise products of MT-maps

We begin by the definition of fan products (see, for example, [1]). Fan products and their projections ($\equiv$ fibrewise products of maps) have the same role in the category Top_Y, as Tychonoff products of spaces have in the category Top.

Definition 3.1. For the collection of continuous maps $p_\alpha : P_\alpha \to Y$, $\alpha \in \mathcal{A}$, the subspace

$$P = \Bigl\{t = \{t_\alpha\} \in \prod\{P_\alpha\colon\ \alpha \in \mathcal{A}\}\colon\ \ p_\alpha t_\alpha = p_\beta t_\beta,\ \forall \alpha, \beta \in \mathcal{A}\Bigr\}$$

of the Tychonoff product $\prod = \prod\{P_\alpha\colon\ \alpha \in \mathcal{A}\}$ is called the fan product of the spaces P_α with respect to the maps p_α, $\alpha \in \mathcal{A}$ and is denoted by $\prod\{P_\alpha \text{ rel } p_\alpha\colon\ \alpha \in \mathcal{A}\}$.

For the projection $pr_\alpha : \prod \to P_\alpha$ of the product $\prod$ onto the factor P_α, the restriction π_α on P will be called the projection of the fan product onto the factor P_α, $\alpha \in \mathcal{A}$. From the definition of fan product we have that, $p_\alpha \circ \pi_\alpha = p_\beta \circ \pi_\beta$ for every α and β in $\mathcal{A}$. Thus one can define a map $p : P \to Y$, called the projection of the fan product, by

$$p = p_\alpha \circ \pi_\alpha, \quad \alpha \in \mathcal{A}.$$

Obviously, the projections p and π_α, $\alpha \in \mathcal{A}$, are continuous.

The projection p is also called the fibrewise product of the maps p_α, $\alpha \in \mathcal{A}$ (since for every point $y \in Y$, the inverse image $p^{-1}y$ is homeomorphic to the Tychonoff product of the fibers $p_\alpha^{-1}y$, $\alpha \in \mathcal{A}$). The fact that p is the fibrewise product of the maps p_α, $\alpha \in \mathcal{A}$, will be denoted by $p = \prod\{p_\alpha\colon \alpha \in \mathcal{A}\}$.

In particular, the fan product P of the spaces X and Z with respect to the maps $f : X \to Y$ and $g : Z \to Y$ will be denoted by $X_f \times_g Z$ and the projections π_α by π_X and π_Z.

We now turn to fibrewise products of *MT*-maps.

Proposition 3.2. *Let the maps $p_i : P_i \to Y$, $i < \omega$, be MT-maps. Consider the projection $p : P = \prod\{P_i \text{ rel } p_i\colon i < \omega\} \to Y$. If p is closed then it is an MT-map. In other words, p has a normal p-development and is a T_0-map.*

Proof. We know that for every point $y \in Y$, the inverse image $p^{-1}y$ is homeomorphic to the Tychonoff product of the fibers $p_i^{-1}y$, $i < \omega$. Let $\mathcal{W}_k^i$, $k < \omega$, be a normal y-development for the map p_i for every $i < \omega$ and consider the sequence

$$\mathcal{W}_k = \left\{\left(\prod_{i<\omega} V_i\right) \cap P\colon\ V_i \neq P_i \quad \text{for } i \leqslant k,\ \text{in which case } V_i \in \mathcal{W}_k^i\right\}$$

of open (in P) covers of $p^{-1}y$.

Now let U_t be an open in P nbd of some point $t = \{t_i\colon i < \omega\} \in p^{-1}y$. Then $p_i t_i = y$ for every $i < \omega$ and there exists a canonical open nbd $G_t = \prod U_i$, with $U_i \neq P_i$ for only a finite number of indices $\alpha(i)$, $i = 1, \ldots, n$, such that $t \in G_t \cap P \subset U_t$. There exists an $O \in N(y)$ and a $k < \omega$ such that $\mathrm{St}(t_{\alpha(i)}, \mathcal{W}_k^{\alpha(i)}) \cap p_{\alpha(i)}^{-1} O \subset U_{\alpha(i)}$, $i = 1, \ldots, n$. Let $m = \max\{k, \alpha(i)\colon i = 1, \ldots, n\}$. From this it follows that $\mathrm{St}(t, \mathcal{W}_m) \cap p^{-1} O \subset G_t \cap P$. We thus see that $\mathcal{W}_k$, $k < \omega$, is a y-development for the map p.

The fact that $\mathcal{W}_k$ is a normal sequence follows from the fact that each $\mathcal{W}_k^i$ is a normal sequence, and the fact that p is a T_0-map follows from the following two facts: (i) a map is a T_0-map if and only if the fibers are T_0-spaces, (ii) the T_0-property for spaces is a multiplicative property. □

With respect to Proposition 3.2 we are interested to know if, at least, the fibrewise product of two *MT*-maps is a closed map. We give an example to show that in general the fibrewise product of two closed maps is not a closed map.

Example 3.3. Let X_i, $i = 1, 2$, be countably compact spaces such that their product is not countably compact and let A be the one-point compactification of a countable discrete space. Let the maps f_i be the projections of the products $X_i \times A$ onto A, for $i = 1, 2$. Then f_i, $i = 1, 2$, are closed maps but their fibrewise product is not since it coincides with the projection of the product $X_1 \times X_2 \times A$ onto A (see [5, Exercise 3.10 A(b)]).

Nevertheless we have the following results for the above mentioned problem.

Corollary 3.4. *If f is a compact MT-map and g is an MT-map, then the fibrewise product $p: X_f \times_g Z \to Y$ is an MT-map.*

Proof. This follows from the fact that in the above hypothesis the map p is a paracompact map and so is closed [3, Theorem 4.5]. □

Proposition 3.5. *If $f: X \to Y$ is the fibrewise product of continuous maps $f_\alpha: X_\alpha \to Y$, $\alpha \in \mathcal{A}$, then*

$$\operatorname{Fr} f^{-1}y \subset \prod \left\{ \operatorname{Fr} f_\alpha^{-1}y\colon \alpha \in \mathcal{A} \right\}, \quad \textit{for every } y \in Y.$$

Proof. Evidently, the set

$$\operatorname{Int} f_\alpha^{-1}y \times \prod \left\{ X_\beta\colon \beta \in \mathcal{A} \setminus \{\alpha\} \right\} \cap X$$

is open in X and is contained in

$$f^{-1}y = \prod \{ f_\alpha^{-1}y\colon \alpha \in \mathcal{A} \}. \quad \square$$

Let us recall that a continuous map $f: X \to Y$ is said to be *peripherically compact* (*peripherically countably compact*) if $\operatorname{Fr} f^{-1}y$ is compact (countably compact) for every $y \in Y$.

Corollary 3.6. *The fibrewise product of closed peripherically compact maps onto a T_1-space is also closed and peripherically compact.*

Proof. Let Y be a T_1-space and $f: X \to Y$ be the fibrewise product of closed peripherically compact maps $f_\alpha: X_\alpha \to Y$, $\alpha \in \mathcal{A}$. It follows from Proposition 3.5 that f is peripherically compact.

Let F be a closed subset of X and $y \notin fF$. This means that

$$F \cap \left(f^{-1}y = \prod \{ f_\alpha^{-1}y\colon \alpha \in \mathcal{A} \} \right) = \emptyset.$$

The spaces $C_\alpha = \operatorname{Fr} f_\alpha^{-1}y$, $\alpha \in \mathcal{A}$, are compact and so we can find a finite subset $\mathcal{B}$ of $\mathcal{A}$ and nbds O_α of C_α, $\alpha \in \mathcal{B}$, such that

$$F \cap \left(\prod \{ O_\alpha\colon \alpha \in \mathcal{B} \} \times \prod \{ X_\alpha\colon \alpha \in \mathcal{A} \setminus \mathcal{B} \} \right) = \emptyset.$$

Since

$$\begin{aligned} &\left(\operatorname{Int} f_\alpha^{-1}y \times \prod \left\{ X_\beta\colon \beta \in \mathcal{A} \setminus \{\alpha\} \right\} \right) \cap X \\ &\quad \subset \prod \{ f_\alpha^{-1}y\colon \alpha \in \mathcal{A} \} = f^{-1}y, \quad \alpha \in \mathcal{A}, \end{aligned}$$

the nbd $U = \prod \{ O_\alpha \cup \operatorname{Int} f_\alpha^{-1}y\colon \alpha \in \mathcal{B} \} \times \prod \{ X_\alpha\colon \alpha \in \mathcal{A} \setminus \mathcal{B} \}$ of $f^{-1}y$ does not intersect F.

Now we can take a nbd V of y such that $f_\alpha^{-1}V \subset O_\alpha \cup \operatorname{Int} f_\alpha^{-1}y$, $\alpha \in \mathcal{B}$. Then $f^{-1}V \subset X \cap U \subset X \setminus F$. □

Remember that a point x of a space X is a *q-point* if there exist nbds U_i of x, $i < \omega$, such that

$$\text{every sequence } x_i \in X,\ x_i \in U_i,\ i < \omega, \text{ has a cluster point in } X. \qquad (*)$$

A space is called a *q-space* if all of its points are q-points. It can be easily seen that $x \in X$ is a q-point if there exist a countably compact set $C \subset X$ and its nbds U_i, $i < \omega$, such that $x \in C$ and for every nbd U of C there exists an $i < \omega$ with $U_i \subset U$. Thus all M-spaces in the sense of Morita [12,13], and all spaces of countable type (in particular, all 1st-countable spaces and all Čech complete spaces) are q-spaces.

Also, recall that a space X is said to be *isocompact* if every closed countably compact subset is compact.

Corollary 3.7. *Let Y be a regular T_1- and q-space, the continuous maps $f_\alpha : X_\alpha \to Y$, $\alpha \in \mathcal{A}$, be regular, prenormal, closed and all the fibers $f_\alpha^{-1} y$, $y \in Y$, $\alpha \in \mathcal{A}$, be isocompact. Then the fibrewise product of f_α, $\alpha \in \mathcal{A}$, is closed and peripherically compact.*

Proof. In Theorem 5.4 and Corollary 5.5 it will be proved that under the above hypothesis, the maps f_α, $\alpha \in \mathcal{A}$, are peripherically compact. □

Corollary 3.8. *Let Y be a regular T_1- and q-space and all the maps $f_i : X_i \to Y$, $i < \omega$, be MT-maps. Then the fibrewise product of f_i, $i < \omega$, is a peripherically compact MT-map.*

4. *MT*-map preimages of metrizable spaces

In this paragraph we give an internal characterization of those spaces that can be mapped by an *MT*-map onto a metrizable space.

Definition 4.1. A map $f : X \to Y$ is called a *Moore map* if it is T_3 and has an f-development.

Definition 4.2. A T_3-space X is called a *DT-space* if there exists a sequence $\{\mathcal{G}_n\colon n \in \omega\}$ of open covers of X such that:

(1) for each $n < \omega$, $\mathcal{G}_{n+1}$ star refines $\mathcal{G}_n$;

(2) the sequence $\{\mathrm{St}(x, \mathcal{G}_n)\colon n < \omega\}$ is a base for $C_x = \bigcap_{n<\omega} \mathrm{St}(x, \mathcal{G}_n)$, that is every open set containing C_x contains some $\mathrm{St}(x, \mathcal{G}_n)$;

(3) for every $x \in X$ there exists a sequence $\{\mathcal{W}_n(x)\colon n < \omega\}$ of open (in X) covers of C_x such that for every $y \in C_x$ and every nbd $U(y)$ of y in X, there exists an $n < \omega$ such that $y \in \mathrm{St}(y, \mathcal{W}_n(x)) \subset U(y)$.

If (3) is strengthened to

(3)* property (3) plus $\mathcal{W}_{n+1}(x)$ star refines $\mathcal{W}_n(x) \wedge (\bigcup \mathcal{W}_{n+1}(x))$ for every $n < \omega$ and every $x \in X$,

then the space X is said to be an *MT-space*.

Theorem 4.3. *A T_3-space X is a DT-space if and only if there exists a metric space M and a Moore map f of X onto M.*

Proof. It is a well known result that from property (1) follows the existence of a pseudometric ρ on X with the following properties:

(i) $\rho(x,z)=0$ if and only if $z \in \bigcap_{n<\omega} \mathrm{St}(x,\mathcal{G}_n)$, and

(ii) the set U is open in the topology generated by ρ if and only if $x \in U \Rightarrow \mathrm{St}(x,\mathcal{G}_n) \subset U$ for some $n<\omega$.

We now define an equivalence relation on X as follows: $x \sim z$ if and only if $\rho(x,z)=0$. Let Y be the quotient space $X/\sim$ and define the function $d: Y \times Y \to \mathbb{R}^+$ by $d(\tilde{x},\tilde{z}) = \rho(x,z)$. It is not difficult to check that d is a metric on Y. We are left to show that the quotient map $f: X \to Y$ is an Moore map. Since $f^{-1}B_d(\tilde{x},\varepsilon) = B_\rho(x,\varepsilon)$ and $B_\rho(x,\varepsilon)$ is open in X by (ii) above, we have that f is continuous. Also, since by the construction of ρ we have that $B_\rho(x, 1/2^{n+1}) \subset \mathrm{St}(x,\mathcal{G}_n)$, we get from property (2) that f is a closed map. Finally, from the fact that for an arbitrary $y=\tilde{x} \in Y$,

$$f^{-1}y = \bigcap_{n<\omega} \mathrm{St}(x,\mathcal{G}_n),$$

from (3) we have that f has a y-development. Therefore, f is a Moore map.

The converse is not difficult to prove and follows directly from the definitions. □

Theorem 4.4. *For a T_3-space X the following are equivalent*:

(1) *the space X is a paracompact DT-space*;

(2) *the space X is an MT-space*;

(3) *there exists a metric space M and an MT-map f of X onto M.*

Proof. (1) $\Rightarrow$ (2) follows from the fact that property (3) of Definition 4.2 can be strengthened to (3)* by the paracompactness of X. (2) $\Rightarrow$ (3) follows from the fact that if X is an *MT*-space, the map f constructed in the proof of Theorem 4.3 is an *MT*-map. Finally, for (3) $\Rightarrow$ (1), we only need to show that X is paracompact, and this follows from the already mentioned result that the paracompact preimage of a paracompact space is paracompact [3]. □

As is seen above, the definition of *MT*-spaces follows on the same lines as that of paracompact M-spaces [12,13]. As we shall see later neither of these classes are contained in each other. For the moment let us stop to consider spaces which are at the same time M- and *MT*-spaces.

Definition 4.5. A space X is said to be a *CMT-space* if it is the compact *MT*-map preimage of a metric space.

Note that a *CMT*-space is an M-space and also an *MT*-space (and so paracompact T_4 and 1st-countable). The next proposition shows that the converse is also true, that is a space X which is an M- and *MT*-space is also a *CMT*-space. Example 2.21 gives a *CMT*-space which is not metrizable.

Proposition 4.6. *If a space X is an M- and MT-space then it is also a CMT-space.*

Proof. The following result is known: If the maps $f_1, f_2, \ldots, f_k$, where $f_i : X \to Y_i$ are closed, Y_1 is a T_1-space and $Y_2, \ldots, Y_k$ are T_3-spaces, then the diagonal $f = f_1 \vartriangle f_2 \vartriangle \cdots \vartriangle f_k$ is closed (see, for example, [5, Proposition 2.3.30]).

Therefore, if $f_1 : X \to M_1$ is an MT-map and $f_2 : X \to M_2$ is a compact map, then $f_1 \vartriangle f_2 : X \to M_1 \times M_2$ is a compact map. It is also not difficult to see that it has an $f_1 \vartriangle f_2$-development and so is a compact MT-map. □

We now turn to products of CMT-spaces.

Proposition 4.7. *If the spaces X_n, $n < \omega$, are CMT-spaces then the product $\prod_{n<\omega} X_n$ is also a CMT-space.*

Proof. Let $f_n : X_n \to M_n$ be compact MT-maps onto metrizable spaces. It is known that the product map

$$f = \prod_{n<\omega} f_n : X = \prod_{n<\omega} X_n \to M = \prod_{n<\omega} M_n$$

is a compact map (see, for example, [5]). We will now show that for an arbitrary point $y = \{y_n\} \in M$, the map f has a countable y-base.

Let $\mathcal{W}_n = \{W_n(k){:}\ k < \omega\}$ be a countable y_n-base for the map f_n and consider the collection $\mathcal{W} = \{\prod_{n<\omega} W_n{:}\ W_n \neq X_n$ for only a finite number of indices, in which case $W_n = W_n(k_n) \in \mathcal{W}_n\}$. This collection is a countable collection of open sets in X. We now show that it is a y-base. Take an arbitrary open nbd V of a point $x = \{x_n\} \in X$, where $x \in f^{-1}y$. There exists a canonical nbd $U = \prod U_n$ of x such that $x \in U \subset V$. Then $U_n \neq X_n$ for only a finite number of indices, say $n(1), \ldots, n(s)$. Let $W_{n(p)}(k_{n(p)})$ and $O_{n(p)}(y_{n(p)}) \in N(y_{n(p)})$ be such that $x_{n(p)} \in W_{n(p)}(k_{n(p)}) \cap f_{n(p)}^{-1} O_{n(p)}(y_{n(p)}) \subset U_{n(p)}$ for $p = 1, \ldots, s$. Then we get that $x \in \prod W_n \cap f^{-1} \prod O_n \subset U$, where $W_{n(p)} = W_{n(p)}(k_{n(p)})$ and $O_{n(p)} = O_{n(p)}(y_{n(p)})$ whenever $n = n(p)$ for some $p = 1, \ldots, s$, otherwise $W_n = X_n$ and $O_n = M_n$. This shows that the collection $\mathcal{W}$ is a y-base for the map f and so by Theorem 2.18, f is an MT-map. □

The proof of the above proposition follows from the fact that if f_n, $n < \omega$, are compact MT-maps, then so is the product $\prod_{n<\omega} f_n$. This is not the case for MT-maps. Consider the following example:

Example 4.8. Let $f_1 = \mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \{0\} \subset \mathbb{R}$, that is f_2 is a constant map. Then f_1 and f_2 are MT-maps but $f_1 \times f_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \{0\}$ is not a closed map (and so is not an MT-map).

Note that in the above example $\mathbb{R} \times \mathbb{R}$ is still an MT-space. This leaves us with the open question of whether the product of two MT-spaces is an MT-space.

Next, as mentioned above, we show that the class of MT-spaces and the class of paracompact M-spaces are distinct from each other and none of the two contains the other.

Any compact space which is not 1st-countable (for example, $\beta\mathbb{N}$ or the LOTS $[0,\omega_1]$), is a paracompact M-space ($\equiv$ paracompact p-space) which is not an MT-space. We now give an example of an MT-space which is not an M-space.

Example 4.9. Let $\Omega=[0,\omega]$ and $\mathbb{R}$ have the usual order and let $X=\mathbb{R}\times\Omega$ have the lexicographic order. Let the topology of X be the order topology plus the following sets as open $\{[(y,\omega),\to[\colon\ y\in\mathbb{R}\}$. Thus X is a GO-space. Consider the projection $f=pr_{\mathbb{R}}\colon X\to Y=\mathbb{R}$, that is $f(y,n)=y$. One can see that for every $y\in Y$ we have that $f^{-1}y$ is a discrete countable space $\Omega_d(y)$. It is clear that f is continuous.

Let us show that f is a closed map. Let F be closed in X and say $y\in Y$, $y\notin f(F)$. Then $f^{-1}y\cap F=\emptyset$ and so there exists a $y_1<y$ with $](y_1,0),(y,1)[\cap F=\emptyset$ and a $y_2>y$ with $](y,\omega),(y_2,\omega)[\cap F=\emptyset$. This implies that $y\in]y_1,y_2[\subset Y$ and $]y_1,y_2[\cap f(F)=\emptyset$ and so f is closed.

Since f is a closed T_3-map with finally compact fibers, it is a Lindelöf map and so is paracompact [3]. This also shows that X is a Lindelöf space (one can also show that X has a σ-minimal base and so is hereditary paracompact). To show that f is an MT-map we are left with constructing a y-development for an arbitrary $y\in Y$. Let $y\in Y$ and consider $\Omega_d(y)$. For each $k<\omega$ we construct an open (in X) cover $\mathcal{G}_k(y)$ of $\Omega_d(y)$. Let us consider the following cases:

(i) for $(y,n)\in\Omega_d(y)$, $0<n<\omega$ let $U_k((y,n))=\{(y,n)\}$,

(ii) for $(y,0)\in\Omega_d(y)$ let $U_k((y,0))=](y\setminus 1/k,\omega),(y,0)]$, and

(iii) for $(y,\omega)\in\Omega_d(y)$ let $U_k((y,\omega))=[(y,\omega),(y+1/k,0)[$.

Now let $\mathcal{G}_k(y)=\{U_k((y,n))\colon\ 0\leqslant n\leqslant\omega\}$ and let $\mathcal{G}(y)=\{\mathcal{G}_k(y)\colon\ k<\omega\}$. It is not difficult to see that $\mathcal{G}(y)$ is a y-development and so f is an MT-map and X an MT-space.

We now show that X is not an M-space. We do this by showing that if X is an M-space then it has a G_δ-diagonal, which would contradict the fact that X is not metrizable.

So, assume that X is a paracompact M-space, then there exists a compact ($\equiv$ perfect) map g from X onto some metrizable space M. There exists a sequence $\{\mathcal{H}_n\colon\ n\in\omega\}$ of open covers of X such that: (i) for each $n<\omega$, $\mathcal{H}_{n+1}$ star refines $\mathcal{H}_n$, (ii) the sequence $\{\mathrm{St}(x,\mathcal{H}_n)\colon\ n<\omega\}$ is a base for $C_x=\bigcap_{n<\omega}\mathrm{St}(x,\mathcal{H}_n)$ and (iii) $C_x=g^{-1}gx$ is compact. Thus $C_x\cap\Omega_d(y)$ is finite for every $y\in Y$. We denote by $\mathcal{H}'_n$ the open cover of X obtained by taking convex components of all the elements of $\mathcal{H}_n$. We further decompose $\mathcal{H}'_n$ in the following way: (i) if the set $U\in\mathcal{H}'_n$ and $U=[(y,n_1),(y,n_2)]$, $0<n_1<n_2<\omega$, then we decompose U into singleton sets $\{(y,n_1)\},\dots,\{(y,n_2)\}$ and (ii) if the set $U\in\mathcal{H}'_n$ and $U=](y_1,*),(y,n)]$, $y_1<y$, $0<n<\omega$, then we decompose U into the sets $](y_1,*),(y,0)]$, $\{(y,1)\},\dots,\{(y,n)\}$. Denote the new open covers by $\mathcal{H}''_n$. We now show that $\{\mathcal{H}''_n\colon\ n\in\omega\}$ is a G_δ-diagonal sequence.

Take an arbitrary $x\in X$. Then $x\in C_x=\bigcap\mathrm{St}(x,\mathcal{H}_n)$ and say $x\in\Omega_d(y_x)$. We will consider three cases:

(i) if $x=(y_x,n)$, $0<n<\omega$ then, since $C_x\cap\Omega_d(y_x)$ is finite, there is a maximum element $(y_x,n_{\max})$ and a minimum element $(y_x,n_{\min})$ in $C_x\cap\Omega_d(y_x)$. Take the convex component of C_x containing x, and say it is $[(y_x,n'_{\min}),(y_x,n'_{\max})]$. One can now easily see that in this case there exists a $k<\omega$ such that $\mathrm{St}(x,\mathcal{H}''_k)=\{x\}$,

(ii) if $x = (y_x, 0)$ and the maximum element in $C_x \cap \Omega_d(y_x)$ is $(y_x, n_{\max})$, then take the convex component of C_x containing x, say $[(y_x, 0), (y_x, n'_{\max})]$. Then again, it is not difficult to see that in this case we have $\bigcap_{k<\omega} \mathrm{St}(x, \mathcal{H}''_k) = \{x\}$,

(iii) finally, if $x = (y_x, \omega)$, there must be an $n_x < \omega$ such that $](y_x, n_x), (y_x, \omega)[\cap C_x = \emptyset$. Then again, it is not difficult to see that $\bigcap_{k<\omega} \mathrm{St}(x, \mathcal{H}''_k) = \{x\}$.

This shows that $\{\mathcal{H}''_n\colon n < \omega\}$ is a G_δ-diagonal sequence which contradicts the fact that X is not metrizable.

Finally, it is not difficult to see that a closed subspace of an *MT*-space is again an *MT*-space but this is not true for an arbitrary subspace. This can be seen by proving that the Sorgenfrey Line S is not an *MT*-space, since S is a subspace of the space X in Example 4.9. That the space S is not an *MT*-space can also be deduced from the fact that an *MT*-space with a G_δ-diagonal is metrizable, which will be shown in the next paragraph. Still, we give a direct proof of this fact as it is interesting in itself.

Proposition 4.10. *The Sorgenfrey line S is not an MT-space.*

Proof. We begin by showing that any metrizable subset N of S is countable. The space S is hereditary separable and so N is a separable metrizable space. Thus N is second countable and any second countable subspace of S must be countable.

Say there exists an *MT*-map $f : S \to M$, where M is a metrizable space. In particular, for every $y \in M$, $f^{-1}y$ is a closed metrizable subspace of S and so is countable. We next show that for every $y \in M$ one can choose an $x_y \in f^{-1}y$ such that there exists an $c_y \in S$, $c_y < x_y$ with $]c_y, x_y[\cap f^{-1}y = \emptyset$. For this we take any $a, b \in f^{-1}y$, $a < b$. There exists some c with $a < c < b$ and $c \notin f^{-1}y$, which implies that $[c, c + \varepsilon[\cap f^{-1}y = \emptyset$ for some $\varepsilon > 0$. Consider the point $x = \inf\{z \in f^{-1}y\colon z > c + \varepsilon\}$. Since $f^{-1}y$ is closed in S we have that $x \in f^{-1}y$. By taking $x_y = x$ and $c_y = c$ we have that $]c_y, x_y[\cap f^{-1}y = \emptyset$. Now let $P = \{x_y\colon y \in M\}$, then P is an uncountable subset of S. We now show that P has a development which would contradict the fact that it is not metrizable.

Consider the development $\{\mathcal{W}_n = \{B_\rho(y, 1/n)\colon y \in M\}\colon n < \omega\}$ in M, where ρ is a compatible metric in M. Let $\mathcal{G}_n = f^{-1}\mathcal{W}_n$ and let $\widehat{\mathcal{G}_n}$ be the cover consisting of the convex components of all the elements of $\mathcal{G}_n$. We now show that $\{\widehat{\mathcal{G}_n} \wedge P\colon n < \omega\}$ is a development for P.

Take an arbitrary element $p \in P$ and let U_p be any nbd of p in P, say $U_p = [p, q[\cap P$. Let $p \in f^{-1}y(p)$, that is $p = x_{y(p)}$. One can take the element q such that $q \notin f^{-1}y(p)$ and so there exists a q' such that $[q, q'[\cap f^{-1}y(p) = \emptyset$. For every $x \in f^{-1}y(p)$, $x < p$ we have that $x < c_{y(p)}$. In this case let $U_x = [x, c_{y(p)}[$. For every $x \in f^{-1}y(p)$, $p < x < q$ take $U_x = [x, q[$. Finally, for every $x \in f^{-1}y(p)$, $x > q$ (and so $x > q'$) take $U_x = [x, \to[$. Now denote by

$$V = \bigcup \{U_x\colon x \in f^{-1}y(p)\},$$

this is an open set containing $f^{-1}y(p)$. Since f is closed, there exists a $k < \omega$ such that $f^{-1}y(p) \subset f^{-1}\mathrm{St}(y(p), \mathcal{W}_k) \subset V$. Therefore, $p \in \mathrm{St}(p, \widehat{\mathcal{G}_k}) \subset [p, q[$. Consequently, $\{\widehat{\mathcal{G}_n} \wedge P\colon n < \omega\}$ is a development for P. □

5. *MT*-maps and the G_δ-diagonal

In this paragraph we give some results connected with *MT*-maps and the G_δ-diagonal. In particular, as a corollary to our results, we have that an *MT*-space with a G_δ-diagonal is metrizable. This result can also be deduced from a result obtained in [2], namely that a normal space X with a G_δ-diagonal is metrizable if it is the preimage of a metrizable space under a closed, continuous map having metrizable fibers (cf. Example 2.22).

Lemma 5.1. *Let a continuous map $f: X \to Y$ be regular, $y \in Y$ and a collection of points $x_i \in f^{-1}y$, $i < \omega$, be discrete in $f^{-1}y$. Then there exist disjoint nbds G_i of x_i in X, $i < \omega$.*

Proof. By the regularity of f, we can find disjoint nbds V_i of x_i and W_i of $\{x_j\colon j < \omega, j \neq i\}$ in X, $i < \omega$. Then the nbds $G_1 = V_1$, $G_i = V_i \cap \bigcap\{W_j\colon j < i\}$, $i > 1$, are the desirable nbds since $G_i \cap G_{i+k} \subset V_i \cap W_i = \emptyset$, $1 \leqslant k$. $\square$

Lemma 5.2. *If in addition to the conditions of Lemma* 5.1, *Y is a T_1-space and f is prenormal then we can find a nbd O of y and nbds H_i of x_i, $H_i \subset f^{-1}O$, $i < \omega$, such that the collection $\{H_i\colon i < \omega\}$ is discrete in $f^{-1}O$.*

Proof. Since the set $F = \{x_i\colon i < \omega\}$ is closed in X and $G = \bigcup\{G_i\colon i < \omega\}$ is its nbd, there exists a nbd O of y and disjoint nbds H of F and U of $f^{-1}O \setminus G$ in $f^{-1}O$. Then the nbds $H_i = H \cap G_i$ of x_i, $i < \omega$, form the desirable collection. $\square$

Lemma 5.3. *If in addition to the conditions of Lemma* 5.2, *Y is regular then we can assume the system $\{H_i\colon i < \omega\}$ to be discrete in X.*

Proof. Indeed, if V is a nbd of y and $[V] \subset O$ then we can take $H_i \cap f^{-1}V$ instead of H_i, $i < \omega$. $\square$

Theorem 5.4. *Let a continuous map $f: X \to Y$ be regular, prenormal and closed and Y be a regular T_1- and q-space. Then f is peripherically countably compact.*

Proof. Suppose that there exist $y \in Y$ and a discrete in $\operatorname{Fr} f^{-1}y$ collection of points x_i, $i < \omega$. Take nbds O_i of y, $i < \omega$, having property $(*)$. By Lemma 5.3, we can take a discrete in X collection of nbds H_i of x_i, $i < \omega$. Since $x_i \in \operatorname{Fr} f^{-1}y$ and Y is a regular T_1-space, we can find $t_i \in H_i$ and nbds V_{i+1} of y, $i < \omega$, such that

$$ft_1 \in O_1 \setminus \{y\}, \quad [V_{i+1}] \subset O_{i+1} \setminus \{ft_1, \ldots, ft_i\}, \quad ft_{i+1} \in V_{i+1} \setminus \{y\},\ i < \omega.$$

Then the set $A = \{t_i\colon i < \omega\}$ is discrete in X, fA has a cluster point t and, evidently, $t \in \bigcap\{[V_{i+1}]\colon i < \omega\}$. Thus $t \notin fA$ and so the set fA is not closed. But this contradicts the discreteness of A and the closedness of f since $[A] \subset \bigcup\{f^{-1}ft_i\colon i < \omega\} = f^{-1}fA$. $\square$

Corollary 5.5. *Under the conditions of Theorem* 5.4, *if all the fibers are isocompact* (*in particular, metrizable*) *then f is peripherically compact.*

Since any MT-map is closed, regular, prenormal and all of its fibers are metrizable we have:

Corollary 5.6. *Let $f : X \to Y$ be an MT-map and Y be a regular T_1- and q-space. Then f is peripherically compact.*

Below, for a continuous map $f : X \to Y$ and $E \subset Y$, we denote by $I_f(E) = \bigcup\{\operatorname{Int} f^{-1}y\colon\ y \in E\}$ and by $Y_0 = \{y \in Y\colon\ \operatorname{Int} f^{-1}y \neq \emptyset\}$. We say that $I_f = I_f(Y)$ has property $\boldsymbol{P}$ if it contains an F_σ-set F such that $F \cap f^{-1}y \neq \emptyset$ for every $y \in Y_0$. Note that if the space X is perfect then I_f has property $\boldsymbol{P}$.

Lemma 5.7. *If $f : X \to Y$ is an MT-map from a space X with a G_δ-diagonal onto a metrizable space Y, then I_f has property $\boldsymbol{P}$.*

Proof. From Corollary 5.6 we have that f is peripherically compact. Choose a point $x(y)$ for every $y \in Y_0$ and consider the subspace $X_0 = (X \setminus I_f) \cup \{x(y)\colon\ y \in Y_0\}$. Then X_0 is closed in X and $f_0 = f|_{X_0}$ is a perfect map from X_0 onto Y. Therefore, the subspace X_0 is an M-space with a G_δ-diagonal and so is metrizable. Thus $F = \{x(y)\colon\ y \in Y_0\}$ is an F_σ-set in X_0 and so in X. □

Lemma 5.8. *If a continuous map $f : X \to Y$ is closed and the set I_f has property $\boldsymbol{P}$, then fI_f is the union of a countable collection of closed and discrete in Y sets.*

Proof. Let $F \subset I_f$ satisfy the hypothesis of the lemma. Then F is the union of closed in X sets F_i, $i < \omega$. Let $D_i = fF_i$. Then D_i, $i < \omega$, are closed in Y and for every $E \subset D_i$, the sets $G = F_i \cap f^{-1}E = F_i \cap I_f(E)$ and $F_i \setminus G = F_i \cap f^{-1}(D_i \setminus E) = F_i \cap I_f(D_i \setminus E)$ are open in F_i and so are closed in it and in X. Consequently, E is closed in D_i. We have just proved that $fI_f = fF = \bigcup\{D_i\colon\ i < \omega\}$ and that D_i, $i < \omega$ are closed and discrete in Y. □

Let us recall that a space X is perfect if every open subset of X is an F_σ-set.

Corollary 5.9. *If a continuous map $f : X \to Y$ is closed and the space X is perfect then fI_f is the union of a countable collection of closed and discrete in Y sets.*

Proposition 5.10. *Let a continuous map $f : X \to Y$ be closed and peripherically compact, all of its fibers be metrizable and I_f has property $\boldsymbol{P}$. Then there exist a metrizable space M and a perfect map $g : X \to Y \times M$ such that $f = p \circ g$ (where p is the projection of the product $Y \times M$ onto its factor Y), $gI_f \cap g(X \setminus I_f) = \emptyset$ and the restriction $g|_{I_f}$ and $p|_{g(X\setminus I_f)}$ are topological embeddings.*

Proof. By Lemma 5.8, the set $D = fI_f$ is the union of closed and discrete in Y sets D_i, $i < \omega$. Let $I_i = I_f(D_i)$. Evidently, the space $K_i = f^{-1}D_i$ is closed in X and is metrizable. Let ϱ_i be some metric on K_i such that $\varrho_i(f^{-1}y, f^{-1}y') > 1$ if $y, y' \in D_i$, $y \neq y'$.

Let $M_i = \{O_i\} \cup I_i$ and take the system of all open in I_i sets and the sets $O_{in} = \{O_i\} \cup I_{in}$, where

$$I_{in} = \left\{x \in I_i\colon \varrho(x, K_i \setminus I_i) \leqslant \frac{1}{n}\right\}, \quad n < \omega,$$

as a base of the topology in M_i. It is clear that M_i is a regular T_1-space with a σ-locally finite base and so M_i is metrizable. Evidently, the map $\varphi_i : X \to M_i$ coinciding with the identity map of I_i on I_i and with $\varphi_i x = O_i$ for all $x \in X \setminus I_i$ is continuous.

Let $g : X \to Y \times M$, where $M = \prod_{i<\omega} M_i$, be the diagonal product of f and φ_i, $i < \omega$. Then g is continuous. Note that the space M is metrizable. Let p be the projection of the product $Y \times M$ onto its factor Y. It is not difficult to see that $g|_{I_f}$ and $p|_{g(X\setminus I_f)}$ are embeddings. Evidently, the fibers of g are either singleton sets or coincide with the boundaries of fibers of f and so are compact. It remains to prove that g is closed.

Let $z \in Y \times M$ and V be a nbd of $g^{-1}z$. We must find a nbd W of z such that $g^{-1}W \subset V$. If $pz \in Y \setminus fX$ then we can take $W = p^{-1}(Y \setminus fX)$. If $pz \in fX \setminus D$ then $f^{-1}pz = g^{-1}p^{-1}pz = g^{-1}z \subset V$. By the closedness of f, we can find a nbd U of pz such that $f^{-1}U \subset V$. Then we can put $W = p^{-1}U$. Finally, let $pz \in D_i$ for some $i < \omega$. Then $V \cup \operatorname{Int} f^{-1}pz$ is a nbd of $f^{-1}pz$ and so there exists a nbd U of pz such that $f^{-1}U \subset V$. We can assume that $U \cap D_i = \{pz\}$. Since the space $\operatorname{Fr} f^{-1}pz$ is compact, there exists $n < \omega$ such that $I_{in} \cap f^{-1}pz \subset V$. Then we can take $W = p^{-1}U \cap q_i^{-1}O_{in}$, where q_i is the projection of the product $Y \times \prod_{i<\omega} M_i$ onto the factor M_i. □

Corollary 5.11. *A perfect MT-space is a CMT-space.*

Corollary 5.12. *A countable product of MT-spaces having property **P** is a CMT-space.*

Since the product of an M-space and a metric space is an M-space and the preimage of an M-space under a perfect map is an M-space we have the following result.

Corollary 5.13. *If $f : X \to Y$ is an MT-map onto a T_1- and M-space and I_f has property **P** (in particular, if X is perfect) then X is also an M-space.*

Finally, since a Tychonoff M-space with a G_δ-diagonal is metrizable, we have:

Corollary 5.14. *An MT-space with a G_δ-diagonal is metrizable.*

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