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Eleven-dimensional supergravity from filtered subdeformations of the Poincaré superalgebra

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ELEVEN-DIMENSIONAL SUPERGRAVITY FROM FILTERED SUBDEFORMATIONS OF THE POINCARÉ SUPERALGEBRA

JOSÉ FIGUEROA-O'FARRILL AND ANDREA SANTI

ABSTRACT. We summarise recent results concerning the classification of filtered deformations of graded subalgebras of the Poincaré superalgebra in eleven dimensions, highlighting what could be considered a novel Lie-algebraic derivation of eleven-dimensional supergravity.

1. INTRODUCTION

In a recent paper [1] we have taken a first step in a Lie-algebraic approach to the classification problem of supersymmetric supergravity backgrounds. In passing we provided what could be considered a Lie-algebraic derivation of eleven-dimensional supergravity. We think that this result is interesting in its own right, but it is hidden in the middle of a somewhat technical paper and hence we fear that the message might be lost. Therefore the purpose of this short note is to summarise the results of [1] and discuss their significance in as few words as possible. Readers who are pressed for time might consider reading this short note before tackling the longer paper.

This note is organised as follows. In Section 2 we define the basic objects: $\mathbb{Z}$-graded subalgebras of the eleven-dimensional Poincaré superalgebra. In Section 3 we explain the problem we set out to solve in [1]: namely, the classification of filtered deformations of such subalgebras, and summarise our results. That is the content of the Theorem below. In Section 4 we mention one of the auxiliary results obtained on the way to proving the Theorem: the calculation of a Spencer cohomology group, which shows how the 4-form emerges in this approach. Finally, in Section 5 we make some remarks about the results and try to discuss their significance.

2. GRADED SUBALGEBRAS OF THE POINCARÉ SUPERALGEBRA

We start with an eleven-dimensional lorentzian vector space (V, η), of "mostly minus" signature. You may think of it as $\mathbb{R}^{1,10}$ with $\eta = \mathrm{diag}(+1, \underbrace{-1, \cdots, -1}_{10})$. We let $\mathfrak{so}(V) \cong \mathfrak{so}(1,10)$ denote the Lorentz Lie algebra. This is the Lie algebra of η-antisymmetric linear transformations of V; that is, $A \in \mathfrak{so}(V)$ if for all $v, w \in V$,

$$\eta(Av, w) = -\eta(v, Aw) . \tag{1}$$

If we think of (V, η) as a flat lorentzian manifold, the Lie algebra of isometries is the Poincaré Lie algebra

$$\mathfrak{p} = \mathfrak{so}(V) \ltimes V , \tag{2}$$

where V acts by infinitesimal translations. The Lie brackets are

$$[A, B] = AB - BA \qquad [A, v] = Av \qquad \text{and} \qquad [v, w] = 0 , \tag{3}$$

for all $A, B \in \mathfrak{so}(V)$ and $v, w \in V$. If we grade $\mathfrak{p}$ by declaring that $\mathfrak{so}(V)$ has degree 0 and V has degree[1] -2, we see that $\mathfrak{p}$ is a $\mathbb{Z}$-graded Lie algebra.

Let us now introduce supersymmetry. Associated to (V, η) we have the Clifford algebra $C\ell(V, \eta) \cong C\ell(1, 10)$. Our conventions for the Clifford algebra are such that

$$v^2 = -\eta(v, v)\mathbb{1} \qquad \forall v \in V , \tag{4}$$

and with our choice of η, we have an isomorphism of real associative algebras

$$C\ell(V, \eta) \cong \mathrm{Mat}_{32}(\mathbb{R}) \oplus \mathrm{Mat}_{32}(\mathbb{R}) , \tag{5}$$

from where one reads that $C\ell(V, \eta)$ has precisely two inequivalent irreducible Clifford modules $S_\pm$, which are real and of dimension 32. They are distinguished by the action of the centre. If we let

[1]the reason for this seemingly bizarre degree will become clear in a moment

$\Gamma_{11} \in C\ell(V, \eta)$ denote the volume form, then $\Gamma_{11}^2 = +\mathbb{1}$ and is central, so that it acts like a scalar multiple of the identity on an irreducible Clifford module. That scalar multiple is ± 1 on $S_\pm$. We let $S = S_-$ from now on.

On S we have a symplectic inner product $\langle -, - \rangle$ which obeys

$$\langle v \cdot s_1, s_2 \rangle = -\langle s_1, v \cdot s_2 \rangle \,, \tag{6}$$

for all $v \in V$ and $s_1, s_2 \in S$ and where $\cdot$ stands for the Clifford action. It follows from equation (6) that $\mathfrak{so}(V)$ preserves $\langle -, - \rangle$. In fact, S is a real symplectic irreducible representation of $\mathfrak{so}(V)$: the spinor representation.

The Poincaré algebra $\mathfrak{p}$ admits a supersymmetric extension: a Lie superalgebra $\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$ with $\mathfrak{s}_{\bar{0}} = \mathfrak{p}$ and $\mathfrak{s}_{\bar{1}} = S$. The Lie brackets extend the Lie brackets (3) of $\mathfrak{p}$ by

$$[A, s] = As \qquad [v, s] = 0 \qquad \text{and} \qquad [s_1, s_2] = \kappa(s_1, s_2) \,, \tag{7}$$

where κ, being symmetric, is determined by its value on the diagonal, where it coincides with the Dirac current of $s \in S$, defined by

$$\eta(\kappa(s, s), v) = \langle v \cdot s, s \rangle \,, \tag{8}$$

for all $s \in S$ and $v \in V$. Relative to an η-orthonormal basis $e_\mu = (e_0, e_1, \cdots, e_9, e_\natural)$ for V, and writing

$$\langle s, e_\mu \cdot s \rangle = \overline{s} \Gamma_\mu s \,, \tag{9}$$

we see that

$$\kappa(s, s) = -\overline{s} \Gamma^\mu s e_\mu \,, \tag{10}$$

where the minus sign comes from equation (6). The Poincaré superalgebra $\mathfrak{s}$ is actually a $\mathbb{Z}$-graded super-extension of the Poincaré algebra $\mathfrak{p}$: we simply let S have degree -1 and write

$$\mathfrak{s} = \mathfrak{s}_0 \oplus \mathfrak{s}_{-1} \oplus \mathfrak{s}_{-2} = \mathfrak{so}(V) \oplus S \oplus V \,. \tag{11}$$

We remark that the $\mathbb{Z}$-grading and the parity are compatible in that the parity is the degree modulo 2. This will be the case for all the $\mathbb{Z}$-graded Lie superalgebras we shall consider.

Now let $\mathfrak{h}_0 < \mathfrak{so}(V)$ be any subalgebra of the Lorentz Lie algebra, $S' \subset S$ be a subspace of S which is stabilised (as a subspace) by $\mathfrak{h}_0$, and let $V' \subset V$ be the subspace of V spanned by the Dirac currents of all the spinors $s \in S'$. Equivariance guarantees that $\mathfrak{h}_0$ also stabilises V' as a subspace. Then $\mathfrak{h} = \mathfrak{h}_0 \oplus S' \oplus V'$ is a $\mathbb{Z}$-graded subalgebra of the Poincaré superalgebra $\mathfrak{s}$ with respect to the restriction to $\mathfrak{h}$ of the Lie brackets in equations (3) and (7). In this note we will restrict to the special subalgebras with $S' = S$, so that $V' = V$ (in fact, if $\dim S' > 16$ then $V' = V$ by the results of [2]). Hence from now on we will consider $\mathbb{Z}$-graded subalgebras $\mathfrak{h}$ of $\mathfrak{s}$ which differ only in degree zero; that is,

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_{-1} \oplus \mathfrak{h}_{-2} = \mathfrak{h}_0 \oplus S \oplus V \subset \mathfrak{so}(V) \oplus S \oplus V = \mathfrak{s} \,. \tag{12}$$

3. Filtered deformations

Every $\mathbb{Z}$-graded Lie superalgebra admits a canonical filtration. In the case of the Lie superalgebra $\mathfrak{h}$ in equation (12), this is a filtration

$$F^\bullet \mathfrak{h}: \qquad \mathfrak{h} = F^{-2}\mathfrak{h} \supset F^{-1}\mathfrak{h} \supset F^0\mathfrak{h} \supset 0 \,, \tag{13}$$

where $F^{-1}\mathfrak{h} = \mathfrak{h}_{-1} \oplus \mathfrak{h}_0$, $F^0\mathfrak{h} = \mathfrak{h}_0$. We can extend it to $F^i\mathfrak{h} = \mathfrak{h}$ for all $i < -2$ and $F^i\mathfrak{h} = 0$ for all $i > 0$. The Lie brackets are such that $[F^i\mathfrak{h}, F^j\mathfrak{h}] \subset F^{i+j}\mathfrak{h}$.

Now, any filtered Lie superalgebra $F^\bullet$ gives rise to an associated graded Lie superalgebra $\mathrm{gr}_\bullet F$, defined by

$$\mathrm{gr}_i F = F^i / F^{i+1} \,, \tag{14}$$

whose Lie brackets are the components of the Lie brackets of $F^\bullet$ which have zero filtration degree. Indeed, it follows from the fact that F is filtered, that $[\mathrm{gr}_i F, \mathrm{gr}_j F] = \mathrm{gr}_{i+j} F$. In the case of the filtered Lie superalgebra $F^\bullet \mathfrak{h}$ associated to $\mathfrak{h}$ in equation (12), the associated graded $\mathrm{gr}_\bullet F\mathfrak{h}$ is isomorphic to $\mathfrak{h}$ itself.

Of course there may be other filtered Lie superalgebras whose associated graded algebra is again isomorphic to $\mathfrak{h}$. They are called **filtered deformations** of $\mathfrak{h}$. We shall be concerned with **nontrivial** filtered deformations: namely, those that are not themselves isomorphic to $\mathfrak{h}$.

What does this mean in practice? A filtered deformation of $\mathfrak{h}$ has the same underlying vector space as $\mathfrak{h}$ but the Lie brackets are obtained by adding to the Lie brackets of $\mathfrak{h}$ terms with positive filtration degree. Since the $\mathbb{Z}$-grading is compatible with the parity, they always have even filtration degree.

For the case at hand, we can be much more explicit. A filtered deformation of $\mathfrak{h}$ in equation (12) is given by

$$
\begin{aligned}
[A, B] &= AB - BA & [s, s'] &= \kappa(s, s') + t\gamma(s, s') \\
[A, s] &= As & [v, s] &= t\beta(v, s) \\
[A, v] &= Av + t\lambda(A, v) & [v, w] &= t\tau(v, w) + t^2\rho(v, w) ,
\end{aligned}
\tag{15}
$$

for all $A, B \in \mathfrak{h}_0$, $v, w \in V$ and $s, s' \in S$, where we have introduced an inessential parameter t to keep track of the filtration degree and where the new brackets are

$$
\begin{aligned}
\lambda &: \mathfrak{h}_0 \otimes V \to \mathfrak{h}_0 & \rho &: \Lambda^2 V \to \mathfrak{h}_0 \ . \\
\tau &: \Lambda^2 V \to V \\
\beta &: V \otimes S \to S \\
\gamma &: \odot^2 S \to \mathfrak{h}_0
\end{aligned}
\tag{16}
$$

Those in the first column have filtration degree 2 and the one in the second column has filtration degree 4.

Of course, $\lambda, \tau, \rho, \beta, \gamma$ are not arbitrary, but must satisfy the Jacobi identity (for all t). In a recent paper [1] we solved this problem and our solution is contained in the Theorem below. But first some notation. By $\mathrm{CSO}(V)$ we mean the Lie group of homothetic linear transformations of (V, η); that is, $\mathrm{CSO}(V) = \mathbb{R}^\times \times \mathrm{SO}(V)$, acting on V by $v \mapsto \alpha L v$, where $\alpha \in \mathbb{R}^\times$ and $L \in \mathrm{SO}(V)$. Recall, as well, that $\theta \in \Lambda^p V$ is said to be **decomposable** if $\theta = v_1 \wedge \cdots \wedge v_p$ for some $v_1, \cdots, v_p \in V$.

The main result of [1] can now be summarised as follows.

Theorem. *The only subalgebras $\mathfrak{h} \subset \mathfrak{s}$ (in the class in equation (12)) which admit nontrivial filtered deformations are those for which $\mathfrak{h}_0 = \mathfrak{so}(V) \cap \mathfrak{stab}(\varphi)$, where $\varphi \in \Lambda^4 V$ is nonzero and decomposable, and $\mathfrak{stab}(\varphi)$ is the Lie algebra of its stabiliser in $\mathrm{GL}(V)$. In the nontrivial filtered deformations, the maps λ and τ are zero and β, γ, ρ are given explicitly in terms of φ by*

$$
\begin{aligned}
\beta(v, s) &= v \cdot \varphi \cdot s - 3\varphi \cdot v \cdot s \\
\gamma(s, s)v &= -2\kappa(s, \beta(v, s)) \\
\rho(v, w)s &= [\beta_v, \beta_w]s ,
\end{aligned}
\tag{17}
$$

where $\beta_v s = \beta(v, s)$. Moreover, if $\varphi, \varphi' \in \Lambda^4 V$ are decomposable and in the same orbit of $\mathrm{CSO}(V)$, then the resulting filtered deformations are isomorphic.

Notice that from equation (17) it follows that we can reabsorb the parameter t in equation (15) into φ, since both β and γ are linear in φ, whereas ρ is quadratic in φ.

There are precisely three $\mathrm{CSO}(V)$-orbits of decomposable nonzero 4-forms in $\Lambda^4 V$. Every nonzero decomposable $\varphi = v_1 \wedge v_2 \wedge v_3 \wedge v_4$ defines a 4-plane $\Pi \subset V$ — namely, the real span of the (v_i) — and conversely, every 4 plane $\Pi \subset V$ gives rise to an $\mathbb{R}^\times$ orbit of decomposable φ by taking the wedge product of any basis. Two different bases will give φ's which are related by the determinant of the change of basis, so in the same $\mathbb{R}^\times$ orbit.

The inner product η restricts to Π in one of three ways:

(a) $(\Pi, \eta|_\Pi)$ is a lorentzian 4-plane,
(b) $(\Pi, \eta|_\Pi)$ is a euclidean 4-plane, or
(c) $(\Pi, \eta|_\Pi)$ is a degenerate plane.

We can choose representative φ's for each case, respectively:

(a) $\varphi = e_0 \wedge e_1 \wedge e_2 \wedge e_3$,
(b) $\varphi = e_7 \wedge e_8 \wedge e_9 \wedge e_\natural$, or
(c) $\varphi = e_+ \wedge e_1 \wedge e_2 \wedge e_3$,

where $e_+ = e_0 + e_\natural$. The resulting filtered Lie superalgebras are precisely the Killing superalgebras of the maximally supersymmetric backgrounds of eleven-dimensional supergravity, respectively:

(a) the family of Freund–Rubin backgrounds $\mathrm{AdS}_4 \times S^7$ [3],
(b) the family of Freund–Rubin backgrounds $S^4 \times \mathrm{AdS}_7$ [4], or
(c) the Kowalski-Glikman pp-wave [5].

4. Spencer cohomology

The proof of the Theorem, broken down into a number of preliminary results, occupies most of [1] and we invite those interested to read the details of the proof in that paper. That being said, it might be worth spending a few words on one aspect of the calculation, since it is this aspect which justifies the title of this note.

Like almost every other algebraic deformation problem, the classification of filtered deformations is governed by an underlying cohomology theory. In this case, it is Spencer cohomology, which is intimately linked to the Chevalley–Eilenberg [6] cohomology of the supertranslation ideal $\mathfrak{m}$ of $\mathfrak{h}$ with values in $\mathfrak{h}$ itself viewed as an $\mathfrak{m}$-module by restricting the adjoint representation. Let us be more precise.

Let $\mathfrak{m} = \mathfrak{h}_{-1} \oplus \mathfrak{h}_{-2} = S \oplus V$ denote the supertranslation ideal of $\mathfrak{h}$. (Notice that $\mathfrak{h} \subset \mathfrak{s}$ could be taken to be $\mathfrak{s}$ itself.) Restricting the adjoint representation of $\mathfrak{h}$ to $\mathfrak{m}$, $\mathfrak{h}$ becomes an $\mathfrak{m}$-module and we may consider the Chevalley–Eilenberg complex $C^\bullet(\mathfrak{m}; \mathfrak{h})$. Because the Lie superalgebra $\mathfrak{h}$ is $\mathbb{Z}$-graded, the differential has zero degree and hence $C^\bullet(\mathfrak{m}; \mathfrak{h})$ admits a decomposition into subcomplexes $C^{\bullet, \bullet}(\mathfrak{m}; \mathfrak{h})$ labelled by the degree. The subcomplex $C^{d, \bullet}(\mathfrak{m}; \mathfrak{h})$ is the Spencer complex of degree d and its cohomology $H^{d, \bullet}(\mathfrak{m}; \mathfrak{h})$ is the Spencer cohomology of degree d. Furthermore, the Spencer differential is $\mathfrak{h}_0$-equivariant and hence the Spencer cohomology is naturally an $\mathfrak{h}_0$-module.

Infinitesimal filtered deformations are classified by the Spencer cohomology group $H^{2,2}(\mathfrak{m}; \mathfrak{h})$. In [1] we compute this group by bootstrapping the simpler calculation of $H^{2,2}(\mathfrak{m}; \mathfrak{s})$, which yields an $\mathfrak{so}(V)$-module isomorphism

$$H^{2,2}(\mathfrak{m}; \mathfrak{s}) \cong \Lambda^4 V , \tag{18}$$

with the cocycle corresponding to $\varphi \in \Lambda^4 V$ given by $\beta \oplus \gamma$ in equation (17).

In particular, we see that the 4-form *emerges* from the calculation. If we put $\varphi = \frac{1}{24} F$, for F the 4-form flux in eleven-dimensional supergravity, we recognise at once the cochain β as the zeroth order term in the gravitino connection. In other words, we have in effect *derived* the gravitino connection

$$D_\nu s = \nabla_\nu s - \beta_\nu s = \nabla_\nu s - \nu \cdot \varphi \cdot s + 3\varphi \cdot \nu \cdot s \tag{19}$$

from the Spencer cohomology of the Poincaré superalgebra. The gravitino connection encodes all the information necessary to define the notion of a supersymmetric background. Indeed, a background is one for which the gamma-trace of the curvature of D vanishes [7] and it is supersymmetric precisely when D has nonzero kernel. Therefore one might be so bold as to say that, at least insofar as the bosonic backgrounds of the theory are concerned, we have re-derived eleven-dimensional supergravity. Of course, to actually re-derive the supergravity action, requires climbing the same mountain that Cremmer, Julia and Scherk did in [8], albeit departing from slightly higher ground.

5. Concluding remarks

It is possible to show that the Killing superalgebra (KSA) of *any* supersymmetric background of eleven-dimensional supergravity is a filtered deformation of a $\mathbb{Z}$-graded subalgebra of the Poincaré superalgebra $\mathfrak{s}$. (The proof will appear in a forthcoming paper.) It is thus not surprising that we should find the KSAs of the maximally supersymmetric backgrounds among the filtered deformations of a subalgebra of the form $\mathfrak{h}$ in equation (12). However it is important to emphasise that we did *not* set out to classify KSAs. That is a different problem, since the maps $\lambda, \tau, \rho, \beta, \gamma$ are then already partially determined *a priori*. So the surprising fact, albeit growing less surprising in the telling, is that we find *nothing else* but the KSAs.

Our derivation in [1] of the maximally supersymmetric backgrounds is to be contrasted with the standard derivation. This starts from the same initial data: namely, the Poincaré superalgebra $\mathfrak{s}$. Then one first finds evidence, again purely by representation-theoretic arguments, of the existence of a supergravity theory in eleven dimensions [9], one then constructs the supergravity theory [8] (which includes discovering D) and then solves the D-flatness equations [10]. Compared to the classical route to the result, one could perhaps argue that our recent approach enjoys a certain conceptual economy.

Finally, although we restricted ourselves in [1] to eleven-dimensional supergravity, we have done similar calculations in other dimensions, which will be the subject of forthcoming papers. In particular, in collaboration with Paul de Medeiros, we have performed this analysis for $N = 1$ $d = 4$ supergravity and can reproduce the Killing spinor equations in [11]. Our results have appeared in [12].

Acknowledgments

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MAXWELL INSTITUTE AND SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, PETER GUTHRIE TAIT ROAD, EDINBURGH EH9 3FD, UNITED KINGDOM