LOOMIS–SIKORSKI THEOREM AND STONE DUALITY FOR EFFECT ALGEBRAS WITH INTERNAL STATE

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Abstract. Recently Flaminio and Montagna, [FlMo], extended the language of MV-algebras by adding a unary operation, called a state-operator. This notion is introduced here also for effect algebras. Having it, we generalize the Loomis–Sikorski Theorem for monotone σ-complete effect algebras with internal state. In addition, we show that the category of divisible state-morphism effect algebras satisfying (RDP) and countable interpolation with an order determining system of states is dual to the category of Bauer simplices Ω such that $\partial_e\Omega$ is an F-space

1. Introduction

The famous Loomis–Sikorski Theorem was proved independently by two authors, Sikorski and Loomis, [Sik, Loo], after the Second World War, and nowadays it has many serious applications in different areas of mathematics. Roughly speaking it states that every σ-complete Boolean algebra is a σ-algebra of subsets of a set up to some modulo, or, precisely every σ-complete Boolean algebra is a σ-epimorphic image of some σ-algebra of subsets. It can be rewritten also in the form that our σ-algebra is practically an appropriate system of $[0,1]$-valued functions; in our case it is a system of characteristic functions, where the Boolean operations on the set of functions are defined by points.

This was extended also for σ-complete MV-algebras in [Dvu1, Mun, BaWe] showing that every σ-complete MV-algebra is a σ-epimorphic image of a system of $[0,1]$-valued functions, called a *tribe*, where again MV-operations on functions in the tribe are defined by points.

In the Nineties, the theory of quantum structures was enriched by new structures, called *effect algebras*, see [FoBe]. They are inspired by the mathematical

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foundations of quantum mechanics (for an overview on effect algebras, see [DvPu]) as well as by many valued features of quantum mechanical measurements. One of the most important examples of effect algebras studied in quantum mechanics is the system $\mathcal{E}(H)$ of all Hermitian operators A on a Hilbert space H such that $O \le A \le I$, where O and I are the zero and identity operator on H. The category of effect algebras contains Boolean algebras, orthomodular lattices, orthomodular posets and orthoalgebras.

The Loomis–Sikorski Theorem was extended also for monotone σ-complete effect algebras by the present authors in [BCD].

The notion of a state, an analogue of probability measure, is a basic notion for quantum structures. It is motivated by the notion of a state in quantum mechanics. The set of all states on an effect algebra can be a good source of information on the given system, and it will be also deeply used in our paper.

Recently, the notion of a state was generalized by [FlMo] to an algebraically defined notion for MV-algebras. They enlarged the language of MV-algebras introducing a unary operation, τ, called an *internal state* or a *state-operator*. Such MV-algebras are called *state MV-algebras*. These algebras are now intensively studied e.g. in [DiDv, DDL1, DDL2, DDL3].

One of important properties of a state-operator τ on an MV-algebra is $\tau^2 = \tau$, idempotency. Inspired by this, in the present paper, we introduce notions of a (i) state-operator for effect algebras as an endomorphism $\tau : E \to E$ such that $\tau \circ \tau = \tau$, (ii) a strong state-operator, and (iii) a state-morphism-operator with some additional properties coming from state MV-algebras. The later two coincide with ones for MV-algebras. We add it to the language of effect algebras as an internal state and they will form a so-called *state effect algebras*. Moreover, given an integer n, we introduce an n-state-operator as an endomorphism $\tau : E \to E$ that in n-potent, i.e., $\tau^n = \tau$, and the couple (E, τ) is said to be *n-state effect algebra*.

Besides the presentation of basic properties of state effect algebras, we present the following two main results:

(1) We generalize the Loomis–Sikorski Theorem for monotone σ-complete n-state effect algebras with the Riesz Decomposition Property ((RDP) for short) showing that it is always a σ-monotone epimorphic image of an effect tribe with (RDP), an appropriate system of $[0, 1]$-valued functions that is a σ-complete effect algebra with pointwise defined effect algebraic operations, and with an n-state-operator induced by a function.
(2) We show that the category of divisible state-morphism effect algebras satisfying (RDP) and countable interpolation with an order determining system of states is dual to the category of Bauer simplices whose objects are pairs (Ω, g), where $\Omega \neq \emptyset$ is a Bauer simplex such that $\partial_e\Omega$ is an F-space (any two disjoint open F_σ subsets of Ω have disjoint closures) and $g : \Omega \to \Omega$ is a continuous function such that $g^n = g$. This is a Stone Duality Type Theorem, and it generalizes the famous result by Stone [Sto] that says that the category of Boolean algebras is dual to the category of Stone spaces (= compact Hausdorff topological space with a base consisting of clopen sets).

The paper is organized as follows. The basic properties of effect algebras are gathered in Section 2. The main object of our study, state-operators and n-state-operators on effect algebras, are introduced in Section 3. Our study will use facts on Choquet simplices and their connection to effect algebras and this is pointed out in Section 4. Section 5 characterizes n-state-operators. Our first goal, the Loomis–Sikorski Theorem for n-state-operators on monotone σ-complete effect algebras, is described in Section 6. Finally, Section 7 presents the second main goal, it gives Stone Type Dualities for some categories of effect algebras and the categories of Bauer simplices whose boundary is an F-space.

2. Elements of Effect Algebras

An *effect algebra* is by [FoBe] a partial algebra $E = (E; +, 0, 1)$ with a partially defined operation $+$ and two constant elements 0 and 1 such that, for all $a, b, c \in E$,

(i) $a + b$ is defined in E if and only if $b + a$ is defined, and in such a case $a + b = b + a$;
(ii) $a + b$ and $(a + b) + c$ are defined if and only if $b + c$ and $a + (b + c)$ are defined, and in such a case $(a + b) + c = a + (b + c)$;
(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a + a' = 1$;
(iv) if $a + 1$ is defined in E, then $a = 0$.

If we define $a \le b$ if and only if there exists an element $c \in E$ such that $a + c = b$, then $\le$ is a partial ordering on E, and we write $c := b - a$. It is clear that $a' = 1 - a$ for any $a \in E$.

For a comprehensive source on the theory of effect algebras, we recommend [DvPu]. A *state* on an effect algebra E is any mapping $s: E \to [0, 1]$ such that (i) $s(1) = 1$, and (ii) $s(a + b) = s(a) + s(b)$ whenever $a + b$ is defined in E. We denote by $\mathcal{S}(E)$ the set of all states on E. It can happen that $\mathcal{S}(E)$ is empty, see e.g. [DvPu, Ex 4.2.4]. A state s is said to be *extremal* if $s = \lambda s_1 + (1 - \lambda)s_2$ for $\lambda \in (0, 1)$ implies $s = s_1 = s_2$. By $\partial_e \mathcal{S}(E)$ we denote the set of all extremal states of $\mathcal{S}(E)$ on E. We say that a net of states, $\{s_\alpha\}$, on E *weakly converges* to a state, s, on E if $s_\alpha(a) \to s(a)$ for any $a \in E$. In this topology, $\mathcal{S}(E)$ is a compact Hausdorff topological space and every state on E lies in the weak closure of the convex hull of the extremal states as it follows from the Krein-Mil'man Theorem, [Goo, Thm 5.17].

Let $G = (G; +, 0)$ be an Abelian po-group (= partially ordered group). An element $u \in G$ is said to be a *strong unit* if given $g \in G$, there is an integer $n \ge 1$ such that $g \le nu$. If we set $\Gamma(G, u) = [0, u]$ and endow it with the restriction of the group addition, $+$, then $\Gamma(G, u) := (\Gamma(G, u); +, 0, u)$ is an effect algebra.

An effect algebra that is either of the form $\Gamma(G, u)$ for some element $u \ge 0$ or is isomorphic with some $\Gamma(G, u)$ is called an *interval effect algebra*.

Let u be a positive element of an Abelian po-group G. The element u is said to be *generative* if every element $g \in G^+$ is a group sum of finitely many elements of $\Gamma(G, u)$, and $G = G^+ - G^+$. Such an element is a strong unit [DvPu, Lem 1.4.6] for G and the couple (G, u) is said to be a *po-group with generative strong unit*. For example, if u is a strong unit of an interpolation po-group G, then u is generative. The same is true for I and $\mathcal{E}(H) := \Gamma(\mathcal{B}(H), I)$.

Let E be an effect algebra and H be an Abelian (po-) group. A mapping $p: E \to H$ that preserves $+$ is called an *H-valued measure* on E.

Remark 2.1. If E is an interval effect algebra, then there is a po-group G with a generative strong unit u such that $E \cong \Gamma(G, u)$, and every H-valued measure $p: \Gamma(G, u) \to H$ can be extended to a group-homomorphism ϕ from G into H. If H is a po-group, then ϕ is a po-group-homomorphism. Then ϕ is unique and (G, u) is also unique up to isomorphism of unital (po-) groups, see [DvPu, Cor 1.4.21]; the element u is said to be a *universal strong unit* for $\Gamma(G,u)$ and the couple (G,u) is said to be a *unigroup*.

We recall that an effect algebra E satisfies the *Riesz Decomposition Property* ((RDP) in abbreviation) if $x_1+x_2=y_1+y_2$ implies there exist four elements $c_{11},c_{12},c_{21},c_{22}\in E$ such that $x_1=c_{11}+c_{12}$, $x_2=c_{21}+c_{22}$, $y_1=c_{11}+c_{21}$, and $y_2=c_{12}+c_{22}$. Equivalently, [DvPu, Lem 1.7.5], E has (RDP) iff $x\le y_1+y_2$ implies that there exist two elements $x_1,x_2\in E$ with $x_1\le y_1$ and $x_2\le y_2$ such that $x=x_1+x_2$.

We say that an Abelian po-group G is an *interpolation group*, if given x_1,x_2,y_1,y_2 in G such that $x_i\le y_j$ for all i,j, there exists z in G such that $x_i\le z\le y_j$ for all i,j. Equivalently, [Goo, Prop 2.1], G is an interpolation group iff an analogous property as (RDP) for effect algebras holds also for $G^+=\{g\in G: g\ge 0\}$.

Remark 2.2. (1) If E is an effect algebra satisfying (RDP), then E is an interval effect algebra. In such a case, there is a unique (up to isomorphism of unital po-groups) interpolation unital po-group (G,u) such that $E\cong\Gamma(G,u)$. Moreover, u is a universal strong unit for E. Conversely, if (G,u) is an interpolation unital po-group, then $\Gamma(G,u)$ satisfies (RDP), and u is a universal strong unit for $\Gamma(G,u)$, see [Rav] ([DvPu, Thm 1.7.17]).

(2) We note that the identity operator I on a Hilbert space H is a universal strong unit for $\Gamma(\mathcal{B}(H),I)$, [DvPu, Cor 1.4.25] that does not satisfy (RDP).

We recall that an *MV-algebra* is an algebra $(A;\oplus,^*,0)$ of signature $\langle 2,1,0\rangle$, where $(A;\oplus,0)$ is a commutative monoid with neutral element 0, and for all $x,y\in A$

(i) $(x^*)^*=x$,
(ii) $x\oplus 1=1$, where $1=0^*$,
(iii) $x\oplus(x\oplus y^*)^*=y\oplus(y\oplus x^*)^*$.

We define also two additional total operations $\odot$ and $\ominus$ on A via $x\odot y:=(x^*\oplus y^*)^*$ and $x\ominus y=x\odot y^*$.

If (G,u) is an Abelian ℓ-group (= lattice ordered group) with a strong unit $u\ge 0$, then $(\Gamma(G,u);\oplus,^*,0)$ is a prototypical example of an MV-algebra, where Γ is the Mundici functor, where $\Gamma(G,u):=[0,u]$ is endowed with the MV-operations $g_1\oplus g_2:=(g_1+g_2)\wedge u$, $g^*:=u-g$, because by [Mun], every MV-algebra is isomorphic to some $\Gamma(G,u)$.

If on an MV-algebra A we define a partial operation, $+$, by $a+b$ is defined in A iff $a\le b^*$, and we set then $a\oplus b:=a\oplus b$. Then $(A;+,0,1)$ is an interval effect algebra with (RDP).

An *ideal* of an effect algebra E is a non-empty subset I of E such that (i) $x\in E$, $y\in I$, $x\le y$ imply $x\in I$, and (ii) if $x,y\in I$ and $x+y$ is defined in E, then $x+y\in I$. We denote by $\mathcal{I}(E)$ the set of all ideals of E. An ideal I is said to be a *Riesz ideal* if, for $x\in I$, $a,b\in E$ and $x\le a+b$, there exist $a_1,b_1\in I$ such that $x=a_1+b_1$ and $a_1\le a$ and $b_1\le b$.

For example, if E has (RDP), then any ideal of E is Riesz.

We say that a poset E is an *antilattice* if joins and meets exist only for comparable elements. For example, every linearly ordered set is an antilattice. According to [Rav, Thm 2.12] or [Dvu2, Thm 7.2], every effect algebra with (RDP) is a subdirect product of antilattice effect algebras with (RDP).

For example, if H is a Hilbert space, then $\mathcal{B}(H)$ is an antilattice [LuZa], but $\mathcal{E}(H) = \Gamma(\mathcal{B}(H), I)$ is not. In fact, if P_M and P_N are orthogonal projectors onto subspaces M and N of H, then $P_M \vee P_N$ exists in $\mathcal{E}(H)$ and equals $P_{M\vee N}$, [Dvu0], where $M \vee N$ denotes the join in the complete lattice of all closed subspaces of H, whereas their join in $\mathcal{B}(H)$ fails when they are not comparable.

If $E = \Gamma(G, u)$ for some effect algebra with (RDP), then all joins and meets from E are the same also in G.

We recall that if (G, u) is an Abelian unital po-group, then a *state* on it is any mapping $s : G \to \mathbb{R}$ such that (i) $s(g) \geq 0$ for any $g \geq 0$, (ii) $s(g_1+g_2) = s(g_1)+(g_2)$ for all $g_1, g_2 \in G$, and (iii) $s(u) = 1$. A state s is *extremal* if from $s = \lambda s_1 + (1-\lambda)s_2$ for $\lambda \in (0, 1)$ it follows $s = s_1 = s_2$. We denote by $\mathcal{S}(G, u)$ and by $\partial_e\mathcal{S}(G, u)$ the sets of all states and all extremal states on (G, u). We can also introduce the weak topology on $\mathcal{S}(G, u)$. We have that $\mathcal{S}(G, u)$ is always nonempty, [Goo, Cor 4.4], whenever $u > 0$. Due to the Krein–Mil'man Theorem, [Goo, Thm 5.17], every state on (G, u) is a weak limit of a net of convex combinations of extremal states on (G, u). If we set $\Gamma(G, u) = [0, u]$, then the restriction of any state on (G, u) onto $\Gamma(G, u)$ is a state on $\Gamma(G, u)$. We recall that if u is a strong unit and G is an interpolation group, in particular, an ℓ-group, or more general a unigroup, then every state on $\Gamma(G, u)$ can be uniquely extended to a state on (G, u). Moreover, this correspondence is an affine homeomorphism (affine means that it preserves all convex combinations).

We say that a po-group G is *Archimedean* if for $x, y \in G$ such that $nx \leq y$ for all positive integers $n \geq 1$, then $x \leq 0$.

Remark 2.3. It is possible to show that a unital group (G, u) is Archimedean iff $G^+ = \{g \in G : \ s(g) \geq 0 \text{ for all } s \in \mathcal{S}(G, u)\}$, [Goo, Thm 4.14], or equivalently, $\Gamma(G, u)$ has an *order determining* system of states, $\mathcal{S}$, i.e., $f \leq g$ iff $s(f) \leq s(g)$ for any $s \in \mathcal{S}$. In a similar way we define an order determining system of states on an effect algebra E. If $E = \Gamma(G, u)$ and (G, u) is a unigroup, then E has an order determining system of state iff (G, u) has it. In particular, $\partial_e\mathcal{S}(E)$ is order determining iff so is $\mathcal{S}(E)$.

An analogous result holds also for some effect algebras, see Proposition 4.1 below.

3. State-Operators and n-State-Operators on Effect Algebras

In this section we introduce state-operators, n-state-operators, strong state-operators, and state-morphism-operators on effect algebras and we present their basic properties. We show that in the case of MV-algebras, a strong state-operator and a state-morphism-operator coincide with state-operators and state-morphism-operators defined for MV-algebras, Propositions 3.5–3.6.

Let E and F be two effect algebras. A mapping $h : \ E \to F$ is said to be a *homomorphism* if (i) $h(a+b) = h(a)+h(b)$ whenever $a+b$ is defined in E, and (ii) $h(1) = 1$. In particular, we have $h(a') = h(a)'$ for each $a \in E$, $h(0) = 0$, $h(a) \leq h(b)$ whenever $a \leq b$ and then $h(b-a) = h(b)-h(a)$. A bijective homomorphism h such that h^{-1} is a homomorphism is said to be an *isomorphism* of E and F.

Let E be an effect algebra. An endomorphism $\tau : E \to E$ such that $\tau^2 = \tau$ is said to be a *state-operator* or an *internal state* and the couple (E, τ) is said to be a *state effect algebra* with internal state.

An endomorphism $\tau : E \to E$ such that

$$\tau(\tau(a) \vee \tau(b)) = \tau(a) \vee \tau(b) \tag{3.1}$$

whenever $\tau(a) \vee \tau(b)$ is defined in E is said to be a *strong state-operator* on E, and the couple (E, τ) is called a *strong state effect algebra*.

From (3.1) we see that, for any $a \in E$, $\tau^2(a) = \tau(\tau(a) \vee \tau(a)) = \tau(a) \vee \tau(a) = \tau(a)$, i.e., any strong state-operator is a state-operator on E.

If a strong-state-operator τ satisfies also $\tau(a \vee b) = \tau(a) \vee \tau(b)$ whenever $a \vee b$ is defined in E, τ it is called a *state-morphism-operator* on E, and the couple (E, τ) is said to be a *state-morphism effect algebra*. An endomorphism τ is a state-morphism-operator iff $\tau^2 = \tau$ and τ preserves all existing joins in E.

Finally, we generalize the just defined notions as follows. Given an integer $n \geq 1$, an endomorphism $\tau : E \to E$ is said to be an *n-state-operator* if τ is n-potent, i.e. $\tau^n = \tau$. The couple (E, τ) is said to be an *n-state effect algebra*. An n-state-operator τ is said to be an n-state-morphism-operator if τ preserves all existing joins in E, and the couple (E, τ) is said to be an *n-state-morphism effect algebra*.

We recall that a state s on E is *discrete* if there is an integer $n \geq 1$ such that $s(E) \subseteq \{0, 1/n, \ldots, n/n\}$.

We have that if $s \in \mathcal{S}(E)$, then $s \circ \tau \in \mathcal{S}(E)$, and if s is discrete, so is $s \circ \tau$.

We say that a state-operator τ satisfies the *extremal state property*, (ESP) for short, if $s \circ \tau \in \partial_e \mathcal{S}(E)$, for any $s \in \partial_e \mathcal{S}(E)$, and we say also that (E, τ) satisfies (ESP).

Let τ be an endomorphism on E. We denote by

$$\mathrm{Ker}(\tau) = \{a \in E : \tau(a) = 0\}$$

the *kernel* of τ. An endomorphism τ is *faithful* if $\mathrm{Ker}(\tau) = \{0\}$. An ideal I of E is said to be a *τ-ideal* if $\tau(I) \subseteq I$. For example, $\mathrm{Ker}(\tau)$ is a τ-ideal.

Example 3.1. (1) The couple (E, id_E) is a state-morphism effect algebra with (ESP).

(2) Let F be an effect algebra and let $E = F \times F$. We define two operators on E by

$$\tau_1(a, b) = (a, a), \quad \text{and} \quad \tau_2(a, b) = (b, b), \quad (a, b) \in F \times F.$$

Then τ_1 and τ_2 are state-morphism-operators on E that preserve all joins and meets existing in E:

In fact, if $\partial_e \mathcal{S}(F) = \emptyset$, then $\partial_e \mathcal{S}(E) = \emptyset$ and both τ_1 and τ_2 trivially satisfy (ESP).

Assume that $\partial_e \mathcal{S}(F) = \{s_t : t \in T\}$ for some index set $T \neq \emptyset$. Define $m_t^1(a, b) = s_t(a)$ and $m_t^2(a, b) = s_t(b)$ for all $(a, b) \in F \times F$ and each $t \in T$. If m is an extremal state on E, then either $m(1, 0) = 1$ or $m(0, 1) = 1$. In the first case, $s(a) := m(a, 0)$ is a state on F. It is extremal, otherwise, $s(a) = \lambda_1 s_1(a) + \lambda_2 s_2(a)$ for some states s_1, s_2 on E. If we define $m_i(a, b) := s_i(a)$, then m_1, m_2 are states on E and $m(a, b) = \lambda_1 m_1(a, b) + \lambda_2 m_2(a, b)$ which is impossible, hence $s(a)$ is an extremal state on F so that $s = s_t$ for some $t \in T$, and $m(a, b) = m(a) = s_t(a) = m_t^1(a, b)$.

Similarly, in the second case, $m = m_t^2$ for some $t \in T$. Consequently, $\partial_e \mathcal{S}(E) = \{m_t^j : t \in T, j = 1, 2\}$.

Check: $m_t^1(\tau_1(a, b)) = m_t^1(a, a) = s_t(a) = m_t^1(a, b)$ and $m_t^2(\tau_1(a, b)) = m_t^2(a, a) = s_t(a) = m_t^1(a, b)$. Therefore, τ_1 as well as τ_2 are state-morphism-operators with (ESP).

Lemma 3.2. *Let τ be an endomorphism of an effect algebra E such that $\tau^2 = \tau$. Then*

(i) *If τ is a strong state-operator, then τ preserves all existing meets from E of the form $\tau(a) \wedge \tau(b)$.*
(ii) *The set $\tau(E)$ is an effect subalgebra, $\tau(E) = \{a \in E : \tau(a) = a\}$, and τ on $\tau(E)$ is the identity on $\tau(E)$. If τ is also a strong state-operator, then if $\tau(a) \vee \tau(b) \in E$, then $\tau(a) \vee \tau(b) \in \tau(E)$.*
(iii) *If E satisfies* (RDP)*, then so does $\tau(E)$.*
(iv) *If τ is faithful, then $a < b$ entails $\tau(a) < \tau(b)$.*
(v) *If τ is faithful, then either $\tau(a) = a$ or $\tau(a)$ and a are not comparable.*
(vi) *If E is linear and τ faithful, then $\tau(a) = a$ for any $a \in E$.*
(vii) *If E is an antilattice effect algebra, then τ preserves all existing meets and joins.*
(viii) *If $\tau : E \to E$ is faithful then τ is a strong state-operator.*

Proof. (i) Passing to negation, we see that τ preserves all existing meets in $\tau(a) \wedge \tau(b) \in E$.

(ii) Since $a \in \tau(E)$ iff $a = \tau(b)$ for some $b \in E$, we have $\tau(a) = \tau(\tau(b)) = \tau(b) = a$.

Let $\tau(a) + \tau(b)$ be defined in E. Then $\tau(\tau(a) + \tau(b)) = \tau(\tau(a)) + \tau(\tau(b)) = \tau(a) + \tau(b) \in \tau(E)$. It is clear now that the restriction of τ onto $\tau(E)$ is the identity on $\tau(E)$.

Now if τ is a strong state-operator, the statement follows from (3.1).

(iii) Let E satisfy (RDP) and let $\tau(a_1) + \tau(a_2) = \tau(b_1) + \tau(b_2)$. There are four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $\tau(a_1) = c_{11} + c_{12}$, $\tau(a_2) = c_{21} + c_{22}$, $\tau(b_1) = c_{11} + c_{21}$ and $\tau(b_2) = c_{12} + c_{22}$. Then $\tau(a_1) = \tau(\tau(a_1)) = \tau(c_{11} + c_{12}) = \tau(c_{11}) + \tau(c_{12})$, similarly for $\tau(a_2), \tau(b_1), \tau(b_2)$ proving that the elements $\tau(c_{11}), \tau(c_{12}), \tau(c_{21}), \tau(c_{22}) \in \tau(E)$ yield that $\tau(E)$ satisfies (RDP).

(iv) Suppose that $a < b$ and $\tau(a) = \tau(b)$. Then $\tau(b - a) = \tau(b) - \tau(a) = \tau(a) - \tau(a) = 0$ giving $b - a = 0$ so that $a = b$, a contradiction.

(v) Assume that $\tau(a) \neq a$ and let $\tau(a)$ and a be comparable. Then either $a < \tau(a)$ or $\tau(a) < a$. By (iv) we have $\tau(a) < \tau(a)$ that is impossible.

(vi) It follows directly from (v).

(vii) Let $a \vee b$ be defined in E. Then a and b are comparable. So are $\tau(a)$ and $\tau(b)$ so that $\tau(a \vee b) = \tau(a) \vee \tau(b)$. Going to negations, we see that τ preserves also meets.

(viii) Assume that $d = \tau(a) \vee \tau(b)$ is defined in E. We show that $\tau(d) = d$. Check, $\tau(d) \geq \tau(\tau(a)) = \tau(a)$ and $\tau(d) \geq \tau(\tau(b)) = \tau(b)$. This yields $\tau(d) \geq \tau(a) \vee \tau(b) = d$. Hence, we have $\tau(\tau(d) - d) = \tau(\tau(d)) - \tau(d) = 0$, i.e. $\tau(d) - d = 0$ and $\tau(d) = d$. □

Proposition 3.3. *Let $E = \Gamma(G, u)$ be an interval effect algebra, where (G, u) is a unital po-group with universal strong unit for E. Let n be a fixed integer.*

Every endomorphism τ with $\tau^n = \tau$ on E can be uniquely extended to an n-potent po-group homomorphism τ_u on (G, u), i.e. $\tau_u^n = \tau_u$. Conversely, the restriction of any n-potent po-group homomorphism of (G, u) to E gives an n-potent endomorphism on E.

Proof. Let τ be an n-potent endomorphism on E. Since $E = \Gamma(G, u)$, the mapping $\tau : E \to E$ is in fact a G-valued measure on E. By Remark 2.1, there is a unique extension, $\tau_u : G \to G$ of τ that is a po-group-homomorphism.

Now we show that $\tau_u^n = \tau_u$. Since every element $x \in G^+$ is expressible via $x = x_1 + \cdots + x_n$, where $x_1, \ldots, x_n \in E$, we have $\tau_u^n(x) = \tau_u^n(x_1) + \cdots + \tau_u^n(x_n) = \tau^n(x_1) + \cdots + \tau^n(x_n) = \tau(x_1) + \cdots + \tau(x_n) = \tau_u(x_1) + \cdots + \tau_u(x_n)$. Every element $x \in G$ is of the form $x = x_1 - x_2$, where $x_1, x_2 \in G^+$, then $\tau_u^n(x) = \tau_u^n(x_1) - \tau_u^n(x_2) = \tau_u(x_1) - \tau_u(x_2) = \tau_u(x)$. $\square$

Proposition 3.4. *If E is a linear effect algebra with* (RDP)*, then every endomorphism on E preserves joins and if s is an extremal state, then $s \circ \tau$ is an extremal state on E.*

Proof. Suppose that $E = \Gamma(G, u)$ for some unital po-group (G, u). Then (G, u) is a unital ℓ-group and a state s on E is extremal iff its extension on (G, u), denoted also by s, is extremal. By [Goo, Thm 12.18], s is extremal iff $s(g \wedge h) = \min\{s(g), s(h)\}$ for all $g, h \in G^+$ iff $s(g \wedge h) = \min\{s(g), s(h)\}$ for all $g, h \in E$.

Since E is linear, τ trivially preserves all joins and meets in E. Hence, if s is an extremal state on E, then $s \circ \tau((a \wedge b)) = s(\tau(a) \wedge \tau(b)) = \min\{s(\tau(a)), s(\tau(b))\}$ proving that $s \circ \tau$ is an extremal state on E. $\square$

According to [FlMo], a *state MV-algebra* is a couple (A, τ), where A is an MV-algebra and τ is a unary operator on A (an internal state or an *MV-state-operator*) satisfying for each $x, y \in A$:

$(1)_{MV}$ $\tau(0) = 0$,
$(2)_{MV}$ $\tau(x^*) = (\tau(x))^*$,
$(3)_{MV}$ $\tau(x \oplus y) = \tau(x) \oplus \tau(y \odot (x \odot y)^*)$,
$(4)_{MV}$ $\tau(\tau(x) \oplus \tau(y)) = \tau(x) \oplus \tau(y)$.

In [FlMo] it is shown that for any state MV-algebra we have (i) $\tau(\tau(x)) = \tau(x)$, (ii) $\tau(1) = 1$, (iii) if $x \leqslant y$, then $\tau(x) \leqslant \tau(y)$, (iv) $\tau(x \oplus y) \leq \tau(x) \oplus \tau(y)$, (v) $\tau(x + y) = \tau(x) + \tau(y)$, (vi) $\tau(\tau(x) \vee \tau(y)) = \tau(x) \vee \tau(y)$, and (vii) the image $\tau(A)$ is the domain of an MV-subalgebra of A and $(\tau(A), \tau)$ is a state MV-subalgebra of (A, τ).

According to [DiDv], an *MV-state-morphism-operator* on an MV-algebra A is any endomorphism $\tau : A \to A$ such that $\tau^2 = \tau$. Every MV-state-morphism-operator is a state-operator.

Proposition 3.5. *Let A be an MV-algebra. If τ is an MV-state-operator on the MV-algebra A, then τ is a strong state-operator on A taken as an effect algebra with the partial addition $+$ derived from $\oplus$.*

Conversely, if τ is a strong state-operator on the derived effect algebra A, then τ is an MV-state-operator on the MV-algebra A.

Proof. Let τ be an MV-state-operator on the MV-algebra A. Due to the basic properties of τ described just before this proposition, we see that τ is a strong state-operator on the effect algebra A derived from the MV-algebraic structure. Moreover, τ preserves all joins and meets in $\tau(A)$.

Conversely, let τ be a strong state-operator on the effect algebra A. Then $(1)_{MV}$ and $(2)_{MV}$ hold.

We have $x \oplus y = x + ((x \oplus y) \ominus x) = x + (y \wedge x^*)$ so that $\tau(x \oplus y) = \tau(x) + \tau(y \wedge x^*) = \tau(x) \oplus \tau(y \odot (x \odot y)^*)$ that gives $(3)_{MV}$.

In addition, $\tau(\tau(x)\oplus\tau(y)) = \tau(\tau(x)+\tau(y)\wedge\tau(x)^*) = \tau(\tau(x)+\tau(y)\wedge\tau(x^*)) = \tau(\tau(x)) + \tau(\tau(y)\wedge\tau(x^*)) = \tau(x)+\tau(y)\wedge\tau(x^*) = \tau(x)\oplus\tau(y)$ and this proves $(4)_{MV}$. □

Proposition 3.6. *Let A be an MV-algebra. If τ is an MV-state-morphism-operator on the MV-algebra A, then τ is a state-morphism-operator with* (ESP) *on A taken as an effect algebra with the partial addition $+$ derived from $\oplus$.*

Conversely, if τ is a state-morphism-operator on the effect algebra derived from an MV-algebra A, then τ is an MV-state-morphism-operator on the MV-algebra A. In addition, τ is with (ESP) *on the effect algebra A.*

Proof. Let τ be an MV-state-morphism-operator on A. Due to Proposition 3.5, τ is a strong state-operator on the effect algebra A that preserves all joins and meets.

Let s be an extremal state, that $s\circ\tau$ is also an extremal state because $s\circ\tau(a\wedge b) = s(\tau(a\wedge b)) = s(\tau(a)\wedge\tau(b)) = \min\{s(\tau(a)), s(\tau(b))\}$ for all $a,b\in A$.

Conversely, let τ be a state-morphism-operator on the effect algebra A. Then τ preserves all joins and meets in A, so that τ is an MV-state-morphism-operator on the MV-algebra A. Due to the first part of the present proof, τ satisfies (ESP) on the effect algebra A. □

4. Effect-Clans and Choquet Simplices

In this section, we show a close connection between the state spaces of effect algebras and Choquet simplices.

An *effect-clan* is a system $\mathcal{E}$ of $[0,1]$-valued functions on $\Omega\neq\emptyset$ such that (i) $1\in\mathcal{E}$, (ii) $f\in\mathcal{E}$ implies $1-f\in\mathcal{E}$, and (iii) if $f,g\in\mathcal{E}$ and $f(\omega)\le 1-g(\omega)$ for any $\omega\in\Omega$, then $f+g\in\mathcal{E}$. Then the effect-clan $\mathcal{E}$ is an effect algebra that is not necessarily a Boolean algebra nor an MV-algebra.

If E is an effect-clan of characteristic functions on Ω, then E satisfies (RDP) iff $E_0 = \{A\subseteq\Omega : \chi_A\in E\}$ is an algebra of subsets of Ω. For example, if Ω is a finite set with an even number of elements, the set of all characteristic functions of subsets of Ω with an even number of subsets is an effect-clan where (RDP) fails: $\chi_{\{1,2\}},\chi_{\{1,3\}}\in E$ but $\chi_{\{1\}}\notin E$.

Proposition 4.1. *Let $E=\Gamma(G,u)$, where (G,u) is a unigroup. The following statements are equivalent:*

(i) *$\mathcal{S}(E)$ is order determining.*
(ii) *E is isomorphic to some effect-clan.*
(iii) *G is Archimedean.*

Proof. (i) ⇒ (ii). Given $a\in E$, let $\hat a$ be a function from $\mathcal{S}(E)$ into the real interval $[0,1]$ such that $\hat a(s) := s(a)$ for any $s\in\mathcal{S}(E)$, and let $\hat E = \{\hat a : \ a\in E\}$. We endow $\hat E$ with pointwise addition, so that $\hat E$ is an effect-clan. Since $\mathcal{S}(E)$ is order determining, the mapping $a\mapsto\hat a$ is an isomorphism and $\hat E$ is Archimedean.

(ii) ⇒ (iii). Let $\hat E$ be any effect-clan isomorphic with E, and let $a\mapsto\hat a$ be such an isomorphism. Let $G(\hat E)$ be the po-group generated by $\hat E$. Then $G(\hat E)$ consists of all functions of the form $\hat a_1+\cdots+\hat a_n-\hat b_1-\cdots-\hat b_m$, and $\hat 1$ is its strong unit. Due to the categorical equivalence, (G,u) and $(G(\hat E),\hat 1)$ are isomorphic. But $(G(\hat E),\hat 1)$ is Archimedean, so is (G,u).

(iii) $\Rightarrow$ (i). Due to [Goo, Thm 4.14], $G^+ = \{g \in G:\ s(g) \geq 0 \text{ for all } s \in \mathcal{S}(G,u)\}$, which means that $\mathcal{S}(G,u)$ is order determining. Hence, the restrictions of all states on (G,u) onto $E=\Gamma(G,u)$ imply $\mathcal{S}(E)$ is order determining. $\square$

We say a poset E is *monotone σ-complete* provided that for every ascending (descending) sequence $x_1 \leq x_2 \leq \cdots$ ($x_1 \geq x_2 \geq \cdots$) in E which is bounded above (below) in E has a supremum (infimum) in E.

An *effect-tribe* on a set $\Omega \neq \emptyset$ is any system $\mathcal{T} \subseteq [0,1]^\Omega$ such that (i) $1 \in \mathcal{T}$, (ii) if $f \in \mathcal{T}$, then $1-f \in \mathcal{T}$, (iii) if $f,g \in \mathcal{T}$, $f \leq 1-g$, then $f+g \in \mathcal{T}$, and (iv) for any sequence $\{f_n\}$ of elements of $\mathcal{T}$ such that $f_n \nearrow f$ (pointwise), then $f \in \mathcal{T}$, i.e. if $f_n(\omega) \nearrow f(\omega)$ for every $\omega \in \Omega$, then $f \in \mathcal{T}$. It is evident that any effect-tribe is a monotone σ-complete effect algebra.

Now let Ω be a compact Hausdorff space. Then $\mathrm{C}(\Omega)$, the system of all real-valued continuous functions on Ω, is an ℓ-group (we recall that, for $f,g \in \mathrm{C}(\Omega)$, $f \leq g$ iff $f(x) \leq g(x)$ for any $x \in \Omega$), and it is Dedekind σ-complete iff Ω is basically disconnected (the closure of every open F_σ subset of Ω is open), see [Goo, Lem 9.1]. In such a case,

$$C_1(\Omega) := \Gamma(\mathrm{C}(\Omega), 1_\Omega)$$

is a σ-complete MV-algebra with respect to the MV-operations that are defined by points.

Let Ω be a convex subset of a real vector space V. A point $x \in \Omega$ is said to be *extreme* if from $x = \lambda x_1 + (1-\lambda)x_2$, where $x_1, x_2 \in \Omega$ and $0 < \lambda < 1$ we have $x = x_1 = x_2$. By $\partial_e \Omega$ we denote the set of extreme points of Ω.

Let Ω and Ω_1 be convex spaces. A mapping $f:\ \Omega \to \Omega_1$ is said to be *affine* if, for all $x,y \in \Omega$ and any $\lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) = \lambda f(x) + (1-\lambda) f(y)$.

Given a compact convex set $\Omega \neq \emptyset$ in a topological vector space, we denote by $\mathrm{Aff}(\Omega)$ the collection of all real-valued affine continuous functions on Ω. Of course, $\mathrm{Aff}(\Omega)$ is a po-subgroup of the po-group $\mathrm{C}(\Omega)$ of all continuous real-valued functions on Ω, hence it is an Archimedean unital po-group with the strong unit 1.

We recall that a *convex cone* in a real linear space V is any subset C of V such that (i) $0 \in C$, (ii) if $x_1, x_2 \in C$, then $\alpha_1 x_1 + \alpha_2 x_2 \in C$ for any $\alpha_1, \alpha_2 \in \mathbb{R}^+$. A *strict cone* is any convex cone C such that $C \cap -C = \{0\}$, where $-C = \{-x:\ x \in C\}$. A *base* for a convex cone C is any convex subset Ω of C such that every non-zero element $y \in C$ may be uniquely expressed in the form $y = \alpha x$ for some $\alpha \in \mathbb{R}^+$ and some $x \in \Omega$.

We recall that in view of [Goo, Prop 10.2], if Ω is a non-void convex subset of V, and if we set

$$C = \{\alpha x:\ \alpha \in \mathbb{R}^+,\ x \in \Omega\},$$

then C is a convex cone in V, and Ω is a base for C iff there is a linear functional f on V such that $f(\Omega) = 1$ iff Ω is contained in a hyperplane in V which misses the origin.

Any strict cone C of V defines a partial order $\leq_C$ via $x \leq_C y$ iff $y - x \in C$. It is clear that $C = \{x \in V:\ 0 \leq_C x\}$. A *lattice cone* is any strict convex cone C in V such that C is a lattice under $\leq_C$.

A *simplex* in a linear space V is any convex subset Ω of V that is affinely isomorphic to a base for a lattice cone in some real linear space. A simplex Ω in a locally convex Hausdorff space is said to be (i) *Choquet* if Ω is compact, and (ii) *Bauer* if Ω and $\partial_e \Omega$ are compact.

Choquet and Bauer simplices are very important for our study because (i) if E is with (RDP), then $\mathcal{S}(E)$ is a Choquet simplex, [Goo, Thm 10.17]; if Ω is a convex compact subset of a locally convex Hausdorff space, then (ii) Ω is a Choquet simplex iff $(\mathrm{Aff}(\Omega),1)$ is an interpolation po-group, [Goo, Thm 11.4], (iii) $\mathcal{S}(E)$ is a Bauer simplex whenever E is an MV-algebra (Example 4.2 below gives an effect algebra with (RDP) that is not MV-algebra), and (iv) Ω is a Bauer simplex iff $(\mathrm{Aff}(\Omega),1)$ is an ℓ-group, [Goo, Thm 11.21].

Let $\mathcal{S}(E) \neq \emptyset$. Given $a \in E$, we define the mapping $\hat a : \mathcal{S}(E) \to [0,1]$ by $\hat a(s) := s(a)$, $s \in \mathcal{S}(E)$, and let $\widehat{E} := \{\hat a : a \in A\}$. Then $\widehat{E}$ is an effect-clan if, e.g., $\mathcal{S}(E)$ is order determining (see Remark 2.3 and Proposition 4.1; in such a case, E and $\hat E$ are isomorphic), or E is an MV-algebra, or E is a monotone σ-complete effect algebra with (RDP), see Theorem 5.2 below, and in these cases the natural mapping $\psi(a) := \hat a$, $(a \in E)$ is a homomorphism of E onto $\widehat{E}$.

In general, $\widehat{E}$ is not necessarily an effect-clan:

Example 4.2. *There is an effect algebra with* (RDP) *such that* $\widehat{E}$ *is not an effect-clan.*

Proof. Let $\mathbb{Q}$ be the set of all rational numbers and let $G = \mathbb{Q}\times\mathbb{Q}$ be ordered by the strict ordering, i.e. $(g_1,g_2) \le (h_1,h_2)$ iff $g_1 < h_1$ and $g_2 < h_2$ or $g_1 = h_1$ and $g_2 = h_2$. If we set $u = (1,1)$, (G,u) is a unital po-group with interpolation.

If we set $s_0(g,h) := h$ and $s_1(g,h) := g$, then s_0 and s_1 are states on (G,u). We claim $\partial_e\mathcal{S}(G,u) = \{s_0,s_1\}$.

Let $s \in \partial_e\mathcal{S}(G,u)$. Since $s(1,1) = 1$, we have $1 = s(\frac{k}{k},\frac{k}{k}) = ks(\frac{1}{k},\frac{1}{k})$, i.e., $s(\frac{1}{k},\frac{1}{k}) = \frac{1}{k}$ for any integer $k \ge 1$. Hence, $s(g,g) = g$ for any $g \in \mathbb{Q}^+$ and consequently for any $g \in \mathbb{Q}$.

Now let $g < h$ be two rational numbers. Take a sequence of rational numbers $\{\delta_i\} \searrow 0$. Then

$$\begin{aligned} s(g-\delta_i, g-\delta_i) \le \quad & s(g,h) \quad \le s(h+\delta_i, h+\delta_i),\\ g-\delta_i \le \quad & s(g,h) \quad \le h-\delta_i, \end{aligned}$$

so that

$$g \le s(g,h) \le h.$$

In a similar way, $g \le s(h,g) \le h$. Therefore, $1 = s(1,1) = s(1,0) + s(0,1)$ and $\lambda := s(1,0)$ and $1-\lambda = s(0,1)$ are positive numbers.

If $\lambda = 0$, then $s(1/k,0) = 0$ for each integer $k \ge 1$, so that $s(g,0) = 0$ for any $g \in \mathbb{Q}^+$ and for any $g \in \mathbb{Q}$. Hence, $s(g,h) = s(0,h) = h = s_0(g,h)$. In a similar way, if $\lambda = 1$, then $s(g,h) = g = s_1(g,h)$. If $0 < \lambda < 1$, we define $s_0'(g,h) = s(0,h)/\lambda$ and $s_1'(g,h) := s(g,0)/(1-\lambda)$, s_0' and s_1' are states on (G,u) such that $s_0' = s_0$ and $s_1' = s_1$ and $s = s_\lambda := \lambda s_0 + (1-\lambda)s_1$, $\lambda \in [0,1]$, so that $\partial_e\mathcal{S}(G,u) = \{s_0,s_1\}$.

Let us set $E = \Gamma(G,u)$. Then E is an effect algebra with (RDP), and $\mathcal{S}(E) = \{s_\lambda' : \lambda \in [0,1]\}$ where s_λ' is the restriction of s_λ to E. We next define $\widehat{E} = \{\hat a : a \in E\}$. We assert that this $\widehat{E}$ is not an effect-clan.

Indeed, let $a = (0.3,0.3)$, $b = (0.7,0.4)$. Then $\hat a(s_\lambda') = 0.3\lambda + 0.3(1-\lambda) = 0.3 \le 1 - \hat b(s_\lambda') = 0.3\lambda + 0.6(1-\lambda)$ for any $\lambda \in [0,1]$, but there is no $c \in E$ such that $\hat c = \hat a + \hat b$ because $\hat a(s_0') + \hat b(s_0') = 1$ and $\hat a(s_1') + \hat b(s_1') = 0.7$. $\square$

We note that the former example shows that $\mathcal{S}(E)$ is a Bauer simplex that is not order determining but it is *separating*, i.e. $s(a)=s(b)$ for any state s on E implies $a=b$. In addition, (G,u) is not Archimedean, see Proposition 4.1.

Let us define

$$A(\mathcal{S}(E)) := \Gamma(\mathrm{Aff}(\mathcal{S}(E)),1).$$

Hence, $A(\mathcal{S}(E))$ is an effect algebra with (RDP) whenever $\mathcal{S}(E)$ is a Choquet simplex, in particular E satisfies (RDP), and E can be converted into an MV-algebra when $\mathcal{S}(E)$ is a Bauer simplex.

Nevertheless not every state space $\mathcal{S}(E)$ is a Bauer simplex, we recall that according to a delicate result of Choquet [Alf, Thm I.5.13], $\partial_e\mathcal{S}(E)$ is always a Baire space in the relative topology induced by the topology of $\mathcal{S}(E)$, i.e. the Baire Category Theorem holds for $\partial_e\mathcal{S}(E)$.

The following lemma [BSW, Lem 7], [Wri, Cor 3] will play a crucial role in our investigation for the Loomis–Sikorski Theorem.

Lemma 4.3. *Let $\{a_i\}$ be a monotone descending sequence of nonnegative functions in $\mathrm{Aff}(\Omega)$, where Ω is a convex compact set and let $a(x)=\lim_i a_i(x)$ for any $x\in\Omega$. Then $\bigwedge_i a_i = 0$ in $\mathrm{Aff}(\Omega)$ if and only if $\{x\in\partial_e\Omega:\ a(x)>0\}$ is a meager subset in the relative topology of $\partial_e\Omega$.*

5. Characterization of n-State-Operators on Effect Algebras

We remind the reader that a state s on E is *discrete* if there is an integer $n\ge 1$ such that $s(E)\subseteq\{0,1/n,\dots,n/n\}$.

Proposition 5.1. *Let E be an effect algebra with $\mathcal{S}(E)\neq\emptyset$ and let τ be an n-state-operator on E. Then there is an affine continuous function $g:\mathcal{S}(E)\to\mathcal{S}(E)$ such that $g^n=g$, and $g(s)(E)\subseteq s(E)$ for any discrete state $s\in\mathcal{S}(E)$.*

Let

$$A(E)=\{f\in\Gamma(\mathrm{Aff}(\mathcal{S}(E)),1): f(s)\in s(E) \text{ for all discrete } s\in\partial_e\mathcal{S}(E)\}. \tag{4.1}$$

Then $A(E)$ is an effect-clan and the mapping $\tau_g: A(E)\to A(E)$ defined by $\tau_g(f)=f\circ g$, $f\in A(E)$, is an n-state-operator on $A(E)$.

Suppose that $\widehat{E}$ is an effect algebra. If we define $\widehat{\tau}$ as a mapping from $\widehat{E}$ into itself such that $\widehat{\tau}(\widehat{a}) := \widehat{\tau(a)}$ $(a\in E)$, then $\widehat{\tau}$ is a well-defined n-state-operator on $\widehat{E}$ that is the restriction of τ_g.

Conversely, if $g:\mathcal{S}(E)\to\mathcal{S}(E)$ is an arbitrary affine and continuous function such that $g^n=g$, and $g(s)(E)\subseteq s(E)$ for any discrete state $s\in\mathcal{S}(E)$, then the mapping $\tau_g: A(E)\to A(E)$, defined by $\tau_g(f):=f\circ g$, $f\in A(E)$, is an n-state-operator.

Proof. If $s\in\mathcal{S}(E)$, then $s\circ\tau\in\mathcal{S}(E)$. Therefore, the mapping $g:\mathcal{S}(E)\to\mathcal{S}(E)$ defined by $g(s)=s\circ\tau$, $s\in\mathcal{S}(E)$, is a well-defined mapping and affine.

Moreover, g is continuous because if $s_\alpha\to s$, then we have $\lim_\alpha g(s_\alpha)(a) = \lim_\alpha s_\alpha(\tau(a)) = s(\tau(a)) = g(s)(a)$ for any $a\in E$.

¿From the construction of g we have $g^n=g$ because if $n=1$, this is clear and if $n\ge 2$, $g^n(s)=g^{n-1}(g(s))=g^{n-1}(s\circ\tau)=s\circ\tau^n=s\circ\tau=g(s)$.

Let s be a discrete state on E. Then $s(E)\subseteq\{0,1/n,\dots,n/n\}$ for some $n\ge 1$ and whence $s(\tau(E))\subseteq\{0,1/n,\dots,n/n\}$.

It is clear that $A(E)$ is an effect-clan. Take $f \in A(E)$. Then f is a continuous function taking values in the interval $[0,1]$. To verify that $\tau_g(f) \in A(E)$ we have to show that $\tau_g(f)(s) \in s(E)$ for any discrete extremal state s on E. Check: $\tau_g(f)(s) = f(g(s)) = f(s \circ \tau) \in (s \circ \tau)(E)) \subseteq s(E)$ due to the just above proved statement. Hence, $\tau_g(f)$ is again an element of $A(E)$. It is easy to verify that τ_g is an n-potent endomorphism from $A(E)$ into itself.

Now we show that $\widehat{\tau}$ is a well-defined operator on $\widehat{E}$. Assume $\hat{a} = \hat{b}$. This means $s(a) = s(b)$ for any $s \in \mathcal{S}(E)$. Hence, $s(\tau(a)) = g(s)(a) = g(s)(b) = s(\tau(b))$, so that $\widehat{\tau(a)} = \widehat{\tau(b)}$ and finally $\widehat{\tau}(\hat{a}) = \hat{a} \circ g = \hat{b} \circ g = \widehat{\tau}(\hat{b})$. Since $\widehat{E}$ is a subalgebra of $A(E)$, $\widehat{\tau}$ is the restriction of τ_g.

Finally, let $g : \mathcal{S}(E) \to \mathcal{S}(E)$ be an affine and continuous function such that $g^n = g$, $g(s)(E) \subseteq s(E)$ for any discrete state $s \in \mathcal{S}(E)$. If $f \in A(E)$, then $\tau_g(f) := f \circ g \in \mathrm{Aff}(\mathcal{S}(E))$. If s is an extremal discrete state, then $f(g(s)) \in g(s)(E) \subseteq s(E)$ for any discrete state $s \in \partial_e \mathcal{S}(E)$ so that $\tau_g(f) \in A(E)$ and τ_g is an n-state-operator on $A(E)$. $\square$

The following principal representation theorem for monotone σ-complete effect algebras with (RDP) follows from [Goo, Cor 16.15] and using the Ravindran representation theorem [Rav], see also Remark 2.2.

Theorem 5.2. *Let E be a nontrivial monotone σ-complete effect algebra with* (RDP)*. Then E is isomorphic with $A(E)$ defined by* (4.1) *and $\widehat{E} = A(E)$.*

We say that an effect algebra E is *weakly divisible*, if given an integer $n \geq 1$, there is an element $v \in E$ such that $n \cdot v := v + \cdots + v = 1$. In such a case, E has no extremal discrete state. We notice that according to (4.1), if E is a weakly divisible effect algebra that is monotone σ-complete, it has no discrete extremal state, therefore, E is *divisible*, that is, given $a \in E$ and $n \geq 1$, there is an element $v \in E$ such that $n \cdot v = a$. Consequently, for monotone σ-complete effect algebras, E, with (RDP) the notions of weak divisibility and divisibility, as well as the property that E admits no discrete (extremal) state, coincide.

We say that an n-state-operator τ on an effect algebra E is *monotone σ-complete* if whenever $a_i \nearrow a$, that is $a_i \leq a_{i+1}$ for any $i \geq 1$ and $a = \bigvee_i a_i$, then $\tau(a) = \bigvee_i \tau(a_i)$. We recall that if τ is a monotone σ-complete n-state-morphism-operator, then it preserves all existing countable suprema and infima existing in E, and we call it a *σ-complete n-state-morphism-operator.*

Let $f : \mathcal{S}(E) \to [0,1]$ be any function; we set $N(f) := \{s \in \partial_e \mathcal{S}(E) : f(s) \neq 0\}$.

Now we present a characterization of σ-complete n-state-operators on monotone σ-complete effect algebras with (RDP).

Theorem 5.3. *Let τ be a monotone σ-complete n-state-operator on a monotone σ-complete effect algebra E with* (RDP)*. Then there is an affine continuous function g defined on $\mathcal{S}(E)$ into itself such that $g^n = g$, $g(s)(E) \subseteq s(E)$ for any discrete extremal state s on E and $\widehat{\tau}(\hat{a}) = \hat{a} \circ g$, $a \in E$.*

Conversely, let g be an affine continuous function on $\mathcal{S}(E)$ into itself such that $g^n = g$, $g(s)(E) \subseteq s(E)$ for any discrete extremal state s. Then the mapping τ_g defined on $\widehat{E}$ by $\tau_g(\hat{a}) := \hat{a} \circ g$, $a \in E$, is a monotone σ-complete n-state-operator on $\widehat{E}$.

In addition, if $\tilde{\tau}_g$ is defined on E via $\tilde{\tau}_g(a) = \tau_g(\hat{a})$, $a \in E$, then $\tilde{\tau}_g$ is a monotone σ-complete n-state-operator on E, and $g(s) = s \circ \tilde{\tau}_g$, $s \in \mathcal{S}(E)$.

Proof. Since E is monotone σ-complete, due to Theorem 5.2, E is isomorphic to $A(E)$ as defined by (4.1). By Proposition 5.1, there is an affine and continuous function $g : \mathcal{S}(E) \to \mathcal{S}(E)$ such that $g^n = g$ and $g(s)(E) \subseteq S(E)$ for any discrete extremal state s on E, and the mapping $\tau_g : A(E) \to A(E)$ defined by $\tau_g(f) := f \circ g$, $f \in A(E)$, is an n-state-operator on $A(E)$.

In what follows, we show that τ_g is monotone σ-complete.

Assume that $a = \bigvee_i a_i$, for $a_i \nearrow a$, or equivalently, $\hat a = \bigvee_i \hat a_i$. Then $\hat a_i \circ g \le \hat a_{i+1} \circ g \le \hat a \circ g$.

If $a_0(s) = \lim_i \hat a_i(s)$, $s \in \mathcal{S}(E)$, i.e. a_0 is a point limit of continuous functions on a compact Hausdorff space, due to Lemma 4.3, the set $N(a_0 - \hat a)$ is a meager set. Similarly, $N(\hat a \circ g - a_0 \circ g)$ is a meager set. If $h = \bigvee_i \hat a_i \circ g$, then $h \le \hat a \circ g$. Since $N(h - \hat a \circ g) \subseteq N(h - a_0 \circ g) \cup N(a_0 \circ g - \hat a \circ g)$, this yields that $N(h - \hat a \circ g)$ is a meager set. Due to the Baire Category Theorem that says that no non-empty open set of a compact Hausdorff space can be a meager set, we have $N(h - \hat a \circ g) = \emptyset$, that is $h = \hat a \circ g$.

Finally, let $a \in E$ and $s \in \mathcal{S}(E)$. Then $(s \circ \tilde\tau_g)(a) = s(\tilde\tau_g(a)) = s(\tau_g(\hat a)) = \hat a(g(s)) = g(s)(a)$, that is $g(s) = s \circ \tilde\tau_g$ for any $s \in \mathcal{S}(E)$. □

6. Loomis–Sikorski Theorem for n-State Effect Algebras

In the present section we will formulate and prove the first main result of the paper.

Let E be a monotone σ-complete effect algebra with (RDP). By Theorem 5.2, $\widehat E = A(E)$ but $\widehat E$ is not necessarily an effect-tribe. Let $\mathcal{T}(E)$ be the effect-tribe of functions from $[0,1]^{\mathcal{S}(E)}$ generated by $\widehat E = A(E)$.

Proposition 6.1. *Let E be a monotone σ-complete effect algebra with* (RDP) *and let g be an affine continuous function on $\mathcal{S}(E)$ into itself such that $g^n = g$ and $g(s) \in s(E)$ for any discrete $s \in \partial_e \mathcal{S}(E)$. Then the operator $\mathcal{T}_g$ defined on $\mathcal{T}(E)$ by $\mathcal{T}_g(f) = f \circ g$, $f \in \mathcal{T}(E)$, is a monotone σ-complete n-state-operator on $\mathcal{T}(E)$ and is the unique extension of the monotone σ-complete n-state-operator τ_g on $A(E)$ defined by $\tau_g(f) = f \circ g$, $f \in A(E)$.*

Proof. First of all we show that $\mathcal{T}_g$ is a well-defined operator on $\mathcal{T}(E)$, that is, if $f \in \mathcal{T}(E)$, then $f \circ g \in \mathcal{T}(E)$. Let $\mathcal{T}'$ be the set of all $f \in \mathcal{T}(E)$ such that $f \circ g \in \mathcal{T}(E)$. Then $\mathcal{T}'$ contains $A(E) = \widehat E$ and if $f \in \mathcal{T}'$, then $1 - f \in \mathcal{T}'$. Now let $f_1, f_2 \in \mathcal{T}'$ be such that $f_1 \le f_2'$, then $f_1 + f_2$ belongs to $\mathcal{T}'$. Hence, if $\{f_i\}$ is a sequence of monotone functions from $\mathcal{T}'$, then, for $f = \lim_i f_i$, we have $f \circ g = \lim_i f_i \circ g \in \mathcal{T}'$. This implies that $\mathcal{T}'$ is an effect-tribe generated by $A(E)$, consequently, $\mathcal{T}' = \mathcal{T}(E)$ and $\mathcal{T}_g$ is a monotone σ-complete n-state-operator on $\mathcal{T}(E)$ that is an extension of τ_g.

Now if τ is any monotone σ-complete n-state-operator on $\mathcal{T}(E)$ that is an extension of τ_g, then again the set of elements $f \in \mathcal{T}(E)$ such that $\tau(f) = \mathcal{T}_g(f)$ is a tribe containing $A(E)$. Thus, it has to be $\mathcal{T}(E)$ and $\tau = \mathcal{T}_g$. □

Let (E_1, τ_1) and (E_2, τ_2) be n-state effect algebras. A homomorphism $h : E_1 \to E_2$ is said to be a *state-homomorphism* if $h \circ \tau_1 = \tau_2 \circ h$. Similarly, we define a *monotone σ-complete state-homomorphism* if (E_1, τ_1) and (E_2, τ_2) are monotone σ-complete state effect algebras and h is a state-homomorphism such that if $a_i \nearrow a$, then $h(a_i) \nearrow h(a)$.

We now generalize the Loomis–Sikorski Theorem for monotone σ-complete n-state effect algebras.

Theorem 6.2 (Loomis–Sikorski Theorem). *Let (E,τ) be a monotone σ-complete n-state effect algebra with* (RDP). *Then there are a monotone σ-complete n-state effect algebra $(\mathcal{T},\mathcal{T}_g)$, where $\mathcal{T}$ is an effect-tribe of functions from $[0,1]^\Omega$ satisfying* (RDP)*, a function $g:\Omega\to\Omega$ such that $g^n=g$ and $f\circ g\in\mathcal{T}$ for any $f\in\mathcal{T}$, such that $\mathcal{T}_g(f):=f\circ g$, $f\in\mathcal{T}$, is a monotone σ-complete n-state-operator on $\mathcal{T}$. Moreover, there is a monotone σ-complete state-homomorphism h from $\mathcal{T}$ onto E such that $h\circ\mathcal{T}_g=\tau\circ h$.*

Proof. Let E be a monotone σ-complete effect algebra with a monotone σ-complete n-state-operator τ. We isomorphically embed E onto $\widehat{E}$. We set $\Omega=\mathcal{S}(E)$, then Ω is a compact Hausdorff topological space and $\widehat{E}=A(E)$ by Theorem 5.2. Let $\mathcal{T}(E)$ be the effect-tribe of functions from $[0,1]^\Omega$ that is generated by $\widehat{E}$. According to Theorem 5.3, the function $g:\mathcal{S}(E)\to\mathcal{S}(E)$ defined by $g(s)=s\circ\tau$, $s\in\mathcal{S}(E)$, is continuous and $g^n=g$. The mapping $\mathcal{T}_g:\mathcal{T}(E)\to\mathcal{T}(E)$ defined by $\mathcal{T}_g(f)=f\circ g$, $f\in\mathcal{T}(E)$, is by Proposition 6.1 a monotone σ-complete n-state-operator on $\mathcal{T}(E)$ and by the same Proposition, it is a unique extension of the monotone σ-complete n-state-operator τ_g on $\mathcal{E}$ defined by $\tau_g(\hat a)=\hat a\circ g$, $a\in E$.

Let us denote by $\mathcal{T}$ the class of all functions $f\in[0,1]^{\mathcal{S}(E)}$ such that there is an element $a\in E$ with a meager set $N(f-\hat a):=\{s\in\partial_e\mathcal{S}(E): f(s)\neq\hat a(s)\}$; in which case we write $f\sim a$.

If a_1 and a_2 are two elements of E such that $f\sim a_1$ and $f\sim a_2$, then $N(\hat a_1-\hat a_2)\subseteq N(f-\hat a_1)\cup N(f-\hat a_2)$ proving that $N(\hat a_1-\hat a_2)$ is a meager set. $\hat a_1$ and $\hat a_2$ are continuous functions, and due to the Baire Category Theorem, we have $\hat a_1=\hat a_2$.

Therefore, the mapping $h:\mathcal{T}\to E$ defined by $h(f)=a$ if $f\sim a$ is a well-defined mapping.

In what follows, we show that $\mathcal{T}$ is an effect-clan with (RDP) such that $\mathcal{T}=\mathcal{T}(E)$ and h is a monotone σ-complete n-state-homomorphism on $A(E)$.

Let f_1 and f_2 be two functions from $\mathcal{T}$ with $f_1\le f_2$. Choose $b_1,b_2\in E$ such that $f_i\sim b_i$ for $i=1,2$. We assert that $b_1\le b_2$.

Indeed, we have $\{s\in\partial_e\mathcal{S}(E):\ 0<\hat b_1(s)-\hat b_2(s)\}\subseteq N(f_1-\hat b_1)\cup N(f_2-\hat b_2)$.

The Baire Category Theorem applied to $\partial_e\mathcal{S}(E)$ implies that no nonempty open set of $\partial_e\mathcal{S}(E)$ can be a meager set, whence $s(b_1)\le s(b_2)$ for any $s\in\partial_e\mathcal{S}(E)$, and consequently $\hat s(b_2-b_1)\ge 0$ for any $s\in\mathcal{S}(E)$. Because our $\mathrm{Aff}(\partial_e\mathcal{S}(E),1)$ is Archimedean, this yields that the set $\partial_e\mathcal{S}(E)$ is order determining that entails $b_1\le b_2$.

It is clear that the set $\mathcal{T}$ is closed under the formation of complements $f\mapsto 1-f$, and it contains $\{\hat a:\ a\in E\}$.

If $f,g\in\mathcal{T}$, $f\le 1-g$, and $N(f-\hat a),N(g-\hat b)$ are meager subsets of $\partial_e\mathcal{S}(E)$, then $a\le b'$, so that $a+b\in E$. Hence, $N(f+g-\widehat{(a+b)})$ is meager, i.e., $f+g\in\mathcal{T}$, and $\mathcal{T}$ is an effect-clan.

To show that $\mathcal{T}$ is an effect-tribe it is necessary to verify that $\mathcal{T}$ is closed under limits of non-decreasing sequences from $\mathcal{T}$. Let $\{f_i\}$ be a non-decreasing sequence of elements from $\mathcal{T}$. For any f_i, choose by (i) a unique $b_i\in E$ such that $N(f_i-\hat b_i)$ is a meager subset of $\partial_e\mathcal{S}(E)$.

Denote by $f = \lim_i f_i$, $b = \bigvee_{i=1}^\infty b_i$, $b_0 = \lim_i \hat b_i$. Then $b \in E$ and $\hat b \in \mathcal{T}$. We have

$$N(f - \hat b) \subseteq N(f - b_0) \cup N(\hat b - b_0)$$

and $N(f - b_0) = \{s \in \partial_e \mathcal{S}(E) :\ f(s) < b_0(s)\} \cup \{s \in \partial_e \mathcal{S}(E) :\ b_0(s) < f(s)\}$.

If $s \in \{s \in \partial_e \mathcal{S}(E) :\ f(s) < b_0(s)\}$, then there is an integer $i \geq 1$ such that $f(s) < \hat b_i(s) \leq b_0(s)$. Hence $f_i(s) \leq f(s) < \hat b_i(s) \leq b_0(s)$ so that $s \in \{s \in \partial_e \mathcal{S}(E) : f_i(s) < \hat b_i(s)\}$.

Similarly we can prove that if $s \in \{s \in \partial_e \mathcal{S}(E) :\ b_0(s) < f(s)\}$, then there is an integer $i \geq 1$ such that $s \in \{s \in \partial_e \mathcal{S}(E) :\ \hat b_i(s) < f_i(s)\}$.

These two cases yield

$$N(f - b_0) \subseteq \bigcup_{i=1}^\infty N(\hat b_i - f_i)$$

which is a meager subset of $\partial_e \mathcal{S}(E)$.

Since $b = \bigvee_{i=1}^\infty b_i$, we conclude that $\bigwedge_i b_i' = b'$ and $\bigwedge_i (b_i' - b') = 0$. Due to Lemma 4.3, we have that $N(\hat b - b_0)$ is a meager subset of $\partial_e \mathcal{S}(E)$. Hence, $f \in \mathcal{T}$.

Consequently, we have proved that $\mathcal{T}$ is an effect-tribe. Since $\mathcal{T}$ contains $A(E)$, we have $\mathcal{T} = \mathcal{T}(E)$.

Now we concentrate to show that $\mathcal{T}$ satisfies (RDP). Let $f \leq g + h$, i.e., $f(s) \leq g(s) + h(s)$ for any $s \in \mathcal{S}(E)$. By (ii) and (i) there are unique elements $a, b, c \in E$ such that $f \sim a$, $g \sim b$ and $h \sim c$, and $a \leq b + c$. Since E satisfies (RDP), there are two elements $b_1, c_1 \in E$ such that $a = b_1 + c_1$ and $b_1 \leq b$ and $c_1 \leq c$. Consequently, there is a meager set K of $\partial_e \mathcal{S}(E)$ such that $f(s) = \hat a(s)$, $g(s) = \hat b(s)$ and $h(s) = \hat c(s)$ for any $s \in \mathcal{S}(E) \setminus K$. For the functions $f|_K$, $g|_K$ and $h|_K$ defined on K (i.e. the system of all $[0,1]$-valued functions on K), (RDP) trivially holds, i.e., there are two functions g_0 and h_0 defined on K such that $f(s) = g_0(s) + h_0(s)$, $g_0(s) \leq g(s)$ and $h_0(s) \leq h(s)$ for any $s \in K$. Let us define functions g_1 and h_1 on $\mathcal{S}(E)$ by $g_1(s) = \hat b_1(s)$, $h_1(s) = \hat c_1(s)$ for any $s \in \mathcal{S}(E) \setminus K$ and $g_1(s) = g_0(s)$, $h_1(s) = h_0(s)$ for any $s \in K$. Then $f = g_1 + h_1$, $g_1 \leq g$ and $h_1 \leq h$, and $g_1 \sim a_1$ and $h_1 \sim b_1$, which proves that $\mathcal{T}$ has (RDP).

Due to the definition of $\mathcal{T}$ and the previous steps, the mapping $h :\ \mathcal{T} \to E$ defined by $h(f) = b$ iff $f \sim b$ is a surjective and monotone σ-complete homomorphism from $\mathcal{T}$ onto E.

Finally, let $f \in \mathcal{T}$ and $a \in E$ be such $h(f) = a$. Then $f \sim a$ so that $N(f - \hat a)$ is a meager set. Then $N(f \circ g - \hat a \circ g) = g^{-1}(N(f - \hat a))$ is also meager. By Theorem 5.3, we have $h(\mathcal{T}_g(f)) = \tau(a) = \tau(h(f))$. ☐

7. Stone Dualities and F-spaces

We present the second main result of the paper, Stone Dualities between some categories of effect algebras and F-spaces, Theorem 7.9.

We say that a topological space Ω is an *F-space* if any two disjoint open F_σ subsets of Ω have disjoint closures. For example, every basically disconnected compact Hausdorff space is an F-space.

We say that a poset E satisfies the *countable interpolation property* provided that for any two sequences $\{x_i\}$ and $\{y_j\}$ of elements of E such that $x_i \leq y_j$ for all i, j, there exists an element $z \in E$ such that $x_i \leq z \leq y_j$ for all i, j.

Theorem 7.1. *Let Ω be a Bauer simplex. The following statements are equivalent.*

(i) $\Gamma(\mathrm{Aff}(\Omega),1)$ *is a weakly divisible effect algebra with countable interpolation.*
(ii) $\Gamma(\mathrm{C}(\partial_e\Omega),1)$ *is a weakly divisible effect algebra with countable interpolation*
(iii) $\partial_e\Omega$ *is an F-space.*

Proof. It is clear that both $\Gamma(\mathrm{Aff}(\Omega),1)$ and $\Gamma(\mathrm{C}(\Omega),1)$ are weakly divisible as well as divisible effect algebras.

(i) $\Rightarrow$ (ii). Assume that $\{f_n\}$ and $\{g_m\}$ are two sequences of continuous functions from $\Gamma(\mathrm{C}(\partial_e\Omega),1)$ such that $f_n \le g_m$. Let $\tilde f_n$ and $\tilde g_m$ be unique extensions to affine continuous functions on Ω of f_n and g_m, respectively. Then $\tilde f_n \le \tilde g_m$ for all n,m. The countable interpolation on $\Gamma(\mathrm{Aff}(\Omega),1)$ yields that there is an affine function $h \in \Gamma(\mathrm{Aff}(\Omega),1)$ such that $\tilde f_n \le h \le \tilde g_m$ for all n,m. If h_0 is the restriction of h onto $\partial_e\Omega$, then $f_n \le h_0 \le g_m$ for all n,m so that $\Gamma(\mathrm{C}(\partial_e\Omega),1)$ satisfies countable interpolation.

(ii) $\Rightarrow$ (i). Since Ω is a Bauer simplex, and consequently a Choquet one, $\mathrm{Aff}(\Omega)$ is an interpolation group, [Goo, Thm 11.4]. Let $\{f_n\}$ and $\{g_m\}$ be two sequences of continuous affine functions from $\Gamma(\mathrm{Aff}(\Omega),1)$ such that $f_n \le g_m$ for each n,m. Since the functions are continuous, there is a continuous function $h \in \mathrm{C}(\partial_e\Omega)$ such that $f_n(x) \le h(x) \le g_m(x)$ for all n,m and $x \in \partial_e\Omega$. By the Tietze Theorem, [Alf, Prop II.3.13], h can be uniquely extended to an affine function $\tilde h$. Since $f_n(x) \le \tilde h(x) \le g_m(x)$ for all $x \in \partial_e\Omega$, by [Goo, Cor 5.20], this implies $f_n(x) \le \tilde h(x) \le g_m(x)$ for any $x \in \Omega$.

(ii) $\Leftrightarrow$ (iii) According to [See, Thm 1.1], a compact Hausdorff topological space K is an F-space iff $\mathrm{C}(K)$ satisfies countable interpolation. By [Goo, Prop 16.3], $\Gamma(\mathrm{C}(\partial_e\Omega),1)$ satisfies countable interpolation iff $(\mathrm{C}(\partial_e\Omega),1)$ satisfies countable interpolation. $\square$

Remark 7.2. Here it is necessary to point out that not every MV-algebra satisfying countable interpolation is σ-complete. Due to the Nakano Theorem [Goo, Cor 9.3], if Ω is a compact Hausdorff space, then the MV-algebra $\Gamma(\mathrm{C}(\Omega),1)$ is σ-complete iff Ω is basically disconnected. Every such a basically disconnected space Ω can be expressed as a union of two nonempty clopen subsets, so that Ω is not connected. But due to [GiHe] or [Goo, p. 280], there exists an F-space, Ω_0, that is connected, so that it is not basically disconnected. By [See], $\Gamma(\mathrm{C}(\Omega_0),1)$ is an MV-algebra that satisfies countable interpolation, but due to the Nakano Theorem it is not σ-complete.

If K is a compact Hausdorff topological space, let $\mathcal{B}(K)$ be the Borel σ-algebra of K generated by all open subsets of K. Let $\mathcal{M}_1^+(K)$ denote the set of all probability measures, that is, all positive regular σ-additive Borel measures μ on $\mathcal{B}(K)$. We recall that a Borel measure μ is called regular if

$$\inf\{\mu(O):\ Y\subseteq O,\ O \text{ open}\} = \mu(Y) = \sup\{\mu(C):\ C\subseteq Y,\ C \text{ closed}\}$$

for any $Y \in \mathcal{B}(K)$.

Let $x \in K$ and let δ_x be the Dirac measure concentrated at the point $x \in K$, i.e., $\delta_x(Y) = 1$ iff $x \in Y$, otherwise $\delta_x(Y) = 0$; then every Dirac measure is a regular Borel probability measure. Moreover, [Goo, Prop 5.24], the mapping

$$\epsilon : x \mapsto \delta_x \tag{7.1}$$

gives a homeomorphism of K onto $\partial_e\mathcal{M}_1^+(K)$.

Hence, if K is an F-space that is connected, see Remark 7.2, then $\Omega := \mathcal{M}_1^+(K)$ gives a Bauer simplex whose boundary $\partial_e\Omega$ is a connected compact Hausdorff F-space. Moreover, $\Gamma(\mathrm{Aff}(\Omega),1)$ gives by Theorem 7.1 a divisible lattice ordered effect algebra satisfying monotone interpolation and (RDP) that has an order determining system of states but $\Gamma(\mathrm{Aff}(\Omega),1)$ is not monotone σ-complete, see [DDL3, Thm 4.2].

Theorem 7.3. *Let E be an effect algebra with* (RDP) *and with countable interpolation such that E has an order determining system of states. Then E is isomorphic to $A(E)$, where $A(E)$ is defined by* (4.1)*, E is lattice ordered and $\partial_e\mathcal{S}(E)$ is an F-space.*

Proof. If E has an order determining system of states, $\mathcal{S}$, then $\mathcal{S}(E)$ is also order determining. Moreover, $E = \Gamma(G,u)$ for some interpolation unital po-group (G,u). In view of Remark 2.3, $\mathcal{S}(G,u)$ is also order determining, so that G is Archimedean. Applying [Goo, Thm 16.19(b)], we have that G is an ℓ-group so that E is a lattice. Due to [Goo, Thm 16.14], E is isomorphic with $A(E)$. Hence, $\mathcal{S}(E)$ is a Bauer simplex and by [Goo, Thm 16.22], $\partial_e\mathcal{S}(E)$ is an F-space. $\square$

Remark 7.4. Under the assumptions of Theorem 7.3, we see that E is in fact a semisimple MV-algebra (equivalently this means that $\mathcal{S}(E)$ is order determining) with countable interpolation whose boundary $\partial_e\mathcal{S}(E)$ is an F-space, and vice-versa. Every semisimple MV-algebra satisfying countable interpolation satisfies the condition of Theorem 7.3.

Let $n \ge 1$ be a fixed integer. Let $\mathcal{DSMEA}_n$ be the category of (weakly) divisible state-morphism effect algebras whose objects are couples (E,τ), where E is an effect algebra satisfying (RDP) and countable interpolation with an order determining system of states, and τ is an n-state-morphism-operator on E; and a morphism from (E_1,τ_1) to (E_2,τ_2) is any homomorphism $h : E_1 \to E_2$ that preserves all existing meets and joins in E_1 such that $h\circ\tau_1 = \tau_2\circ h$. We note that $\mathcal{DSMEA}_n$ is a category.

Let $\mathcal{BSF}_n$ be the category of Bauer simplices whose objects are pairs (Ω,g), where $\Omega \neq \emptyset$ is a Bauer simplex such that $\partial_e\Omega$ is an F-space, and $g : \Omega \to \Omega$ is an affine continuous function such that $g^n = g$, $g : \partial_e\Omega \to \partial_e\Omega$. Morphisms from (Ω_1,g_1) into (Ω_2,g_2) are continuous affine functions $p : \Omega_1 \to \Omega_2$ such that $p : \partial_e\Omega_1 \to \partial_e\Omega_2$ and $p\circ g_1 = g_2\circ p$. Then $\mathcal{BSF}_n$ is also a category.

Now we reformulate the substantial part of Proposition 5.1 for state-morphism-operators on lattice ordered effect algebras with (RDP) and with an ordering system of states.

Proposition 7.5. *Let τ be an n-state-morphism on a lattice ordered effect algebra satisfying* (RDP) *and countable interpolation and with an ordering system of states. Then τ satisfies* (ESP) *and there is an affine continuous function g from $\mathcal{S}(E)$ into itself such that it maps $\partial_e\mathcal{S}(E)$ into itself, $g^n = g$, $g(s)(E) \subseteq S(E)$ for any discrete state s on E. Moreover, the mapping $\tau_g : A(E) \to A(E)$ defined by $\tau_g(f) = f\circ g$, $f \in A(E)$, where $A(E)$ is defined by* (4.1)*, is an n-state-morphism-operator on $A(E)$. In addition, (E,τ) and $(A(E),\tau_g)$ are isomorphic n-state-morphism effect algebras.*

Proof. The mapping $\psi : a \mapsto \hat{a}$, defined by $\hat{a}(s) := s(a)$, $a \in E$, $(s \in \mathcal{S}(E))$ is by Theorem 7.3 an isomorphism from E into the effect-clan $A(E)$ defined by (4.1). Since E is in fact an MV-algebra, τ satisfies (ESP) property, and by Proposition

5.1, the mapping $g : \mathcal{S}(E) \to \mathcal{S}(E)$, defined by $g(s) := s \circ \tau$, $s \in \mathcal{S}(E)$, is affine and continuous, $g^n = g$, and it maps any extremal state on E into an extremal state. Moreover, $g(s)(E) \subseteq s(E)$ for any discrete extremal state s.

The mapping ψ is an isomorphism of effect algebras. We show that it also preserves all existing meets and joins in E. We know already from Theorem 7.3 and Remark 7.4 that E is in fact an MV-algebra. Then for any discrete state s on E we have $\psi(a \wedge b)(s) = s(a \wedge b) = \min\{s(a), s(b)\} = \min\{\psi(a)(s), \psi(b)(s)\}$ due to basic properties of extremal states on MV-algebras. Hence $(\psi(a) \wedge \psi(b))(s) = \psi(a \wedge b)(s)$ for any discrete state s. Since $\psi(a) \wedge \psi(b) \in A(E)$, we have that $(\psi(a) \wedge \psi(b))(s) = \psi(a \wedge b)(s)$ for any state s on E.

Therefore, the mapping τ_g is a well-defined state-operator on $A(E)$. For all $f_1, f_2 \in A(E)$ we have $(f_1 \wedge f_2)(s) = \min\{f_1(s), f_2(s)\}$ for any $s \in \mathcal{S}(E)$. Hence, $(\tau_g(f_1 \wedge f_2))(s) = (f_1 \wedge f_2)(g(s)) = \min\{f_1(g(s)), f_2(g(s))\} = (\tau_g(f_1) \wedge \tau_g(f_2))(s)$. So that τ_g is an n-state-morphism-operator on $A(E)$.

Now it is easy to verify that $\psi \circ \tau = \tau_g \circ \psi$ proving that (E, τ) and $(A(E), \tau_g)$ are isomorphic n-state-morphism effect algebras because ψ preserves all existing meets and joins in E. $\square$

Define a morphism $S : \mathcal{DSMEA}_n \to \mathcal{BSF}_n$ by $S(E, \tau) = (\mathcal{S}(E), g)$, where g is an affine continuous function from $\mathcal{S}(E) \to \mathcal{S}(E)$ such that $g(s) = s \circ \tau$, $s \in \mathcal{S}(E)$, $g^n = g$ that is guaranteed by Proposition 7.5.

Proposition 7.6. *The function $S : \mathcal{DSMEA}_n \to \mathcal{BSF}_n$ defined by $S(E, \tau) = (\mathcal{S}(E), g)$ is a contravariant functor from $\mathcal{DSMEA}_n$ into $\mathcal{BSF}_n$.*

Proof. Let (E, τ) be an object from $\mathcal{DSMEA}_n$. By Theorem 7.3, (E, τ) is isomorphic with $(A(E), \tau_g)$, where $A(E)$ is defined by (4.1) and g is an affine continuous function on $\Omega := \mathcal{S}(E)$ into itself such that it maps $\partial_e\mathcal{S}(E)$ into itself, $g^n = g$ and $g(s) := s \circ \tau$ for any $s \in \mathcal{S}(A)$.

Let h be any morphism from (E, τ) into (E', τ'). Define a mapping $S(h) : \mathcal{S}(E') \to \mathcal{S}(E)$ by $S(h)(s') := s' \circ h$, $s' \in \mathcal{S}(E')$. Then $S(h)$ is affine, continuous and $g \circ S(h) = S(h) \circ g'$. Indeed, let $s' \in \mathcal{S}(E')$. Then $S(h) \circ g' \circ s' = (g' \circ s') \circ h = g' \circ (s' \circ h) = s' \circ h \circ \tau = s' \circ \tau' \circ h = S(h) \circ s' \circ \tau' = S(h) \circ g'$. $\square$

Given an convex compact Hausdorff topological space $\Omega \neq \emptyset$, let

$$E(\Omega) := \Gamma(\mathrm{Aff}(\Omega), 1). \tag{7.2}$$

Then $E(\Omega)$ is a weakly divisible effect algebra with a determining system of states.

Define a morphism $T : \mathcal{BSF}_n \to \mathcal{DSMEA}_n$ via $T(\Omega, g) = (E(\Omega), \tau_g)$, where $E(\Omega) = \Gamma(\mathrm{Aff}(\Omega), 1)$, $\tau_g(f) := f \circ g$, $f \in E(\Omega)$, and if $p : (\Omega, g) \to (\Omega', g')$, then $T(p)(f) : E(\Omega') \to E(\Omega)$ is defined by $T(p)(f) := f \circ p$, $f \in E(\Omega')$.

Proposition 7.7. *The function $T : \mathcal{BSF}_n \to \mathcal{DSMEA}_n$ is a contravariant functor from $\mathcal{BSF}_n$ to $\mathcal{DSMEA}_n$.*

Proof. If (Ω, g) is an object from $\mathcal{DSMEA}_n$, then $E(\Omega)$ is a (weakly) divisible effect algebra satisfying (RDP) and with an ordering system of states. In addition, by Theorem 7.1, $E(\Omega)$ satisfies countable interpolation. The mapping $\tau_g(f) := f \circ g$, $f \in E(\Omega)$, is by Proposition 7.5 a state-morphism-operator on $E(\Omega)$. Therefore, $T(\Omega, g) = (E(\Omega), \tau_g) \in \mathcal{DSMEA}_n$.

Now let $p : (\Omega, g) \to (\Omega', g')$ be a morphism, i.e. an affine continuous function $p : \Omega \to \Omega'$ such that $p : \partial_e\Omega \to \partial_e\Omega'$ and $p \circ g = g' \circ p$. We assert $\tau_g \circ T(p) = T(p) \circ \tau_{g'}$.

Check: for any $f \in E(\Omega')$, we have $\tau_g \circ T(p) \circ f = \tau_g \circ (T(p) \circ f) = \tau_g \circ (f \circ p) = (f \circ p) \circ g = f \circ (p \circ g) = f \circ (g' \circ p) = (f \circ g') \circ p = T(p) \circ (f \circ g') = T(p) \circ (\tau_{g'} \circ f) = T(p) \circ \tau_{g'} \circ f$. □

Remark 7.8. It is worthy to remark that due to [Goo, Thm 7.1], if Ω is a compact convex subset of a locally convex Hausdorff space, then the evaluation mapping $p:\ \Omega \to \mathcal{S}(E(\Omega))$ defined by $p(x)(f) = f(x)$ for all $f \in E(\Omega)$ $(x \in \Omega)$ is an affine homeomorphism of Ω onto $\mathcal{S}(E(\Omega))$.

Theorem 7.9 (Stone Duality Theorem)**.** *The categories $\mathcal{BSF}_n$ and $\mathcal{DSMEA}_n$ are dual.*

Proof. We show that the conditions of [Mac, Thm IV.1] are fulfilled, i.e. $T \circ S(E,\tau) \cong (E,\tau)$ and $S \circ T(\Omega, g) \cong (\Omega, g)$ for all $(E,\tau) \in \mathcal{DSMEA}_n$ and $(\Omega, g) \in \mathcal{BSF}_n$.

(i) Propositions 7.6–7.7 entail that if $(E,\tau) \in \mathcal{DSMEA}_n$, then $T \circ S(E,\tau) = T(\mathcal{S}(E), g) = (E(\mathcal{S}(E)), \tau_g) \cong (E,\tau)$.

(ii) Now let (Ω, g) be any object from $\mathcal{BSF}_n$. By Remark 7.8, Ω and $\mathcal{S}(E(\Omega))$ are affinely homeomorphic under the evaluation mapping $p : \Omega \to \mathcal{S}(E(\Omega))$. We assert $p \circ g = g' \circ p$.

Let $x \in \Omega$ and $f \in E(\Omega)$ be arbitrary. Then $s = p(x)$ is a state from $\mathcal{S}(E(\Omega))$. The function $g' : \mathcal{S}(E(\Omega)) \to \mathcal{S}(E(\Omega))$ is defined by the property $g'(s) = s \circ \tau_g$. Since $g'(s) = g'(p(x))$, we get

$$\begin{aligned}(g' \circ p)(x)(f) &= g'(p(x))(f) = (g'(s))(f) = (s \circ \tau_g)(f)\\ &= p(x) \circ (\tau_g(f)) = p(x) \circ (f \circ g) = f(g(x)).\end{aligned}$$

On the other hand,

$$(p \circ g)(x)(f) = p(g(x))(f) = f(g(x))$$

that proves $p \circ g = g' \circ p$. Hence, the categories $\mathcal{BSF}_n$ and $\mathcal{DSMEA}_n$ are dual. □

References

[Alf] E.M. Alfsen, *"Compact Convex Sets and Boundary Integrals"*, Springer-Verlag, Berlin, 1971.

[BaWe] G. Barbieri, H. Weber, *Measures on clans and on MV-algebras*, in: Handbook of Measure Theory, E. Pap (Editor), Elsevier Science, Amsterdam, 2002, Vol. II, pp. 911–945.

[BCD] D. Buhagiar, E. Chetcuti, A. Dvurečenskij, *Loomis-Sikorski representation of monotone σ-complete effect algebras,* Fuzzy Sets and Systems **157** (2006), 683–690.

[BSW] L.J. Bunce, K. Saitô, and J.D.M. Wright, *A Dixmier-Schaefer-Zhang theorem for operator algebras,* Proc. Amer. Math. Soc. **127** (1999), 2975–2979.

[DiDv] Di Nola, A., Dvurečenskij, A.: *State-morphism MV algebras,* J. Pure Appl. Logic **161** (2009), 161–173. DOI: 10.1016/j.apal.F.05.003

[DDL1] A. Di Nola, A. Dvurečenskij, A. Lettieri, *On varieties of MV-algebras with internal states,* Inter. J. Approx. Reasoning. DOI: 10.1016/j.ijar.2010.01.017

[DDL2] A. Di Nola, A. Dvurečenskij, A. Lettieri, *On the Loomis–Sikorski theorem for MV-algebras with internal state,* submitted.

[DDL3] A. Di Nola, A. Dvurečenskij, A. Lettieri, *Stone duality type theorems for MV-algebras with internal state,* submitted.

[Dvu0] A. Dvurečenskij, *Measures and $\perp$-decomposable measures on effects of a Hilbert space,* Atti Sem. Mat. Fis. Univ. Modena **45** (1997), 259–288.

[Dvu1] A. Dvurečenskij, *Loomis–Sikorski theorem for σ-complete MV-algebras and ℓ-groups*, J. Austral. Math. Soc. Ser. A **68** (2000), 261–277.

[Dvu2] A. Dvurečenskij, *Perfect effect algebras are categorically equivalent with Abelian interpolation po-groups,* J. Austral. Math. Soc. **82** (2007), 183–207.

[DvPu] A. Dvurečenskij, S. Pulmannová, *"New Trends in Quantum Structures"*, Kluwer Acad. Publ., Dordrecht, Ister Science, Bratislava, 2000.

[FlMo] T. Flaminio, F. Montagna, *MV-algebras with internal states and probabilistic fuzzy logic*, Inter. J. Approx. Reasoning **50** (2009), 138–152.

[FoBe] D.J. Foulis, M.K. Bennett, *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[GiHe] L. Gillman, M. Henriksen, *Rings of continuous functions in which every finitely generated ideal is principal,* Trans. Amer. Math. Soc. **82** (1956), 366–391.

[Goo] K.R. Goodearl, *"Partially Ordered Abelian Groups with Interpolation",* Math. Surveys and Monographs No. 20, Amer. Math. Soc., Providence, Rhode Island, 1986.

[Loo] L.H. Loomis, *On the representation of σ-complete Boolean algebras*, Bull. Amer. Math. Soc. **53** (1947), 757–760.

[LuZa] W.A.J. Luxemburg, A.C. Zaanen, *"Riesz Spaces, I,"* North-Holland, Amsterdam, London, 1971.

[Mac] S. MacLane, *"Categories for the Working Mathematician",* Springer-Verlag, New York, 1971.

[Mun] D. Mundici, *Interpretation of AF C^*-algebras in Łukasiewicz sentential calculus,* J. Funct. Anal. **65** (1986), 15–63.

[Mun] D. Mundici, *Averaging the truth-value in Łukasiewicz logic*, Studia Logica **55** (1995), 113–127.

[Rav] K. Ravindran, *On a structure theory of effect algebras*, PhD theses, Kansas State Univ., Manhattan, Kansas, 1996.

[See] G.L. Seever, *Measures on F-spaces,* Trans. Amer. Math. Soc. **133** (1968), 267–280.

[Sik] R. Sikorski, *"Boolean Algebras",* Springer–Verlag, Berlin, Heidelberg, New York, 1964.

[Sto] M.H. Stone, *The theory of representations of Boolean algebras,* Trans. Amer. Math. Soc. **40** (1936), 37–111.

[Wri] J.D.M. Wright, *On approximating concave functions by convex functions,* Bull. London Math. Soc. **5** (1973), 221–222.