Mem. Fac. Sci. Eng. Shimane Univ.
Series B: Mathematical Science
39 (2006), pp. 1–13

CONNECTIVE SPACES

JOSEPH MUSCAT AND DAVID BUHAGIAR

Communicated by Takuo Miwa

(Received: November 7, 2005)

Abstract. A new category of connective spaces is defined, which includes topological spaces and simple graphs, and generalizes the concept of connectedness. Not every connective space has a compatible topology; those that do are characterized by compatible partial orders.

1. Connective Spaces

1.1. **Introduction.** As a topological concept connectedness is of somewhat different character than most other important properties, such as the covering properties, studied in the category TOP. Its aim is to topologically explain the intuitive notion of continuity of a point set. Roughly speaking, a connected space is one which cannot be represented as the sum of two pieces far from each other.

The contemporary definition of connectedness is that a topological space X is connected if it cannot be represented as the sum $X_1 \oplus X_2$ of two non-empty subspaces of X, that is it cannot be decomposed into two disjoint non-empty closed (open) subsets. This definition was introduced by Jordan in 1893 [4] for the class of compact subsets of the plane. The definition for general topological spaces was given by Hausdorff in 1914 [3] although the notion was given earlier by Lennes [7]. A systematic study of connected topological spaces was carried out by Hausdorff [3] and by Knaster and Kuratowski [5]. If a space is not connected then it is said to be disconnected. An excellent exposition of connected spaces is given in [6].

Another form of connectedness is path-connectedness. A topological space X is said to be path-connected if for any two points x and y in X there exists a continuous function f from the unit interval $[0, 1]$ to X such that $f(0) = x$ and $f(1) = y$ (This function is called a path from x to y). Every path-connected space is connected. Examples of connected spaces that are not path-connected include the extended long line. However, subsets of the real line $\mathbb{R}$ are connected if, and only if, they are path-connected; these subsets are the intervals of $\mathbb{R}$. Also, open subsets of $\mathbb{R}^n$ or $\mathbb{C}^n$ are connected if, and only if, they are path-connected. Additionally, connectedness and path-connectedness are the same for finite topological spaces.

A related notion is that of connectedness in simple graphs. A graph is said to be connected if there is a path consisting of edges from any point to any other point in the graph. A graph that is not connected is said to be disconnected.

The aim of this article is to axiomatize certain properties of connected sets and form a category of connective spaces as defined in the next section.

1.2. **Definition.**

Definition 1.1. A **connective space** is a set X together with a collection of subsets $\mathcal{C}$, such that the following axioms hold:

(i) $\forall C \subseteq \mathcal{C} \quad \bigcap C \neq \emptyset \implies \bigcup C \in \mathcal{C}$,
(ii) $\forall x \in X \quad \{x\} \in \mathcal{C}$,
(iii) Given any nonempty sets $A, B \in \mathcal{C}$ with $A \cup B \in \mathcal{C}$ then $\exists x \in A \cup B : \{x\} \cup A \in \mathcal{C}$ and $\{x\} \cup B \in \mathcal{C}$,
(iv) if $A, B, C_i \in \mathcal{C}$ are disjoint and $A \cup B \cup \bigcup_{i \in I} C_i \in \mathcal{C}$ then $\exists J \subseteq I,\ A \cup \bigcup_{j \in J} C_j \in \mathcal{C}$ and $B \cup \bigcup_{i \in I-J} C_i \in \mathcal{C}$.

The collection $\mathcal{C}$ is called the **connective structure** or **connectology** of X, and its elements are called the **connected sets** of X. One can add the axiom that the empty set is connected although this follows from (i). Connected sets with two elements are called *edges*. Spaces that satisfy (i) and (ii) only, will be called *c-spaces*, and the corresponding $\mathcal{C}$ a *c-structure*.

1.2.1. *Examples.*

- The real line $\mathbb{R}$ together with the intervals is a connective space.
- Connective structures are partially ordered by set inclusion $\mathcal{C}_1 \subseteq \mathcal{C}_2$; $\mathcal{C}_2$ is called *coarser/weaker* than $\mathcal{C}_1$ and the latter *finer/stronger* than the former. The weakest connectology is the power set, $\mathcal{P}(X)$, called the *totally connected* space on X; the strongest consists of the single points and $\emptyset$, and is called the *totally disconnected* space on X.
- Finite simple graphs are connective spaces on finite sets, with the connected sets being the 'edge-connected' subgraphs. It will be shown later that every connective spaces on a finite set is a graph.
- Topological spaces, with the connected sets defined as usual by the non-existence of non-trivial open partitions, are connective spaces. Axiom (iv) is an equivalent formulation of a theorem of Kuratowski [6]. Finer topologies (with more open sets) induce finer connectologies (with less connected sets).

  A topology is an additional structure on top of a connectology: later it will be shown that homeomorphic topological spaces must have 'isomorphic' connectologies, but the converse is false. For example, the totally disconnected connectology can arise from several non-homeomorphic topologies on the same set (e.g. $\mathbb{Q}$ and $\mathbb{N}$ have 'isomorphic' connectologies but different topologies.)
- There are connective spaces that are not topologizable. In fact there are graphs such as the pentagon C_5 (or any C_n with $n \geq 5$ odd), which do not admit any topology having the same connected sets.

- The cofinite connective space on a set has connected sets consisting of $\emptyset$, the singletons and the infinite subsets.
- The connected sets of X which contain a particular point $x \in X$, together with all the singletons, form a connective space, called the connective structure of X **rooted** at the point x.

1.3. Touching Sets.

Definition 1.2. A point x is said to **touch** a set A when there is a non-empty subset $C \subseteq A$ such that $\{x\} \cup C$ is connected. Subsets A and B are said to touch when there is a point $x \in A \cup B$ which touches both A and B.

The set of points touching a set A will be denoted by $t(A)$. The following are some trivial consequences:

Proposition 1.1.

(i) *$A \subseteq t(A)$;*
(ii) *When A is connected, x touches $A \iff \{x\} \cup A$ is connected;*
(iii) *For $A, B \neq \emptyset$ connected, $A \cup B$ connected $\iff A, B$ touch;*
(iv) *If A touches $B \subseteq C$ then A touches C;*
(v) *$A \subseteq B \implies t(A) \subseteq t(B)$;*
(vi) *$t(\bigcap_i A_i) \subseteq \bigcap_i t(A_i)$.*

Proposition 1.2. *There is a partition on X of maximally connected sets, called **components**. Every non-empty connected set is contained in a unique component. Components do not touch, and contain all points that touch them.*

Proof. The relation defined by $\{x, y\} \subseteq C$ for some $C \in \mathcal{C}$ is reflexive by axiom (ii), trivially symmetric and transitive by axiom (i). Moreover,

$$\{x, y\} \subseteq C \in \mathcal{C} \iff y \in \bigcup\{C \in \mathcal{C} : x \in C\},$$

so that the component of x is the union of all the connected sets containing x. This shows that it is maximally connected and that every connected set with a point x must be in its component.

If components C and D touch then their union $C \cup D$ would also be connected, and this is possible only when $C = D$. Finally, if x touches a component C, then $C \cup \{x\}$ is connected, so that $x \in C$ by maximality of components. □

Note that these components agree with the distinct definitions of "connected components" in graphs and in topological spaces.

Axiom (i) can be replaced by the following: *if $C_i, i \in I$ are non-empty connected sets and $\exists i_0 \in I$ such that C_{i_0} and C_i touch $\forall i \in I$, then $\bigcup_{i \in I} C_i$ is connected.* Simply note that the sets $C_i \cup C_0$ have non-empty intersection so that their union is connected. The following is also true: *a set C_0 is connected if, and only if, every cover of non-empty connected sets touching C_0 has connected union.* (Proof: for the converse, take the cover of touching components.)

Axiom (iv) is equivalent to the either of the following statements due to Kuratowski [6]:

(a) *If* $A \subseteq B$ *are both connected and* C *is maximally connected in* $B - A$*, then* $B - C$ *is connected;*

(b) *If* A_i*,* B_j *are finite families of connected sets with* $\bigcup_i A_i \cup \bigcup_j B_j$ *connected, then* $\exists i, j\ A_i \cup B_j$ *connected.* (The proof is by induction on the sizes of the families);

(c) *If* $A_i, 0 \leq i \leq N$*, is a finite family of connected sets with* $\bigcup_{i=0}^N A_i$ *connected then the* $A_i, 1 \leq i$*, can be rearranged so that* $A_0 \cup \bigcup_{i=1}^n A_i$ *is connected for* $n = 1, \ldots, N$*.*

Proposition (a) is a rewording of axiom (iv), while a careful reading of Kuratowski's proofs shows that (b) and (c) do not use strictly topological properties of connected sets, but only proposition (a). A particular case of (c) is the following:

$$A, B, C \in \mathcal{C}, \quad A \cup B \cup C \in \mathcal{C} \implies A \cup B \in \mathcal{C} \text{ OR } A \cup C \in \mathcal{C}. \tag{1}$$

Proposition 1.3. *Every finite connective space is a simple graph.*

Proof. Let $A = \{a_1, \ldots, a_n\}$ be a connective space. Assume, by induction, that every connective space with m elements is a graph for $m < n$. Let $C \subseteq A$ be any non-empty connected subset. We need to show that any two vertices in C are connected by a path of edges. Let a, b be any two vertices in C. The connective subspace $C - \{a\}$ is a graph, by induction, and applying statement (b) to the vertex a and the vertices of $C - \{a\}$, we deduce that $\{a, c\}$ is connected (an edge) for some $c \in C - \{a\}$. Hence C is a connected graph, and there must be path of edges joining a to b. Conversely, every edge-connected subset of A is connected, by repeated use of axiom (i), so that the connected subsets of A are precisely the edge-connected ones. □

1.4. Morphisms.

Definition 1.3. A function $f : X \to Y$ on connective spaces, is called **c-continuous** or **catenuous** when it maps connected sets of X to connected sets of Y. A **catenomorphism** is an isomorphism i.e. a bijective map $f : X \to Y$ for which f and f^{-1} are c-continuous.

One can also define *c-continuity at a point* when the map preserves the connective structure rooted at that point.

Any function which maps components of X to constants is obviously c-continuous; for example, every function on a totally disconnected space. Hence X is connected if, and only if, every c-continuous map into the totally disconnected space $\{0, 1\}$ is constant.

These are indeed morphisms because when $f : X \to Y$ and $g : Y \to Z$ are c-continuous, then so is $g \circ f$; and the identity maps are obviously c-continuous.

The c-continuous mappings on graphs are precisely the graph homomorphisms i.e. those that preserve edges or join adjacent vertices.

Continuous mappings between topological spaces are c-continuous, but the converse is false e.g. $f(x) = \sin(1/x)$, $f(0) = 0$ is c-continuous on $\mathbb{R}$ but not continuous. Homeomorphisms are catenomorphisms; in fact the standard examples in textbooks showing that the circle, the real line and the real plane are not homeomorphic, in effect show that they are not catenomorphic.

It is not hard to show that for $\mathbb{R}$ with the usual topology, order-preserving c-continuous maps are continuous; and that bijective c-continuous maps are order-preserving.

The space of c-continuous functions from X to $\mathbb{C}$ is an algebra, since addition and multiplication are continuous on $\mathbb{C}^2$.

1.5. Subspaces.

Definition 1.4. A subset $Y \subseteq X$ can be given the connectivity structure $\mathcal{C}_Y := \{A \in \mathcal{C}_X : A \subseteq Y\}$.

It is easily checked that this is a connectology. The restriction of a c-continuous map to a subspace remains c-continuous.

This definition agrees with that of subgraphs, and is consistent with the connected sets of topological subspaces with the relative topology.

1.6. Quotient Spaces.

Definition 1.5. A partition of connected sets, $X = \bigcup_i E_i$ induces a connective quotient space, with a collection of equivalence classes being connected when their union in X is connected.

Proof. Axioms (i), (ii) and (iv) are trivially satisfied. For axiom (iii), let A, B be connected sets in the quotient space, with $A \cup B$ connected. Then they are both connected as subsets of X, so that there is a point $x \in X$ with $\{x\} \cup A$ and $\{x\} \cup B$ connected in X. But $x \in E_j$ for some j, implying that $E_j \cup A$ and $E_j \cup B$ are connected in X, and hence in the quotient space. □

The map $\phi : x \mapsto [x]$ where $x \in E_i = [x]$, is c-continuous, as follows: let C be a non-empty connected set in X; then $\bigcup \phi C = \bigcup_{x \in C} [x] = \bigcup_{x \in C} C \cup [x]$ is connected in X by axiom (i).

The partition of components induces a totally disconnected quotient space.

1.7. Bases.

Proposition 1.4. *Any collection of sets $\mathcal{B}$ can generate a c-structure defined as the unique smallest c-structure containing $\mathcal{B}$.*

Proof. Let $\mathcal{C} = \bigcap \{\mathcal{D} \text{ c-structure} : \mathcal{B} \subseteq \mathcal{D}\}$. Note that $\mathcal{P}(X)$ is a valid $\mathcal{D}$. $\mathcal{C}$ is a c-structure since let $C_i \in \mathcal{C}$ such that $\bigcap_i C_i \neq \emptyset$. Then for any $\mathcal{D}$, $C_i \in \mathcal{D}$, hence $\bigcup_i C_i \in \mathcal{D}$ and $\bigcup_i C_i \in \mathcal{C}$. Similarly the points are in any c-structure $\mathcal{D}$ and hence must be in $\mathcal{C}$. Moreover $\mathcal{B}$ is in $\mathcal{C}$ since it is in all the $\mathcal{D}$. □

The sets in $\mathcal{B}$ will be called *basic* connected sets. In the above examples, the closed bounded intervals generate the connective structure of $\mathbb{R}$; the edges generate that of graphs.

Definition 1.6. A **chain** of connected sets is a map C from a finite index set I to $\mathcal{C}$ such that each set C_i intersects its successor C_{i+1}.

We will sometimes abuse the notation and speak of C_i as the chain. Chains can obviously be concatenated and reversed. Moreover induction on axiom (i) shows that chains are connected sets.

Proposition 1.5. *The non-trivial (i.e. non-empty non-punctal) connected sets of a c-structure generated by $\mathcal{B}$ are characterized by the condition that any two points of such a connected set C can be joined by a finite chain of basic connected sets in C i.e. for all $x, y \in C$ there is a chain $B : I \mapsto \mathcal{B}$ such that $B_i \subseteq C$, $x \in B_0$ and $y \in B_1$.*

Proof. Let $\mathcal{E}$ be the collection of subsets $C \subseteq X$ for which every two points in C can be joined by a finite chain of basic connected sets in C, together with the set of singletons $\{x\}$.

$\mathcal{E}$ is a c-structure on X: let $C_i \in \mathcal{E}$ with non-empty intersection. Let $x, y \in \bigcup_i C_i$ so that $x \in C_1$, $y \in C_2$, say, and $z \in \bigcap_i C_i$. There then exist chains joining x to z in C_1 and z to y in C_2, and hence their concatenation joins x to y in $C_1 \cup C_2$. It follows that $\bigcup_i C_i \in \mathcal{E}$.

Moreover every basic connected set B satisfies this condition by taking the constant chain $i \mapsto B$, implying $\mathcal{B} \subseteq \mathcal{E}$. Also, $\mathcal{E} \subseteq \mathcal{D}$ for any connective structure $\mathcal{D} \supseteq \mathcal{B}$ since chains are connected. $\mathcal{E}$ must therefore equal the c-structure generated by $\mathcal{B}$. □

1.8. **Products.**

Theorem 1.6. *There is a categorical product of c-spaces.*

Proof. Let X be a set and $(X_i, \mathcal{C}_i)$ connective spaces for all $i \in I$. Let $f_i : X \to X_i$ be a set of maps indexed by I. We define a c-structure $\mathcal{C}$ on X as follows: $C \in \mathcal{C}$ if, and only if, $f_i(C) \in \mathcal{C}_i$ for all $i \in I$. Then $\mathcal{C}$ is indeed a c-structure, for if $\mathcal{D} \subseteq \mathcal{C}$ and $\bigcap \mathcal{D} \neq \emptyset$ then we have that $f_i(A) \in \mathcal{C}_i$ for all $A \in \mathcal{D}$ and $i \in I$, so that $\emptyset \neq f_i(\bigcap \mathcal{D}) \subseteq \bigcap f_i(\mathcal{D})$. Thus, $f_i(\bigcup \mathcal{D}) = \bigcup f_i(\mathcal{D}) \in \mathcal{C}_i$ for all $i \in I$ and consequently, $\bigcup \mathcal{D} \in \mathcal{C}$. Evidently, $\{x\} \in \mathcal{C}$ since $f_i\{x\} \in \mathcal{C}_i$ for all $i \in I$.

We next show that $\mathcal{C}$ is an initial c-structure on X, that is, for any connective space $(Y, \mathcal{D})$, a map $g : Y \to X$ is c-continuous if, and only if, $f_i \circ g : Y \to X_i$ is c-continuous for every $i \in I$. Indeed, if $g : Y \to X$ is c-continuous then evidently $f_i \circ g : Y \to X$ is c-continuous for every $i \in I$. Conversely, if $f_i \circ g : Y \to X_i$ is c-continuous for every $i \in I$ then for any $A \in \mathcal{D}$ and any $i \in I$ we have that $f_i \circ g(A) \in \mathcal{C}_i$ so that $g(A) \in \mathcal{C}$ and hence, g is c-continuous. It is not difficult to see that this initial c-structure is unique (just take $g = id : (X, \mathcal{C}_1) \to (X, \mathcal{C}_2)$ and vice versa, if both $\mathcal{C}_1$ and $\mathcal{C}_2$ where initial c-structures with respect to $f_i : X \to X_i$, $i \in I$).

One can see that this initial c-structure is the coarsest c-structure on X such that f_i is c-continuous for every $i \in I$.

We now apply the above to the product connective space of a family of c-spaces $(X_i, \mathcal{C}_i)$, $i \in I$. One defines the product c-structure $\mathcal{C} = \prod_{i \in I} \mathcal{C}_i$ on $X = \prod_{i \in I} X_i$ as the initial c-structure with respect to the projections $\pi_i : X \to X_i$. Thus, $A \in \mathcal{C}$ if, and only if, $\pi_i(A) \in \mathcal{C}_i$ for all $i \in I$.

Thus, if $(Y,\mathcal{D})$ is a c-space then $f : Y \to X$ is c-continuous if, and only if, $\pi_i \circ f : Y \to X_i$ is c-continuous for every $i \in I$.

We next see that the product as defined above is categorical. Consider $(X, \pi_i, i \in I)$ where $\pi_i : X \to (X_i, \mathcal{C}_i)$ are the projections. Let $f_i : (Y,\mathcal{D}) \to (X_i,\mathcal{C}_i), i \in I$ be a collection of c-continuous maps. We need to find a c-continuous map $f : (Y,\mathcal{D}) \to (X,\mathcal{C})$ such that $\pi_i \circ f = f_i$. Such a map can be defiined by $f(y) = (f_i(y))_{i\in I} \in X$. Evidently we have $\pi_i \circ f = f_i$, so that for any $A \in \mathcal{D}$, $\pi_i \circ f(A) \in \mathcal{C}_i$ for every $i \in I$ and hence $f(A) \in \mathcal{C}$. This shows that f is c-continuous and that the product is categorical. □

1.9. **Connective Closure.**

Definition 1.7. A set is called **t-closed** when it contains its touching points, $t(F) = F$. The **connective closure** of a set is defined as the smallest t-closed set containing it,

$$\bar{A} := \bigcap\{F \subseteq X : t(F) = F \text{ and } A \subseteq F\}.$$

For example, X and the $\emptyset$ are t-closed, and these are the only t-closed sets in a totally connected space. A space is totally disconnected if, and only if, every set is t-closed.

For graphs, a subset is t-closed if, and only if, it is a union of components of the graph.

Proposition 1.7. *If $f : X \to Y$ is c-continuous and*

(i) *x touches A then $f(x)$ touches fA;*
(ii) *F is t-closed in Y then $f^{-1}F$ is t-closed in X;*

Proof. (i) There is a non-empty subset $C \subseteq A$ such that $\{x\} \cup C$ is connected. Therefore $\{f(x)\} \cup fC$ is connected and $fC \subseteq fA$ is non-empty.

(ii) follows from (i). □

Proposition 1.8.

(i) $A \subseteq t(A) \subseteq \bar{A}$; $\quad t(A) = A \iff \bar{A} = A$;
(ii) *the intersection of t-closed sets is t-closed;*
(iii) $\bar{\bar{A}} = \bar{A}$;
(iv) $A \subseteq B \implies \bar{A} \subseteq \bar{B}$;
(v) $\overline{\bigcup_i \bar{A}_i} = \overline{\bigcup_i A_i}$;
(vi) $f\bar{A} \subseteq \overline{fA}$.

Proof. To prove (i) let $F \supseteq A$ be t-closed; then $t(A) \subseteq t(F) = F$ and hence $t(A) \subseteq \bar{A}$. Moreover when a set A is t-closed, $\bar{A} = A$. Part (ii) follows from proposition 1.1(vi), and this in turn implies (iii). For (iv) and (v) note that $A \subseteq B \implies \bigcap_{A\subseteq F} F \subseteq \bigcap_{B\subseteq F} F$, and $\bigcup_i A_i \subseteq F \implies A_i \subseteq F$ and use (i) and (iv). Also with the same reasoning, if F is t-closed and contains fA then $f^{-1}F$ is t-closed and contains A, so that $\bar{A} \subseteq \bigcap_F f^{-1}F$, which proves (vi). □

Corollary 1.9. *The connective closure of a set A in a t-closed subspace B is that induced by the closure in X i.e. $\bar{A}^B = \bar{A} \cap B$.*

Note that $\overline{A \cup B} \neq \bar{A} \cup \bar{B}$ in general, even for topological spaces e.g. $\mathbb{R}^+ = \mathbb{Q}^+ \cup \mathbb{Q}'^+$ but $0 \in \overline{\mathbb{R}^+}$ and $0 \notin \overline{\mathbb{Q}^+} \cup \overline{\mathbb{Q}'^+}$. In general $\bar{A}$ includes not just the touching points of A but also the touching points of $t(A)$ etc.

Proposition 1.10.

(i) *A connected $\implies$ $t(A)$ connected;*
(ii) *A connected $\implies$ $\bar{A}$ connected;*
(iii) *the components of X are t-closed, t-open and non-touching;*
(iv) *for A, B non-empty connected sets, $\bar{A} \cap \bar{B} \neq \emptyset \iff \bar{A} \cup \bar{B}$ is connected.*
(v) *for A, B connected, $t(A \cup B) = t(A) \cup t(B)$, and $\overline{A \cup B} = \bar{A} \cup \bar{B}$.*

Proof. (i) For each point x touching A, the set $A \cup \{x\}$ is connected and intersects the connected set A. So their union $t(A)$ is also connected by axiom (i).

(iii) Let C be a component. If x touches $X - C$ then $\{x\} \cup B$ is connected for some non-empty set $B \subseteq X - C$. But then $\{x\} \cup B \subseteq C$ (since components are maximally connected), unless $x \in X - C$. The rest was proved in 1.2

(ii) Let B be the component of A in $\bar{A}$. Therefore, B is t-closed in $\bar{A}$, and hence in X. This forces $\bar{A} = B$ which is connected.

(iv) is a restatement that two connected sets have a connected union precisely when they touch.

(v) Let x touch $A \cup B$. Then $\{x\} \cup A \cup B$ or $\{x\} \cup A$ or $\{x\} \cup B$ is connected. In any case, by (1), $\{x\} \cup A$ or $\{x\} \cup B$ is connected, which proves the first part. Next, for A, B connected, $t(\bar{A} \cup \bar{B}) = t(\bar{A}) \cup t(\bar{B}) = \bar{A} \cup \bar{B}$; this, combined with proposition 1.8(v), proves the second part. □

Note that if A is connected and $A \subseteq B \subseteq \bar{A}$ then B need not be connected. For example, in graphs, let A be a single vertex; then $\bar{A}$ is its component, and may contain several disconnected subgraphs that include A.

1.9.1. *Associated Topologies.* Every connective space gives rise to at least two topologies:

- A topology generated by the t-closed sets; in this case the closed sets would be intersections of finite unions of t-closed sets, and will be called *c-closed* sets.
- A weaker topology generated by the t-closed connected sets; the closed sets would be intersections of finite unions of t-closed connected sets, and will be called *s-closed* sets.

Hence we get a c-closure $\widetilde{A}$ and an s-closure $\widehat{A}$ of a set A, with *c-closureA* $\subseteq \bar{A} \subseteq \widehat{A}$.

In general, these topologies may not be compatible with the connective structure. For example for an infinite totally disconnected space, the associated s-topology is the cofinite one, with the s-closed sets being the finite subsets, resulting in the cofinite connective space. One consequence is that c-continuous maps need not be continuous relative to either of these topologies, although catenomorphisms are necessarily homeomorphisms.

For graphs both these topologies are equal to the topology generated by the components.

1.9.2. *Other Constructs.* One can define the t-interior, t-boundary, t-compactness analogously to the topological definitions, replacing 'open' and 'closed' by 't-open' and 't-closed'; and similarly for c-interior and s-interior etc. Several parallel statements are true e.g. every set induces a partition of X into its c-interior, c-exterior and c-boundary; c-continuous maps preserve t-compact sets; t-closed subsets of t-compact sets are t-compact. Note however that for the real line at least, the t-compact sets are the countable subsets (consider the c-open cover consisting of the cosets of the rationals.)

Analogues of the separation axioms are also interesting, in particular the analogue of the topological T_1 axiom, here termed

Axiom C_1: *Distinct points are disconnected.*

This is equivalent to saying that distinct points do not touch or that $\overline{\{x\}} = \{x\}$ or that finite sets are c-closed. A totally disconnected space is obviously C_1.

1.10. **Pathconnectedness.** Every connective structure has two important substructures:

(a) The graph substructure of **edge**-connected sets, that is, sets such that any two elements have a path of edges connecting them. Equivalently, a set A is edge-connected when for any two points $a, b \in A$, there is a c-continuous function $f : \mathbb{N} \to A$ which covers a and b. Here $\mathbb{N}$ is taken to be the connective space consisting of the natural numbers together with all its intervals $[m, n]$ (plus $\emptyset$ and $\mathbb{N}$).

(b) The substructure of **path**-connected sets, namely those sets A such that for any two points $a, b \in A$, there is a c-continuous function $f : \mathbb{R} \to A$ which covers a and b.

Proposition 1.11. *The path-connected structure and the graph structure are connectologies.*

Proof. Axiom (i). Let C_i be path-connected sets in X with a common point $z \in C_i$, $\forall i$. Then for any two points in their union, say $x \in C_0$, $y \in C_1$ we get a c-continuous path from x to z in C_0 and from z to y in C_1; joining these paths gives the desired c-continuous path from x to y in $C_0 \cup C_1$.

Axiom (ii) is trivially true using the constant paths. For axiom (iii), consider two path-connected sets A and B, whose union is also path-connected. If A and B are not disjoint, the proof is immediate; so assume they are disjoint. Pick any two points $x \in A$ and $y \in B$; there is a c-continuous path f from x to y in $A \cup B$. Consider the non-empty set $W = \{t : f(t) \in A\}$; W has an upperbound since $f(\beta) = y \notin A$, and hence a least upperbound α, with $f(\alpha) = z$. Now $B \cup \{z\}$ is path-connected by $f : [\alpha, \beta] \to B$ which joins z to y, and hence to all other points of B. If $z \in A$ then the assertion is proved. If instead $z \in B$, then take a sequence $a_n \to \alpha$ with $a_n \in A$, and c-continuous paths $\phi_n : [a_n, a_{n+1}) \to A$ with $\phi_{n+1}(a_n) = \phi_n(a_{n+1})$. Replace the original path by $\phi : [0, \alpha) \to A$ defined by

$\phi(x) = \phi_n(x)$ for $x \in [a_n, a_{n+1})$, still c-continuous. Then ϕ joins x to z in A, and $A \cup \{z\}$ is path-connected.

Axiom (iv). Let M be a path-component of A in $X - B$. Then M must contain any path-connected sets C_i wholly. Let $x \in B$ and $y \in X - M$. Then there is a c-continuous path joining x to y in X. The set $W = \{t : f[t, 1] \subseteq X - M - B\}$ has a lower-bound of 0, hence a greatest lowerbound α, with $f(\alpha) \notin M$. Hence $f(\alpha) \in B$, implying that x is joined to $f(\alpha)$ to y inside $X - M$. Hence $X - M$ is path-connected.

The proof for edge-connectedness is similar and simpler. □

1.11. **Further Work.** The study of those properties of connective spaces that are preserved by catenomorphisms is still uncharted. The following is a short sample list of possible areas of research:

Regular connective spaces are those for which all the rooted connective structures are catenomorphic.

Transitive connective spaces are those such that for any two points $x, y \in X$ there is an automorphism of X that sends x to y.

A connective space X is called n**-connected** when there are n points in X which disconnect X when removed. X is called homogeneously n-connected when removing *any* n points disconnects it. For example, the real line is homogeneously 1-connected. A totally connected space is the only space which remains connected when any subset is removed.

The **dimension** of a connected space may be defined inductively as follows: X has dimension 0 when it is a singleton; it has dimension $n + 1$ when for $X - A$ to be disconnected with A a connected subset, A must be at least n-dimensional.

One can imagine extending the definitions of homology classes (using c-continuous paths), simply-connected spaces, Euler number etc.

1.11.1. *Open Problems.*

(i) Suppose the connective structure of $X - \{x\}$ is given for all x up to catenomorphism, is that of X uniquely reconstructible? (This is called the reconstruction conjecture in graph theory, and is currently unproved.) Disconnected spaces are obviously uniquely reconstructible.

(ii) Which connective spaces are embeddable in a given space X? This is open even for graphs.

(iii) Which connective spaces are topologisable (with a compatible topology)? It can be shown that locally connected T_3 spaces are topologisable using the s-topology.

2. Topological Spaces

It is a standard theorem in topological spaces that the addition of a limit point to a connected set, leaves it connected; in our terminology this becomes,

Proposition 2.1. *A limit point of a connected set is a touching point; t-closed connected sets are closed.*

The converse is false in general e.g. one vertex in the graph K_2 has a touching vertex, but no limit point.

As a corollary, the s-topology is weaker than the topology of the space (s-closed sets are closed).

Proposition 2.2. *Every topological space has a T_0 topology with the same connected sets.*

Proof. Suppose we're given a topological space with topology $\mathcal{T}$. Consider the equivalence relation $x \sim y$ defined by $\forall U \in \mathcal{T},\ x \in U \iff y \in U$. It induces a partition on X; well-order the equivalence classes. Let $\mathcal{S}$ be generated from $\mathcal{T}$ together with the open sets $U = V - \{x_0, \ldots, x_n\}$ for $x_0, \ldots$ in an equivalence class. If $x \not\sim y$ then x can be separated from y or vice-versa; if $x \sim y$, they belong to an equivalence class, and one of the additional open sets can separate them. Hence $\mathcal{S}$ is T_0.

Disconnected sets remain disconnected under $\mathcal{S}$ because this topology is finer. Now let A be disconnected in $\mathcal{S}$, using open sets U_1 and U_2. There cannot be an equivalence class part of which intersects $U_1 \cap A$ and also $U_2 \cap A$, else we would get $x \sim y$ with $x \in U_1$ and $y \in U_2$, and hence $x \in U_2$. Equivalence classes must therefore lie completely in U_1 or completely in U_2 or in neither. Hence U_1 and U_2 are open sets in $\mathcal{T}$, and A is disconnected in $\mathcal{T}$. □

Proposition 2.3. *For T_1 topological spaces,*

(i) *touching points are limit points,*
(ii) *the t-closure of a set is a subset of its topological closure,*
(iii) *closed sets are t-closed, and open sets are t-open,*
(iv) *when A is connected, $t(A) = \bar{A} = \tilde{A}$ and it is t-closed $\iff$ it is closed.*

Proof. In T_1 topological spaces, a point and a distinct closed set are disconnected. This implies that a point x in the exterior of a set A cannot touch it. Hence touching points are limit points (but not conversely in general), and closed sets are t-closed; also open sets are t-open. Therefore the intersection of all the t-closed sets containing A, i.e $\bar{A}$, is contained in the intersection of all the closed sets containing A, i.e. its topological closure. (iv) follows from the previous proposition. □

It follows that the c-topology is stronger than the given topology (closed sets are t-closed, hence c-closed).

Note: The converses are in general false e.g. the set $\{1/n : n \in \mathbb{N}\}$ in $\mathbb{R}$ is t-closed but not closed.

Proposition 2.4. *Let X and Y be catenomorphic topological spaces. Then X is $T_1 \iff Y$ is T_1.*

Proof. Let $\phi : X \to Y$ be a catenomorphism. Suppose X is T_1, and consider any two distinct points in Y. Then they map to distinct points in X, and are disconnected in X, hence in Y. □

However, if X is T_3 then Y need not be T_3. For example, consider $\mathbb{R}$ with the topology generated by the standard open intervals together with $(-1, 1) - \{1/n\}$; it is not T_3. As it is finer than the standard topology, non-intervals are disconnected. Intervals however are connected since let U, V be a disconnection; take $x \in U$, $z \in V$, let $y = \sup U \cap [x, z]$; then $y \notin U$ since open sets do not include their endpoints in this topology, and, similarly, $y \notin V$. Thus it is catenomorphic to the standard real line.

2.1. **Topologizable Graphs.** It has been shown [2] that two-colourable (i.e. bipartite) locally-finite graphs (including trees) are essentially uniquely topologisable by taking white vertices to be open and black vertices to be closed.

More generally, it is known [1, 8] that T_0 spaces with the property that $\bigcap_i U_i$ is open for any open sets U_i, are in 1-1 correspondence with the partial orders on the vertices. Since this property holds for topologies on finite sets, we have

Proposition 2.5. *A finite graph admits a compatible topology if, and only if, it admits a partial order on its vertices, compatible with its edges, in the sense that two vertices are comparable if, and only if, they are identical or adjacent.*

Proof. By proposition 2.2, we can assume that the compatible topology is T_0. Consider the relation $x \leq y$, which is defined to hold when every open set that includes y includes x as well. It is easy to show reflexivity and transitivity, while anti-symmetry follows from the T_0 axiom. Non-comparable vertices are, by definition, disconnected and vice-versa, distinct non-adjacent vertices are disconnected, hence non-comparable.

Conversely, given a compatible partial order, let $\mathcal{T}$ be the topology generated from the closed sets $F_a := \{x : x \leq a\}$. Then non-comparable vertices a, b are disconnected by the closed sets F_a and F_b. Now let $a < b$ be distinct comparable vertices, and let F be any closed set containing b; then there are a finite number of vertices c_{ij} with $F = \bigcup_i \bigcap_j F_{c_{ij}}$. But $b \in F$ implies $\exists i \, \forall j \, b \leq c_{ij}$, and hence $a \in F$; thus every closed set that contains b contains a and $\{a, b\}$ is connected. Thus comparable vertices are adjacent, and the given topology is compatible with the graph structure. □

Bipartite graphs have an obvious compatible partial order.

2.2. **Additional Structures.**

Definition 2.1. A **connective group** is a group G with a connective structure such that the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are c-continuous.

It follows that right/left multiplication and inversion are catenomorphisms. This implies,

Proposition 2.6. *If A, B are connected sets then gA, Ag, A^{-1} and AB are connected.*

The connective structure rooted at e determines the connective structure by translation. In particular a group homomorphism is c-continuous when it is c-continuous at e.

Proposition 2.7. *The component of e is catenomorphic to any other component, and is a normal subgroup, G_e. G/G_e is then a totally disconnected group.*

Proof. The map $g : [e] \to [g]$ defined by $h \mapsto gh$ is the required catenomorphism; $g^{-1}Cg$ is connected for any $g \in G$ and any connected set C, and fixes e, so that $g^{-1}G_e g \subseteq G_e$. □

Conversely, for any normal subgroup N of G, there is a connective space with $G_e = N$.

For a finite "group-graph" acting c-continuously on itself, if $\{e, a\}$ is an edge, then so are $\{e, g^{-1}ag\}$ and $\{e, g^{-1}a^{-1}g\}$ for any $g \in G$. This means that the identity is adjacent to whole conjugacy classes and their inverses; and knowledge of which conjugacy classes is sufficient to generate the graph, since if the edge $\{e, a\}$ is mapped under translations to the edge $\{hg, hag\}$ with one vertex being e, then there are two possibilities: either $hg = e$ and the edge is in fact $\{e, g^{-1}ag\}$ or $hag = e$ and the edge is $\{g^{-1}a^{-1}g, e\}$.

Definition 2.2. A **connective vector space** is a vector space over the real numbers (more generally over a connected field) with a connective structure such that addition and scalar multiplication are c-continuous.

Proposition 2.8. *If A and B are connected then so is $A + B$; if $I \subseteq \mathbb{R}$ is an interval and $A \subseteq X$ connected, then IA is connected.*

In particular, $x + A$, λA, line segments $[x, y]$, linear subspaces, convex sets, star shapes and X itself are connected.

References

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Pub. vol. 25 (third edition), New York, 1967.
[2] A. Bretto and A. Faisant, *Topologies on bipartite graphs*, Rend. del Sem. Mat. di Messina Ser.II, Supp. al n.8 (2002), 11–18.
[3] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig, 1914.
[4] C. Jordan, *Cours d'analyse, I*, Paris, 1893.
[5] B. Knaster and K. Kuratowski, *Sur les ensembles connexes*, Fund. Math. **2** (1921), 206–256.
[6] K. Kuratowski, *Topology*, vol. II, New York, 1968.
[7] N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, Amer. Journ. of Math. **33** (1911), 287–326.
[8] A. K. Steiner, *The lattice of topologies: Structure and complementation*, Trans. Amer. Soc. **122** (1966), 379–397.

Joseph Muscat, Mathematics Department, University of Malta
E-mail address: joseph.muscat@um.edu.mt

David Buhagiar, Mathematics Department, University of Malta
E-mail address: david.buhagiar@um.edu.mt