Robust Model Predictive Control: Robust Control Invariant Sets and Efficient Implementation

Chengyuan Liu

Department of Electrical and Electronic Engineering
Imperial College London

This thesis is submitted for the degree of
Doctor of Philosophy

March 2017

I would like to dedicate this thesis to my family

Declaration of Originality

I hereby confirm that this thesis is the result of my own research work. The ideas and results of other people have been properly referenced.

Chengyuan Liu
March 2017

Copyright Declaration

Acknowledgements

I would like to express my sincere gratitude to my supervisor Dr. Imad M. Jaimoukha for providing me this opportunity to learn deeply in robust control. His patient guidance and encouragement are essential in my research route. I was inspired by his creative ideas and passionate attitude to research. His knowledge and valuable advices always inspired me to go further in my study.

I would like to thank Dr. Paola Falugi, Dr. Eric C. Kerrigan, Dr. Ming Ge and Dr. Zhe Feng for their help and valuable advices. It is great enjoyment to discuss control knowledge with them and I can always get many novel ideas from them. I also want to take this chance to thank my friends Cheng Cheng, Jingye Sun, Fangjing Hu, Mingyang Sun, Yilun Zhou, Boli Chen, and Chen Wang. Their help and accompany provides me further strength to go on my journal of life. I am gratitude for the countless enjoyment they shared with me in my free time. Particularly thanks to Dr. Jingjing Jiang who provides me many precious advices in both study and life. I can always learned something from our discussion.

At last, I would like to thank my family for their selfless dedication. I cannot reach this stage without their encouragement. Words cannot express my appreciation of all the love and support I obtained from my parents and my brother.

Abstract

Robust model predictive control (RMPC) is widely used in industry. However, the online computational burden of this algorithm restricts its development and application to systems with relatively slow dynamics. We investigate this problem in this thesis with the overall aim of reducing the online computational burden and improving the online efficiency.

In RMPC schemes, robust control invariant (RCI) sets are vitally important in dealing with constraints and providing stability. They can be used as terminal (invariant) sets in RMPC schemes to reduce the online computational burden and ensure stability simultaneously. To this end, we present a novel algorithm for the computation of full-complexity polytopic RCI sets, and the corresponding feedback control law, for linear discrete-time systems subject to output and initial state constraints, performance bounds, and bounded additive disturbances. Two types of uncertainty, structured norm-bounded and polytopic uncertainty, are considered. These algorithms are then extended to deal with systems subject to asymmetric initial state and output constraints.

Furthermore, the concept of RCI sets can be extended to invariant tubes, which are fundamental elements in tube based RMPC scheme. The online computational burden of tube based RMPC schemes is largely reduced to the same level as model predictive control for nominal systems. However, it is important that the constraint tightening that is needed is not excessive, otherwise the performance of the MPC design may deteriorate, and there may even not exist a feasible control law. Here, the algorithms we proposed for RCI set approximations are extended and applied to the problem of reducing the constraint tightening in tube based RMPC schemes.

In order to ameliorate the computational complexity of the online RMPC algorithms, we propose an online-offline RMPC method, where a causal state feedback structure on the controller is considered. In order to improve the efficiency of the online computation, we calculate the state feedback gain offline using a semi-definite program (SDP). Then we propose a novel method to compute the control perturbation component online. The online op-

timization problem is derived using Farkas' Theorem, and then approximated by a quadratic program (QP) to reduce the online computational burden. A further approximation is made to derive a simplified online optimization problem, which results in a large reduction in the number of variables.

Numerical examples are provided that demonstrate the advantages of all our proposed algorithms over current schemes.

Table of contents

List of figures

List of tables

Notation

$\mathscr{R}$	The set of all real numbers
$\mathscr{R}^m$	The set of all real valued m-dimensional (column) vectors
$\mathscr{R}^{m\times n}$	The set of all $m \times n$ matrices with real entries
$\mathscr{R}_+$	The set of all positive real numbers
$(\cdot)^T$	The transpose of a matrix
$A \succ 0$	The symmetric matrix A is positive definite
$A \prec 0$	The symmetric matrix A is negative definite
$\mathscr{D}_+^m$	The set of positive semidefinite diagonal matrices of dimension $m \times m$
$\mathscr{S}_+^m$	The set of positive definite symmetric matrices of dimension $m \times m$
I_m	The $m \times m$ identity matrix
e_i	The ith column of I_m, where m is defined by the context
$0_{m\times n}$	The $m \times n$ null matrix, with the dimensions omitted when defined by the context
e	The vector of ones with the dimension deduced from the context
$\mathscr{I}_m$	$\mathscr{I}_m = \{1,\ldots,m\}$ for an integer $m \geq 1$
$\mathscr{P}(P,b,x_c)$	The polytope $\{x \in \mathscr{R}^n : -b \leq P(x-x_c) \leq b\}$ with center $x_c \in \mathscr{R}^n$, $P \in \mathscr{R}^{m\times n}$, $0 < b \in \mathscr{R}^m$, and $m \geq n$
$\mathscr{P}(P,b)$	The polytope $\mathscr{P}(P,b,0)$
$\mathscr{E}(Q,x_c)$	The ellipsoid $\{x \in \mathscr{R}^n : (x-x_c)^T Q(x-x_c) \leq 1\}$ with center $x_c \in \mathscr{R}^n$, and $Q \in \mathscr{S}_+^n$
$\mathscr{E}(Q)$	The ellipsoid $\mathscr{E}(Q,0)$
$\rho(\cdot)$	The spectral radius of a matrix
$\oplus$	Minkowski sum of two sets, that is $A \oplus B = \{a+b \vert a \in A, b \in B\}$
$\ominus$	Minkowski difference of two sets, that is $A \ominus B = \{c \vert c \oplus B \subseteq A\}$
$\otimes$	Kronecker Product
$vec(\cdot)$	The vectorization of a matrix
$\lvert A\rvert$	The matrix of the absolute values of the elements of the real matrix A

In addition, the comparison between two vectors is taken in element-wise with $a \geq b$ (also for $a > b$, $a \leq b$, and $a < b$). The block diagonal matrix with the ith diagonal block A_i is denoted as $diag(A_1,\ldots,A_m)$. A congruence transformation T for the matrix inequality $A \prec 0$ implies pre- and post- multiplication by T and T^T for the inequality to deduce $TAT^T \prec 0$. A

Schur complement corresponds to the result that $C - B^T A^{-1} B \prec 0$ is equivalent to $\begin{bmatrix} A & B \\ \star & C \end{bmatrix} \prec 0$ if $A = A^T \prec 0$ and $C = C^T$, where $\star$ denotes a term deduced from symmetry.

Acronyms

LMI	Linear Matrix Inequality
MPC	Model Predictive Control
NLMI	Non-linear Matrix Inequality
QP	Quadratic Program
RMPC	Robust Model Predictive Control
RCI	Robust Control Invariant
SDP	Semi-definite Program

Chapter 1

Introduction

Robust model predictive control (RMPC) has been widely employed in industrial processes, due mainly to its ability to deal with hard constraints compared to other control algorithms. However, its large online computational demands remain an obstacle to its adoption for fast systems. This thesis aims to reduce the online computational burden of this control method so that its use can be extended to systems with faster dynamics. A commonly used method to achieve this is to treat a robust control invariant (RCI) set as the terminal set in RMPC schemes [12] so that the control law may be switched to a pre-computed feedback controller once the online controller has steered the state inside the RCI set. Since the system state cannot in general converge to the origin in the presence of disturbances and uncertainties, the RCI set is usually used as a substitute for the origin and acts as the objective state set as stated in [40]. Recent developments in computational methods for RMPC design are reviewed in the following sections. In addition, tube based RMPC schemes as well as combined online/offline computational approaches are considered.

1.1 Background and Motivation

1.1.1 Computation of Robust ControI Invariant Sets

RCI sets determine a bounded region to which the system state can be confined, for all possible disturbances/uncertainties, through the application of a feedback control law [12, 13]. Therefore, RCI sets are widely used in the analysis and design of robust control schemes for disturbed or uncertain systems. In particular, these sets are of primary importance in establishing the stability and recursive feasibility of RMPC Schemes, see e.g. [38], [64] and the references therein. Invariant sets also form an important part of the tube based RMPC schemes [31], [37]. Furthermore, they serve as suitable target sets in robust time-optimal control schemes [26], [39].

Due to their great application in robust control, the problem of efficient computation of RCI sets has been studied extensively over the past few decades, see [12] and [13] for a comprehensive literature survey. The two main RCI set structures considered in the literature are Polytopic and Ellipsoidal [29]. Computational methods for ellipsoidal RCI sets are investigated in [30] and [36]. [9] and [48] provide algorithms for the calculation of polytopic RCI sets. For these structures, the problem of computing the minimal- as well as the maximal-volume RCI set is important and, in most cases, intractable. For example, it is shown in [29] that the exact computation of the minimal invariant set for uncertain systems involves the Minkowski's sum of infinitely many terms, which leads to intractability unless the system dynamics are nilpotent [39]. Therefore, most of the research has been focused on efficient computation of suitable inner/outer approximations to the maximal/minimal RCI set, which is discussed in the following paragraphs.

In [52], a method to compute an outer approximation of the minimal invariant set has been proposed for linear systems with additive disturbances. This approach was subsequently extended in [51] to the degenerate disturbance case. However, both these schemes employ a constant, pre-computed feedback control law which can lead to excessive conservatism. In [27] and [22], in contrast, methods are derived to compute control laws which render a fixed set invariant. It is clear that in order to optimize the size of the invariant set, the best approach is to simultaneously consider both the feedback control law and RCI set as decision variables in the optimization. In this regard, [60] presents a method to approximate both an ellipsoidal invariant set as well as the feedback control law. However, as discussed in [12], generally the polytopic representation is not only less conservative than its ellipsoidal counterpart but also more naturally captures the physical constraints on state and control variables. Therefore, we focus on Polytopic RCI sets in this thesis.

In [6] and [8], the invariance conditions for the computation of polytopic RCI sets for discrete time systems without uncertainty or disturbances have been presented. [21] [42] and [66] provide invariance conditions for polytopic RCI sets in the presence of additive disturbances. These conditions are extended to the parametric uncertainty case in [61], [7] and [41]. An algorithm, initially proposed in [25], computes the RCI set by iteratively imposing auxiliary constraints. This method has been extended to systems with polytopic uncertainty in [48]. Algorithms for computing RCI sets for linear systems with polytopic uncertainty and disturbances are also reported in [2] (for variable controllers) and [10] (for fixed controllers).

More recently, research in the literature has focused on the computation of low-complexity polytopic RCI sets, in which the the maximum number of faces of the polytope is equal to twice the dimension of the state-space. This is due to their computational advantages for the associated control schemes as well as their reduced conservatism in comparison to ellipsoidal RCI sets (see [14] for details). In [32], necessary and sufficient conditions are derived for the existence of a low-complexity polyhedral RCI set for discrete-time systems with uncertainty, and the set is computed, for a given feedback control law, by solving a quadratic optimization problem. A unified approach to determine the RCI set is proposed in [18]. An algorithm that optimizes both the polytopic RCI set and the feedback controller simultaneously is proposed in [17] for nonlinear systems, though the computation complexity for obtaining such an RCI set is substantially increased owing to the large number of variables involved. In [62], an efficient method was proposed to compute a hyper-rectangular RCI set (which is a special case of low-complexity polytopic set) and the corresponding control law in one step through a single semi-definite programming (SDP) problem. However, the hyper-rectangular set structure is, in general, conservative. Finally, in [14], a new algorithm to compute both the low-complexity RCI set as well as the corresponding control law has been proposed for systems with polytopic uncertainties.

Although less conservative than the ellipsoidal set, the low-complexity polytopic RCI set structure is still restrictive due to the restriction on the number of faces of the polytope. In this thesis, we address this issue by proposing a novel algorithm to efficiently compute full-complexity RCI sets, where arbitrarily large number of faces can be specified for the polytope, thereby enabling a more accurate inner/outer approximations to the maximal/minimal RCI sets (see Chapter 2 and 3 for details).

1.1.2 Robust Model Predictive Control Algorithm

Model predictive control (MPC) has been widely used for the industrial control of constrained systems. Good literature surveys about the algorithm and its application can be found in [1], [4], and [38]. Traditional MPC algorithms usually calculate a control sequence online by solving an optimization problem at each time step, and only implement the first control action. A new control sequence is computed again at the next time step. The optimization problem involving polyhedral constraints and a quadratic objective function, is solved iteratively at each time step. There are two main methods considered in MPC algorithm. Open-loop MPC utilizes the inherent robustness of the nominal MPC to obtain a sequence of control actions, examples are given in [68]. This method is not widely employed because it cannot predict the state trajectories resulting from the disturbances and uncertainties. This can be avoided by using feedback MPC [54], in which a control policy, which is a sequence of control laws, is treated as the decision variable. However, the computational burden of feedback MPC is very heavy and the process procedure is relatively slow and not applicable for fast dynamic systems with the presence of disturbances and uncertainties. Many methods are proposed in the literature to ease this problem. One of these methods is explicit MPC, which is an offline MPC algorithm that computes the control action offline and forms a lookup table. When implemented online, control actions are chosen based on the lookup table. [5] present a method to form the lookup table with a state feedback control law. This method cannot be implemented for large scale systems since the size of the lookup table will increase exponentially with the number of dimension of the system and the predictive horizon. Many other fast MPC algorithms have been proposed in recent years, for example, the primal barrier method [44], which transforms the constrained quadratic programming (QP) problem to an unconstrained log-barrier optimization problem. This method is not widely used because of several drawbacks. [67] provides some variations of this method. Fixing the structure of the controller, e.g. using state feedback or causal state feedback structure, is another way to improve the process speed.

Due to the presence of state and input constraints and the disturbances, infeasibility may result. Some constraint tightening methods were developed to solve this problem. [35] present a method involving some constraint restrictions to achieve robustness. These constraint restrictions are computed offline. Hence, no extra online computational burden is added to the MPC algorithm. On the basis of this method, [37] presented the novel concept of tube MPC, in which a piecewise affine control law is used and a tube, rather than a state trajectory, is designed to guide the system. This method presents a novel application of the minimal RCI set and reduces the computational burden of a disturbed system to the level of a nominal system and increases the robustness further. The invariant tube and related control law are computed offline and a nominal MPC algorithm is employed under the constraint of the tube. The computational complexity of this method increases linearly in the predictive horizon length, rather than exponentially as in the traditional RMPC algorithm. [31] provides a detailed description of this method and gives some variations. In [40], this method has been improved by considering the initial state of the nominal system as an optimizer, and the invariant tube and the objective RCI set (terminal set) are used simultaneously. [55]

provides a method to scale the rigid tubes and form a series of homothetic tubes, which achieves less conservativeness on online constraint handling. Based on this, [53] improves the scaling dynamic of the tubes and the computation of the terminal constraint set and introduces an equi-normalization process to relax the conditions for the application of the tube based RMPC algorithm. The existence and characterization of the minimal tube is verified in [53].

Tube based RMPC algorithms are computationally tractable due to the application of the tube. Accurate calculation of the tube is very important. In this thesis, we improved the computation method of the tube and provide approximate optimal controller and observer gains simultaneously, which reduce conservativeness of this algorithm. Details are given in Chapter 4.

In order to implement MPC algorithms in the presence of disturbances, min-max techniques are used and the optimization is reduced to a convex optimization problem involving linear matrix inequalities (LMIs) [30]. Although this method provides robust solutions in the presence of uncertainties, SDP problems slow the process speed further compared to QP problems. [63] presents two methods to calculate controllers with causal state feedback structure online by solving SDP problems involving LMIs. These methods improve the robustness and tractability of the MPC algorithm, but the online computational burden is large and the process speed is slow. In order to improve the process speed, we present an online-offline MPC method in this thesis. Linear discrete-time systems subject to additive bounded disturbance are considered. The online computational complexity is reduced by calculating part of the control law offline, and transforming the SDP problem into a QP problem using Farkas' Theorem for online optimization. Finally, a simplified online algorithm is proposed to reduce the computational burden further. The efficiency of these methods is tested by numerical examples. Details are given in Chapter 5.

1.2 Challenges and Contributions

1.2.1 Research Challenges

The most important challenges addressed in this thesis are listed below:

- *Full-complexity polytopic structure*. A tractable algorithm for the computation of full-complexity polytopic RCI sets is developed in this thesis in order to provide better accuracy compared with the structures considered in the literature.
- *Initial state and performance constraints*. Maximal RCI sets are required to be sufficiently small so that the performance is acceptable once the system states are included. Minimal RCI sets are often required to be sufficiently large so that the initial state set can be contained. Incorporating these constraints in the calculation process, which is currently missing in the literature, is significantly important and is presented in this thesis.
- *Asymmetric constraints*. The computation of RCI sets for systems subject to asymmetric constraints is seldom addressed in the literature, even though such constraints are not uncommon in real systems. This thesis develops a class of such algorithms.
- *Optimized observer gain*. The observer gain and the feedback control gain are normally chosen and fixed before online process in tube based MPC scheme. Optimizing the gains simultaneously with the invariant sets, which we carry out, provides better control accuracy and less conservativeness.
- *online-offline separation*. Due to the heavy computational burden of traditional online RMPC and the excessive conservativeness of the offline RMPC scheme, transferring part of the online computational burden offline, which we carry out, reduces the online computational burden without introducing excessive conservativeness.

1.2.2 Contributions

This thesis aims to reduce the online computational burden of RMPC scheme. Three methods are considered: treating an RCI set as a terminal set in RMPC schemes and thereby switching from the online RMPC scheme to a feedback control scheme, with a pre-computed fixed control law once the system states enter the RCI set; applying the RCI set computational algorithms thus developed to tube based RMPC schemes; implementing a combined online-offline RMPC scheme.

Chapter 2 presents a novel algorithm for the computation of full-complexity polytopic robust control invariant (RCI) sets, and the corresponding feedback control law, for linear discrete-time systems subject to output, initial state and performance constraints, with additive disturbances. Structured norm-bounded and polytopic uncertainties are also considered. The proposed scheme allows arbitrarily large number of faces for the invariant polytope which enables less conservative inner/outer approximations to the maximal/minimal RCI sets. The nonlinearities associated with the computation of such an RCI set structure are overcome through the application of corollary of Elimination Lemma. An initial full-complexity inner/outer approximation to the maximal/minimal RCI set as well as the feedback gain are computed through convex/LMI optimization. The volume of this initial invariant set is then iteratively optimized (minimized/maximized) based on a Newton-like update. The algorithm, based on convex/LMI optimizations, is shown to yield larger/smaller volume inner/outer approximations to maximal/minimal RCI sets as compared to other schemes from the literature.

In Chapter 3, asymmetric system constraints are considered. The center of the RCI set is shifted from the origin to provide more flexibility. Farkas' Theorem is first used to derive necessary and sufficient conditions for the existence of an admissible polytopic RCI set in the form of nonlinear matrix inequalities. Corollaries of Elimination Lemma are then used to derive sufficient conditions, in the form of linear matrix inequalities, for the existence of the solution. An optimization algorithm to approximate maximal/minimal RCI set is also proposed. The algorithm improves the accuracy of the computed maximal/minimal RCI set, while keeps the same computational complexity compared with the literature.

In Chapter 4, the tube based RMPC scheme proposed in [37] and [31] is improved by calculating the minimal approximations to the invariant tube. The algorithms proposed in Chapter 2 are modified to approximate the tube and optimizing the feedback control gain and the observer gain simultaneously. This RMPC scheme gives an effective compromise between computational complexity and optimality.

Chapter 5 proposes an online-offline RMPC scheme for linear discrete-time systems with bounded additive disturbance. We consider a causal state feedback structure on the controller, which comprises of a causal state feedback gain and a control perturbation component. The state feedback gain is calculated offline via SDP, the control perturbation component is computed online. Farkas' Theorem is applied to guarantee the satisfaction of the system constraints for all disturbances. The online optimization problem is derived as a QP problem to reduce the online computational burden. Further approximations are made to derive a simplified online optimization problem, which results in a large reduction in the number of variables.

1.2.3 Publications

Part of the research results in this thesis are based on the following papers which have been published or under review.

- Chengyuan Liu and Imad M. Jaimoukha, "The Computation of Full-complexity Polytopic Robust Control Invariant Sets", in proceedings of the IEEE Conference on Decision and Control, Osaka, Japan, 2015, pp. 6233-6238.
- Chengyuan Liu, Imad M. Jaimoukha and Furqan Tahir, "Full-complexity Polytopic Robust Control Invariant Sets for Uncertain Linear Systems", IEEE Transactions on Automatic Control (under review).

1.3 Techniques and Preliminaries

1.3.1 Major Definitions and Techniques

Some basic definitions and mathematic lemmas and theorem are listed in this section.

In this thesis, we consider a linear discrete time system with disturbances

$$\begin{bmatrix} x^+ \\ f \\ z \end{bmatrix} = \begin{bmatrix} A & B & B_w \\ C & D & D_w \\ C_2 & D_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \\ w \end{bmatrix}, \tag{1.1}$$

with

$$\begin{bmatrix} x^+ \\ f \\ z \end{bmatrix} \in \begin{bmatrix} \mathscr{R}^n \\ \mathscr{R}^{m_f} \\ \mathscr{R}^{m_2} \end{bmatrix}, \qquad \begin{bmatrix} x \\ u \\ w \end{bmatrix} \in \begin{bmatrix} \mathscr{R}^n \\ \mathscr{R}^{n_u} \\ \mathscr{R}^{n_w} \end{bmatrix},$$

where x is the current state, x^+ is the successor state and u and w denote the current control and disturbance signals, respectively, and where the distribution matrices in (1.1) have appropriate dimensions, and where the pair (A,B) is assumed to be stabilizable. We assume that the disturbance w belongs to a bounded polytope

$$\mathscr{W} := \mathscr{P}(V,d) \tag{1.2}$$

where

$$V \in \mathscr{R}^{m_w \times n_w}, \qquad 0 < d \in \mathscr{R}^{m_w} \tag{1.3}$$

are given. The output constraint signal f is given as $f \in \mathscr{F} \subseteq \mathscr{R}^{m_f}$. Note that we use a general form of constraint which involves all the constraints on states, outputs and inputs and their linear combinations.

The initial system state x_0 is required to be contained in a set, that is

$$x_0 \in \mathscr{X}_0 \subseteq \mathscr{R}^n. \tag{1.4}$$

In the sequel, we consider another type of constraint related to the signals z defined in (1.1).

Next, we introduce the concept of control invariance. The concept of an invariant set was introduced in [43], and was subsequently extended in many different ways, e.g. [3], [16], and [24]. In this thesis, we consider the following definition, given in [12] and [49].

Definition 1.1 (Robust Control Invariant Set)**.** *A set $\mathscr{P} \subseteq \mathscr{R}^n$ is a robust control invariant set for system* (1.1) *if $x^+ \in \mathscr{P}, \forall w \in \mathscr{W}$ for every $x \in \mathscr{P}$ and the output constraints $f \in \mathscr{F}$ are satisfied for every $x \in \mathscr{P}$, $\forall w \in \mathscr{W}$.*

While previous work on the computation of RCI sets uses Farkas' Lemma, we will use the following version of Farkas' Theorem instead since expressing the constraints in quadratic form will be shown to offer computational advantages.

Theorem 1.1 (Farkas' Theorem)**.** *[50] Suppose that $\mathscr{C} \subseteq \mathscr{R}^n$ and $f_1, \ldots, f_m : \mathscr{R}^n \to \mathscr{R}$ are convex and satisfy the Slater condition [20]. Let $f_0 : \mathscr{R}^n \to \mathscr{R}$ and define the system*

$$\mathscr{S} : \{f_0(x) < 0; f_j(x) \leq 0, j = 1, \ldots, m; x \in \mathscr{C}\}.$$

Consider the statements

(1) $\exists y_1, \ldots, y_m \geq 0 : f_0(x) + \sum_{j=1}^{m} y_j f_j(x) \geq 0, \forall x \in \mathscr{C}$.

(2) $\mathscr{S}$ is not solvable.

Then $(1) \Rightarrow (2)$*. Furthermore, if f_0 is convex, then* $(1) \Leftrightarrow (2)$*.*

The following version of the Elimination Lemma [15] will be used to deal with the nonlinearities.

Lemma 1.1 (Elimination Lemma)**.** *Let* $Q = Q^T \in \mathscr{R}^{n\times n}$*,* $R \in \mathscr{R}^{n\times m}$ *and* $S \in \mathscr{R}^{n\times p}$*. Let* $R_\perp$ *and* $S_\perp$ *be orthogonal complements of R and S, respectively, and consider the following two statements:*

(1) $\exists Y \in \mathscr{Y} \subseteq \mathscr{R}^{m\times p}: \; Q + RYS^T + SY^TR^T \succ 0.$

(2) $R_\perp^T QR_\perp \succ 0$ *and* $S_\perp^T QS_\perp \succ 0.$

Then $(1) \Rightarrow (2)$*. Furthermore, if* $\mathscr{Y} = \mathscr{R}^{m\times p}$ *then* $(1) \Leftrightarrow (2)$*.*

The next result follows from Farkas' Theorem and is used for norm–bounded structured uncertainties.

Lemma 1.2. *[23] Given* $T_1 = T_1^T \in \mathscr{R}^{r\times r}, T_2 \in \mathscr{R}^{r\times n_p}, T_3 \in \mathscr{R}^{n_q\times r}$ *and* $\mathbf{\Delta} \subseteq \mathscr{R}^{n_p\times n_q}$*. Define the subspace*

$$\mathscr{B} = \{(S,T,R) \in \mathscr{S}_+^{n_p} \times \mathscr{S}_+^{n_q} \times \mathscr{R}^{n_p\times n_q} : S\Delta = \Delta T,\; \Delta R^T + R\Delta^T = 0 \; \forall \Delta \in \mathbf{\Delta}\},$$

and consider the statements:

(1) $\exists (S,T,R) \in \mathscr{B} : \begin{bmatrix} T_1 - T_2ST_2^T & T_3^T + T_2R \\ \star & T \end{bmatrix} \succ 0.$

(2) $T_1 + T_2\Delta T_3 + (T_2\Delta T_3)^T \succ 0 \; \forall \Delta \in \mathbf{\Delta}, \|\Delta\| \leq 1.$

Then $(1) \Rightarrow (2)$*. Furthermore, if* $n_p = n_q$ *and* $\mathbf{\Delta} = \mathscr{R}^{n_p\times n_p}$*, then* $(1) \Leftrightarrow (2)$*.*

1.3.2 Preliminaries on MPC

Consider the linear discrete-time system (1.1) and regarding the MPC scheme, define the terminal constraint and cost signals

$$\begin{bmatrix} f(N) \\ z(N) \end{bmatrix} = \begin{bmatrix} \bar{C} & \bar{D}_w \\ \bar{C}_2 & 0 \end{bmatrix} \begin{bmatrix} x(N) \\ w(N) \end{bmatrix}, \tag{1.5}$$

where $x(k)$ denote the system state at kth step and N is the prediction horizon. The other symbols denote the appropriate transition and distribution matrices. The initial state $x_0 = x(0)$ for system (1.1) is constrained as

$$x_0 \in \mathscr{X}_0 := \{x_0 \in \mathscr{R}^n | \underline{x}_0 \leq x_0 \leq \bar{x}_0\} \tag{1.6}$$

where $\underline{x}_0 < 0 < \bar{x}_0 \in \mathscr{R}^n$ are given.

We define the objective function as

$$J = \sum_{k=0}^{N} z(k)^T z(k), \tag{1.7}$$

It is required that for all $k \in \{0, 1, \ldots N-1\}$, there exist a $u(k)$ such that the constraints:

$$\begin{aligned} f(k) \in \mathscr{F} &:= \{f \in \mathscr{R}^{m_f} | f \leq \bar{f},\ \forall w \in \mathscr{W}\}, \\ f(N) \in \mathscr{F}_N &:= \{f \in \mathscr{R}^{m_f} | f \leq \bar{f}_N,\ \forall w \in \mathscr{W}\}, \end{aligned} \tag{1.8}$$

are satisfied, and an upper bound γ on the cost function J, that is

$$J \leq \gamma,\ \forall w \in \mathscr{W},$$

is minimized.

Define the system state, constraint signal, cost signal, input, and disturbance sequences as

$$x := \begin{bmatrix} x(1) \\ x(2) \\ \vdots \\ x(N) \end{bmatrix}, f := \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N) \end{bmatrix}, z := \begin{bmatrix} z(0) \\ z(1) \\ \vdots \\ z(N) \end{bmatrix}, u := \begin{bmatrix} u(0) \\ u(1) \\ \vdots \\ u(N-1) \end{bmatrix}, w := \begin{bmatrix} w(0) \\ w(1) \\ \vdots \\ w(N) \end{bmatrix}, \tag{1.9}$$

respectively. We assume that at the current time step, the system state x_0 is measured and the input $u(0)$ can be calculated based on our MPC algorithm proposed in this thesis.

By iterating the system dynamics (1.1), we derive the following relations

$$\begin{bmatrix} x \\ f \\ z \end{bmatrix} = \begin{bmatrix} \Phi & G & G_w \\ \Phi_f & G_f & G_{fw} \\ \Phi_z & G_z & G_{zw} \end{bmatrix} \begin{bmatrix} x_0 \\ u \\ w \end{bmatrix}, \tag{1.10}$$

where the distribution matrices in (1.10) can be deduced from (1.1) and the definitions in (1.9).

The controller is defined in the causal state feedback structure, which includes a causal state feedback part that depends on the state trajectory and a control perturbation component. Therefore, we have

$$u = \bar{K}_0 x_0 + \bar{K}x + \bar{v} \tag{1.11}$$

where $\bar{K}_0 \in \mathscr{R}^{N\cdot n_u \times n}$ is the current state feedback gain, $\bar{K} \in \mathscr{R}^{N\cdot n_u \times N\cdot n}$ is the future feedback gain and $\bar{v} \in \mathscr{R}^{N\cdot n_u}$ is the control perturbation sequence. The structure of the matrix $[\bar{K}_0 \ \bar{K}]$ is block lower triangular, where the block dimension is $n_u \times n$, which ensures causality [63].

Substituting the state dynamic in (1.10) into (1.11) gives

$$u = K_0 x_0 + KG_w w + v \tag{1.12}$$

with $K_0 = (I - \bar{K}G)^{-1}(\bar{K}_0 + \bar{K}\Phi)$, $K = (I - \bar{K}G)^{-1}\bar{K}$ and $v = (I - \bar{K}G)^{-1}\bar{v}$. Note that K_0 and K have the same causal block structure as $\bar{K}_0$ and $\bar{K}$, and that $\bar{v}$, $\bar{K}_0$ and $\bar{K}$ are uniquely determined by v, K_0 and K. Substituting (1.12) into the constraint and cost signal dynamics in (1.10) gives

$$\begin{bmatrix} f \\ z \end{bmatrix} = \begin{bmatrix} E_f^{K_0} & G_f & E_{fw}^K \\ E_z^{K_0} & G_z & E_{zw}^K \end{bmatrix} \begin{bmatrix} x_0 \\ v \\ w \end{bmatrix},$$

with

$$\begin{aligned} E_f^{K_0} &= \Phi_f + G_f K_0, \quad & E_{fw}^K &= G_f K G_w + G_{fw}, \\ E_z^{K_0} &= \Phi_z + G_z K_0, \quad & E_{zw}^K &= G_z K G_w + G_{zw}. \end{aligned}$$

Then the objective function (1.7) can be rewritten as

$$J = z^T z = J_0 + J_1, \tag{1.13}$$

where

$$\begin{aligned} J_0 &= (E_z^{K_0} x_0 + E_{zw}^K w)^T (E_z^{K_0} x_0 + E_{zw}^K w), \\ J_1 &= v^T G_z^T G_z v + 2(G_z^T E_z^{K_0} x_0 + G_z^T E_{zw}^K w)^T v. \end{aligned}$$

The bounds on the disturbance sequence can be rewritten as

$$w \in \mathbb{W} := \{w \in \mathscr{R}^{(N+1)\cdot n_w} | -\bar{w} \leq w \leq \bar{w}\}, \tag{1.14}$$

where $\bar{w} := e \otimes \bar{w}$, and the constraint (1.8) becomes

$$f \in \mathbb{F} := \{f \in \mathscr{R}^{(N+1)\cdot m_f} | f \leq \bar{f},\ \forall w \in \mathbb{W}\}, \tag{1.15}$$

where $\bar{f} := e \otimes \bar{f}$.

Our aim is to minimize the objective function (1.13) in order to obtain an optimal controller of structure (1.12) under the constraints (1.15). When minimizing the objective function (1.13), the unmeasured disturbances become a barrier. In the literature, min-max methods and SDP optimization are the two mainly used algorithms. Although they provide valuable results, there is scope to improve the efficiency of the algorithms. Our algorithms, proposed in the following chapters, which improve the computational efficiency compared with the literature, are approximations of the following SDP method presented in [63].

Theorem 1.2 (SDP method)**.** *With all definitions as above, the following optimization problem will provide optimal feasible $K_0 \in \mathscr{R}^{N\cdot n_u \times n}$, $K \in \mathscr{R}^{N\cdot n_u \times N\cdot n}$, $v \in \mathscr{R}^{N\cdot n_u}$:*

$$\min_{K_0, K, v} \gamma \tag{1.16}$$

$$s.t. \begin{bmatrix} D_c & 0 & (E_{zw}^K)^T \\ \star & \gamma - \bar{w}^T D_c \bar{w} & v^T G_z^T + x_0^T (E_z^{K_0})^T \\ \star & \star & I \end{bmatrix} \succ 0;$$

$$\begin{bmatrix} D_e^i & -(E_{fw}^K)^T e_i \\ \star & 2e_i^T(\bar{f} - E_f^{K_0} x_0 - G_f v) - \bar{w}^T D_e^i \bar{w} \end{bmatrix} \succ 0,\ \forall i \in \mathscr{I}_{(N+1)\cdot m_f}.$$

where $D_c, D_e^i \in \mathscr{D}_+^{(N+1)\cdot n_w}$.

This theorem can be proved based on the results in [63]. It gives sufficient conditions for the existence of a control law of the form (1.12) that guarantees an upper bound on the cost function in (1.13) and gives necessary and sufficient conditions for satisfying the constraints in (1.15) for all w satisfying (1.14). These conditions are in the form of LMIs and provide an SDP algorithm to minimize the upper bound.

Chapter 2

Computation of Robust Control Invariant Sets

Robust control invariant (RCI) sets are important in control analysis. These sets can be used as the target set of the model predictive control (MPC) scheme. They also play an important role in the design of tube based MPC scheme.

In this chapter, we propose a novel algorithm to efficiently compute full-complexity RCI sets, where arbitrarily large number of faces can be specified for the polytope, thereby enabling less conservative and more accurate inner/outer approximations to the maximal/minimal RCI sets than the ellipsoidal and low-complexity polytopic approximations. The proposed method considers linear discrete-time systems subject to additive disturbances and computes an initial full-complexity inner/outer approximation to the maximal/minimal RCI set as well as the feedback gain through convex/LMI optimization. The nonlinearities associated with the computation of such an RCI set structure are overcome through the application of corollary of Elimination Lemma. Structured norm-bounded and polytopic uncertainty are considered separately. An update algorithm is then proposed that iteratively increases/reduces the volume of inner/outer bounding ellipsoids on this initial invariant set, and computes the corresponding feedback gain to obtain improved approximations to the maximal/minimal RCI set. The update algorithm is convex and is based on a Newton-like update. Both conservatism and computational complexity are therefore reduced compared with the published literature. Through numerical examples, it is shown that the proposed algorithm can result in a substantially improved RCI set (volume-wise) as compared to some of the other schemes in the literature.

2.1 Robust control invariant Set

Standard procedures of calculating admissible RCI sets require a pre-defined structure for computational tractability. A full-complexity polytopic structure, which provides higher precision than the previous published structures, will be considered in this thesis. This has the form

$$\mathscr{P}(P,b) = \{x \in \mathscr{R}^n : -b \leq Px \leq b\},$$

where $0 < b \in \mathscr{R}^m$, $P \in \mathscr{R}^{m \times n}$ and $m \geq n$; m can be chosen based on the required accuracy. Note that for $m = n$, $\mathscr{P}(P,b)$ reduces to a low-complexity polytope (see e.g. [14, 65]).

The requirements on an RCI set [49] include invariance and output constraint satisfaction. For the system and disturbances described by (1.1)-(1.3), set $\mathscr{P}(P,b)$ and for given $f \in \mathscr{F} := \mathscr{P}(I_{m_f}, \bar{f})$, where $0 < \bar{f} \in \mathscr{R}^{m_f}$, these can be written as

$$\left\{ \begin{array}{c} x \in \mathscr{P}(P,b) \\ w \in \mathscr{W} \end{array} \right\} \Rightarrow x^+ \in \mathscr{P}(P,b), \qquad \text{(Invariance)} \qquad (2.1)$$

$$\left\{ \begin{array}{c} x \in \mathscr{P}(P,b) \\ w \in \mathscr{W} \end{array} \right\} \Rightarrow f \in \mathscr{F}, \qquad \text{(Output constraint)} \qquad (2.2)$$

respectively. Since RCI sets are in general not unique, maximal (minimal) RCI sets are defined in terms of unions (intersections) of all such sets. For RCI sets of a pre–defined structure, this definition is modified to optimize the volume of these sets.

Maximal RCI sets are associated with target sets in the state–space and are required to be large, since they are associated with switching from on–line to off–line control once the state is inside the set. Since a direct characterization of the volume of a polytope is not feasible when $m > n$, we introduce an inner bounding ellipsoid $\mathscr{E}(\underline{Q})$, require

$$\exists \underline{Q} \in \mathscr{S}_+^n : \mathscr{E}(\underline{Q}) \subset \mathscr{P}(P,b) \qquad \text{(Inner bounding ellipsoid)} \qquad (2.3)$$

and maximize $\log\det(\underline{Q}^{-1})$, since the volume of the ellipsoid $\mathscr{E}(\underline{Q})$ is proportional to $\det(\underline{Q}^{-1})$. Maximal RCI sets are also required to be sufficiently small so that the performance is acceptable once the state is inside. While the volume is to some extent limited by the output constraint requirement, we further impose a performance constraint. With $z(k)$ denoting the current cost signal, where z defined in (1.1), and r a required performance level, we require

$$x \in \mathscr{P}(P,b) \Rightarrow J_{\mathscr{H}} := \sum_{k=0}^{\infty} \|z(k)\|^2 < r^2. \qquad (\mathscr{H}_2 \text{ constraint}) \tag{2.4}$$

Minimal RCI sets are associated with initial states, and are required to be small. To this end, we introduce an outer bounding ellipsoid $\mathscr{E}(\bar{Q})$, such that

$$\exists \bar{Q} \in \mathscr{S}_+^n : \mathscr{P}(P,b) \subset \mathscr{E}(\bar{Q}) \qquad \text{(Outer bounding ellipsoid)} \tag{2.5}$$

and minimize $\log\det(\bar{Q}^{-1})$. Minimal sets are also often required to include an initial state set as defined in (1.4) [31] and so, for given $\mathscr{X}_0 := \mathscr{P}(P_0, b_0)$, where $P_0 \in \mathscr{R}^{m_0 \times n}$ and $0 < b_0 \in \mathscr{R}^{m_0}$, we require

$$\mathscr{P}(P_0, b_0) \subseteq \mathscr{P}(P,b). \qquad \text{(Initial condition constraint)} \tag{2.6}$$

For given system (1.1), sets $\mathscr{W}, \mathscr{F}, \mathscr{P}(P_0, b_0)$, parameter r and $m \geq n$, and with $\Psi := \mathscr{R}^{m \times n} \times \mathscr{R}^m \times \mathscr{R}^{n_u \times n}$, we present convex algorithms, based on semidefinite programs (SDP), to solve the optimizations:

$$\max_{\substack{(P,b,K) \in \Psi \\ (2.1),(2.2),(2.3),(2.4)}} \log\det \underline{Q}^{-1}, \tag{2.7}$$

$$\min_{\substack{(P,b,K) \in \Psi \\ (2.1),(2.2),(2.5),(2.6)}} \log\det \bar{Q}^{-1}. \tag{2.8}$$

A triple $(P,b,K) \in \Psi$ satisfying either of the constraints in (2.7) or (2.8) will be called admissible.

Remark 2.1. *Systems subject to asymmetric constraints and a more general form of the full-complexity polytopic set $\mathscr{P}(P,b,x_c) = \{x \in \mathscr{R}^n : -b \leq P(x - x_c) \leq b\}$ will be considered in Chapter 3. The nonlinearities caused by the variable x_c lead to mathematical difficulties in both the initial linearization and the update procedure. Since it is more common, we consider this specific form $\mathscr{P}(P,b)$ and the symmetric constraint first in this Chapter for clarity.*

2.2 Nonlinear formulation

In this section, we derive conditions, in the form of nonlinear matrix inequalities (NLMIs), for the admissibility of $(P,b,K) \in \Psi$. While previous work uses Farkas' Lemma, we will use the Farkas' Theorem (Theorem 1.1) instead since expressing the constraints in quadratic form will be shown to offer computational advantages.

The following result uses Theorem 1.1 to derive conditions, in the form of NLMIs, for the existence of an admissible triple $(P,b,K) \in \Psi$ [34].

Theorem 2.1. *Let all definitions be as above and denote*

$$A^K := A + BK, \qquad C^K := C + DK, \qquad C_2^K := C_2 + D_2K.$$

Then for $(P,b,K) \in \Psi$ *we have:*

1. *The invariance condition* (2.1) *is satisfied if and only if*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} D_i \in \mathscr{D}_+^m \\ W_i \in \mathscr{D}_+^{m_w} \end{bmatrix} : \left[\begin{array}{c|c} 2e_i^T b - b^T D_i b - d^T W_i d & e_i^T P \begin{bmatrix} B_w & A^K \end{bmatrix} \\ \hline \star & \begin{bmatrix} V^T W_i V & 0 \\ \star & P^T D_i P \end{bmatrix} \end{array}\right] \succ 0. \tag{2.9}$$

2. *The output constraint condition* (2.2) *is satisfied if and only if,*

$$\forall j \in \mathscr{I}_{m_f}, \exists \begin{bmatrix} E_j \in \mathscr{D}_+^m \\ G_j \in \mathscr{D}_+^{m_w} \end{bmatrix} : \begin{bmatrix} 2e_j^T \bar{f} - b^T E_j b - d^T G_j d & e_j^T D_w & e_j^T C^K \\ \star & V^T G_j V & 0 \\ \star & \star & P^T E_j P \end{bmatrix} \succ 0. \tag{2.10}$$

3. *The inner bounding ellipsoid condition* (2.3) *is satisfied if and only if*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \mu_i > 0 \\ \underline{Q} \in \mathscr{S}_+^n \end{bmatrix} : \begin{bmatrix} 2e_i^T b - \mu_i & e_i^T P \\ \star & \mu_i \underline{Q} \end{bmatrix} \succ 0. \tag{2.11}$$

4. *The* $\mathscr{H}_2$*-norm condition* (2.4) *is satisfied if*

$$\exists \begin{bmatrix} Q \in \mathscr{S}_+^n \\ D_z \in \mathscr{D}_+^m \end{bmatrix} : \begin{bmatrix} Q & 0 & A^K \\ \star & rI & C_2^K \\ \star & \star & Q^{-1} \end{bmatrix} \succ 0, \quad \begin{bmatrix} Q & I \\ \star & P^T D_z P \end{bmatrix} \succ 0, \quad r > b^T D_z b. \tag{2.12}$$

5. *The outer bounding ellipsoid condition* (2.5) *is satisfied if*

$$\exists \begin{bmatrix} \bar{D} \in \mathscr{D}_+^m \\ \bar{Q} \in \mathscr{S}_+^n \end{bmatrix} : P^T \bar{D} P - \bar{Q} \succ 0, \quad 1 > b^T \bar{D} b. \tag{2.13}$$

6. *The initial state constraint condition* (2.6) *is satisfied if and only if*

$$\forall i \in \mathscr{I}_m, \exists F_i \in \mathscr{D}_+^{m_0} : \begin{bmatrix} 2e_i^T b - b_0^T F_i b_0 & e_i^T P \\ \star & P_0^T F_i P_0 \end{bmatrix} \succ 0. \tag{2.14}$$

Hence solutions to the optimizations (2.7) *and* (2.8) *can be obtained by solving the nonlinear SDPs*

$$\max_{\substack{(P,b,K)\in\Psi \\ (2.9),(2.10),(2.11),(2.12)}} \log\det \underline{Q}^{-1}, \tag{2.15}$$

$$\min_{\substack{(P,b,K)\in\Psi \\ (2.9),(2.10),(2.13),(2.14)}} \log\det \bar{Q}^{-1}, \tag{2.16}$$

respectively.

Proof. The proof is an application of Farkas' Theorem. In more detail:

1. Condition (2.1) is equivalent to the requirement that for all $i \in \mathscr{I}_m$,

$$\left\{ \begin{array}{c} (e_j^T P x)^2 - (e_j^T b)^2 \leq 0, \forall j \in \mathscr{I}_m \\ (e_k^T V w)^2 - (e_k^T d)^2 \leq 0, \forall k \in \mathscr{I}_{m_w} \end{array} \right\} \Rightarrow 2e_i^T \left(b - P((A+BK)x + B_w w)\right) \geq 0.$$

   The result then follows from Theorem 1.1 based on the following identity

$$\begin{aligned} 2e_i^T(P((A+BK)x+B_w w)-b) = &-(b^T D_i b - x^T P^T D_i P x) \\ &-(d^T W_i d - w^T V^T W_i V w) \\ &-a^T N_i a \end{aligned} \tag{2.17}$$

where $a^T := \begin{bmatrix} -1 & w^T & x^T \end{bmatrix}$, and

$$N_i := \begin{bmatrix} 2e_i^T b - b^T D_i b - d^T W_i d & e_i^T P B_w & e_i^T P(A+BK) \\ \star & V^T W_i V & 0 \\ \star & \star & P^T D_i P \end{bmatrix}.$$

For $D_i \in \mathscr{D}_+^m$ and $W_i \in \mathscr{D}_+^{m_w}$, the first and second terms on the RHS of (2.17) are non-positive for all $x \in \mathscr{P}(P,b,x_c)$ and all $w \in \mathscr{W}$, it follows that the invariance condition is satisfied if $N_i \succ 0$, which proves the sufficiency of (2.9). Necessity follows from Farkas' Theorem.

2. The proof is similar to Part 1) and follows from Theorem 1.1 and some manipulations.

3. Condition (2.3) is equivalent to the requirement that for all $i \in \mathscr{I}_m$,

$$x^T \underline{Q} x - 1 \leq 0 \Rightarrow 2e_i^T(b - Px) \geq 0.$$

The result then follows from Theorem 1.1.

4. For any $x \in \mathscr{P}(P,b)$, a minor extension of the results in [30] gives the first inequality in (2.12) and

$$r - x^T Q^{-1} x \geq 0 \tag{2.18}$$

as sufficient conditions for $J_{\mathscr{H}} < r^2$. Theorem 1.1 then gives the second and third inequalities in (2.12) as sufficient conditions for (3.18) to be satisfied for all $x \in \mathscr{P}(P,b)$.

5. Condition (2.5) is equivalent to the requirement that

$$(e_j^T Px)^2 - (e_j^T b)^2 \leq 0 \; \forall j \in \mathscr{I}_m \Rightarrow 1 - x^T \bar{Q} x \geq 0.$$

The result then follows from Theorem 1.1.

6. Condition (2.6) is equivalent to the requirement that for all $i \in \mathscr{I}_m$,

$$(e_j^T P_0 x)^2 - (e_j^T b_0)^2 \leq 0 \; \forall j \in \mathscr{I}_{m_0} \Rightarrow 2e_i^T(b - Px) \geq 0.$$

The result then follows from Theorem 1.1.

Finally, (2.15) and (2.16) follows from 1-6 above. □

Remark 2.2. *We have opted for strict inequalities in* (2.9)-(2.14) *in order to avoid numerical difficulties associated with optimality. It follows that, in common with other LMI problems [58], the algorithms resulting from the use of Theorem 2.1 may become badly conditioned near optimality.*

2.3 Linearization and Initial Computation

While Theorem 2.1 gives necessary and sufficient conditions for the the triple (P,b,K) to be admissible, the conditions are nonlinear. The nonlinearities have three forms: the first occurs in the term $e_i^T PA^K$, the second in the terms $P^T D_i P$, $P^T E_j P$, $P^T D_z P$, $P^T \bar{D} P$, $b^T D_i b$, $b^T E_j b$, and $b^T D_z b$, and the third in the terms $\mu_i \underline{Q}$ and Q^{-1}. In the low-complexity case, the variable P is square and can be assumed nonsingular, and this forms the basis for efficient approximation procedures in [65]. In the full-complexity case treated here, we propose a linearization algorithm involving the computation of an initial solution. An update algorithm is then presented in the next section.

We set

$$\mathscr{P}(P,b) = \mathscr{P}(P_r X, b_r) = \{x \in \mathscr{R}^n : -b_r \leq P_r X x \leq b_r\}$$

as an initial full-complexity inner/outer approximation to the maximal/minimal RCI set, where b_r and P_r are given (see Remark 2.3 below), and where $X \in \mathscr{R}^{n \times n}$ is a variable used to reshape (rotate and scale) the polyhedral set $\mathscr{P}(P_r, b_r)$.

The following is a corollary of the Elimination Lemma (Lemma 1.1) and is used for the initial linear solution.

Corollary 2.1. *[34] Given $T \in \mathscr{S}_+^n, E \in \mathscr{R}^{n \times p}, F \in \mathscr{R}^{p \times m}, Z \in \mathscr{S}_+^m$ and $\mathscr{Y} \subseteq \mathscr{R}^{p \times p}$. Consider the statements:*

(1) $M := \begin{bmatrix} T & EY & 0 \\ \star & Y^T + Y & F \\ \star & \star & Z \end{bmatrix} \succ 0$ *holds for some* $Y \in \mathscr{Y}$.

(2) $N := \begin{bmatrix} T & EF \\ \star & Z \end{bmatrix} \succ 0.$

Then $(1) \Rightarrow (2)$. *Furthermore, if* $\mathscr{Y} = \mathscr{R}^{p\times p}$, *then* $(1) \Leftrightarrow (2)$.

Proof. Write M as $M = Q + RYS^T + SY^TR^T$ where

$$\left[\begin{array}{c|c} Q & R \\ \hline S^T & \star \end{array}\right] = \left[\begin{array}{ccc|c} T & 0 & 0 & E \\ 0 & 0 & F & I \\ 0 & F^T & Z & 0 \\ \hline 0 & I & 0 & \star \end{array}\right].$$

Since

$$R_\perp = \begin{bmatrix} I & 0 \\ -E^T & 0 \\ 0 & I \end{bmatrix}, \quad S_\perp = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix},$$

are orthogonal complements of R and S, respectively, the result follows from the Elimination Lemma upon noting that $R_\perp^T Q R_\perp = N$ and

$$S_\perp^T Q S_\perp = \begin{bmatrix} T & 0 \\ 0 & Z \end{bmatrix}.$$

□

The following result gives sufficient conditions for the admissibility of the triple (P_rX, b_r, K) in the form of LMIs by using Corollary 2.1.

Theorem 2.2. *[34] Let all the definitions be as above and let* $P = P_rX$ *and* $b = b_r$, *where* $P_r \in \mathscr{R}^{m\times n}$ *and* $b_r \in \mathscr{R}^m$ *are given and where* $X \in \mathscr{R}^{n\times n}$. *Denote*

$$\hat{X} := X^{-1}, \qquad \hat{K} := KX^{-1}, \qquad \hat{A} := A\hat{X} + B\hat{K},$$
$$\hat{C} := C\hat{X} + D\hat{K}, \qquad \hat{C}_2 := C_2\hat{X} + D_2\hat{K}.$$

Then

1. *Condition* (2.9), *hence* (2.1), *is satisfied if*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \lambda_i > 0 \\ \hat{D}_i \in \mathscr{D}_+^m \\ \hat{W}_i \in \mathscr{D}_+^{m_w} \end{bmatrix} : \begin{bmatrix} 2\lambda_i e_i^T b_r - b_r^T \hat{D}_i b_r - d^T \hat{W}_i d & \lambda_i e_i^T P_r & 0 & 0 \\ \star & \hat{X} + \hat{X}^T & B_w & \hat{A} \\ \star & \star & V^T \hat{W}_i V & 0 \\ \star & \star & \star & P_r^T \hat{D}_i P_r \end{bmatrix} \succ 0. \qquad (2.19)$$

2. *Condition* (2.10)*, hence* (2.2)*, is satisfied if and only if*

$$\forall j \in \mathscr{I}_{m_f}, \exists \begin{bmatrix} E_j \in \mathscr{D}_+^m \\ G_j \in \mathscr{D}_+^{m_w} \end{bmatrix} : \begin{bmatrix} 2e_j^T \bar{f} - b_r^T E_j b_r - d^T G_j d & e_j^T D_w & e_j^T \hat{C} \\ \star & V^T G_j V & 0 \\ \star & \star & P_r^T E_j P_r \end{bmatrix} \succ 0. \quad (2.20)$$

3. *Condition* (2.11)*, hence* (2.3)*, is satisfied if*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \hat{\mu}_i > 0 \\ \gamma_i > 0 \\ \underline{Q}^{-\frac{1}{2}} \in \mathscr{S}_+^n \end{bmatrix} : \begin{bmatrix} 2\gamma_i e_i^T b_r - \hat{\mu}_i & \gamma_i e_i^T P_r & 0 \\ \star & \hat{X} + \hat{X}^T & \underline{Q}^{-\frac{1}{2}} \\ \star & \star & \hat{\mu}_i I_n \end{bmatrix} \succ 0. \quad (2.21)$$

4. *Condition* (2.12)*, hence* (2.4)*, is satisfied if*

$$\exists \begin{bmatrix} \zeta > 0 \\ \hat{Q} \in \mathscr{S}_+^n \\ \hat{D}_z \in \mathscr{D}_+^m \end{bmatrix} : \begin{bmatrix} \hat{Q} & 0 & \hat{A} \\ \star & (2-\zeta) r I & \hat{C}_2 \\ \star & \star & \hat{X} + \hat{X}^T - \hat{Q} \end{bmatrix} \succ 0, \begin{bmatrix} \hat{Q} & \hat{X} \\ \star & P_r^T \hat{D}_z P_r \end{bmatrix} \succ 0, \zeta r > b_r^T \hat{D}_z b_r. \quad (2.22)$$

5. *Condition* (2.13)*, hence* (2.5)*, is satisfied if*

$$\exists \begin{bmatrix} \bar{D} \in \mathscr{D}_+^m \\ \bar{Q}^{-1} \in \mathscr{S}_+^n \end{bmatrix} : \begin{bmatrix} \bar{Q}^{-1} & \hat{X} \\ \star & P_r^T \bar{D} P_r \end{bmatrix} \succ 0, \quad 1 > b_r^T \bar{D} b_r. \quad (2.23)$$

6. *Condition* (2.14)*, hence* (2.6)*, is satisfied if and only if*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \nu_i > 0 \\ \hat{F}_i \in \mathscr{D}_+^{m_0} \end{bmatrix} : \begin{bmatrix} 2\nu_i e_i^T b_r - b_0^T \hat{F}_i b_0 & \nu_i e_i^T P_r & 0 \\ \star & \hat{X} + \hat{X}^T & I_n \\ \star & \star & P_0^T \hat{F}_i P_0 \end{bmatrix} \succ 0. \quad (2.24)$$

Hence initial solutions to the optimizations (2.7) *and* (2.8) *can be obtained by solving the convex SDPs*

$$\max_{\substack{\hat{X}, \hat{K} \\ (2.19),(2.20),(2.21),(2.22)}} \log\det \underline{Q}^{-\frac{1}{2}}, \quad (2.25)$$

$$\min_{\substack{\hat{X}, \hat{K} \\ (2.19),(2.20),(2.23),(2.24)}} trace(\bar{Q}^{-1}), \quad (2.26)$$

respectively.

Proof. The proof consists in manipulating each of (2.9)-(2.14) into the form of statement (2) of Corollary 2.1 and then use the corollary to show that (2.19)-(2.24), after some manipulation, correspond to statement (1) with an appropriate Y, and are therefore sufficient conditions for (2.9)-(2.14) and hence for (2.1)-(2.6), respectively. In detail:

1. Applying Corollary 2.1 on (2.9) (with $E = e_i^T P_r X$ and $Y = \lambda_i X^{-1}$), effecting a Schur complement and the congruence $diag(\lambda_i^{\frac{1}{2}}, \lambda_i^{-\frac{1}{2}} I_n, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{\frac{1}{2}} X^{-1})$ shows that (2.19) implies (2.9) upon the redefinitions
$$\hat{D}_i := \lambda_i D_i,\ \hat{W}_i := \lambda_i W_i. \tag{2.27}$$

2. Effecting the congruence $diag(1, I_{n_w}, X^{-T})$ on (2.10) shows that it is equivalent to (2.20).

3. Applying Corollary 2.1 (with $E = e_i^T P_r X$ and $Y = \gamma_i X^{-1}$) on (2.11), implementing the congruence $diag(\gamma_i^{\frac{1}{2}}, \gamma_i^{-\frac{1}{2}} I_n, \gamma_i^{\frac{1}{2}} \underline{Q}^{-\frac{1}{2}})$ shows that (2.21) implies (2.11) upon the redefinition
$$\hat{\mu}_i := \gamma_i \mu_i. \tag{2.28}$$

4. Effecting the congruence $diag(\zeta^{-\frac{1}{2}} I_n, \zeta^{\frac{1}{2}} X^{-T})$ shows that the second inequalities in (2.12) and (2.22) are equivalent while the third inequality in (2.22) is ζ times the third inequality in (2.12). Effecting the congruence $diag(I_n, 1, Q)$ and applying Corollary 2.1 on the first inequality in (2.12) (with $F = Q$ and $Y = \zeta X^{-1}$) followed by a Schur complement, the congruence $diag(\zeta^{-\frac{1}{2}} I_n, \zeta^{-\frac{1}{2}}, \zeta^{-\frac{1}{2}} I_n)$ shows that the first inequality in (2.22) implies the first inequality in (2.12) since $\zeta^{-1} \geq 2 - \zeta$ for all $\zeta > 0$ upon the redefinitions
$$\hat{Q} := \zeta^{-1} Q,\ \hat{D}_z := \zeta D_z. \tag{2.29}$$

5. For the first inequality in (2.13), effecting the congruence X^{-T} and then using a Schur complement shows that (2.23) is equivalent to (2.13).

6. Applying Corollary 2.1 (with $E = e_i^T P_r X$ and $Y = \nu_i X^{-1}$) on (2.14) and implementing the congruence $diag(\nu_i^{\frac{1}{2}}, \nu_i^{-\frac{1}{2}} I_n, \nu_i^{\frac{1}{2}} I_n)$ shows that (2.24) implies (2.14) upon the

redefinition

$$\hat{F}_i := \nu_i F_i. \tag{2.30}$$

Finally, (2.25) and (2.26) follows from 1-6 above and the fact that $\det(Z) \leq \left(\frac{\operatorname{trace}(Z)}{n}\right)^n$ for any $n \times n$ positive definite matrix Z. $\square$

Note that the optimizations derived in the above theorem have variables $\hat{X}$, $\hat{K}$, $\underline{Q}^{-\frac{1}{2}}$, ζ, $\hat{Q}$, $\hat{D}_z$, $\bar{D}$, $\bar{Q}^{-1}$; λ_i, $\hat{D}_i$, $\hat{W}_i$, $\hat{\mu}_i$, γ_i, ν_i, $\hat{F}_i$ for $i \in \mathscr{I}_m$; E_j, G_j for $j \in \mathscr{I}_{m_f}$.

Remark 2.3. *The conservatism introduced by the linearization in Theorem 2.2, compared to Theorem 2.1, can be traced back to the use of Corollary 2.1 and to the choice of the initial polytope $\mathscr{P}(P_r, b_r)$. Note that we restrict $\mathscr{Y}$ for a tractable solution. Although this restriction can be relaxed, the resulting optimization becomes nonlinear. We present one possible relaxation in Theorem 2.4 below, where the nonlinearity involves a scalar variable and is thus tractable. In our examples, we used the vector of ones for b_r and the regular polytope with $2m$ faces for P_r. Since $P = P_r X$, then X provides scaling and rotational degrees of freedom. Another possible choice of $\mathscr{P}(P_r, b_r)$ is outlined in the next theorem.*

While the existence of polytopic RCI sets is not in general guaranteed - see [12, 8] for more details - the next result, based on an idea in [14], gives conditions which guarantee the existence of the initial solution in Theorem 2.2 satisfying the invariance and inner bounding ellipsoidal conditions (2.1) and (2.3), respectively, (which correspond to conditions (2.19) and (2.21) in Theorem 2.2, respectively) in the special case of no disturbances, uncertainties, $\mathscr{H}_2$, output or initial state constraints.

Theorem 2.3. *Suppose that $\|A + BK\|^2 < n^{-1}$ and let $m = ln$ where l is any integer greater than 0. Let $b_r \in \mathscr{R}^m$ be the vector of ones and let $P_r = \begin{bmatrix} U_1^T & \cdots & U_l^T \end{bmatrix}^T \in \mathscr{R}^{m \times n}$ for any orthogonal $U_j \in \mathscr{R}^{n \times n}$. Then there exist $\hat{X} \in \mathscr{R}^{n \times n}$ and $\hat{K} \in \mathscr{R}^{n_u \times n}$ such that*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \lambda_i > 0 \\ \hat{D}_i \in \mathscr{D}_+^m \end{bmatrix} : L_i := \begin{bmatrix} 2\lambda_i e_i^T b_r - b_r^T \hat{D}_i b_r & \lambda_i e_i^T P_r & 0 \\ \star & \hat{X} + \hat{X}^T & A\hat{X} + B\hat{K} \\ \star & \star & P_r^T \hat{D}_i P_r \end{bmatrix} \succ 0,$$

and (2.21) *are satisfied.*

Proof. Let $\hat{X} = I_n, \hat{K} = K$ and $\underline{Q} = I_n$, and, for every $i \in \mathscr{I}_m$, define the unique integers l_i and n_i such that $1 \leq l_i \leq l, 1 \leq n_i \leq n, i = (l_i - 1)n + n_i$ and define $\hat{D}_i = \frac{1}{n}(e_{l_i} e_{l_i}^T) \otimes I_n$ and

$\lambda_i = \hat{\mu}_i = \gamma_i = 1$, where $\otimes$ denotes the Kronecker product. Then

$$L_i = \begin{bmatrix} 1 & e_{n_i}^T U_{l_i} & 0 \\ \star & 2I & A+BK \\ \star & \star & \frac{1}{n} I_n \end{bmatrix}.$$

Using an upper Schur complement,

$$\begin{aligned} L_i \succ 0 &\Leftrightarrow \begin{bmatrix} I + U_{l_i}^T (I - e_{n_i} e_{n_i}^T) U_{l_i} & A+BK \\ \star & \frac{1}{n} I_n \end{bmatrix} \succ 0 \\ &\Leftrightarrow \begin{bmatrix} I & A+BK \\ \star & \frac{1}{n} I_n \end{bmatrix} + \begin{bmatrix} U_{l_i}^T (I - e_{n_i} e_{n_i}^T) U_{l_i} & 0 \\ \star & 0 \end{bmatrix} \succ 0. \end{aligned}$$

Therefore $L_i \succ 0$ since

$$\|A+BK\|^2 < n^{-1} \Leftrightarrow \begin{bmatrix} I & A+BK \\ \star & \frac{1}{n} I_n \end{bmatrix} \succ 0,$$

and $U_{l_i}^T (I - e_{n_i} e_{n_i}^T) U_{l_i} \succeq 0$. A similar procedure proves that (2.21) is satisfied. $\square$

Remark 2.4. *Note that if (A,B) is controllable, there exists $Q \in \mathscr{S}_+^n$ such that $(A+BK)^T Q (A+BK) \prec Q/n$. Letting $Q = T^{-T} T^{-1}$, effecting the similarity transformation $(A,B) \to (T^{-1}AT, T^{-1}B)$, and redefining the problem appropriately, shows that we can assume, without loss of generality, that $\|A+BK\|^2 < n^{-1}$. Although Theorem 2.3 can be applied to obtain an initial solution, our experience indicates that the solution of Theorem 2.2 gives a much better initial solution. We only give it to show that, under certain conditions, the existence of our initial solution is guaranteed.*

The following theorem provides a possible relaxation for the restriction on $\mathscr{Y}$ (see Remark 2.3).

Theorem 2.4. *Let all the definitions be as above, condition* (2.9)*, hence* (2.1)*, is satisfied if*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \alpha > 0 \\ \lambda_i > 0 \\ \hat{D}_i \in \mathscr{D}_+^m \\ \hat{W}_i \in \mathscr{D}_+^{m_w} \end{bmatrix} : \begin{bmatrix} 2\lambda_i e_i^T b_r - b_r^T \hat{D}_i b_r - d^T \hat{W}_i d & \lambda_i e_i^T P_r - \alpha e_i^T P_r \hat{X}^T & 0 & 0 & 0 \\ \star & \hat{X} + \hat{X}^T & 2\hat{X} v_i \alpha & B_w & \hat{A} \\ \star & \star & 2\lambda_i \alpha & 0 & 0 \\ \star & \star & \star & V^T \hat{W}_i V & 0 \\ \star & \star & \star & \star & P_r^T \hat{D}_i P_r \end{bmatrix} \succ 0. \tag{2.31}$$

Proof. Implementing Corollary 2.1 for the invariance condition (2.9), with

$$E = e_i^T P_r X,\ Y = \lambda_i X^{-1} - X^{-1} v_i \alpha v_i^T X^{-T},$$

where v_i is any vector such that $e_i^T P_r v_i \neq 0$ (otherwise, it is not difficult to show that (2.31) offers no advantage over (2.19)). For simplicity, we choose $v_i = P_r^T e_i / \|P_r^T e_i\|$. Then effecting the congruence

$$diag(\lambda_i^{\frac{1}{2}}, \lambda_i^{-\frac{1}{2}} I_n, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{\frac{1}{2}} X^{-1})$$

and Schur complement for the $(2,2)$ entry shows that (2.31) implies (2.9) upon the redefinitions $\hat{D}_i = \lambda_i D_i$, $\hat{W}_i = \lambda_i W_i$. □

This relaxation introduces a variable α which makes the resulting invariance condition nonlinear. Note that, in the limit, as $\alpha \to 0$, then (2.31) (after removing the third block rows and columns) is the same as (2.19). It follows that if (2.19) is not feasible, it may be possible that (2.31) is feasible for some $\alpha > 0$. This may reduce the conservatism to some extent. Since the NLMI in (2.31) is linear for a given α and since α is a scalar, it is tractable to use a gridding on α and a simple search algorithm to find a feasible solution to (2.31).

2.4 Update Computation Algorithm

Once an admissible initial triple $(P, b, K) \in \Psi$ is obtained, this section presents an algorithm to update the solution based on the following result.

Lemma 2.1. *Let $\boldsymbol{L}, L \in \mathscr{R}^{m\times n}$ and $\boldsymbol{D}, D \in \mathscr{S}_+^m$. Define*

$$\mathscr{L}_{\boldsymbol{L},\boldsymbol{D}}^{L,D} := \boldsymbol{L}^T D^{-1} L + L^T D^{-1} \boldsymbol{L} - L^T D^{-1} \boldsymbol{D} D^{-1} L, \tag{2.32}$$

$$\mathscr{N}_{\boldsymbol{L},\boldsymbol{D}} := \boldsymbol{L}^T \boldsymbol{D}^{-1} \boldsymbol{L}. \tag{2.33}$$

Then $\mathscr{N}_{\boldsymbol{L},\boldsymbol{D}} \succeq \mathscr{L}_{\boldsymbol{L},\boldsymbol{D}}^{L,D}$ and $\mathscr{N}_{L,D} = \mathscr{L}_{L,D}^{L,D}$. Hence,

$$\{\exists L \in \mathscr{R}^{m\times n}, D \in \mathscr{S}_+^m : \mathscr{N}_{L,D} \succ 0\} \Rightarrow \{\exists \boldsymbol{L} \in \mathscr{R}^{m\times n}, \boldsymbol{D} \in \mathscr{S}_+^m : \mathscr{N}_{\boldsymbol{L},\boldsymbol{D}} \succeq \mathscr{L}_{\boldsymbol{L},\boldsymbol{D}}^{L,D} \succ 0\}.$$

Proof. The proof follows from the identity

$$\mathscr{N}_{\boldsymbol{L},\boldsymbol{D}} = \mathscr{L}_{\boldsymbol{L},\boldsymbol{D}}^{L,D} + (\boldsymbol{L} - \boldsymbol{D} D^{-1} L)^T \boldsymbol{D}^{-1} (\boldsymbol{L} - \boldsymbol{D} D^{-1} L). \tag{2.34}$$

$\square$

Remark 2.5. *Note that if $\mathscr{M}_{\mathbf{L},\mathbf{D}}$ is any linear matrix function of $\mathbf{L}$ and $\mathbf{D}$, then the linear matrix equation $\mathscr{M}_{\mathbf{L},\mathbf{D}} + \mathscr{L}_{\mathbf{L},\mathbf{D}}^{L,D} = 0$ is the Newton update for the nonlinear matrix* equation *$\mathscr{M}_{\mathbf{L},\mathbf{D}} + \mathscr{N}_{\mathbf{L},\mathbf{D}} = 0$ from the initial approximation L, D [47].*

The next result extends this idea to derive Newton–like updates for the nonlinear matrix inequalities of Theorem 2.1 starting from the initial approximations given in Theorem 2.2.

Theorem 2.5. *With all definitions as above and $\mathscr{N}_{\cdot,\cdot}$ and $\mathscr{L}_{\cdot,\cdot}^{\cdot,\cdot}$ as defined in* (2.32) *and* (2.33)*, respectively, let $(\boldsymbol{P},\boldsymbol{b},\boldsymbol{K}) \in \Psi$. Then*

1. *Suppose that $(P,b,K,D_i,W_i), \forall i \in \mathscr{I}_m$ satisfy* (2.9)*. Then*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \boldsymbol{D}_i^{-1} \in \mathscr{D}_+^m \\ \boldsymbol{W}_i \in \mathscr{D}_+^{m_w} \end{bmatrix} : \left[\begin{array}{c|c} \mathscr{M}_i(\boldsymbol{D}_i^{-1},\boldsymbol{b},\boldsymbol{W}_i) + \mathscr{L}_{L_i(\boldsymbol{P},\boldsymbol{K}),F_i(\boldsymbol{D}_i^{-1})}^{L_i(P,K),F_i(D_i^{-1})} & \star \\ \hline EL_i(\boldsymbol{P},\boldsymbol{K}) & I_n \end{array}\right] \succ 0, \qquad (2.35)$$

where $E = [\,-I_n \;\; I_n \;\; 0], F_i(D_i) = \mathrm{diag}(I_n, I_n, D_i)$,

$$\mathscr{M}_i(\boldsymbol{D}_i^{-1},\boldsymbol{b},\boldsymbol{W}_i) = \begin{bmatrix} \boldsymbol{D}_i^{-1} & \boldsymbol{b} & 0 & 0 \\ \star & 2e_i^T\boldsymbol{b} - d^T\boldsymbol{W}_i d & 0 & 0 \\ \star & \star & V^T\boldsymbol{W}_i V & 0 \\ \star & \star & \star & 0 \end{bmatrix},$$

and

$$L_i(P,K) = \begin{bmatrix} 0 & P^Te_i & 0 & 0 \\ 0 & 0 & B_w & A^K \\ 0 & 0 & 0 & P \end{bmatrix}.$$

Furthermore, (2.9) *and* (2.1) *are satisfied by*

$$(P,b,K,D_i,W_i) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{K},\boldsymbol{D}_i,\boldsymbol{W}_i).$$

2. *Suppose that* $(P,b,K,E_j,G_j), \forall j \in \mathscr{I}_{m_f}$ *satisfy* (2.10). *Then*

$$\forall j \in \mathscr{I}_{m_f}, \exists \begin{bmatrix} \boldsymbol{E}_j^{-1} \in \mathscr{D}_+^m \\ \boldsymbol{G}_j \in \mathscr{D}_+^{m_w} \end{bmatrix} : \begin{bmatrix} \boldsymbol{E}_j^{-1} & \boldsymbol{b} & 0 & 0 \\ \star & 2e_j^T \bar{f} - d^T \boldsymbol{G}_j d & e_j^T D_w & e_j^T C^{\boldsymbol{K}} \\ \star & \star & V^T \boldsymbol{G}_j V & 0 \\ \star & \star & \star & \mathscr{L}_{\boldsymbol{P},\boldsymbol{E}_j^{-1}}^{P,E_j^{-1}} \end{bmatrix} \succ 0. \quad (2.36)$$

Furthermore, (2.10) *and* (2.2) *are satisfied by*

$$(P,b,K,E_j,G_j) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{K},\boldsymbol{E}_j,\boldsymbol{G}_j).$$

3. *Suppose that* $(P,b,\mu_i,\underline{Q}), \forall i \in \mathscr{I}_m$ *satisfy* (2.11). *Then*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \boldsymbol{\mu}_i > 0 \\ \underline{\boldsymbol{Q}}^{-\frac{1}{2}} \in \mathscr{S}_+^n \end{bmatrix} : \begin{bmatrix} 2e_i^T \boldsymbol{b} - \boldsymbol{\mu}_i & e_i^T \boldsymbol{P} & 0 \\ \star & 2\mu_i \underline{Q} & \mu_i \underline{Q}\underline{\boldsymbol{Q}}^{-\frac{1}{2}} \\ \star & \star & \boldsymbol{\mu}_i I_n \end{bmatrix} \succ 0. \quad (2.37)$$

Furthermore, (2.11) *and* (2.3) *are satisfied by*

$$(P,b,\mu_i,\underline{Q}) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{\mu}_i,\underline{\boldsymbol{Q}}).$$

4. *Suppose that* (P,b,K,Q,D_z) *satisfy* (2.12). *Then*

$$\exists \begin{bmatrix} \boldsymbol{Q} \in \mathscr{S}_+^n \\ \boldsymbol{D}_z^{-1} \in \mathscr{D}_+^m \end{bmatrix} : \begin{bmatrix} \boldsymbol{Q} & 0 & A^{\boldsymbol{K}} \\ \star & rI_{m_2} & C_2^{\boldsymbol{K}} \\ \star & \star & \mathscr{L}_{I_n,\boldsymbol{Q}}^{I_n,Q} \end{bmatrix} \succ 0, \begin{bmatrix} \boldsymbol{Q} & I_n \\ \star & \mathscr{L}_{\boldsymbol{P},\boldsymbol{D}_z^{-1}}^{P,D_z^{-1}} \end{bmatrix} \succ 0, \begin{bmatrix} \boldsymbol{D}_z^{-1} & \boldsymbol{b} \\ \star & r \end{bmatrix} \succ 0. \quad (2.38)$$

Furthermore, (2.12) *and* (2.4) *are satisfied by*

$$(P,b,K,Q,D_z) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{K},\boldsymbol{Q},\boldsymbol{D}_z).$$

5. *Suppose that* $(P,b,\bar{Q},\bar{D})$ *satisfy* (2.13). *Then*

$$\exists \begin{bmatrix} \bar{\boldsymbol{D}}^{-1} \in \mathscr{D}_+^m \\ \bar{\boldsymbol{Q}} \in \mathscr{S}_+^n \end{bmatrix} : \mathscr{L}_{\boldsymbol{P},\bar{\boldsymbol{D}}^{-1}}^{P,\bar{D}^{-1}} - \bar{\boldsymbol{Q}} \succ 0, \begin{bmatrix} \bar{\boldsymbol{D}}^{-1} & \boldsymbol{b} \\ \star & 1 \end{bmatrix} \succ 0. \quad (2.39)$$

Furthermore, (2.13) *and* (2.5) *are satisfied by*

$$(P,b,\bar{Q},\bar{D}) := (\boldsymbol{P},\boldsymbol{b},\bar{\boldsymbol{Q}},\bar{\boldsymbol{D}}).$$

6. *Suppose that* $(P,b,F_i), \forall i \in \mathscr{I}_m$ *satisfy* (2.14)*. Then*

$$\forall i \in \mathscr{I}_m, \exists \boldsymbol{F}_i \in \mathscr{D}_+^{m_0} : \begin{bmatrix} 2e_i^T \boldsymbol{b} - b_0^T \boldsymbol{F}_i b_0 & e_i^T \boldsymbol{P} \\ \star & P_0^T \boldsymbol{F}_i P_0 \end{bmatrix} \succ 0. \tag{2.40}$$

Furthermore, (2.14) *and* (2.6) *are satisfied by*

$$(P,b,F_i) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{F}_i).$$

Hence, if $\underline{Q}^{-\frac{1}{2}}$ *and* $\bar{Q}$ *are solutions to the optimization problems in* (2.25) *and* (2.26)*, respectively, then*

$$\left(\max_{\substack{\boldsymbol{P},\boldsymbol{b},\boldsymbol{K} \\ (2.35),(2.36),(2.37),(2.38)}} \log\det \underline{\boldsymbol{Q}}^{-\frac{1}{2}} \right) \geq \log\det \underline{Q}^{-\frac{1}{2}}, \tag{2.41}$$

$$\left(\min_{\substack{\boldsymbol{P},\boldsymbol{b},\boldsymbol{K} \\ (2.35),(2.36),(2.39),(2.40)}} -\log\det \bar{\boldsymbol{Q}} \right) \leq -\log\det \bar{Q}. \tag{2.42}$$

Proof. The proof is essentially an application of Lemma 2.1, congruences, Schur complements and some re–definitions to show that (2.9)-(2.14) imply (2.35)-(2.40), which in turn imply (2.9)-(2.14) (with bold variables) and therefore (2.1)-(2.6) (with bold variables), respectively, from Theorem 2.1. In more detail:

1. Effecting an upper Schur complement on $b^T D_i b$, a manipulation shows that (2.9) is equivalent to

$$\mathscr{M}_i(D_i^{-1},b,W_i) + \mathscr{N}_{L_i(P,K),F_i(D_i^{-1})} - L_i(P,K)^T E^T E L_i(P,K) \succ 0. \tag{2.43}$$

The result follows by applying Lemma 2.1 on the second term and a Schur complement on the third.

2. This follows by applying a Schur complement on $b^T E_j b$ and Lemma 2.1 on $\mathscr{N}_{P,E_j^{-1}} = P^T E_j P$ in (2.10).

3. This follows by applying Lemma 2.1 on $\mathscr{N}_{I,(\mu_i \underline{Q})^{-1}} = \underline{Q}\mu_i$ and taking a Schur complement in (2.11).

4. The result follows by applying Lemma 2.1 on $\mathscr{N}_{I,Q} = Q^{-1}$ and $\mathscr{N}_{P,D_z^{-1}} = P^T D_z P$ and a Schur complement on the third inequality in (2.12).

5. The second inequality in (2.39) and (2.13) are equivalent using a Schur complement argument. The result follows by applying Lemma 2.1 on the term $\mathscr{N}_{P,\bar{D}^{-1}} = P^T \bar{D} P$ in the first inequality in (2.13).

6. The result is trivially satisfied since (2.14) is linear in the variables.

Finally, (2.41) and (2.42) follow from 1-6 above. □

Note that the variables in the optimizations of the above theorem are emphasised in bold.

Remark 2.6. *Note that taking $\boldsymbol{D}_i^{-1}, \boldsymbol{E}_j^{-1}, \boldsymbol{D}_z^{-1}, \bar{\boldsymbol{D}}^{-1}$ as the update variables allows us to use Lemma 2.1 to ensure recursive feasibility, that is, the volume of the updated inner/outer approximation to the maximal/minimal RCI set is at least as good (large for maximal and small for minimal sets) as that of the previous set. It also allows us to use $\boldsymbol{b}$ as a variable, thus improving the updated solution. Our numerical experience, part of which is reported below, as well as Remark 2.5, suggest quadratic convergence, although a formal proof of this is beyond the scope of this work.*

2.5 Solution algorithm

The following algorithm summarizes our solution.

Algorithm 2.1. *Given system* (1.1), $\mathscr{F} = \mathscr{P}(I_{m_f}, \bar{f})$ *and sets* $\mathscr{W} = \mathscr{P}(V,d), \mathscr{P}(P_0, b_0)$ *and parameter r.*

1. **Initial data***: Choose $m \geq n$, initial polytope $\mathscr{P}(P_r, b_r)$ and tolerance level tol.*

2. **Initial solution**

(a) *Use Theorem 2.2 to solve the convex SDPs in* (2.25) *or* (2.26).

(b) *Define* D_i, W_i, Q, D_z *and* μ_i *from* (2.27)-(2.30) *so that* (2.9)-(2.10), (2.11) *or* (2.13), (2.12) *or* (2.14) *are satisfied.*

3. **Update** *Solve the optimizations in* (2.41) *or* (2.42).

4. **Stopping condition**

(a) *If* $\det(\underline{\boldsymbol{Q}}^{-1}) - \det(\underline{Q}^{-1}) \leq tol$ *(for maximization) or* $\det(\bar{Q}^{-1}) - \det(\bar{\boldsymbol{Q}}^{-1}) \leq tol$ *(for minimization), stop.*

(b) *Else update* $Z := \mathbf{Z}$, *where* $\mathbf{Z}$ *denotes a variable in the optimizations in* (2.41) *or* (2.42), *and go to step 3.*

5. **End**

2.6 Norm-Bounded Uncertainty

Previous sections provide an efficient algorithm to calculate the approximate maximal/minimal RCI set for discrete time system subject to additive disturbance. In this section, we extend these results to systems subject to structured norm-bounded uncertainty and additive disturbance as follows:

$$\begin{bmatrix} x^+ \\ f \\ z \end{bmatrix} = \begin{bmatrix} \tilde{A} & \tilde{B} & B_w \\ C & D & D_w \\ C_2 & D_2 & 0 \end{bmatrix} \begin{bmatrix} x \\ u \\ w \end{bmatrix}, \qquad u = Kx, \tag{2.44}$$

with

$$\begin{bmatrix} x^+ \\ f \\ z \end{bmatrix} \in \begin{bmatrix} \mathscr{R}^n \\ \mathscr{R}^{m_f} \\ \mathscr{R}^{m_2} \end{bmatrix}, \qquad \begin{bmatrix} x \\ u \\ w \end{bmatrix} \in \begin{bmatrix} \mathscr{R}^n \\ \mathscr{R}^{n_u} \\ \mathscr{R}^{n_w} \end{bmatrix},$$

where $\tilde{A}$ and $\tilde{B}$ belong to the norm-bounded structured uncertainty set:

$$\Omega := \{(\tilde{A}, \tilde{B}) : [\tilde{A}\ \tilde{B}] = [A\ B] + B_p \Delta [C_q\ D_q],\ \|\Delta\| \leq 1,\ \Delta \in \mathbf{\Delta} \subseteq \mathscr{R}^{n_p \times n_q}\}, \tag{2.45}$$

where A, B, B_p, C_q and D_q are given matrices with appropriate dimensions, and where it is assumed that the subspace $\mathscr{B}$ associated with the structured subspace $\mathbf{\Delta}$ defined in Lemma 1.2 may be characterized. All the other notations and constraints are defined as before.

The following subsections will provide nonlinear and linearized conditions for the existence triple $(P, b, K) \in \Psi$ for uncertain systems, and the optimization algorithm to obtain the approximate maximal/minimal RCI set.

2.6.1 Robust control invariant Set

The invariance and $\mathscr{H}_2$ constraint requirements become

$$\left\{\begin{array}{c} x \in \mathscr{P}(P,b) \\ w \in \mathscr{W} \\ (\tilde{A},\tilde{B}) \in \Omega \end{array}\right\} \Rightarrow x^+ \in \mathscr{P}(P,b) \qquad \text{(Invariance)} \tag{2.46}$$

$$\left\{\begin{array}{c} x \in \mathscr{P}(P,b) \\ (\tilde{A},\tilde{B}) \in \Omega \end{array}\right\} \Rightarrow J := \sum_{k=0}^{\infty} \|z_k\|^2 < r^2 \qquad (\mathscr{H}_2 \text{ constraint}) \tag{2.47}$$

respectively. All the other requirements are the same as (2.2)-(2.6) since they do not involve the uncertain parameters $(\tilde{A}, \tilde{B})$.

For given system (2.44), sets $\Omega, \mathscr{W}, \mathscr{F}, \mathscr{P}(P_0, b_0)$, parameter r and $m \geq n$, and with $\Psi := \mathscr{R}^{m\times n} \times \mathscr{R}^m \times \mathscr{R}^{n_u \times n}$, we present convex algorithms, based on semidefinite programs (SDP), to solve the optimizations:

$$\max_{\substack{(P,b,K)\in\Psi \\ (2.46),(2.2),(2.3),(2.47)}} \log\det \underline{Q}^{-1}, \tag{2.48}$$

$$\min_{\substack{(P,b,K)\in\Psi \\ (2.46),(2.2),(2.5),(2.6)}} \log\det \bar{Q}^{-1}. \tag{2.49}$$

A triple $(P, b, K) \in \Psi$ satisfying either of the constraints in (2.48) or (2.49) will be called admissible.

2.6.2 Nonlinear formulation

The following theorem uses Theorem 1.1 and Lemma 1.2 to derive invariance and $\mathscr{H}_2$ constraint conditions, in the form of NLMIs, for the existence of an admissible triple $(P,b,K) \in \Psi$.

Theorem 2.6. *Let all definitions be as above and denote*

$$A^K := A + BK, \qquad C^K := C + DK,$$
$$C_q^K := C_q + D_q K, \qquad C_2^K := C_2 + D_2 K.$$

Then for $(P,b,K) \in \Psi$ we have:

1. *The invariance condition* (2.46) *is satisfied if (and only if when $n_p = n_q$ and $\mathbf{\Delta} = \mathscr{R}^{n_p \times n_p}$)*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} D_i \in \mathscr{D}_+^m \\ W_i \in \mathscr{D}_+^{m_w} \\ (S_i,T_i,R_i) \in \mathscr{B} \end{bmatrix} : \left[\begin{array}{c|c} 2e_i^T b - b^T D_i b - d^T W_i d & e_i^T P \begin{bmatrix} B_w & B_p S_i & B_p R_i & A^K \end{bmatrix} \\ \hline \star & \begin{bmatrix} V^T W_i V & 0 & 0 & 0 \\ \star & S_i & 0 & 0 \\ \star & \star & T_i & C_q^K \\ \star & \star & \star & P^T D_i P \end{bmatrix} \end{array}\right] \succ 0. \tag{2.50}$$

2. *The $\mathscr{H}_2$–norm condition* (2.4) *is satisfied if*

$$\exists \begin{bmatrix} Q \in \mathscr{S}_+^n \\ D_z \in \mathscr{D}_+^m \\ (S,T,R) \in \mathscr{B} \end{bmatrix} : \begin{bmatrix} Q - B_p S B_p^T & 0 & B_p R & A^K \\ \star & rI & 0 & C_2^K \\ \star & \star & T & C_q^K \\ \star & \star & \star & Q^{-1} \end{bmatrix} \succ 0, \begin{bmatrix} Q & I \\ \star & P^T D_z P \end{bmatrix} \succ 0, r > b^T D_z b. \tag{2.51}$$

Hence solutions to the optimizations (2.48) *and* (2.49) *can be obtained by solving the nonlinear SDPs*

$$\max_{\substack{(P,b,K)\in\Psi \\ (2.50),(2.10),(2.11),(2.51)}} \log\det \underline{Q}^{-1}, \tag{2.52}$$

$$\min_{\substack{(P,b,K)\in\Psi \\ (2.50),(2.10),(2.13),(2.14)}} \log\det \bar{Q}^{-1}, \tag{2.53}$$

respectively.

Proof. The proof is an application of Lemma 1.2 and Farkas' Theorem. In more detail:

1. Condition (2.46) is equivalent to the requirement that for all $i \in \mathscr{I}_m$ and for all $(\tilde{A}, \tilde{B}) \in \Omega$,

$$\left\{ \begin{array}{c} (e_j^T Px)^2 - (e_j^T b)^2 \le 0, \forall j \in \mathscr{I}_m \\ (e_k^T Vw)^2 - (e_k^T d)^2 \le 0, \forall k \in \mathscr{I}_{m_w} \end{array} \right\} \Rightarrow 2e_i^T \left(b - P((\tilde{A} + \tilde{B}K)x + B_w w) \right) \ge 0.$$

   Similar to the proof of Part 1 of Theorem 2.1, applying Farkas' Theorem and then, additionally, employing Lemma 1.2 as well as some manipulations gives the result.

2. Similar to the proof of Part 4 of Theorem 2.1, for any $x \in \mathscr{P}(P, b)$, a minor extension of the results in [30] gives the first inequality in (2.51) and

$$r - x^T Q^{-1} x \ge 0 \tag{2.54}$$

   as sufficient conditions for $J < r^2, \forall (\tilde{A}, \tilde{B}) \in \Omega$. Theorem 1.1 then gives the second and third inequalities in (2.51) as sufficient conditions for (2.54) to be satisfied for all $x \in \mathscr{P}(P, b)$.

Finally, (2.52) and (2.53) follows from 1-2 above and Theorem 2.1. □

2.6.3 Linearization and Initial Computation

The following result gives sufficient invariance and $\mathscr{H}_2$ constraint conditions for the admissibility of the triple $(P_r X, b_r, K)$ in the form of LMIs by using Corollary 2.1.

Theorem 2.7. *[34] Let all the definitions be as above and let $P = P_r X$ and $b = b_r$, where $P_r \in \mathscr{R}^{m \times n}$ and $b_r \in \mathscr{R}^m$ are given and where $X \in \mathscr{R}^{n \times n}$. Denote*

$$\hat{X} := X^{-1}, \qquad \hat{K} := KX^{-1}, \qquad \hat{A} := A\hat{X} + B\hat{K},$$
$$\hat{C} := C\hat{X} + D\hat{K}, \qquad \hat{C}_q := C_q\hat{X} + D_q\hat{K}, \qquad \hat{C}_2 := C_2\hat{X} + D_2\hat{K}.$$

Then

1. *Condition* (2.50)*, hence* (2.46)*, is satisfied if*

$$\forall i\in\mathscr{I}_m,\exists\begin{bmatrix}\lambda_i>0\\ \hat{D}_i\in\mathscr{D}_+^m\\ \hat{W}_i\in\mathscr{D}_+^{m_w}\\ (\hat{S}_i,\hat{T}_i,\hat{R}_i)\in\mathscr{B}\end{bmatrix}:\begin{bmatrix}2\lambda_i e_i^T b_r-b_r^T\hat{D}_i b_r-d^T\hat{W}_i d & \lambda_i e_i^T P_r & 0 & 0 & 0\\ \star & \hat{X}+\hat{X}^T-B_p\hat{S}_iB_p^T & B_w & B_p\hat{R}_i & \hat{A}\\ \star & \star & V^T\hat{W}_iV & 0 & 0\\ \star & \star & \star & \hat{T}_i & \hat{C}_q\\ \star & \star & \star & \star & P_r^T\hat{D}_iP_r\end{bmatrix}\succ 0. \tag{2.55}$$

2. *Condition* (2.51)*, hence* (2.47)*, is satisfied if*

$$\exists\begin{bmatrix}\zeta>0\\ \hat{Q}\in\mathscr{S}_+^n\\ \hat{D}_z\in\mathscr{D}_+^m\\ (\hat{S},\hat{T},\hat{R})\in\mathscr{B}\end{bmatrix}:\begin{bmatrix}\hat{Q}-B_p\hat{S}B_p^T & 0 & B_p\hat{R} & \hat{A}^K\\ \star & (2-\zeta)rI & 0 & \hat{C}_2\\ \star & \star & \hat{T} & \hat{C}_q\\ \star & \star & \star & \hat{X}+\hat{X}^T-\hat{Q}\end{bmatrix}\succ 0,$$

$$\begin{bmatrix}\hat{Q} & \hat{X}\\ \star & P_r^T\hat{D}_zP_r\end{bmatrix}\succ 0,\ \zeta r>b_r^T\hat{D}_zb_r. \tag{2.56}$$

Hence initial solutions to the optimizations (2.48) *and* (2.49) *can be obtained by solving the convex SDPs*

$$\max_{\substack{\hat{X},\hat{K}\\ (2.55),(2.20),(2.21),(2.56)}} \log\det \underline{Q}^{-\frac{1}{2}}, \tag{2.57}$$

$$\min_{\substack{\hat{X},\hat{K}\\ (2.55),(2.20),(2.23),(2.24)}} trace(\bar{Q}^{-1}), \tag{2.58}$$

respectively.

Proof. The proof consists in manipulating each of (2.50)-(2.51) into the form of statement (2) of Corollary 2.1 and then use the corollary to show that (2.55)-(2.56), after some manipulation, correspond to statement (1) with an appropriate Y, and are therefore sufficient conditions for (2.50)-(2.51) and hence for (2.46)-(2.47), respectively. In detail:

1. Applying Corollary 2.1 on (2.50) (with $E=e_i^TP_rX$ and $Y=\lambda_iX^{-1}$), effecting a Schur complement and the congruence $diag(\lambda_i^{\frac{1}{2}},\lambda_i^{-\frac{1}{2}}I_n,\lambda_i^{\frac{1}{2}}I_{n_w},\lambda_i^{-\frac{1}{2}}I_{n_q},\lambda_i^{\frac{1}{2}}X^{-1})$ shows that

(2.55) implies (2.50) upon the redefinitions

$$\hat{D}_i := \lambda_i D_i,\ \hat{W}_i := \lambda_i W_i,\ \hat{S}_i := \lambda_i^{-1} S_i,\ \hat{T}_i := \lambda_i^{-1} T_i,\ \hat{R}_i := \lambda_i^{-1} R_i. \tag{2.59}$$

2. Effecting the congruence $diag(\zeta^{-\frac{1}{2}} I_n, \zeta^{\frac{1}{2}} X^{-T})$ shows that the second inequalities in (2.51) and (2.56) are equivalent while the third inequality in (2.56) is ζ times the third inequality in (2.51). Effecting the congruence $diag(I_n, 1, I_{n_q}, Q)$ and applying Corollary 2.1 on the first inequality in (2.51) (with $F = Q$ and $Y = \zeta X^{-1}$) followed by a Schur complement, the congruence $diag(\zeta^{-\frac{1}{2}} I_n, \zeta^{-\frac{1}{2}}, \zeta^{-\frac{1}{2}} I_{n_q}, \zeta^{-\frac{1}{2}} I_n)$ shows that the first inequality in (2.56) implies the first inequality in (2.51) since $\zeta^{-1} \geq 2 - \zeta$ for all $\zeta > 0$ upon the redefinitions

$$\hat{Q} := \zeta^{-1} Q,\ \hat{D}_z := \zeta D_z,\ \hat{S} := \zeta^{-1} S,\ \hat{R} := \zeta^{-1} R,\ \hat{T} := \zeta^{-1} T. \tag{2.60}$$

Finally, (2.57) and (2.58) follows from 1-2 above and Theorem 2.2, and the fact that $\det(Z) \leq \left(\frac{\mathrm{trace}(Z)}{n}\right)^n$ for any $n \times n$ positive definite matrix Z. □

Remark 2.7. *Note that the relaxation method stated in Theorem 2.4 can be extended for systems subject to norm-bounded structured uncertainties by employing Lemma 1.2, although we omit the details.*

2.6.4 Update Computation Algorithm

The following theorem gives the update procedure on the invariance and $\mathcal{H}_2$ constraint once an admissible initial triple $(P, b, K) \in \Psi$ is obtained.

Theorem 2.8. *With all definitions as above and $\mathcal{N}_{\cdot,\cdot}$ and $\mathcal{L}_{\cdot,\cdot}^{\cdot,\cdot}$ as defined in* (2.32) *and* (2.33)*, respectively, let $(\boldsymbol{P}, \boldsymbol{b}, \boldsymbol{K}) \in \Psi$. Then*

1. *Suppose that $(P, b, K, D_i, W_i, S_i, T_i, R_i), \forall i \in \mathcal{I}_m$ satisfy* (2.50)*. Then*

$$\forall i \in \mathcal{I}_m, \exists \begin{bmatrix} \boldsymbol{D}_i^{-1} \in \mathcal{D}_+^m \\ \boldsymbol{W}_i \in \mathcal{D}_+^{m_w} \\ (\boldsymbol{S}_i, \boldsymbol{T}_i, \boldsymbol{R}_i) \in \mathcal{B} \end{bmatrix} : \left[\begin{array}{c|c} \begin{array}{c} \mathcal{M}_i(\boldsymbol{D}_i^{-1}, \boldsymbol{b}, \boldsymbol{W}_i, \boldsymbol{S}_i, \boldsymbol{T}_i, \boldsymbol{K}) \\ + \mathcal{L}_{L_i(\boldsymbol{P}, \boldsymbol{K}, \boldsymbol{S}_i, \boldsymbol{R}_i), F_i(\boldsymbol{D}_i^{-1})}^{L_i(P,K,S_i,R_i), F_i(D_i^{-1})} \end{array} & \star \\ \hline EL_i(\boldsymbol{P}, \boldsymbol{K}, \boldsymbol{S}_i, \boldsymbol{R}_i) & I_n \end{array}\right] \succ 0, \tag{2.61}$$

where $E = [\,-I_n \;\; I_n \;\; 0],\, F_i(D_i) = \mathrm{diag}(I_n, I_n, D_i)$,

$$\mathscr{M}_i(\boldsymbol{D}_i^{-1}, \boldsymbol{b}, \boldsymbol{W}_i, \boldsymbol{S}_i, \boldsymbol{T}_i, \boldsymbol{K}) = \begin{bmatrix} \boldsymbol{D}_i^{-1} & \boldsymbol{b} & 0 & 0 & 0 & 0 \\ \star & 2e_i^T\boldsymbol{b} - d^T\boldsymbol{W}_i d & 0 & 0 & 0 & 0 \\ \star & \star & V^T\boldsymbol{W}_i V & 0 & 0 & 0 \\ \star & \star & \star & \boldsymbol{S}_i & 0 & 0 \\ \star & \star & \star & \star & \boldsymbol{T}_i & C_q^{\boldsymbol{K}} \\ \star & \star & \star & \star & \star & 0 \end{bmatrix},$$

and

$$L_i(P, K, S_i, R_i) = \begin{bmatrix} 0 & P^T e_i & 0 & 0 & 0 & 0 \\ 0 & 0 & B_w & B_p S_i^T & B_p R_i & A^K \\ 0 & 0 & 0 & 0 & 0 & P \end{bmatrix}.$$

Furthermore, (2.50) *and* (2.46) *are satisfied by*

$$(P, b, K, D_i, W_i, S_i, T_i, R_i) := (\boldsymbol{P}, \boldsymbol{b}, \boldsymbol{K}, \boldsymbol{D}_i, \boldsymbol{W}_i, \boldsymbol{S}_i, \boldsymbol{T}_i, \boldsymbol{R}_i).$$

2. *Suppose that* $(P, b, K, Q, D_z, S, T, R)$ *satisfy* (2.51). *Then*

$$\exists \begin{bmatrix} \boldsymbol{Q} \in \mathscr{S}_+^n \\ \boldsymbol{D}_z^{-1} \in \mathscr{D}_+^m \\ (\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{R}) \in \mathscr{B} \end{bmatrix} : \begin{bmatrix} \boldsymbol{Q} - B_p\boldsymbol{S}B_p^T & 0 & B_p\boldsymbol{R} & A^{\boldsymbol{K}} \\ \star & rI & 0 & C_2^{\boldsymbol{K}} \\ \star & \star & \boldsymbol{T} & C_q^{\boldsymbol{K}} \\ \star & \star & \star & \mathscr{L}_{I,\boldsymbol{Q}}^{I,Q} \end{bmatrix} \succ 0,$$
$$\begin{bmatrix} \boldsymbol{Q} & I \\ \star & \mathscr{L}_{\boldsymbol{P}, \boldsymbol{D}_z^{-1}}^{P, D_z^{-1}} \end{bmatrix} \succ 0, \begin{bmatrix} \boldsymbol{D}_z^{-1} & \boldsymbol{b} \\ \star & r \end{bmatrix} \succ 0. \tag{2.62}$$

Furthermore, (2.51) *and* (2.47) *are satisfied by*

$$(P, b, K, Q, D_z, S, T, R) := (\boldsymbol{P}, \boldsymbol{b}, \boldsymbol{K}, \boldsymbol{Q}, \boldsymbol{D}_z, \boldsymbol{S}, \boldsymbol{T}, \boldsymbol{R}).$$

Hence, if $\underline{Q}^{-\frac{1}{2}}$ and $\bar{Q}$ are the solutions to the optimization problems in (2.57) *and* (2.58)*, respectively, then*

$$\left(\max_{\substack{\boldsymbol{P},\boldsymbol{b},\boldsymbol{K}\\(2.61),(2.36),(2.37),(2.62)}} \log\det \underline{\boldsymbol{Q}}^{-\frac{1}{2}}\right) \geq \log\det \underline{Q}^{-\frac{1}{2}}, \tag{2.63}$$

$$\left(\min_{\substack{\boldsymbol{P},\boldsymbol{b},\boldsymbol{K}\\(2.61),(2.36),(2.39),(2.40)}} -\log\det \bar{\boldsymbol{Q}}\right) \leq -\log\det \bar{Q}. \tag{2.64}$$

Proof. The proof is essentially an application of Lemma 2.1, congruences, Schur complements and some redefinitions to show that (2.50)-(2.51) imply (2.61)-(2.62), which in turn imply (2.50)-(2.51) (with bold variables) and therefore (2.46)-(2.47) (with bold variables), respectively, from Theorem 2.6. In more detail:

1. Effecting an upper Schur complement on $b^T D_i b$, a manipulation shows that (2.50) is equivalent to

$$\mathscr{M}_i(D_i^{-1}, b, W_i, S_i, T_i, K) + \mathscr{N}_{L_i(P,K,S_i,R_i),F_i(D_i^{-1})} - L_i(P,K,S_i,R_i)^T E^T E L_i(P,K,S_i,R_i) \succ 0. \tag{2.65}$$

   The result follows by applying Lemma 2.1 on the second term and a Schur complement on the third.

2. The result follows by applying Lemma 2.1 on $\mathscr{N}_{I,Q} = Q^{-1}$ and $\mathscr{N}_{P,D_z^{-1}} = P^T D_z P$ and a Schur complement on the third inequality in (2.62).

Finally, (2.63) and (2.64) follow from 1-2 above and Theorem 2.5. □

2.6.5 Solution algorithm

The following algorithm summarizes our solution for systems subject to norm-bounded structured uncertainty.

Algorithm 2.2. *Given system* (2.44)*, $\mathscr{F} = \mathscr{P}(I_{m_f}, \bar{f})$ and sets $\Omega, \mathscr{W} = \mathscr{P}(V,d), \mathscr{P}(P_0, b_0)$ and parameter r.*

1. **Initial data**: *Choose* $m \geq n$, *initial polytope* $\mathscr{P}(P_r, b_r)$ *and tolerance level* tol.

2. **Initial solution**

   (a) *Use Theorem 2.2 and Theorem 2.7 to solve the convex SDPs in* (2.57) *or* (2.58).

   (b) *Define* $D_i, W_i, S_i, T_i, R_i, Q, D_z, S, T, R$ *and* μ_i *from* (2.59)-(2.60) *so that* (2.50) *and* (2.10), (2.11) *or* (2.13), (2.51) *or* (2.14) *are satisfied.*

3. **Update** *Solve the optimizations in* (2.63) *or* (2.64).

4. **Stopping condition**

   (a) *If* $\det(\underline{\boldsymbol{Q}}^{-1}) - \det(\underline{Q}^{-1}) \leq tol$ *(for maximization) or* $\det(\bar{Q}^{-1}) - \det(\bar{\boldsymbol{Q}}^{-1}) \leq tol$ *(for minimization), stop.*

   (b) *Else update* $Z := \boldsymbol{Z}$, *where* $\boldsymbol{Z}$ *denotes a variable in the optimizations* (2.63) *or* (2.64), *and go to step 3.*

5. **End**

2.7 Polytopic Uncertainty

Since our algorithms are linear, in the case of polytopic uncertainty, that is, for system (2.44) if $(\tilde{A}, \tilde{B}) \in \Omega$ where

$$\Omega := \{(\tilde{A}, \tilde{B}) : [\tilde{A}\ \tilde{B}] = \sum_{l=1}^{p} \alpha_i [A_l\ B_l], \sum_{l=1}^{p} \alpha_i = 1, \alpha_i \geq 0\},$$

and where $A_l \in \mathscr{R}^{n \times n}$ and $B_l \in \mathscr{R}^{n \times n_u}$ are given matrices for all $l \in \mathscr{I}_p$, all our algorithms are applicable except that (A, B) are replaced by (A_l, B_l) and the constraints need to be satisfied for all $l \in \mathscr{I}_p$.

Remark 2.8. *Note that Theorem 2.3 can be extended to quadratically controllable systems [57], which would allows us to obtain an initial solution for systems subject to polytopic uncertainties.*

2.8 Examples

This section presents numerical examples that illustrate the algorithms developed in this chapter.

2.8.1 Example 1

In this example, we demonstrate the improvement in the RCI set accuracy as we increase the complexity of polytope. Consider the following discrete-time system from [11] with $V = 1$, $\bar{f} = e$, $d = 1$ and:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D_w = 0.$$

There is no $\mathscr{H}_2$ constraint for the system, and we calculate the approximate maximal RCI set. The following table gives the final value of $\log\det \underline{Q}^{-1}$ computed using Algorithm 2.1 as the complexity of the polytope (measured by m) increases.

m	2	3	6	10
$\log\det \underline{Q}^{-1}$	0.6825	1.2668	1.2680	1.2684

Table 2.1 Improvement in the RCI set accuracy for increasing m.

The volume for $m = 3$ is much larger than for $m = 2$, although for $m > 3$, there is no significant increase in the volume. The resulting approximation to the maximal RCI set when $m = 2$ is shown in yellow (dashed border), and the final polytopic RCI set with $m = 3$ is shown in red (solid border) in Fig.2.1. The box in white color and dash-dot border shows the state constraints. For illustration, a state trajectory, which starts from the edge of the final RCI set is shown (dashed blue line), and the red cross marks represent the system states. Only one state trajectory for $m = 3$ is plotted for clarity.

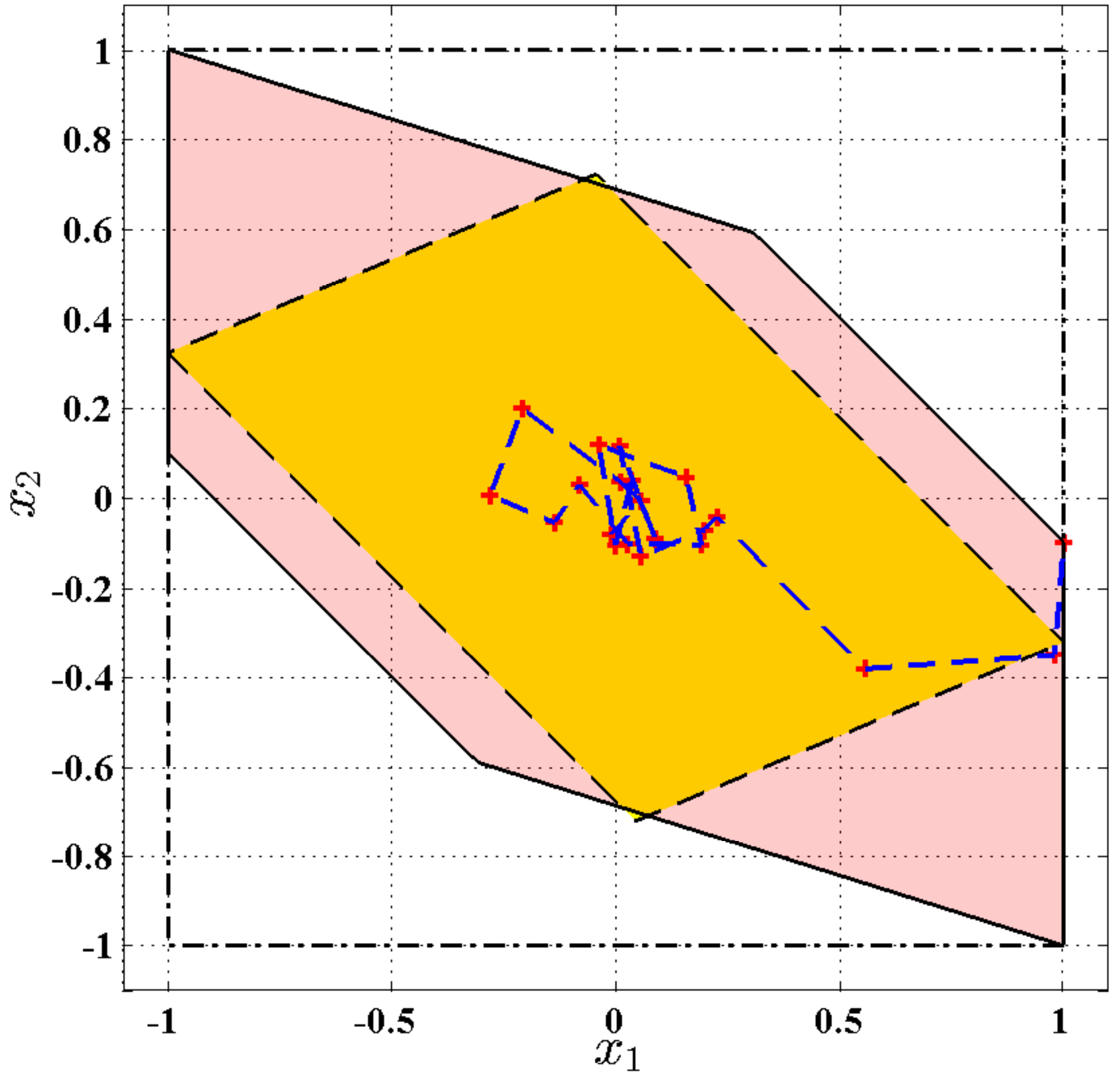


Fig. 2.1 Maximal polytopic RCI set with $m = 2$ (dashed border), $m = 3$ (solid border), state constraints set (dash–dot border) and a typical state trajectory (crosses).

2.8.2 Example 2

Consider the following example from [69] (originally from [28]) with $C = 0$, $D = 1$, $D_w = 0$, $\bar{f} = e$ and:

$$A = \begin{bmatrix} 0.8876 & \text{-}0.5555 \\ 0.5555 & 1.5542 \end{bmatrix}, \qquad B = \begin{bmatrix} -0.1124 \\ 0.5555 \end{bmatrix}.$$

Note that the system is without disturbance and state constraints. Here, the actuator saturation is treated as input constraints, which corresponds to the first case in [69]. The magenta ellipsoid in Fig.2.2 is the ellipsoidal approximation to the maximal RCI set solved by the method proposed in [69] and [28]. Our initial maximized full-complexity polytopic RCI set (shown in yellow with dashed border) gives better precision than the ellipsoidal set. The volume of the resulting RCI set is increased by implementing the update procedure in Section 2.4, and the final RCI set is shown in red (with solid border) in Fig.2.2. An illustrative state trajectory (shown with dashed blue line and red cross mark) starts from the margin of the final RCI set and stays inside the set for all future times. Without the presence of disturbances, the state trajectory converges to the origin under the effect of the state feedback controller.

2.8.3 Example 3

This example illustrates our approach for a system subject to polytopic uncertainty. Consider the double integrator example in [48]. We set $m = 30$ and $\mathscr{P}(P_r, b_r)$ a regular hexacontagon. Figure 2.3 shows the initial (in yellow, with solid border) and final (in red, with solid border) inner approximation to the RCI set, with the final control law given as $K = [-0.0794 \quad -0.0781]$. The blue cross marks and the dashed green line shows the trajectory of the system states under the feedback control law (only one trajectory is shown for clarity). The white box with dash-dot border shows the output constraints. Figure 2.5 displays the convergence of $\log\det \underline{\boldsymbol{Q}}^{-\frac{1}{2}}$ with the update times N (blue, dashed line).

For comparison, the low-complexity ($m = n = 2$) inner approximation to the RCI set is also shown in Figure 2.3 (in blue color and with dashed border). Note that considering a full-complexity RCI set leads to a much larger volume. [48] gives an optimal solution under the control gain $K = [-0.3 \quad -0.1]$ as shown in green color with dotted border in Figure

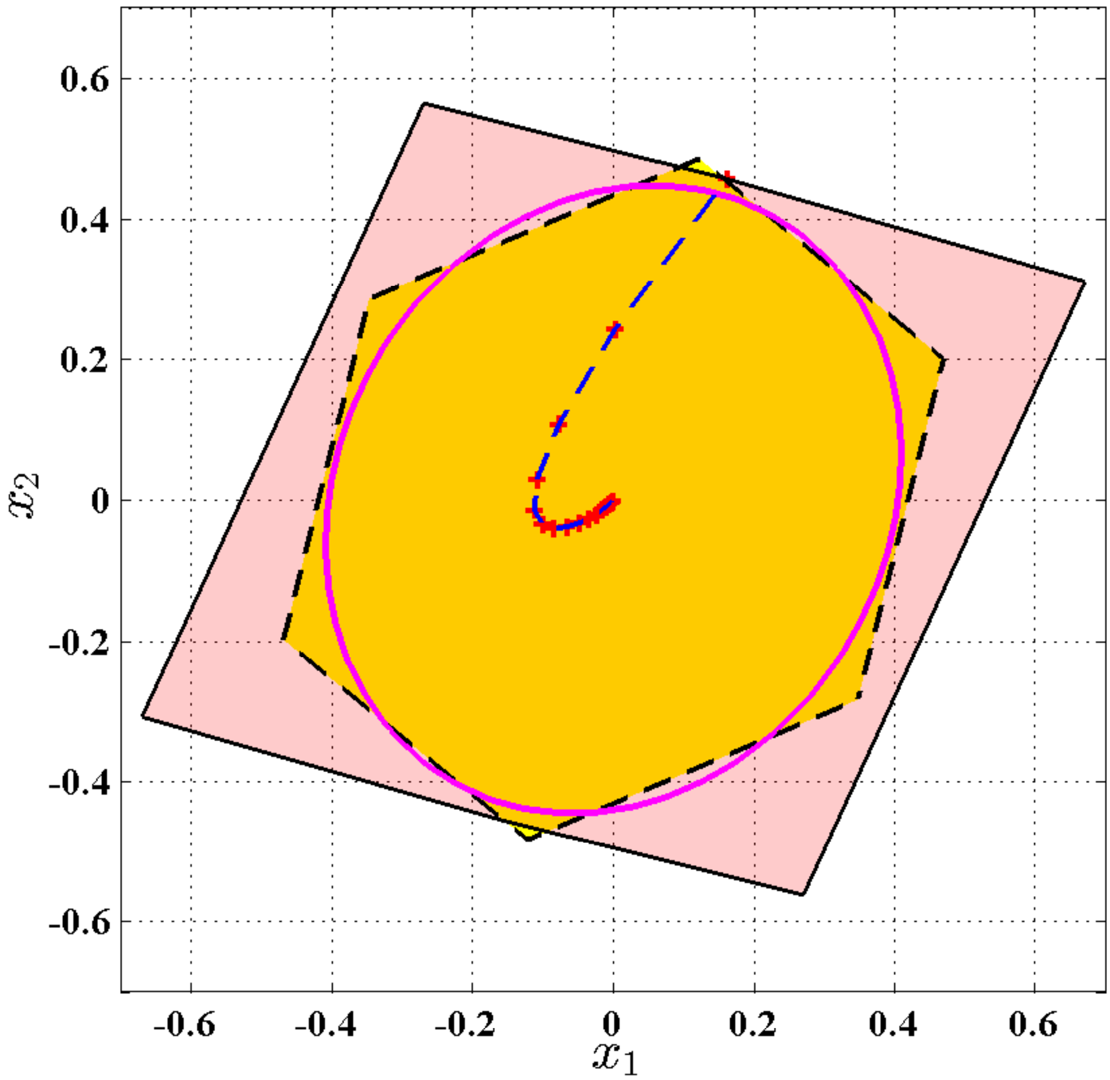


Fig. 2.2 Initial (solid border) and final (dashed border) maximal polytopic RCI sets, ellipsoidal approximation using methods of [69],[28] and a typical state trajectory (crosses).

2.3. Our result shows that the volume of the RCI set can be greatly increased by treating the feedback gain as a decision variable in the optimization.

Let

$$C_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}. \tag{2.66}$$

Using (2.12), the $\mathscr{H}_2$–performance level, defined in (2.4), for the final inner approximation to the maximal RCI set is given by $\gamma = 6.36$. We can improve the performance level by, for example, setting $\gamma = 3$ and incorporating the $\mathscr{H}_2$ constraint condition (2.12) in our algorithm. The final inner approximation to the RCI set with the improved performance requirement is shown in Figure 2.3 (in magenta color and with dash-dot border). This illustrates the compromise between the volume of the RCI set and the expected performance.

2.8.4 Example 4

This example illustrates our approach for a system subject to norm-bounded structured uncertainty. Consider the example of a continuous-time DC motor system with norm–bounded structured uncertainty proposed in [14]. We discretize the system with a sampling period $T = 0.1s$ and express the discrete-time system in the form of (1.1) and the uncertainty in the form of (2.45) with appropriate nominal system (A, B), distribution matrices B_p, C_q and D_q and uncertainty set

$$\boldsymbol{\Delta} = \{diag(\delta_1 I_2, \delta_2),\ \delta_i \in \mathscr{R},\ |\delta_i| \leq 1\}.$$

We also incorporate an additive disturbance and the state and input constraints are integrated into our output constraint by setting

$$B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad C^T = \begin{bmatrix} 0 & 0.1 & 0 \\ 0.1 & 0 & 0 \end{bmatrix},$$

$D^T = \begin{bmatrix} 0 & 0 & 0.1 \end{bmatrix}$, $D_w = 0_{3\times 1}$, $V = 1$, and d, and $\bar{f}$ as vectors of ones with appropriate dimensions.

Set $m = 8$ and $\mathscr{P}(P_r, b_r)$ a regular hexadecagon. Using Algorithm 2.1 to find an outer approximation to the minimal RCI set, we obtain the initial and final sets as shown in Fig-

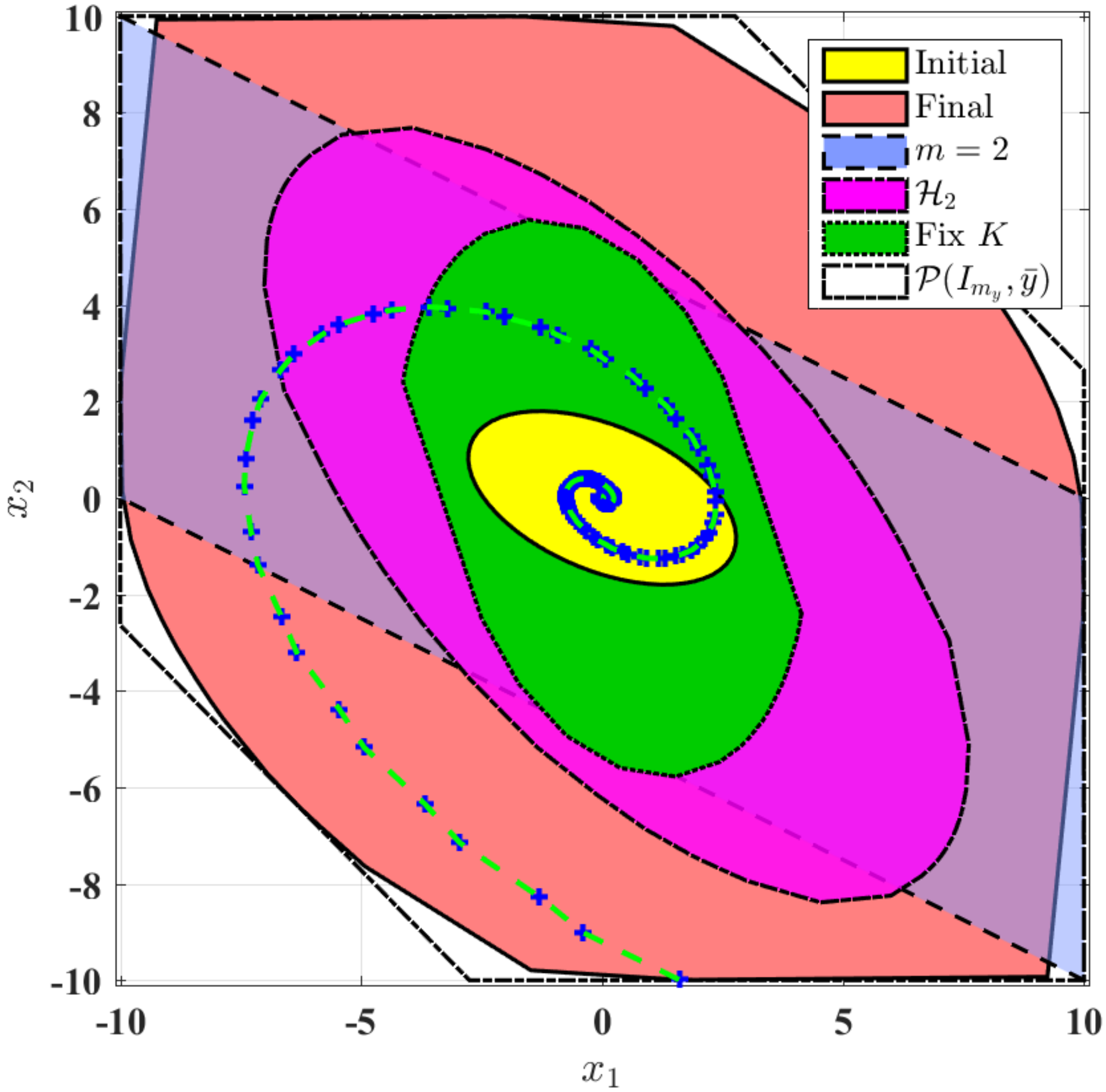


Fig. 2.3 Initial (yellow) and final (red) polytopic RCI sets, final polytopic RCI set with $m = 2$ (blue), final polytopic RCI set with $\mathscr{H}_2$ constraint (magenta), final polytopic RCI set computed by [48] with a fixed controller (green), the output constraint (white) and a typical state trajectory (crosses).

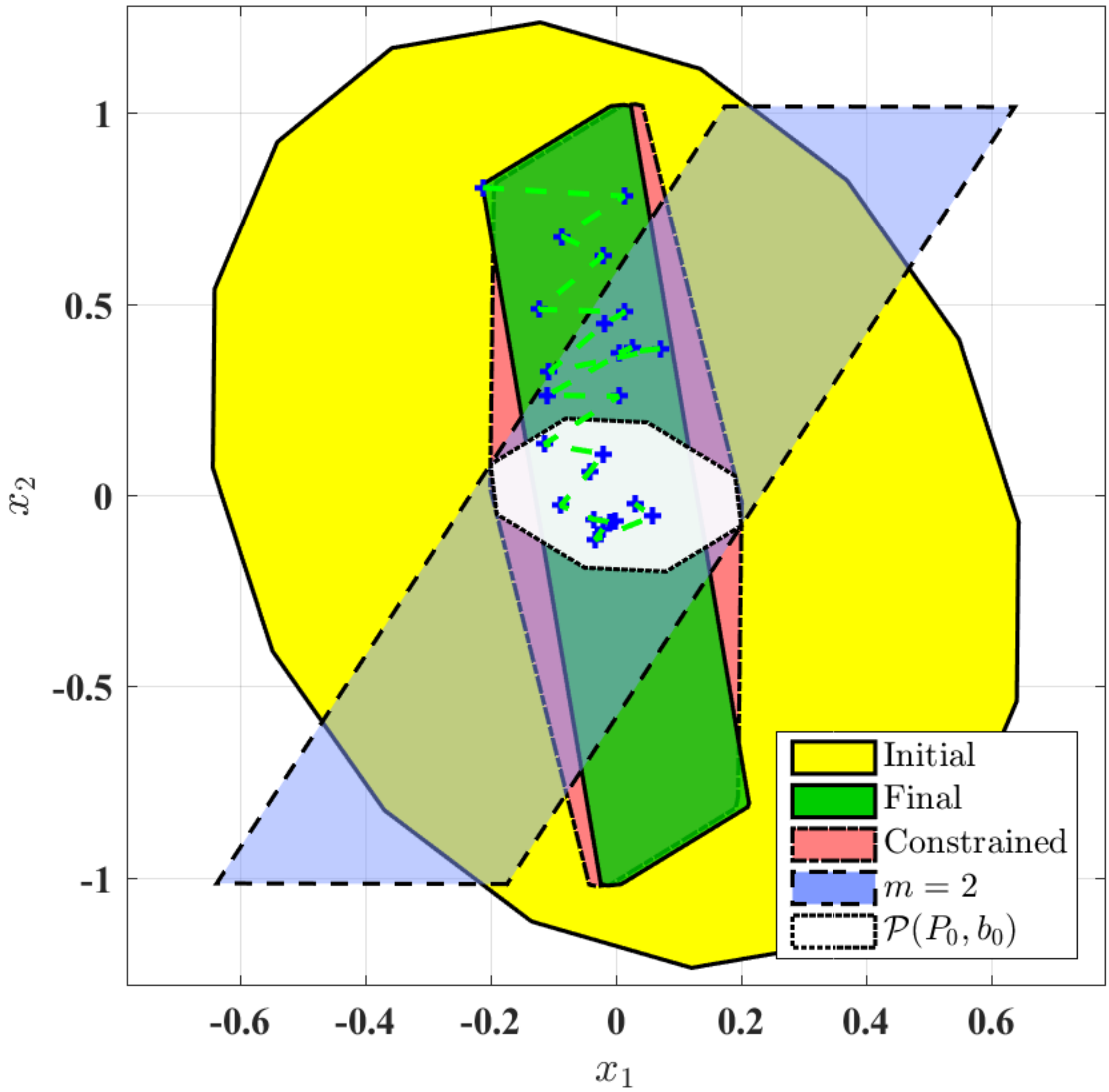


Fig. 2.4 Initial (yellow) and final (green) polytopic RCI sets, the final polytopic RCI set with initial state constraint (red), final polytopic RCI set with $m = 2$ constraint (blue), the initial state constraint (white), and a typical state trajectory (crosses).

ure 2.4 with the final control law as $K = \begin{bmatrix} -9.9707 & -0.0588 \end{bmatrix}$. The convergence rate of $\log\det\bar{\boldsymbol{Q}}^{-1}$ with the update times N is shown in Figure 2.5 (red, solid line).

We can show the effect of including the initial state constraint (2.6) with

$$P_0^T = \begin{bmatrix} 4.167 & 5.159 & 0.397 & 3.571 \\ 4.167 & 0.397 & 5.159 & -3.571 \end{bmatrix},$$

and b_0 as the vector of ones with appropriate dimensions. The final outer approximation to the minimal RCI set subject to the initial state constraint is shown in Figure 2.4 (in red color and with dash-dot border; the initial state constraint is shown in white color with dotted border). Under this requirement, the superiority of using full-complexity RCI set is obvious. For comparison, the outer approximation to the minimal low-complexity ($m = n = 2$) RCI set is also shown in Figure 2.4 (in blue color and with dashed border).

To illustrate the invariance condition, the blue cross marks and the dashed green line shows the trajectory of the system states under the feedback control law (only one trajectory is shown for clarity). The trajectory is representative since it starts from the edge of the set and is produced using the worst case disturbances and uncertainties.

2.9 Conclusion

We have proposed a novel scheme, based on convex/LMI optimizations, for the computation of full-complexity inner/outer approximations to polytopic maximal/minimal RCI sets and the corresponding feedback control law (K) for linear discrete–time systems subject to additive disturbances and output, initial state and performance constraints, as well as model uncertainties.

This chapter first derives necessary and sufficient conditions for the existence of an admissible RCI set and feedback gain matrix, that are, in general, nonlinear and nonconvex. A corollary of Elimination Lemma is then used to relax the problem and obtain sufficient LMI conditions, thus rendering the optimization problem tractable. An initial invariant polytope, and control law K, is first obtained and the set-volume is then iteratively optimized by solving convex/LMI optimizations. These iterations are reminiscent of Newton updates which appears to promote good convergence speed. Furthermore, the proposed scheme is able to handle both structured norm-bounded as well as polytopic model uncertainties.

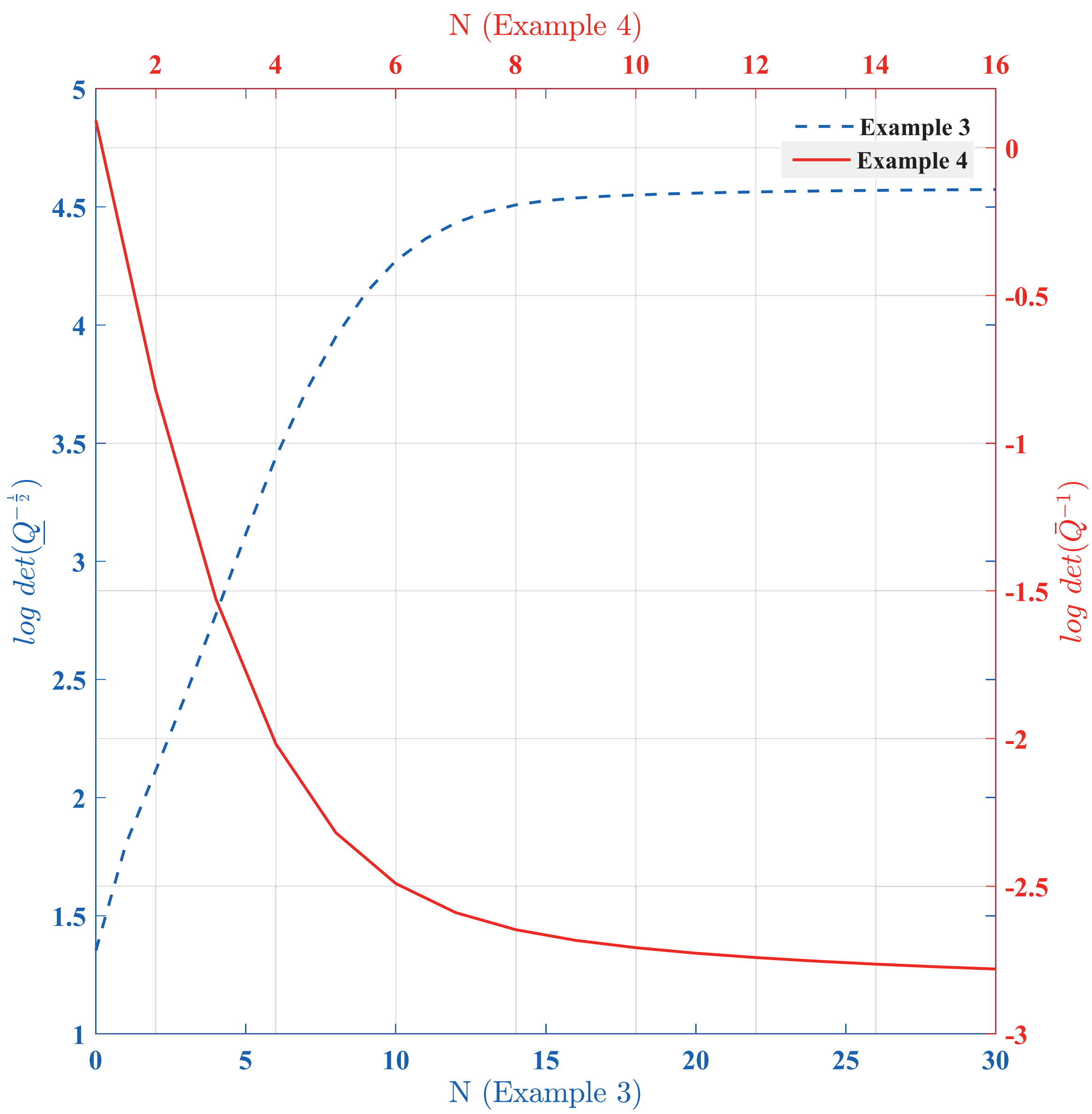


Fig. 2.5 Convergence Rates for Examples 3 (blue) and 4 (red).

Unlike many of the schemes in the literature, the algorithm places no restriction on the complexity of the invariant polytope and allows for arbitrarily large values of m. This, coupled with the fact that K is treated as a variable of optimization, results in larger/smaller inner/outer approximations to the maximal/minimal RCI sets. This is reflected in the results from the numerical examples, which show that the proposed scheme can yield a polytopic RCI set with a substantially improved volume as compared to other schemes from the literature.

Chapter 3

Robust Control Invariant Sets with Asymmetric Constraints

The computation algorithm proposed in the previous chapter considered symmetric constraints only, while in many control cases, system constraints are asymmetric, especially in tracking problems.

In this chapter, consider the linear discrete time system subject to additive disturbance and asymmetric constraints, we propose an algorithm to compute full-complexity RCI sets, where arbitrarily large number of faces can be specified for the polytope and the center of the RCI set is shifted from zero point, thereby enabling a less conservative and more accurate inner/outer approximations to the maximal/minimal RCI sets.

3.1 Problem Description

Consider the linear discrete time system as described in (1.1) subject to asymmetric output and initial state constraints. Since the output constraints are asymmetric, we shift the center of the invariant sets from zero point by defining the invariant sets in the following form:

$$\mathscr{P}(P,b,x_c) = \{x \in \mathscr{R}^n : -b \le P(x-x_c) \le b\},$$

where $0 < b \in \mathscr{R}^m$, $P \in \mathscr{R}^{m\times n}$, $m \ge n$, and $x_c \in \mathscr{R}^n$.

For given system (1.1), disturbance set $\mathscr{W}$ defined in (1.2)-(1.3), (asymmetric) output constraint set $\mathscr{F}_{Asy} := \{f \in \mathscr{R}^{m_f} : \underline{f} \leq f \leq \bar{f}\}$ with $\underline{f} \leq 0 \leq \bar{f} \in \mathscr{R}^{m_f}$, (asymmetric) initial state constraint set $\mathscr{P}_{Asy}(P_0, \bar{b}_0, \underline{b}_0) := \{x \in \mathscr{R}^n : \underline{b}_0 \leq P_0 x \leq \bar{b}_0\}$ with $\underline{b}_0 \leq 0 \leq \bar{b}_0 \in \mathscr{R}^{m_0}$ and $P_0 \in \mathscr{R}^{m_0 \times n}$, $\mathscr{H}_2$ performance bound r and $m \geq n$, and all the other notation as defined in the previous chapter, we present convex SDP based algorithms to solve the following optimizations to obtain an admissible quadruple $(P, b, x_c, K) \in \Phi := \mathscr{R}^{m \times n} \times \mathscr{R}^m \times \mathscr{R}^n \times \mathscr{R}^{n_u \times n}$ and approximate maximal/minimal RCI set $\mathscr{P}(P, b, x_c)$:

$$\max_{(P,b,x_c,K)\in\Phi} \quad \log\det \underline{Q}^{-1} \tag{3.1}$$

$$s.t. \left\{\begin{matrix} x \in \mathscr{P}(P,b,x_c) \\ w \in \mathscr{W} \end{matrix}\right\} \Rightarrow x^+ \in \mathscr{P}(P,b,x_c) \qquad \text{(Invariance)} \tag{3.2}$$

$$\left\{\begin{matrix} x \in \mathscr{P}(P,b,x_c) \\ w \in \mathscr{W} \end{matrix}\right\} \Rightarrow f \in \mathscr{F}_{Asy} \qquad \text{(Output constraint)} \tag{3.3}$$

$$\exists \underline{Q} \in \mathscr{S}_+^n : \mathscr{E}(\underline{Q}, x_c) \subset \mathscr{P}(P,b,x_c) \qquad \text{(Inner bounding ellipsoid)} \tag{3.4}$$

$$x \in \mathscr{P}(P,b,x_c) \Rightarrow J_{\mathscr{H}} := \sum_{k=0}^{\infty} \|z_k\|^2 < r^2 \qquad (\mathscr{H}_2 \text{ constraint}) \tag{3.5}$$

$$\min_{(P,b,x_c,K)\in\Phi} \quad \log\det \bar{Q}^{-1} \tag{3.6}$$

$$s.t. \left\{\begin{matrix} x \in \mathscr{P}(P,b,x_c) \\ w \in \mathscr{W} \end{matrix}\right\} \Rightarrow x^+ \in \mathscr{P}(P,b,x_c) \qquad \text{(Invariance)}$$

$$\left\{\begin{matrix} x \in \mathscr{P}(P,b,x_c) \\ w \in \mathscr{W} \end{matrix}\right\} \Rightarrow f \in \mathscr{F}_{Asy} \qquad \text{(Output constraint)}$$

$$\exists \bar{Q} \in \mathscr{S}_+^n : \mathscr{P}(P,b,x_c) \subset \mathscr{E}(\bar{Q}, x_c) \qquad \text{(Outer bounding ellipsoid)} \tag{3.7}$$

$$\mathscr{P}_{Asy}(P_0, \bar{b}_0, \underline{b}_0) \subseteq \mathscr{P}(P,b,x_c) \qquad \text{(Initial condition constraint)} \tag{3.8}$$

3.2 Nonlinear formulation

In this section, we derive conditions, in the form of NLMIs, for the admissibility of the quadruple $(P, b, x_c, K) \in \Phi$ under asymmetric output and initial state constraints.

The following result derives a nonlinear optimization algorithm to obtain an approximate maximal/minimal RCI set $\mathscr{P}(P, b, x_c)$.

Theorem 3.1. *An admissible quadruple $(P, b, x_c, K) \in \Phi$ and an approximate maximal/minimal RCI set $\mathscr{P}(P, b, x_c)$ can be obtained by solving the nonlinear SDPs*

$$\max_{(P,b,x_c,K)\in\Phi} \log\det \underline{Q}^{-1} \tag{3.9}$$

$$s.t. \begin{bmatrix} f_i(b,D_i,W_i,P,x_c) & e_i^T PB_w & e_i^T PA^K + x_c^T P^T D_i P \\ \star & V^T W_i V & 0 \\ \star & \star & P^T D_i P \end{bmatrix} \succ 0 \ \forall i \in \mathscr{I}_m; \tag{3.10}$$

$$\begin{bmatrix} \bar{g}_j(b,\bar{E}_j,\bar{G}_j,P,x_c) & e_j^T D_w & e_j^T C^K + x_c^T P^T \bar{E}_j P \\ \star & V^T \bar{G}_j V & 0 \\ \star & \star & P^T \bar{E}_j P \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} \underline{g}_j(b,\underline{E}_j,\underline{G}_j,P,x_c) & e_j^T D_w & e_j^T C^K - x_c^T P^T \underline{E}_j P \\ \star & V^T \underline{G}_j V & 0 \\ \star & \star & P^T \underline{E}_j P \end{bmatrix} \succ 0 \ \forall j \in \mathscr{I}_{m_f}; \tag{3.11}$$

$$\begin{bmatrix} 2e_i^T b + 2e_i^T P x_c - \mu_i + x_c^T \mu_i \underline{Q} x_c & e_i^T P + x_c^T \mu_i \underline{Q} \\ \star & \mu_i \underline{Q} \end{bmatrix} \succ 0 \ \forall i \in \mathscr{I}_m; \tag{3.12}$$

$$\begin{bmatrix} Q & 0 & A^K \\ \star & rI & C_2^K \\ \star & \star & Q^{-1} \end{bmatrix} \succ 0, \quad \begin{bmatrix} Q & I \\ \star & P^T D_z P \end{bmatrix} \succ 0, \quad r > b^T D_z b. \tag{3.13}$$

$$\min_{(P,b,x_c,K)\in\Phi} \log\det\bar{Q}^{-1} \tag{3.14}$$

$$s.t. \begin{bmatrix} f_i(b,D_i,W_i,P,x_c) & e_i^T PB_w & e_i^T PA^K + x_c^T P^T D_i P \\ \star & V^T W_i V & 0 \\ \star & \star & P^T D_i P \end{bmatrix} \succ 0 \; \forall i \in \mathscr{I}_m;$$

$$\begin{bmatrix} \bar{g}_j(b,\bar{E}_j,\bar{G}_j,P,x_c) & e_j^T D_w & e_j^T C^K + x_c^T P^T \bar{E}_j P \\ \star & V^T\bar{G}_j V & 0 \\ \star & \star & P^T \bar{E}_j P \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} \underline{g}_j(b,\underline{E}_j,\underline{G}_j,P,x_c) & e_j^T D_w & e_j^T C^K - x_c^T P^T \underline{E}_j P \\ \star & V^T\underline{G}_j V & 0 \\ \star & \star & P^T \underline{E}_j P \end{bmatrix} \succ 0 \; \forall j \in \mathscr{I}_{m_f};$$

$$\begin{bmatrix} 1 - b^T\bar{D}b - x_c^T\bar{Q}x_c + x_c^T P^T \bar{D} P x_c & x_c^T\bar{Q} - x_c^T P^T \bar{D} P \\ \star & P^T\bar{D}P - \bar{Q} \end{bmatrix} \succ 0; \tag{3.15}$$

$$\begin{bmatrix} 2e_i^T b + 2e_i^T P x_c + \bar{b}_0^T F_i \underline{b}_0 & e_i^T P + \frac{1}{2}(\bar{b}_0 + \underline{b}_0)^T F_i P_0 \\ \star & P_0^T F_i P_0 \end{bmatrix} \succ 0 \; \forall i \in \mathscr{I}_m. \tag{3.16}$$

where $D_i, \bar{E}_j, \underline{E}_j, D_z, \bar{D} \in \mathscr{D}_+^m$*;* $W_i, \bar{G}_j, \underline{G}_j \in \mathscr{D}_+^{m_w}$*;* $\mu_i \in \mathscr{R}_+$*;* $\underline{Q}, Q, \bar{Q} \in \mathscr{S}_+^n$*;* $F_i \in \mathscr{D}_+^{m_0}$ *and where*

$$\begin{aligned} A^K &:= A + BK, \\ C^K &:= C + DK, \\ C_2^K &:= C_2 + D_2K, \\ f_i(b,D_i,W_i,P,x_c) &:= 2e_i^T b - b^T D_i b - d^T W_i d + 2e_i^T P x_c + x_c^T P^T D_i P x_c, \\ \bar{g}_j(b,\bar{E}_j,\bar{G}_j,P,x_c) &:= 2e_j^T \bar{f} - b^T \bar{E}_j b - d^T \bar{G}_j d + x_c^T P^T \bar{E}_j P x_c, \\ \underline{g}_j(b,\underline{E}_j,\underline{G}_j,P,x_c) &:= -2e_j^T \underline{f} - b^T \underline{E}_j b - d^T \underline{G}_j d + x_c^T P^T \underline{E}_j P x_c. \end{aligned}$$

Proof. Condition (3.2) is equivalent to the requirement that for all $i \in \mathscr{I}_m$,

$$\left\{ \begin{array}{c} (e_j^T P(x - x_c))^2 - (e_j^T b)^2 \le 0, \forall j \in \mathscr{I}_m \\ (e_k^T V w)^2 - (e_k^T d)^2 \le 0, \forall k \in \mathscr{I}_{m_w} \end{array} \right\} \Rightarrow 2e_i^T\,(b - P((A+BK)x + B_w w - x_c)) \ge 0.$$

The result then follows from Theorem 1.1 based on the following identity

$$\begin{aligned} 2e_i^T(P((A+BK)x + B_w w - x_c) - b) = &-(b - Px + Px_c)^T D_i (b + Px - Px_c) \\ &-(d^T W_i d - w^T V^T W_i V w) \\ &-a^T N_i a \end{aligned} \tag{3.17}$$

where $a^T := \begin{bmatrix} -1 & w^T & x^T \end{bmatrix}$, and

$$N_i := \begin{bmatrix} 2e_i^T b - b^T D_i b - d^T W_i d + 2e_i^T P x_c + x_c^T P^T D_i P x_c & e_i^T P B_w & e_i^T P A^K + x_c^T P^T D_i P \\ \star & V^T W_i V & 0 \\ \star & \star & P^T D_i P \end{bmatrix}.$$

For $D_i \in \mathscr{D}_+^m$ and $W_i \in \mathscr{D}_+^{m_w}$, the first and second terms on the RHS of (3.17) are nonpositive for all $x \in \mathscr{P}(P, b, x_c)$ and all $w \in \mathscr{W}$, it follows that the invariance condition is satisfied if $N_i \succ 0$, which gives the result. This proves the sufficiency of (3.10), necessity follows from Farkas' Theorem.

Similarly, following Theorem 1.1 and some manipulations, we can prove that (3.11) is equivalent to (3.3). The LMIs in (3.11) corresponding to the upper and lower bound respectively.

Condition (3.4) and (3.8) are equivalent to the requirement that for all $i \in \mathscr{I}_m$,

$$(x - x_c)^T \underline{Q}(x - x_c) - 1 \leq 0 \Rightarrow 2e_i^T(b - P(x - x_c)) \geq 0,$$
$$\left\{ \begin{array}{l} (e_j^T P_0 x)^2 - (e_j^T \bar{b}_0)^2 \leq 0 \\ (e_j^T \underline{b}_0)^2 - (e_j^T P_0 x)^2 \leq 0 \end{array} \right\} \forall j \in \mathscr{I}_{m_0} \Rightarrow 2e_i^T(b - P(x - x_c)) \geq 0,$$

respectively. Then (3.12) and (3.16) follow from Theorem 1.1.

For any $x \in \mathscr{P}(P, b, x_c)$, a minor extension of the results in [30] gives the first inequality in (3.13) and

$$r - x^T Q^{-1} x \geq 0 \tag{3.18}$$

as sufficient conditions for $J_{\mathscr{H}} < r^2$. Theorem 1.1 then gives the second and third inequalities in (3.13) as sufficient conditions for (3.18) to be satisfied for all $x \in \mathscr{P}(P, b, x_c)$.

Condition (3.7) is equivalent to the requirement that

$$(e_j^T P(x - x_c))^2 - (e_j^T b)^2 \leq 0 \; \forall j \in \mathscr{I}_m \Rightarrow 1 - (x - x_c)^T \bar{Q}(x - x_c) \geq 0.$$

(3.15) is then obtained follows Theorem 1.1.

Finally, (3.9) and (3.14) follow from the above conditions. $\square$

3.3 Linearization and Initial Computation

While Theorem 3.1 gives optimization algorithms for approximate maximal/minimal RCI set $\mathscr{P}(P,b,x_c)$, and necessary and sufficient conditions for the the quadruple (P,b,x_c,K) to be admissible, the conditions are nonlinear. In addition to the nonlinearity forms in Theorem 2.1, new nonlinear forms appear due to the introduction of the asymmetric constraints and the variable x_c. In this section, we propose a linearization algorithm involving the computation of an initial solution.

We set

$$\mathscr{P}(P,b,x_c) = \mathscr{P}(P_rX,b_r,x_c) = \{x \in \mathscr{R}^n : -b_r \leq P_rX(x-x_c) \leq b_r\}$$

as an initial full-complexity inner/outer approximation to the maximal/minimal RCI set, where b_r and P_r are given (see Remark 2.3), and where $X \in \mathscr{R}^{n\times n}$ is a variable used to reshape (rotate and scale) the polyhedral set defined by P_r and x_c is the centre of the RCI set, which we take to be a variable.

The following two are corollaries of the Elimination Lemma (Lemma 1.1) and are used for the linearization procedure.

Corollary 3.1. *Given $T \in \mathscr{S}_+^n, E \in \mathscr{R}^{n\times p}, F \in \mathscr{R}^{p\times n}$ and $\mathscr{Y} \subseteq \mathscr{R}^{p\times p}$. Consider the statements:*

(1) $M := \begin{bmatrix} T & F^T - EY \\ \star & Y^T + Y \end{bmatrix} \succ 0$ *holds for some $Y \in \mathscr{Y}$.*

(2) $N := T + EF + F^TE^T \succ 0$.

Then $(1) \Rightarrow (2)$. *Furthermore, if $\mathscr{Y} = \mathscr{R}^{p\times p}$, then* $(1) \Leftrightarrow (2)$.

Proof. Write M as $M = Q + RYS^T + SY^TR^T$ where

$$\left[\begin{array}{c|c} Q & R \\ \hline S^T & \star \end{array}\right] = \left[\begin{array}{cc|c} T & F^T & -E \\ F & 0 & I \\ \hline 0 & I & \star \end{array}\right].$$

Since

$$R_\perp = \begin{bmatrix} I \\ E^T \end{bmatrix}, \quad S_\perp = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

are orthogonal complements of R and S, respectively, the result follows from the Elimination Lemma upon noting that $R_\perp^T Q R_\perp = N$ and $S_\perp^T Q S_\perp = T$. □

Corollary 3.2. *Given $T \in \mathscr{S}_+^n, E \in \mathscr{R}^{n\times p}, F \in \mathscr{R}^{p\times m}, H \in \mathscr{R}^{n\times m}, Z \in \mathscr{S}_+^m$ and $\mathscr{Y} \subseteq \mathscr{R}^{p\times p}$. Consider the statements:*

(1) $M := \begin{bmatrix} T & EY & H \\ \star & Y^T + Y & -F \\ \star & \star & Z \end{bmatrix} \succ 0$ *holds for some $Y \in \mathscr{Y}$.*

(2) $N := \begin{bmatrix} T & H + EF \\ \star & Z \end{bmatrix} \succ 0.$

Then $(1) \Rightarrow (2)$*. Furthermore, if $\mathscr{Y} = \mathscr{R}^{p\times p}$, then* $(1) \Leftrightarrow (2)$*.*

Proof. Write M as $M = Q + RYS^T + SY^TR^T$ where

$$\left[\begin{array}{c:c} Q & R \\ \hdashline S^T & \star \end{array}\right] = \left[\begin{array}{ccc:c} T & 0 & H & E \\ 0 & 0 & -F & I \\ H^T & -F^T & Z & 0 \\ \hdashline 0 & I & 0 & \star \end{array}\right].$$

Since

$$R_\perp = \begin{bmatrix} I & 0 \\ -E^T & 0 \\ 0 & I \end{bmatrix}, \quad S_\perp = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix},$$

are orthogonal complements of R and S, respectively, the result follows from the Elimination Lemma upon noting that $R_\perp^T Q R_\perp = N$ and

$$S_\perp^T Q S_\perp = \begin{bmatrix} T & H \\ H^T & Z \end{bmatrix}.$$

□

The following result gives initial solutions to the optimizations (3.1) and (3.6) by deriving sufficient conditions for the admissibility of the quadruple (P_rX, b_r, x_c, K) in the form of LMIs using Corollary 3.1 and Corollary 3.2.

Theorem 3.2. *Let all the definitions be as above and $P = P_rX$ and $b = b_r$, where $P_r \in \mathscr{R}^{m\times n}$ and $b_r \in \mathscr{R}^m$ are given and where $X \in \mathscr{R}^{n\times n}$. The approximate maximal/minimal RCI set $\mathscr{P}(P,b,x_c)$ and $K \in \mathscr{R}^{n_u\times n}$ can be obtained by solving the linear convex SDPs:*

$$\max_{x_c,\hat{X},\hat{K}} \ \log\det \underline{Q}^{-\frac{1}{2}} \tag{3.19}$$

$$s.t. \begin{bmatrix} \hat{l}_i(\lambda_i,\hat{D}_i,\hat{W}_i) & 0 & 0 & x_c^T & x_c^T-\lambda_i e_i^T P_r \\ \star & V^T\hat{W}_iV & 0 & 0 & B_w^T \\ \star & \star & P_r^T\hat{D}_iP_r & 0 & \hat{A}^T \\ \star & \star & \star & \hat{X}+\hat{X}^T & \hat{A}^T \\ \star & \star & \star & \star & \hat{X}+\hat{X}^T \end{bmatrix} \succ 0,\ \forall i \in \mathscr{I}_m; \tag{3.20}$$

$$\begin{bmatrix} 2e_j^T\bar{f}-b_r^T\bar{E}_jb_r-d^T\bar{G}_jd & e_j^TD_w & e_j^T\hat{C} & x_c^T+e_j^T\hat{C} \\ \star & V^T\bar{G}_jV & 0 & 0 \\ \star & \star & P_r^T\bar{E}_jP_r & 0 \\ \star & \star & \star & \hat{X}+\hat{X}^T \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} -2e_j^T\underline{f}-b_r^T\underline{E}_jb_r-d^T\underline{G}_jd & e_j^TD_w & e_j^T\hat{C} & x_c^T-e_j^T\hat{C} \\ \star & V^T\underline{G}_jV & 0 & 0 \\ \star & \star & P_r^T\underline{E}_jP_r & 0 \\ \star & \star & \star & \hat{X}+\hat{X}^T \end{bmatrix} \succ 0,\ \forall j \in \mathscr{I}_{m_f}; \tag{3.21}$$

$$\begin{bmatrix} 2\gamma_ie_i^Tb_r-\hat{\mu}_i & \gamma_ie_i^TP_r & 0 \\ \star & \hat{X}+\hat{X}^T & \underline{Q}^{-\frac{1}{2}} \\ \star & \star & \hat{\mu}_iI_n \end{bmatrix} \succ 0,\ \forall i \in \mathscr{I}_m; \tag{3.22}$$

$$\begin{bmatrix} \hat{Q}-B_p\hat{S}B_p^T & 0 & \hat{A} \\ \star & (2-\zeta)rI & \hat{C}_2 \\ \star & \star & \hat{X}+\hat{X}^T-\hat{Q} \end{bmatrix} \succ 0, \begin{bmatrix} \hat{Q} & \hat{X} \\ \star & P_r^T\hat{D}_zP_r \end{bmatrix} \succ 0, \zeta r > b_r^T\hat{D}_zb_r. \tag{3.23}$$

$$\min_{x_c,\hat{X},\hat{K}} trace(\bar{Q}^{-1}) \tag{3.24}$$

$$s.t. \begin{bmatrix} \hat{l}_i(\lambda_i,\hat{D}_i,\hat{W}_i) & 0 & 0 & x_c^T & x_c^T-\lambda_i e_i^T P_r \\ \star & V^T\hat{W}_iV & 0 & 0 & B_w^T \\ \star & \star & P_r^T\hat{D}_iP_r & 0 & \hat{A}^T \\ \star & \star & \star & \hat{X}+\hat{X}^T & \hat{A}^T \\ \star & \star & \star & \star & \hat{X}+\hat{X}^T \end{bmatrix} \succ 0,\ \forall i \in \mathscr{I}_m;$$

$$\begin{bmatrix} 2e_j^T\bar{f}-b_r^T\bar{E}_jb_r-d^T\bar{G}_jd & e_j^TD_w & e_j^T\hat{C} & x_c^T+e_j^T\hat{C} \\ \star & V^T\bar{G}_jV & 0 & 0 \\ \star & \star & P_r^T\bar{E}_jP_r & 0 \\ \star & \star & \star & \hat{X}+\hat{X}^T \end{bmatrix} \succ 0,$$

$$\begin{bmatrix} -2e_j^T\underline{f}-b_r^T\underline{E}_jb_r-d^T\underline{G}_jd & e_j^TD_w & e_j^T\hat{C} & x_c^T-e_j^T\hat{C} \\ \star & V^T\underline{G}_jV & 0 & 0 \\ \star & \star & P_r^T\underline{E}_jP_r & 0 \\ \star & \star & \star & \hat{X}+\hat{X}^T \end{bmatrix} \succ 0,\ \forall j \in \mathscr{I}_{m_f};$$

$$\begin{bmatrix} \bar{Q}^{-1} & \hat{X} \\ \star & P_r^T\bar{D}P_r \end{bmatrix} \succ 0, \qquad 1 > b_r^T\bar{D}b_r. \tag{3.25}$$

$$\begin{bmatrix} 2\nu_ie_i^Tb_r+\underline{b}_0^T\hat{F}_i\bar{b}_0 & \frac{1}{2}(\bar{b}_0+\underline{b}_0)^TF_iP_0 & x_c^T-\nu_ie_i^TP_r \\ \star & P_0^T\hat{F}_iP_0 & I_n \\ \star & \star & \hat{X}+\hat{X}^T \end{bmatrix} \succ 0,\ \forall i \in \mathscr{I}_m. \tag{3.26}$$

where $\lambda_i, \hat{\mu}_i, \gamma_i, \zeta, \nu_i \in \mathscr{R}_+$; $\hat{D}_i, \bar{E}_j, \underline{E}_j, \hat{D}_z, \bar{D} \in \mathscr{D}_+^m$; $\hat{W}_i, \bar{G}_j, \underline{G}_j \in \mathscr{D}_+^{m_w}$; $\underline{Q}^{-\frac{1}{2}}, \hat{Q}, \bar{Q}^{-1} \in \mathscr{S}_+^n$; $\hat{F}_i \in \mathscr{D}_+^{m_0}$ *and* $\hat{X} := X^{-1}$, $\hat{K} := KX^{-1}$ *are the variables. Furthermore,*

$$\hat{A} := A\hat{X}+B\hat{K}, \qquad \hat{C} := C\hat{X}+D\hat{K},$$

$$\hat{C}_2 := C_2\hat{X}+D_2\hat{K}, \qquad \hat{l}_i(\lambda_i,\hat{D}_i,\hat{W}_i) := 2\lambda_ie_i^Tb_r-b_r^T\hat{D}_ib_r-d^T\hat{W}_id.$$

Proof. The proof is similar to the proof of Theorem 2.2 except that it uses Corollary 3.1 and Corollary 3.2 in addition to Corollary 2.1. In more detail:

Applying the congruence

$$\begin{bmatrix} 1 & 0 & -x_c^T \\ 0 & I_{n_w} & 0 \\ 0 & 0 & I_n \end{bmatrix}$$

and then Corollary 3.1 on (3.10) (with $E^T = \begin{bmatrix} e_i^T P_r X & 0 & 0 \end{bmatrix}$, $F = \begin{bmatrix} x_c - A^K x_c & B_W & A^K \end{bmatrix}$ and $Y = \lambda_i X^{-1}$), then applying Corollary 3.2 (with $E^T = \begin{bmatrix} x_c^T & 0 & 0 \end{bmatrix}$, $F = -(A^K)^T$ and $Y = \lambda_i^{-1} X$), effecting the congruence $diag(\lambda_i^{\frac{1}{2}}, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{\frac{1}{2}} X^{-1}, \lambda_i^{\frac{1}{2}} I_n, \lambda_i^{-\frac{1}{2}} I_n)$ shows that (3.20) implies (3.10) upon the redefinitions

$$\hat{D}_i := \lambda_i D_i,\ \hat{W}_i := \lambda_i W_i. \tag{3.27}$$

Effecting the congruence

$$\begin{bmatrix} 1 & 0 & -x_c^T \\ 0 & I_{n_w} & 0 \\ 0 & 0 & I_n \end{bmatrix}$$

and employing Corollary 3.1 (with $E^T = \begin{bmatrix} -e_j^T C^K & 0 & 0 \end{bmatrix}$ and $Y = X^{-1}$) on the first LMI of (3.11) and effecting the congruence

$$\begin{bmatrix} 1 & 0 & x_c^T \\ 0 & I_{n_w} & 0 \\ 0 & 0 & -I_n \end{bmatrix}$$

for the second LMI, and employing Corollary 3.1 (with $E^T = \begin{bmatrix} e_j^T C^K & 0 & 0 \end{bmatrix}$ and $Y = X^{-1}$) shows that (3.21) is sufficient for (3.11).

For (3.12), effecting the congruence $\begin{bmatrix} 1 & -x_c \\ 0 & I_n \end{bmatrix}$, applying Corollary 2.1 (with $E = e_i^T P_r X$ and $Y = \gamma_i X^{-1}$), implementing the congruence $diag(\gamma_i^{\frac{1}{2}}, \gamma_i^{-\frac{1}{2}} I_n, \gamma_i^{\frac{1}{2}} \underline{Q}^{-\frac{1}{2}})$ shows that (3.22) implies (3.12) upon the redefinition

$$\hat{\mu}_i := \gamma_i \mu_i. \tag{3.28}$$

Effecting the congruence $diag(\zeta^{-\frac{1}{2}} I_n, \zeta^{\frac{1}{2}} X^{-T})$ for the second inequality in (3.13) shows that the second inequalities in (3.13) and (3.23) are equivalent, while the third inequality in (3.23) is ζ times the third inequality in (3.13). Effecting the congruence $diag(I_n, 1, Q)$

and applying Corollary 2.1 on the first inequality in (3.13) (with $F = Q$ and $Y = \zeta X^{-1}$) followed by a Schur complement, the congruence $diag(\zeta^{-\frac{1}{2}} I_n, \zeta^{-\frac{1}{2}}, \zeta^{-\frac{1}{2}} I_n)$ shows that the first inequality in (3.23) implies the first inequality in (3.13) since $\zeta^{-1} \geq 2 - \zeta$ for all $\zeta > 0$ upon the redefinitions

$$\hat{Q} := \zeta^{-1} Q,\ \hat{D}_z := \zeta D_z. \tag{3.29}$$

For (3.25),

Effecting the congruence $\begin{bmatrix} 1 & x_c^T \\ 0 & I_n \end{bmatrix}$ on (3.16) shows that it is equivalent to $1 > b_r^T \bar{D} b_r$ and $X^T P_r^T \bar{D} P_r X - \bar{Q} \succ 0$. Effecting the congruence X^{-T} on the last inequality followed by a Schur complement shows that (3.25) is equivalent to (3.15).

Applying Corollary 3.1 (with $E^T = \begin{bmatrix} e_i^T P_r X & 0 \end{bmatrix}$ and $Y = \nu_i X^{-1}$) on (3.16) and implementing the congruence $diag(\nu_i^{\frac{1}{2}}, \nu_i^{\frac{1}{2}} I_n, \nu_i^{-\frac{1}{2}} I_n)$ shows that (3.26) implies (3.16) upon the redefinition

$$\hat{F}_i := \nu_i F_i. \tag{3.30}$$

Finally, (3.19) and (3.24) follows from the above proof and the fact that $\det(Z) \leq \left(\frac{\mathrm{trace}(Z)}{n}\right)^n$ for any $n \times n$ positive definite matrix Z. □

Remark 3.1. *The conservatism introduced by the linearization in Theorem 3.2, compared to Theorem 3.1, can be traced back to the use of Corollary 2.1, Corollary 3.1 and Corollary 3.2, as well as the choice of the parameters P_r and b_r used to define the initial polytope. Similar to Remark 2.3, we restrict $\mathscr{Y}$ in Corollaries 2.1, 3.1 and 3.2 for a tractable solution. Although this restriction can be relaxed, the resulting optimization becomes nonlinear.*

3.4 Update Computation Algorithm

Once an admissible initial quadruple $(P, b, x_c, K) \in \Phi$ is obtained, this section presents an algorithm to update the solution based on Lemma 2.1 and the following result.

Lemma 3.1. *Let $\boldsymbol{L}, L \in \mathscr{R}^{m\times n}$, $\boldsymbol{M}, M \in \mathscr{R}^{m\times m}$ and $\boldsymbol{D}, D \in \mathscr{S}_+^m$. Define*

$$\mathscr{H}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}}^{L,M,D} := \mathscr{L}_{\boldsymbol{L},\boldsymbol{I}}^{L,I} + \mathscr{L}_{(\boldsymbol{M}^T D^{-1} ML),\boldsymbol{I}}^{(M^T D^{-1} ML),I} - L^T M^T D^{-1} \boldsymbol{D} D^{-1} ML, \tag{3.31}$$

$$\mathscr{F}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}}^{L,M,D} := \mathscr{H}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}}^{L,M,D} - (\boldsymbol{M}^T D^{-1} ML - \boldsymbol{L})^T (\boldsymbol{M}^T D^{-1} ML - \boldsymbol{L}) \tag{3.32}$$

$$\mathscr{G}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}} := \boldsymbol{L}^T \boldsymbol{M}^T \boldsymbol{D}^{-1} \boldsymbol{M} \boldsymbol{L}, \tag{3.33}$$

where $\mathscr{L}_{\cdot,\cdot}^{\cdot,\cdot}$ as defined in (2.33)*. Then $\mathscr{G}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}} \succeq \mathscr{F}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}}^{L,M,D}$ and $\mathscr{G}_{L,M,D} = \mathscr{F}_{L,M,D}^{L,M,D}$. Hence,*

$$\begin{aligned}&\{\exists L \in \mathscr{R}^{m\times n}, M \in \mathscr{R}^{m\times m}, D \in \mathscr{S}_+^m : \mathscr{G}_{L,M,D} \succ 0\}\\ &\qquad \Rightarrow \{\exists \boldsymbol{L} \in \mathscr{R}^{m\times n}, \boldsymbol{M} \in \mathscr{R}^{m\times m}, \boldsymbol{D} \in \mathscr{S}_+^m : \mathscr{G}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}} \succeq \mathscr{F}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}}^{L,M,D} \succ 0\}.\end{aligned}$$

Proof. The proof follows from Lemma 2.1 and the identities

$$\mathscr{G}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}} = \mathscr{L}_{(\boldsymbol{ML}),\boldsymbol{D}}^{(ML),D} + (\boldsymbol{ML} - \boldsymbol{D} D^{-1} ML)^T \boldsymbol{D}^{-1} (\boldsymbol{ML} - \boldsymbol{D} D^{-1} ML)$$

and

$$\begin{aligned}\mathscr{L}_{(\boldsymbol{ML}),\boldsymbol{D}}^{(ML),D} =& \mathscr{H}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}}^{L,M,D} - (\boldsymbol{M}^T D^{-1} ML - \boldsymbol{L})^T (\boldsymbol{M}^T D^{-1} ML - \boldsymbol{L})\\ &+ (\boldsymbol{M} D^{-1} ML - M D^{-1} ML)^T (\boldsymbol{M} D^{-1} ML - M D^{-1} ML) + (\boldsymbol{L} - L)^T (\boldsymbol{L} - L).\end{aligned}$$

□

Remark 3.2. *Note that if $\mathscr{M}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}}$ is any linear matrix function of $\boldsymbol{L}$, $\boldsymbol{M}$ and $\boldsymbol{D}$, then the linear matrix equation $\mathscr{M}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}} + \mathscr{F}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}}^{L,M,D} = 0$ is the Newton update for the nonlinear matrix* equation *$\mathscr{M}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}} + \mathscr{G}_{\boldsymbol{L},\boldsymbol{M},\boldsymbol{D}} = 0$ from the initial approximation L, M, D.*

The next result derive a Newton-like updates for the nonlinear matrix inequalities of Theorem 3.1 starting from the initial approximations given in Theorem 3.2.

Theorem 3.3. *With all definitions as above and $\mathscr{H}_{\cdot,\cdot,\cdot}^{\cdot,\cdot,\cdot}$, $\mathscr{F}_{\cdot,\cdot,\cdot}^{\cdot,\cdot,\cdot}$ and $\mathscr{G}_{\cdot,\cdot,\cdot}$, as defined in* (3.31)*,* (3.32)*, and* (3.33) *respectively, let $(\boldsymbol{P}, \boldsymbol{b}, \boldsymbol{x_c}, \boldsymbol{K}) \in \Phi$. Suppose that the optimizations in* (3.19) *and* (3.24) *have feasible solutions. Then the following optimization problems are feasible and provide Newton-like updates:*

$$\max_{(\boldsymbol{P},\boldsymbol{b},\boldsymbol{x_c},\boldsymbol{K})\in\Phi} \log\det \underline{\boldsymbol{Q}}^{-\frac{1}{2}}, \tag{3.34}$$

$$s.t. \left[\begin{array}{c|cc} \mathscr{M}_i(\boldsymbol{D}_i^{-1},\boldsymbol{b},\boldsymbol{W}_i)+\mathscr{H}_{L_i(\boldsymbol{P},\boldsymbol{x_c},\boldsymbol{K}),M(\boldsymbol{P}),H_i(\boldsymbol{D}_i)^{-1}}^{L_i(P,x_c,K),M(P),H_i(D_i)^{-1}} & \star & \star \\ \hline EM(\boldsymbol{P})L_i(\boldsymbol{P},\boldsymbol{x_c},\boldsymbol{K}) & I_n & \star \\ M(\boldsymbol{P})^T H_i(D_i)M(P)L_i(P,x_c,K)-L_i(\boldsymbol{P},\boldsymbol{x_c},\boldsymbol{K}) & I_n & I_n \end{array}\right] \succ 0,\ \forall i\in\mathscr{I}_m; \tag{3.35}$$

$$\left[\begin{array}{c|c} \bar{\mathscr{E}}_j(\bar{\boldsymbol{E}}_j^{-1},\boldsymbol{b},\bar{\boldsymbol{G}}_j,\boldsymbol{K})+\mathscr{H}_{\bar{H}(\boldsymbol{x_c}),\boldsymbol{P},\bar{\boldsymbol{E}}_j^{-1}}^{\bar{H}(x_c),P,\bar{E}_j^{-1}} & \star \\ \hline \boldsymbol{P}^T\bar{E}_jP\bar{H}(x_c)-\bar{H}(\boldsymbol{x_c}) & I_n \end{array}\right] \succ 0,$$

$$\left[\begin{array}{c|c} \underline{\mathscr{E}}_j(\underline{\boldsymbol{E}}_j^{-1},\boldsymbol{b},\underline{\boldsymbol{G}}_j,\boldsymbol{K})+\mathscr{H}_{\underline{H}(\boldsymbol{x_c}),\boldsymbol{P},\underline{\boldsymbol{E}}_j^{-1}}^{\underline{H}(x_c),P,\underline{E}_j^{-1}} & \star \\ \hline \boldsymbol{P}^T\underline{E}_jP\underline{H}(x_c)-\underline{H}(\boldsymbol{x_c}) & I_n \end{array}\right] \succ 0,\ \forall j\in\mathscr{I}_{m_f}; \tag{3.36}$$

$$\left[\begin{array}{c|cc} \begin{bmatrix} 2e_i^T\boldsymbol{b}-\boldsymbol{\mu}_i & \star \\ \boldsymbol{P}^Te_i & 0 \end{bmatrix} \begin{array}{l} +R_i(\boldsymbol{P},\boldsymbol{x_c})^TJ_i(\mu_i\underline{Q})R_i(P,x_c) \\ +R_i(P,x_c)^TJ_i(\mu_i\underline{Q})R_i(\boldsymbol{P},\boldsymbol{x_c}) \end{array} & \star & \star \\ \hline ER_i(\boldsymbol{P},\boldsymbol{x_c}) & I_n & \star \\ \underline{\boldsymbol{Q}}^{-\frac{1}{2}}ZJ_i(\mu_i\underline{Q})R_i(P,x_c) & 0 & \boldsymbol{\mu}_iI_n \end{array}\right] \succ 0,\ \forall i\in\mathscr{I}_m; \tag{3.37}$$

$$\begin{bmatrix} \boldsymbol{Q} & 0 & A^{\boldsymbol{K}} \\ \star & rI_{m_2} & C_2^{\boldsymbol{K}} \\ \star & \star & \mathscr{L}_{I_n,\boldsymbol{Q}}^{I_n,Q} \end{bmatrix} \succ 0, \quad \begin{bmatrix} \boldsymbol{Q} & I_n \\ \star & \mathscr{L}_{\boldsymbol{P},\boldsymbol{D}_z^{-1}}^{P,D_z^{-1}} \end{bmatrix} \succ 0, \quad \begin{bmatrix} \boldsymbol{D}_z^{-1} & \boldsymbol{b} \\ \star & r \end{bmatrix} \succ 0 \tag{3.38}$$

$$\min_{(\boldsymbol{P},\boldsymbol{b},\boldsymbol{x_c},\boldsymbol{K})\in\Phi} \left(-\log\det\bar{\boldsymbol{Q}}\right) \tag{3.39}$$

$$s.t. \left[\begin{array}{c|cc} \mathscr{M}_i(\boldsymbol{D}_i^{-1},\boldsymbol{b},\boldsymbol{W}_i)+\mathscr{H}^{L_i(P,x_c,K),M(P),H_i(D_i)^{-1}}_{L_i(\boldsymbol{P},\boldsymbol{x_c},\boldsymbol{K}),M(\boldsymbol{P}),H_i(\boldsymbol{D}_i)^{-1}} & \star & \star \\ \hline EM(\boldsymbol{P})L_i(\boldsymbol{P},\boldsymbol{x_c},\boldsymbol{K}) & I_n & \star \\ M(\boldsymbol{P})^T H_i(D_i)M(P)L_i(P,x_c,K)-L_i(\boldsymbol{P},\boldsymbol{x_c},\boldsymbol{K}) & I_n & I_n \end{array}\right] \succ 0,\ \forall i\in\mathscr{I}_m;$$

$$\left[\begin{array}{c|c} \bar{\mathscr{E}}_j(\bar{\boldsymbol{E}}_j^{-1},\boldsymbol{b},\bar{\boldsymbol{G}}_j,\boldsymbol{K})+\mathscr{H}^{\bar{H}(x_c),P,\bar{E}_j^{-1}}_{\bar{H}(\boldsymbol{x_c}),\boldsymbol{P},\bar{\boldsymbol{E}}_j^{-1}} & \star \\ \hline \boldsymbol{P}^T\bar{E}_jP\bar{H}(x_c)-\bar{H}(\boldsymbol{x_c}) & I_n \end{array}\right] \succ 0,$$

$$\left[\begin{array}{c|c} \underline{\mathscr{E}}_j(\underline{\boldsymbol{E}}_j^{-1},\boldsymbol{b},\underline{\boldsymbol{G}}_j,\boldsymbol{K})+\mathscr{H}^{\underline{H}(x_c),P,\underline{E}_j^{-1}}_{\underline{H}(\boldsymbol{x_c}),\boldsymbol{P},\underline{\boldsymbol{E}}_j^{-1}} & \star \\ \hline \boldsymbol{P}^T\underline{E}_jP\underline{H}(x_c)-\underline{H}(\boldsymbol{x_c}) & I_n \end{array}\right] \succ 0,\ \forall j\in\mathscr{I}_{m_f};$$

$$\mathscr{L}^{P,\bar{D}^{-1}}_{\boldsymbol{P},\bar{\boldsymbol{D}}^{-1}}-\bar{\boldsymbol{Q}}\succ 0, \qquad \begin{bmatrix}\bar{\boldsymbol{D}}^{-1} & \boldsymbol{b}\\ \star & 1\end{bmatrix}\succ 0; \tag{3.40}$$

$$\left[\begin{array}{c|c} \mathscr{T}_i(\boldsymbol{P},\boldsymbol{b},\boldsymbol{F}_i)+\mathscr{L}^{S(P),I}_{S(\boldsymbol{P}),I} & \star \\ \hline TS(\boldsymbol{P}) & I_n \end{array}\right]\succ 0,\ \forall i\in\mathscr{I}_m \tag{3.41}$$

where the variables are $\boldsymbol{D}_i^{-1},\bar{\boldsymbol{E}}_j^{-1},\underline{\boldsymbol{E}}_j^{-1},\boldsymbol{D}_z^{-1},\bar{\boldsymbol{D}}^{-1}\in\mathscr{D}_+^m$; $\boldsymbol{W}_i,\bar{\boldsymbol{G}}_j,\underline{\boldsymbol{G}}_j\in\mathscr{D}_+^{m_w}$; $\boldsymbol{\mu}_i\in\mathscr{R}_+$; $\underline{\boldsymbol{Q}}^{-\frac{1}{2}},\boldsymbol{Q},\bar{\boldsymbol{Q}}\in\mathscr{S}_+^n$; $\boldsymbol{F}_i\in\mathscr{D}_+^{m_0}$ *and where* $E=\begin{bmatrix}-I_n & I_n & 0\end{bmatrix}$, $M(P)=diag(I_n,I_n,P)$, $H_i(D_i)=diag(I_n,I_n,D_i)$, $\bar{H}(x_c)=\begin{bmatrix}0 & x_c & 0 & I_n\end{bmatrix}$, $\underline{H}(x_c)=\begin{bmatrix}0 & x_c & 0 & -I_n\end{bmatrix}$, $Z=\begin{bmatrix}0 & -I_n & I_n\end{bmatrix}$, $T=\begin{bmatrix}-I_n & 1\end{bmatrix}$

$$\mathscr{M}_i(\boldsymbol{D}_i^{-1},\boldsymbol{b},\boldsymbol{W}_i)=\begin{bmatrix}\boldsymbol{D}_i^{-1} & \boldsymbol{b} & 0 & 0\\ \star & 2e_i^T\boldsymbol{b}-d^T\boldsymbol{W}_id & 0 & 0\\ \star & \star & V^T\boldsymbol{W}_iV & 0\\ \star & \star & \star & 0\end{bmatrix}, L_i(P,x_c,K)=\begin{bmatrix}0 & P^Te_i & 0 & 0\\ 0 & x_c & B_w & A^K\\ 0 & x_c & 0 & I\end{bmatrix},$$

$$\bar{\mathscr{E}}_j(\bar{\boldsymbol{E}}_j^{-1},\boldsymbol{b},\bar{\boldsymbol{G}}_j,\boldsymbol{K})=\begin{bmatrix}\boldsymbol{E}_j^{-1} & \boldsymbol{b} & 0 & 0\\ \star & 2e_j^T\bar{f}-d^T\bar{\boldsymbol{G}}_jd & e_j^TD_w & e_j^TC^{\boldsymbol{K}}\\ \star & \star & V^T\bar{\boldsymbol{G}}_jV & 0\\ \star & \star & \star & 0\end{bmatrix},$$

and

$$\underline{\mathscr{E}}_j(\underline{\boldsymbol{E}}_j^{-1},\boldsymbol{b},\underline{\boldsymbol{G}}_j,\boldsymbol{K}) = \begin{bmatrix} \underline{\boldsymbol{E}}_j^{-1} & \boldsymbol{b} & 0 & 0 \\ \star & -2e_j^T\underline{f} - d^T\underline{\boldsymbol{G}}_j d & e_j^T D_w & e_j^T C^{\boldsymbol{K}} \\ \star & \star & V^T\underline{\boldsymbol{G}}_j V & 0 \\ \star & \star & \star & 0 \end{bmatrix},$$

$$J_i(\mu_i\underline{Q}) = \begin{bmatrix} I_n & 0 & 0 \\ \star & I_n & 0 \\ \star & \star & \mu_i\underline{Q} \end{bmatrix}, \quad R_i(P,x_c) = \begin{bmatrix} P^T e_i & 0 \\ x_c & 0 \\ x_c & I_n \end{bmatrix}.$$

$$\mathscr{T}_i(\boldsymbol{P},\boldsymbol{b},\boldsymbol{F}_i) = \begin{bmatrix} 2e_i^T\boldsymbol{b} + \bar{b}_0^T\boldsymbol{F}_i\underline{b}_0 & e_i^T\boldsymbol{P} + \frac{1}{2}(\bar{b}_0 + \underline{b}_0)^T\boldsymbol{F}_i P_0 \\ \star & P_0^T\boldsymbol{F}_i P_0 \end{bmatrix}, \quad S_i(P) = \begin{bmatrix} P^T e_i & 0 \\ 1 & 0 \end{bmatrix}.$$

Proof. The proof is essentially an application of Lemma 2.1 and Lemma 3.1, congruences, Schur complements and some re-definitions to show that (3.10)-(3.16) imply (3.35)-(3.41), which in turn imply (3.10)-(3.16) (with bold variables) and therefore (3.2)-(3.8) (with bold variables), respectively, from Theorem 3.1. In more detail:

Suppose that $(P,b,x_c,K,D_i,W_i), \forall i \in \mathscr{I}_m$ satisfy (3.10). Then effecting an upper Schur complement on $b^T D_i b$ on (3.10), a manipulation shows that (3.10) is equivalent to

$$\mathscr{M}_i(D_i^{-1},b,W_i) + \mathscr{G}_{L_i(P,x_c,K),M(P),H_i(D_i)^{-1}} - L_i(P,x_c,K)^T M^T E^T E M L_i(P,x_c,K) \succ 0. \tag{3.42}$$

The result (3.35) follows by applying Lemma 3.1 and Lemma 2.1 on the second term and a Schur complement on the third. Furthermore, (3.10) and (3.2) are satisfied by

$$(P,b,x_c,K,D_i,W_i,) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{x_c},\boldsymbol{K},\boldsymbol{D}_i,\boldsymbol{W}_i).$$

Suppose that $(P,b,x_c,K,\bar{E}_j,\bar{G}_j,\underline{E}_j,\underline{G}_j), \forall j \in \mathscr{I}_{m_f}$ satisfy (3.11). Then (3.36) follows by applying Schur complement on $b^T\bar{E}_j b$ and $b^T\underline{E}_j b$, and Lemma 3.1 on $\mathscr{G}_{\bar{H}(\boldsymbol{x_c}),\boldsymbol{P},\bar{\boldsymbol{E}}_j^{-1}}$ and $\mathscr{G}_{\underline{H}(\boldsymbol{x_c}),\boldsymbol{P},\underline{\boldsymbol{E}}_j^{-1}}$ in (3.11). Furthermore, (3.11) and (3.3) are satisfied by

$$(P,b,x_c,K,\bar{E}_j,\bar{G}_j,\underline{E}_j,\underline{G}_j) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{x_c},\boldsymbol{K},\bar{\boldsymbol{E}}_j,\bar{\boldsymbol{G}}_j,\underline{\boldsymbol{E}}_j,\underline{\boldsymbol{G}}_j).$$

Suppose that $(P,b,x_c,\mu_i,\underline{Q}), \forall i \in \mathscr{I}_m$ satisfy (3.12). Then (3.37) follows by applying Lemma 3.1 on $\mathscr{N}_{R_i(P,x_c),J_i(\mu_i\underline{Q})}$ and taking a Schur complement in (3.12). Furthermore, (3.12) and (3.4) are satisfied by

$$(P,b,x_c,\mu_i,\underline{Q}) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{x_c},\boldsymbol{\mu}_i,\underline{\boldsymbol{Q}}).$$

Suppose that (P,b,K,Q,D_z) satisfy (3.13). Then (3.38) follows by applying Lemma 2.1 on $\mathscr{N}_{I,Q}=Q^{-1}$ and $\mathscr{N}_{P,D_z^{-1}}=P^T D_z P$ and a Schur complement on the third inequality, in (3.13). Furthermore, (3.13) and (3.5) are satisfied by

$$(P,b,K,Q,D_z) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{K},\boldsymbol{Q},\boldsymbol{D}_z).$$

Suppose that $(P,b,\bar{Q},\bar{D})$ satisfy (3.15). Then the inequalities in (3.40) are equivalent with the inequality in (3.15) while using a congruence transformation. The result follows by applying Lemma 2.1 on the term $\mathscr{N}_{P,\bar{D}^{-1}} = P^T \bar{D} P$. Furthermore, (3.15) and (3.7) are satisfied by

$$(P,b,\bar{Q},\bar{D}) := (\boldsymbol{P},\boldsymbol{b},\bar{\boldsymbol{Q}},\bar{\boldsymbol{D}}).$$

Suppose that $(P,b,F_i), \forall i \in \mathscr{I}_m$ satisfy (3.16). Then (3.41) follows by applying Lemma 2.1 on $\mathscr{N}_{S_i(P),I}$ and taking a Schur complement. Furthermore, (3.16) and (3.8) are satisfied by

$$(P,b,F_i) := (\boldsymbol{P},\boldsymbol{b},\boldsymbol{F}_i).$$

Finally, (3.34) and (3.39) follow from above proof. $\square$

Remark 3.3. *Note that taking $\boldsymbol{D}_i^{-1}, \bar{\boldsymbol{E}}_j^{-1}, \underline{\boldsymbol{E}}_j^{-1}, \boldsymbol{D}_z^{-1}, \bar{\boldsymbol{D}}^{-1}$ as variables allows us to use Lemma 3.1 and Lemma 2.1 to ensure recursive feasibility, that is, the volume of the updated inner/outer approximation to the maximal/minimal RCI set is at least as good (large for maximal and small for minimal sets) as that of the previous set, that is*

$$\left(\max_{(\boldsymbol{P},\boldsymbol{b},\boldsymbol{x_c},\boldsymbol{K})\in\Phi} \log\det \underline{\boldsymbol{Q}}^{-\frac{1}{2}}\right) \geq \log\det \underline{Q}^{-\frac{1}{2}},$$

$$\left(\min_{(\boldsymbol{P},\boldsymbol{b},\boldsymbol{x_c},\boldsymbol{K})\in\Phi} -\log\det \bar{\boldsymbol{Q}}\right) \leq -\log\det \bar{Q},$$

if $\underline{Q}^{-\frac{1}{2}}$ and $\bar{Q}$ are solutions of (3.19) *and* (3.24)*. It also allows us to use **b** as a variable, thus improving the updated solution. Our numerical experience, part of which is reported* *below, as well as Remark 2.5, suggest quadratic convergence, although a formal proof of this is beyond the scope of this work.*

3.5 Solution algorithm

The following algorithm summarizes our solution.

Algorithm 3.1. *Given system* (1.1), $\mathscr{F}_{Asy}$ *and sets* $\mathscr{W} = \mathscr{P}(V,d), \mathscr{P}_{Asy}(P_0, \bar{b}_0, \underline{b}_0)$ *and parameter* r.

1. **Initial data***: Choose* $m \geq n$*, initial polytope* $\mathscr{P}(P_r, b_r, x_c)$ *and tolerance level* tol.
2. **Initial solution**
   (a) *Use Theorem 3.2 to solve the convex SDPs in* (3.19) *or* (3.24).
   (b) *Define* D_i, W_i, Q, D_z *and* μ_i *from* (3.27)-(3.30) *so that* (3.10)-(3.11), (3.12) *or* (3.15)*,* (3.13) *or* (3.16) *are satisfied.*
3. **Update** *Solve the optimizations in* (3.34) *or* (3.39).
4. **Stopping condition**
   (a) *If* $\det(\underline{\boldsymbol{Q}}^{-1}) - \det(\underline{Q}^{-1}) \leq tol$ *(for maximization) or* $\det(\bar{Q}^{-1}) - \det(\bar{\boldsymbol{Q}}^{-1}) \leq tol$ *(for minimization), stop.*
   (b) *Else update* $Z := \boldsymbol{Z}$*, where* $\boldsymbol{Z}$ *denotes a variable in the optimizations in* (3.34) *or* (3.39)*, and go to step 3.*
5. **End**

3.6 Examples

3.6.1 Example 1

Consider the model for a car following scenario described in [59] and [45] with the parameters:

$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}.$$

The asymmetric output constraints are defined by the parameters

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad D_w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{f} = \begin{bmatrix} 1.85 \\ 3 \\ 4.85 \\ 2.4 \end{bmatrix}, \quad \underline{f} = \begin{bmatrix} -3 \\ -3 \\ -2.2 \\ -2.4 \end{bmatrix}.$$

With $m = 3$, Figure 3.1 shows the initial (in yellow, with solid border) and final (in red, with dashed border) inner approximations to the maximal RCI set, with the final control law given as $K = [-0.6271 \quad -0.6931]$. The blue cross marks and the dashed green line shows the trajectory of the system states under the feedback control law (only one trajectory is shown for clarity). The white box with dotted border shows the constraints on state and input signals. We also gives the RCI set centered at the origin (in green color and dash-dot border) for comparison. When treating the RCI set as a terminal target set, the polytopic form $\mathscr{P}(P,b,x_c)$ has the advantage of having a larger volume. In this example, the values of $\log\det \underline{Q}^{-\frac{1}{2}}$ are 1.1334 and 0.9390 for $\mathscr{P}(P,b,x_c)$ and $\mathscr{P}(P,b)$, respectively.

3.6.2 Example 2

Consider the example in [46] with

$$A = \begin{bmatrix} 0.98 & 0.72 \\ -0.02 & 0.72 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{f} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}, \quad \underline{f} = \begin{bmatrix} -3 \\ -3 \\ -3 \\ -4 \end{bmatrix}.$$

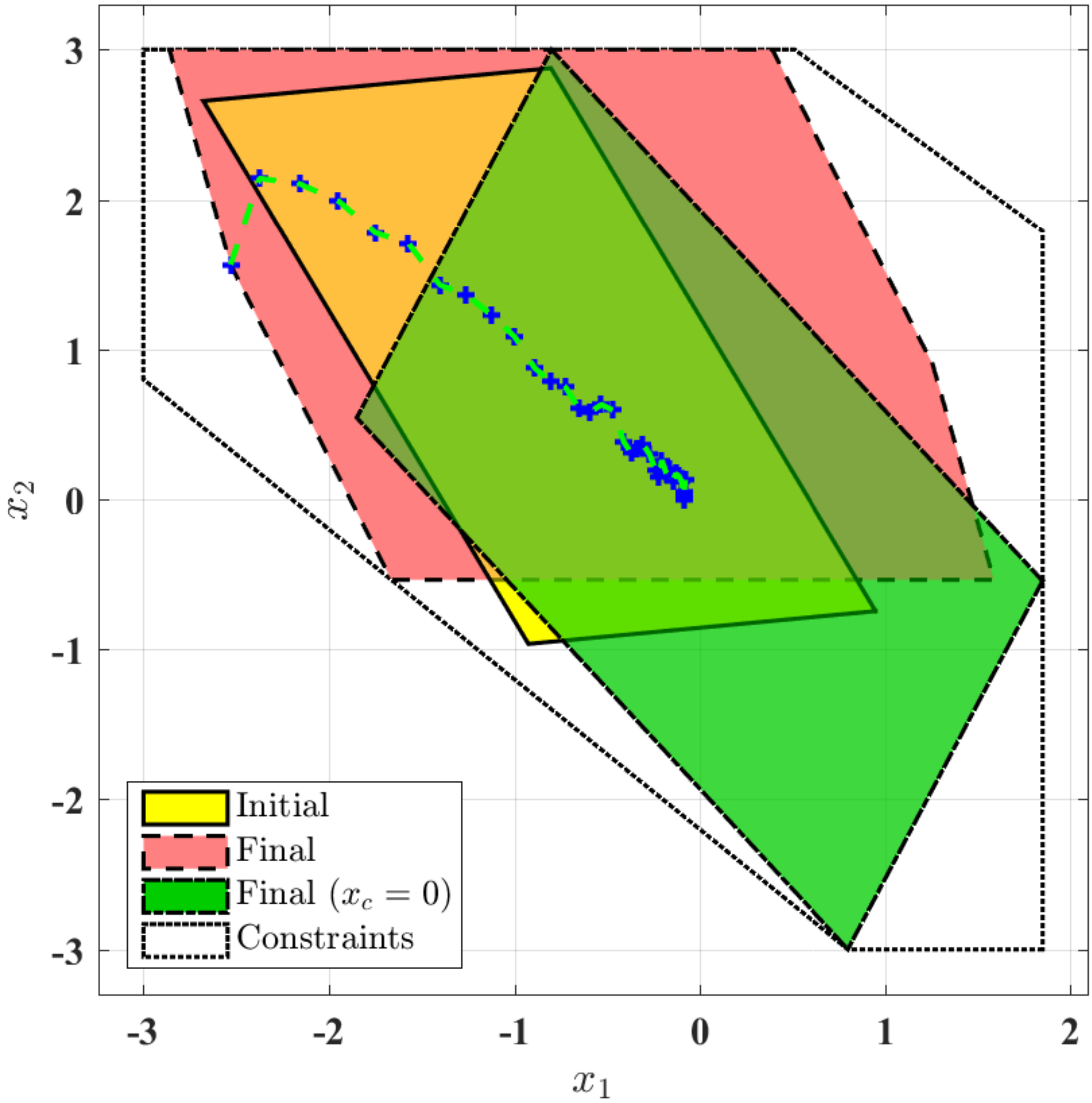


Fig. 3.1 Initial (yellow) and final (red) approximations of maximal RCI sets, final RCI set for $\mathscr{P}(P,b)$ (green), and output constraint (white).

B_u, B_w and V are the identity matrices, D_w is the zero matrix and d is the vector of ones. The initial state constraint is defined by

$$P_0 = \begin{bmatrix} 5 & 5 \\ 7 & 0.4 \\ 0.4 & 7 \\ 6 & -6 \end{bmatrix}, \quad \bar{b}_0 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \underline{b}_0 = \begin{bmatrix} -5 \\ -5 \\ -5 \\ -5 \end{bmatrix}.$$

We choose $m = 5$ and apply Algorithm 3.1 to obtain an outer approximation to the minimal invariant set. Figure 3.2 demonstrates the initial set (in yellow with solid border) and the final set (in red with dashed border). The initial state constraint is shown in white with solid border. One state trajectory is given (in blue cross marks and dashed green line). The final result in the $x_c = 0$ case is given (in blue with dash-dot border). The structure $\mathscr{P}(P,b,x_c)$ gives smaller result with $-\log\det\bar{Q} = -0.9748$ while when using $\mathscr{P}(P,b)$, $-\log\det\bar{Q} = -0.7352$.

3.7 Conclusion

We have proposed a novel scheme, based on a convex SDP algorithm, that can efficiently compute full-complexity polytopic RCI sets and the corresponding control law, whilst taking account of additive disturbances, asymmetric output and initial state constraints as well as $\mathscr{H}_2$ performance constraints in a unified framework. The chapter first derives the nonlinear necessary and sufficient conditions for the existence of an admissible RCI set and feedback gain matrix. Corollaries of the Elimination Lemma are then derived and used to linearize the LMI conditions and render the optimization problem tractable. An initial invariant polytope, and control law K, are first obtained and the set-volume is then iteratively optimized by solving convex/LMI optimizations. In addition to handling arbitrarily large number of faces of the invariant polytope, the iterative algorithm guarantees recursive feasibility. Furthermore, the iterations - based on a Newton-like update - result in an observed quadratic speed of convergence.

Apart from the freedom of choosing arbitrarily large values of m and the optimization for the control gain K, the algorithm proposed in this chapter allows the center x_c of the polytopic RCI set to shift from the origin, which, when the constraints are asymmetric, further improves the inner/outer approximations of the maximal/minimal RCI set compared with

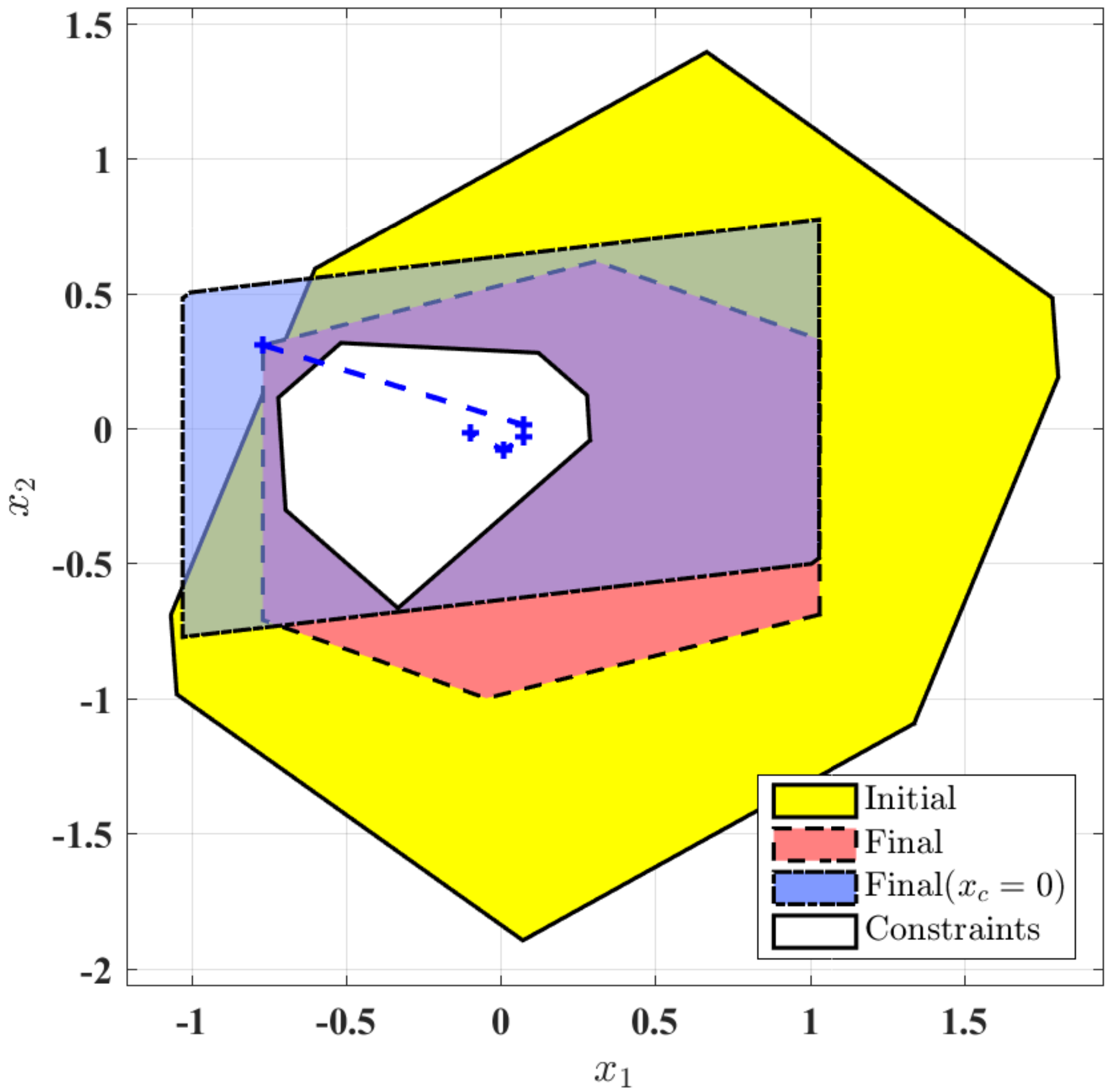


Fig. 3.2 Initial (yellow) and final (red) approximations of minimal RCI sets, final RCI set for $\mathscr{P}(P,b)$ (blue), initial state constraint (white).

many of the schemes in the literature. The efficiency of this algorithm has been illustrated using numerical examples.

Chapter 4

Tube Based Model Predictive Control

Tube based MPC algorithms are computationally tractable due to the use of invariant tubes. Their computational complexity increases linearly in the length of predictive horizon. For linear time-invariant systems, the algorithms proposed in the literature (see [31] and the references therein) often choose a piecewise affine control policy in which the control gain is predefined. When considering the observer for the systems [56], the observer gain is also predefined.

In this chapter, we extended the algorithms proposed in Chapter 2 to use the control and observer gains as variables in the optimization of invariant sets and tubes since the extra degrees of freedom provided by these variables will result in tighter volumes of the tubes. The efficiency of these algorithms are then illustrated by numerical examples.

4.1 Problem Description

Consider the system (1.1) with z as the cost signal and the output $y = Cx + Du + D_w w$, and consider only separate state and input constraints

$$
\begin{aligned}
x \in \mathscr{X} &:= \{x \in \mathscr{R}^n | \underline{f}_x \leq V_x x \leq \bar{f}_x\}, && V_x \in \mathscr{R}^{m_x \times n}, && \underline{f}_x < 0 < \bar{f}_x \in \mathscr{R}^{m_x}, \\
u \in \mathscr{U} &:= \{u \in \mathscr{R}^{n_u} | \underline{f}_u \leq V_u u \leq \bar{f}_u\}, && V_u \in \mathscr{R}^{m_u \times n_u}, && \underline{f}_x < 0 < \bar{f}_u \in \mathscr{R}^{m_u},
\end{aligned}
$$

for clarity. Instead of a linear controller, we will use a tube based MPC controller, which is time-varying and piecewise affine. Implementation of tube based MPC also require the use of a nominal system:

$$\begin{bmatrix} \bar{x}^+ \\ \bar{z}^+ \end{bmatrix} = \begin{bmatrix} A & B \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \tag{4.1}$$

where $\bar{x}, \bar{x}^+ \in \mathscr{R}^n$ are the current and successor nominal system states, respectively, $\bar{z}$ is the nominal cost signal and $\bar{u} \in \mathscr{R}^{n_u}$ is the nominal system input, which will be a variable in the MPC procedure.

We also consider the Luenberger Observer [56]:

$$\begin{bmatrix} \hat{x}^+ \\ \hat{y} \end{bmatrix} = \begin{bmatrix} A & B & L \\ C & D & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ u \\ y - \hat{y} \end{bmatrix},$$

and use the control input

$$u = \bar{u} + K(\hat{x} - \bar{x}),$$

where $\hat{x}, \hat{x}^+ \in \mathscr{R}^{n \times n}$ are the current and successor estimated states, respectively, and $\hat{y} \in \mathscr{R}^{m_y}$ is the estimated output. The matrix $L \in \mathscr{R}^{n \times n_y}$ is the Luenberger Observer gain, and the matrix $K \in \mathscr{R}^{n_u \times n}$ is the feedback gain. In the literature, L is predefined and is normally chosen to satisfy $\rho(A^L) < 1$, where $\rho(\cdot)$ denotes the spectral radius and where $A^L := A - LC$. The feedback gain K is also predefined and similarly chosen to satisfy $\rho(A^K) < 1$, where $A^K := A + BK$. In the next section, we will provide the algorithm to ensure A^K and A^L are stable and furthermore, use them as variables in the optimization of the volume of the tubes.

Define the state estimation and state control error signals as $\tilde{x} := x - \hat{x}$ and $\xi := \hat{x} - \bar{x}$, respectively. Then their dynamics are described by

$$\tilde{x}^+ = A^L \tilde{x} + B_w^L w, \tag{4.2}$$

$$\xi^+ = A^K \xi + \theta, \tag{4.3}$$

where $B_w^L = B_w - LD_w$, $\theta := L(y - \hat{y}) = L(C\tilde{x} + D_w w)$. It follows that

$$u = \bar{u} + K\xi. \tag{4.4}$$

Suppose there exists an RCI set $\mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$ for system (4.2) and an RCI set $\mathscr{P}(P_\xi, b_\xi)$ for system (4.3) so that

$$\left\{\begin{array}{c} \tilde{x} \in \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}}) \\ w \in \mathscr{W} \end{array}\right\} \Rightarrow \tilde{x}^+ \in \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}}), \tag{4.5}$$

$$\left\{\begin{array}{c} \xi \in \mathscr{P}(P_\xi, b_\xi) \\ \theta \in \Theta := L(C\mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}}) \oplus D_w \mathscr{W}) \end{array}\right\} \Rightarrow \xi^+ \in \mathscr{P}(P_\xi, b_\xi). \tag{4.6}$$

Since $x(k) = \hat{x}(k) + \tilde{x}(k)$, it follows that $x(k) \in \mathscr{X}$ is satisfied if $\hat{x}(k) \in \mathscr{X} \ominus \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and $\tilde{x}(k) \in \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$ for all $w(k) \in \mathscr{W}$. Furthermore,

$$\left.\begin{array}{c} \tilde{x}(0) \in \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}}) \\ x(0) \in \{\hat{x}(0)\} \oplus \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}}) \\ w \in \mathscr{W} \end{array}\right\} \Rightarrow x(k) \in \{\hat{x}(k)\} \oplus \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$$

Similarly for the control error and the input, we have $\hat{x}(k) \in \{\bar{x}(k)\} \oplus \mathscr{P}(P_\xi, b_\xi)$ and $u(k) \in \{\bar{u}(k)\} \oplus K\mathscr{P}(P_\xi, b_\xi)$ if $\hat{x}(0) \in \{\bar{x}(0)\} \oplus \mathscr{P}(P_\xi, b_\xi)$ and $\xi(0) \in \mathscr{P}(P_\xi, b_\xi)$ for all $w \in \mathscr{W}$. Hence, the system constraints $x \in \mathscr{X}$, $u \in \mathscr{U}$ are satisfied if

$$\bar{x}(k) \in \bar{\mathscr{X}} := \mathscr{X} \ominus \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}}) \ominus \mathscr{P}(P_\xi, b_\xi) \tag{4.7}$$

$$\bar{u}(k) \in \bar{\mathscr{U}} := \mathscr{U} \ominus K\mathscr{P}(P_\xi, b_\xi) \tag{4.8}$$

In conclusion, we have

$$\left.\begin{array}{l} \hat{x}(k) \in \{\bar{x}(k)\} \oplus \mathscr{P}(P_\xi, b_\xi) \\ u(k) \in \{\bar{u}(k)\} \oplus K\mathscr{P}(P_\xi, b_\xi) \\ \tilde{x}(0) \in \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}}) \\ \xi(0) \in \mathscr{P}(P_\xi, b_\xi) \\ w \in \mathscr{W} \end{array}\right\} \Rightarrow x(k) \in \{\bar{x}(k)\} \oplus \mathscr{P}(P_\xi, b_\xi) \oplus \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$$

The set $\{\bar{x}(k)\} \oplus \mathscr{P}(P_\xi, b_\xi) \oplus \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$ is the invariant tube at kth step. The trajectory of the nominal system state $\bar{x}(k)$ is the center of the tube in which $\hat{x}(k)$ lies. The feedback component $K\xi$ will attempt to steer the trajectory of $\hat{x}(k)$ towards the center of the tube.

Define the objective function of the MPC scheme as $J = \sum_{k=0}^{N} z(k)^T z(k)$. When using tube MPC, we replace the original optimization problem

$$
\begin{aligned}
& min \sum_{k=0}^{N} z(k)^T z(k) \\
& s.t. \\
& x \in \mathscr{X},\ u \in \mathscr{U}
\end{aligned}
$$

by the following nominal MPC optimization

$$
\begin{aligned}
& min \sum_{k=0}^{N} \bar{z}(k)^T \bar{z}(k) \\
& s.t. \\
& \bar{x}(k) \in \bar{\mathscr{X}},\ \bar{u}(k) \in \bar{\mathscr{U}}
\end{aligned}
\tag{4.9}
$$

Thus the MPC problem for a system subject to additive disturbance can be replaced by the nominal MPC problem. The nominal state and input constraints $\bar{x}(k) \in \bar{\mathscr{X}}$ and $\bar{u}(k) \in \bar{\mathscr{U}}$ will ensure the satisfaction of the original state and input constraints $x \in \mathscr{X}$ and $u \in \mathscr{U}$.

4.2 Invariant Tube

Consider the invariance conditions (4.5)-(4.6) for the state estimation and state control error dynamics in (4.2) and (4.3), respectively. From the above analysis, we have that the original constraints $x \in \mathscr{X}$ and $u \in \mathscr{U}$ are tightened by $\mathscr{P}(P_\xi, b_\xi)$ and $\mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$; see (4.7) and (4.8). It follows that minimal invariant set are required to obtain reasonable control performance. To this end, we introduce outer bounding ellipsoids $\mathscr{E}(Q_{\tilde{x}})$ and $\mathscr{E}(Q_\xi)$ for the invariant sets for minimization and require that

$$\exists Q_{\tilde{x}} \in \mathscr{S}_+^n :\ \mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}}) \subset \mathscr{E}(Q_{\tilde{x}}) \tag{4.10}$$

$$\exists Q_\xi \in \mathscr{S}_+^n :\ \mathscr{P}(P_\xi, b_\xi) \subset \mathscr{E}(Q_\xi). \tag{4.11}$$

For system (4.2) and (4.3), sets $\mathscr{W}$, $m \geq n$, and with $\Xi := \mathscr{R}^{m\times n} \times \mathscr{R}^{m} \times \mathscr{R}^{n\times n_y}$ and $\Psi := \mathscr{R}^{m\times n} \times \mathscr{R}^{m} \times \mathscr{R}^{n_u\times n}$, we present convex algorithms to solve the following optimizations:

$$\min_{\substack{(P_{\tilde{x}}, b_{\tilde{x}}, L)\in\Xi \\ (4.5),(4.10)}} \log\det Q_{\tilde{x}}^{-1}, \qquad \min_{\substack{(P_{\xi}, b_{\xi}, K)\in\Psi \\ (4.6),(4.11)}} \log\det Q_{\xi}^{-1}. \tag{4.12}$$

In the following, we provide sufficient conditions for the existence of the invariant sets $\mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$, $\mathscr{P}(P_{\xi}, b_{\xi})$ and the corresponding L, K.

Remark 4.1. *Note that the RCI set in* (4.6) *requires knowledge of the RCI set in* (4.5). *It is outside the scope of this work to take L and K as variables and simultaneously optimize both sets and leave it is a direction for future research. We will, instead, provide algorithms to approximately minimize the volume of the set $\mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$, and then use this set, as well as the corresponding observer gain matrix L, in the approximate minimization of the volume of the set $\mathscr{P}(P_{\zeta}, b_{\zeta})$.*

4.2.1 Invariant Set of Estimation Error

In this subsection, we derive sufficient conditions for the existence of an admissible triple $(P_{\tilde{x}}, b_{\tilde{x}}, L) \in \Xi$. The conditions are first derived in the form of NLMIs. Then we give sufficient conditions for the admissibility of the triple $(P_r X_{\tilde{x}}, b_r, L)$ in the form of LMIs by using Corollary 2.1. Once an admissible initial triple $(P_{\tilde{x}}, b_{\tilde{x}}, L) \in \Xi$ is obtained, we give an algorithm to update the solution.

Theorem 4.1. *The invariance condition* (4.5) *is satisfied if and only if*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} D_i \in \mathscr{D}_{+}^{m} \\ W_i \in \mathscr{D}_{+}^{m_w} \end{bmatrix} : \left[\begin{array}{c|cc} 2e_i^T b_{\tilde{x}} - b_{\tilde{x}}^T D_i b_{\tilde{x}} - d^T W_i d & e_i^T P_{\tilde{x}} B_w^L & e_i^T P_{\tilde{x}} A^L \\ \hline \star & V^T W_i V & 0 \\ \star & \star & P_{\tilde{x}}^T D_i P_{\tilde{x}} \end{array}\right] \succ 0. \tag{4.13}$$

Let $P_{\tilde{x}} = P_r X_{\tilde{x}}$ *and* $b_{\tilde{x}} = b_r$*, where* $P_r \in \mathscr{R}^{m\times n}$ *and* $b_r \in \mathscr{R}^m$ *are given and where* $X_{\tilde{x}} \in \mathscr{R}^{n\times n}$*. Denote* $\hat{L} := X_{\tilde{x}}L, \hat{A} := X_{\tilde{x}}A - \hat{L}C$*. Then condition* (4.13)*, hence* (4.5)*, is satisfied if*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \lambda_i > 0 \\ \hat{D}_i \in \mathscr{D}_+^m \\ \hat{W}_i \in \mathscr{D}_+^{m_w} \end{bmatrix} : \begin{bmatrix} 2\lambda_i e_i^T b_r - b_r^T \hat{D}_i b_r - d^T \hat{W}_i d & e_i^T P_r X_{\tilde{x}} B_w^L & e_i^T P_r \hat{A} & 0 & 0 \\ \star & 2I_{n_w} & 0 & \lambda_i I_{n_w} & 0 \\ \star & \star & X_{\tilde{x}} + X_{\tilde{x}}^T & 0 & \lambda_i I_n \\ \star & \star & \star & V^T \hat{W}_i V & 0 \\ \star & \star & \star & \star & P_r^T \hat{D}_i P_r \end{bmatrix} \succ 0. \tag{4.14}$$

With $\mathscr{N}_{\cdot,\cdot}$ *and* $\mathscr{L}_{\cdot,\cdot}^{\cdot,\cdot}$ *as defined in* (2.32) *and* (2.33)*, respectively, let* $(\boldsymbol{P}_{\tilde{x}}, \boldsymbol{b}_{\tilde{x}}, \boldsymbol{L}) \in \Xi$*. Suppose that* $(P_{\tilde{x}}, b_{\tilde{x}}, L, D_i, W_i), \forall i \in \mathscr{I}_m$ *satisfy* (4.13)*. Then*

$$\forall i \in \mathscr{I}_m, \exists \begin{bmatrix} \boldsymbol{D}_i^{-1} \in \mathscr{D}_+^m \\ \boldsymbol{W}_i \in \mathscr{D}_+^{m_w} \end{bmatrix} : \left[\begin{array}{c|c} \mathscr{M}_i(\boldsymbol{D}_i^{-1}, \boldsymbol{b}_{\tilde{x}}, \boldsymbol{W}_i) + \mathscr{L}_{L_i(\boldsymbol{P}_{\tilde{x}},\boldsymbol{K}),F_i(\boldsymbol{D}_i^{-1})}^{L_i(P_{\tilde{x}},K),F_i(D_i^{-1})} & \star \\ \hline EL_i(\boldsymbol{P}_{\tilde{x}}, \boldsymbol{K}) & I_n \end{array}\right] \succ 0, \tag{4.15}$$

where $E = [\,-I_n \;\; I_n \;\; 0], F_i(D_i) = \mathrm{diag}(I_n, I_n, D_i)$*,*

$$\mathscr{M}_i(\boldsymbol{D}_i^{-1}, \boldsymbol{b}_{\tilde{x}}, \boldsymbol{W}_i) = \begin{bmatrix} \boldsymbol{D}_i^{-1} & \boldsymbol{b}_{\tilde{x}} & 0 & 0 \\ \star & 2e_i^T \boldsymbol{b}_{\tilde{x}} - d^T \boldsymbol{W}_i d & 0 & 0 \\ \star & \star & V^T \boldsymbol{W}_i V & 0 \\ \star & \star & \star & 0 \end{bmatrix},$$

and

$$L_i(P_{\tilde{x}}, L) = \begin{bmatrix} 0 & P_{\tilde{x}}^T e_i & 0 & 0 \\ 0 & 0 & B_w^L & A^L \\ 0 & 0 & 0 & P_{\tilde{x}} \end{bmatrix}.$$

Furthermore, (4.13) *and* (4.5) *are satisfied by*

$$(P_{\tilde{x}}, b_{\tilde{x}}, L, D_i, W_i,) := (\boldsymbol{P}_{\tilde{x}}, \boldsymbol{b}_{\tilde{x}}, \boldsymbol{L}, \boldsymbol{D}_i, \boldsymbol{W}_i).$$

Proof. The proof of (4.13) is an application of Farkas' Theorem. Details are similar to Part 1) of Theorem 2.1.

Applying Corollary 2.1 on (4.13) (with $E = \begin{bmatrix} e_i^T P_r X_{\tilde{x}} B_w^L & e_i^T P_r X_{\tilde{x}} A^L X_{\tilde{x}}^{-1} \end{bmatrix}$, $Y = \lambda_i^{-1} diag(I_{n_w}, X_{\tilde{x}})$), effecting the congruence $diag(\lambda_i^{\frac{1}{2}}, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{\frac{1}{2}} I_n, \lambda_i^{\frac{1}{2}} I_{n_w}, \lambda_i^{\frac{1}{2}} X_{\tilde{x}}^{-1})$ shows that (4.14) implies (4.13) upon the redefinitions

$$\hat{D}_i := \lambda_i D_i,\ \hat{W}_i := \lambda_i W_i. \tag{4.16}$$

Effecting an upper Schur complement on $b_{\tilde{x}}^T D_i b_{\tilde{x}}$, a manipulation shows that (4.13) is equivalent to

$$\mathscr{M}_i(D_i^{-1}, b_{\tilde{x}}, W_i) + \mathscr{N}_{L_i(P_{\tilde{x}},L),F_i(D_i^{-1})} - L_i(P_{\tilde{x}},K)^T E^T E L_i(P_{\tilde{x}},L) \succ 0. \tag{4.17}$$

The result follows by applying Lemma 2.1 on the second term and a Schur complement on the third. $\square$

Theorem 4.2. *Let all definitions be as above, then for* $(P_{\tilde{x}}, b_{\tilde{x}}, L) \in \Xi$*, the outer bounding ellipsoid condition* (4.10) *is satisfied if*

$$\exists \begin{bmatrix} \bar{D} \in \mathscr{D}_+^m \\ Q_{\tilde{x}} \in \mathscr{S}_+^n \end{bmatrix} : P_{\tilde{x}}^T \bar{D} P_{\tilde{x}} - Q_{\tilde{x}} \succ 0,\ 1 > b_{\tilde{x}}^T \bar{D} b_{\tilde{x}}. \tag{4.18}$$

Let $P_{\tilde{x}} = P_r X_{\tilde{x}}$ *and* $b_{\tilde{x}} = b_r$*, where* $P_r \in \mathscr{R}^{m \times n}$ *and* $b_r \in \mathscr{R}^m$ *are given and where* $X_{\tilde{x}} \in \mathscr{R}^{n \times n}$*. Denote* $\hat{L} := X_{\tilde{x}} L, \hat{A} := X_{\tilde{x}} A - \hat{L} C$*. Then condition* (4.18)*, hence* (4.10)*, is satisfied if*

$$\exists \begin{bmatrix} \bar{D} \in \mathscr{D}_+^m \\ Q_{\tilde{x}} \in \mathscr{S}_+^n \end{bmatrix} : \begin{bmatrix} X_{\tilde{x}} + X_{\tilde{x}}^T - Q_{\tilde{x}} & I_n \\ \star & P_r^T \bar{D} P_r \end{bmatrix} \succ 0,\ 1 > b_r^T \bar{D} b_r. \tag{4.19}$$

With $\mathscr{N}_{\cdot,\cdot}$ *and* $\mathscr{L}_{\cdot,\cdot}^{\cdot,\cdot}$ *as defined in* (2.32) *and* (2.33)*, respectively, let* $(\boldsymbol{P}_{\tilde{x}}, \boldsymbol{b}_{\tilde{x}}, \boldsymbol{L}) \in \Xi$*. Suppose that* $(P_{\tilde{x}}, b_{\tilde{x}}, Q_{\tilde{x}}, \bar{D})$ *satisfy* (4.18)*. Then*

$$\exists \begin{bmatrix} \bar{\boldsymbol{D}}^{-1} \in \mathscr{D}_+^m \\ \boldsymbol{Q}_{\tilde{x}} \in \mathscr{S}_+^n \end{bmatrix} : \mathscr{L}_{\boldsymbol{P}_{\tilde{x}}, \bar{\boldsymbol{D}}^{-1}}^{P_{\tilde{x}}, \bar{D}^{-1}} - \boldsymbol{Q}_{\tilde{x}} \succ 0,\ \begin{bmatrix} \bar{\boldsymbol{D}}^{-1} & \boldsymbol{b} \\ \star & 1 \end{bmatrix} \succ 0. \tag{4.20}$$

Furthermore, (4.18) *and* (4.10) *are satisfied by*

$$(P_{\tilde{x}}, b_{\tilde{x}}, Q_{\tilde{x}}, \bar{D}) := (\boldsymbol{P}_{\tilde{x}}, \boldsymbol{b}_{\tilde{x}}, \boldsymbol{Q}_{\tilde{x}}, \bar{\boldsymbol{D}}).$$

Proof. The proof of (4.18) is an application of Farkas' Theorem. Details are similar to Part 5) of Theorem 2.1.

For the first inequality in (4.18), effecting a Schur complement, followed by the congruence $diag(X_{\tilde{x}}, X_{\tilde{x}}^{-T})$ and using (2.34) (with $\boldsymbol{L} := X_{\tilde{x}}$, $\boldsymbol{D} = Q_{\tilde{x}}$, $L = D = I$) and ignoring a positive term, shows that (4.19) implies (4.18).

The second inequality in (4.20) and (4.18) are equivalent using a Schur complement argument. The result follows by applying Lemma 2.1 on the term $\mathcal{N}_{P_{\tilde{x}}, \bar{D}^{-1}} = P_{\tilde{x}}^T \bar{D} P_{\tilde{x}}$ in the first inequality in (4.18). □

The following algorithm summarizes our solution to obtain the approximate minimal invariant set of estimation error $\mathscr{P}(P_{\tilde{x}}, b_{\tilde{x}})$ and the corresponding observer gain L.

Algorithm 4.1. *Given system* (4.2) *and set* $\mathscr{W} = \mathscr{P}(V, d)$.

1. **Initial data**: *Choose* $m \geq n$, *initial polytope* $\mathscr{P}(P_r, b_r)$ *and tolerance level* tol.

2. **Initial solution**

   (a) *Solve the following convex SDP*

$$\min_{\substack{X_{\tilde{x}}, \hat{L} \\ (4.14),(4.19)}} \; -\log\det Q_{\tilde{x}}. \tag{4.21}$$

   (b) *Define* $D_i, W_i, Q_{\tilde{x}}$ *and* $\bar{D}$ *from* (4.16) *so that* (4.13) *and* (4.18) *are satisfied.*

3. **Update** *Solve the following optimization*

$$\min_{\substack{\boldsymbol{P}_{\tilde{x}}, \boldsymbol{b}_{\tilde{x}}, \boldsymbol{L} \\ (4.15),(4.20)}} \; \left(-\log\det \boldsymbol{Q}_{\tilde{x}}\right). \tag{4.22}$$

4. **Stopping condition**

   (a) *If* $\det(Q_{\tilde{x}}^{-1}) - \det(\boldsymbol{Q}_{\tilde{x}}^{-1}) \leq tol$, *stop.*

   (b) *Else update* $Z := \boldsymbol{Z}$, *where* $\boldsymbol{Z}$ *denotes a variable in the optimizations in* (4.22), *and go to step 3.*

5. **End**

4.2.2 Invariant set of the Control Error

Once an admissible and approximated minimal invariant set of the estimation error is obtained, that is a triple $(P_{\tilde{x}}, b_{\tilde{x}}, L) \in \Xi$ is given, we can give the conditions to derive an admissible $(P_\xi, b_\xi, K) \in \Psi$ for system (4.3). Note, however, that the problem has the same form as the problem treated in Chapter 2 with the substitutions:

$$\begin{bmatrix} LD_w & LC \end{bmatrix} \to B_w$$
$$\begin{bmatrix} V & 0 \\ 0 & P_{\tilde{x}} \end{bmatrix} \to V$$
$$\begin{bmatrix} d \\ b_{\tilde{x}} \end{bmatrix} \to d,$$

and therefore, the solution will not be explicitly presented.

4.3 Example

Consider the double integrator with the parameters:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad V_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \bar{f}_x = \begin{bmatrix} 10 \\ 10 \end{bmatrix}, \quad \underline{f}_x = \begin{bmatrix} -50 \\ -50 \end{bmatrix}.$$

and $V_u = 1$, $\bar{f}_u = -\underline{f}_u = 10$. Applying our algorithm proposed above, we obtain the size of the tube as shown in Figure 4.1 in red, which is significantly smaller than the tube calculated by the method used in [31] (shown in blue color). The control and observer gain are $K = \begin{bmatrix} -0.9999 & -1 \end{bmatrix}$ and $L^T = \begin{bmatrix} 1 & 0.3063 \end{bmatrix}$, respectively. In Figure 4.2, the state constraint on x is shown in blue with solid border, the calculated constraint on $\bar{x}$ using our proposed algorithm is shown in red with dashed border, and the calculated constraint on $\bar{x}$ by the method in [31] is shown in light blue with dotted border. The figure shows that our algorithm can provide a much better tightened constraint on nominal system state.

The control performance is shown in Figure 4.3, the real, estimated and nominal system states are shown in red, blue and green, respectively, and the tubes are shown in black. The figure shows that, compared with the methods used in [31] (the tubes and performance are

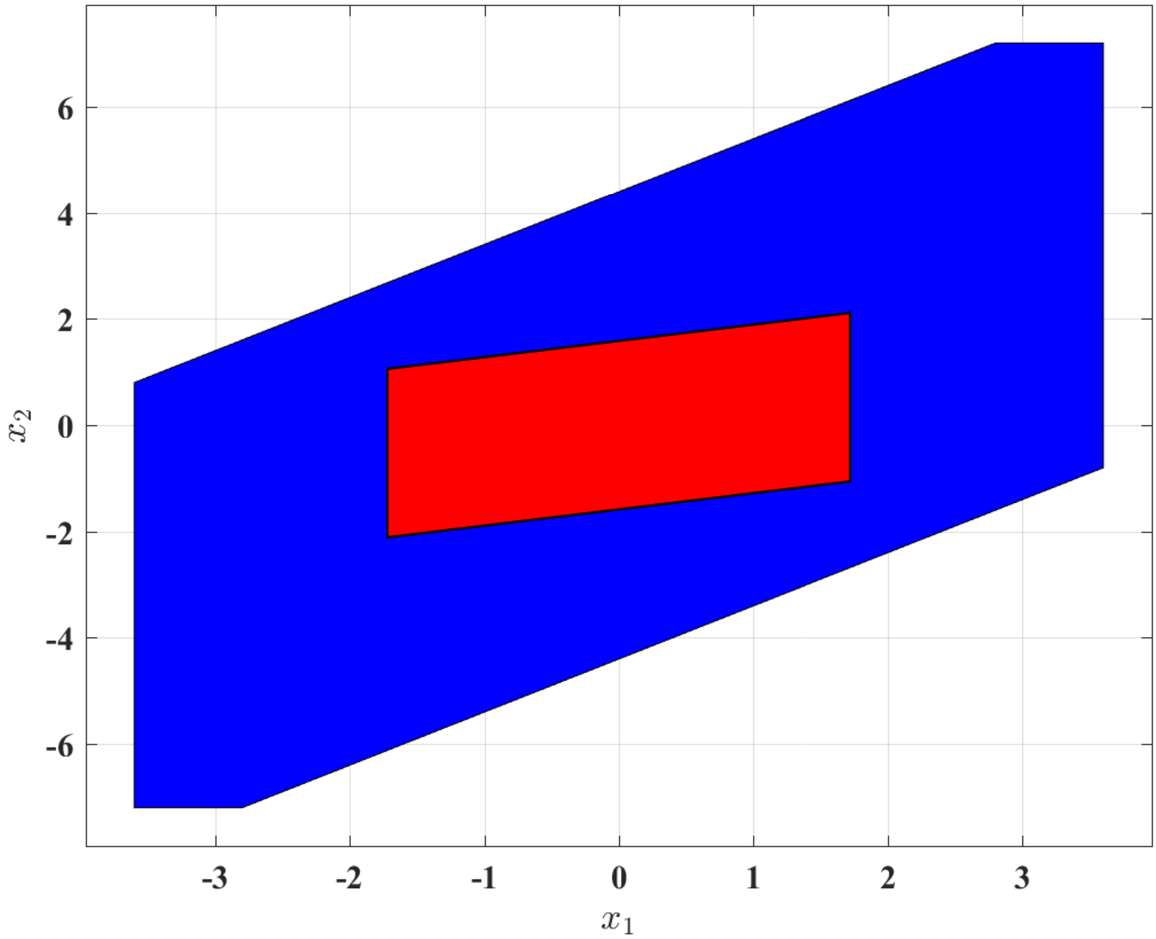


Fig. 4.1 Tube calculated using our algorithm (red) and tube calculated using method in [31].

shown in light blue), our algorithm gives smaller tubes, which results in a better control performance.

4.4 Conclusion

We have extended the approximation algorithm of RCI set proposed in Chapter 2 to the observer case. Nonlinear necessary and sufficient conditions for the existence of an admissible RCI set and observer gain are first derived. The Elimination Lemma and its corollaries are then used to derive the linearizations. An update algorithm, reminiscent of Newton updates, is also proposed.

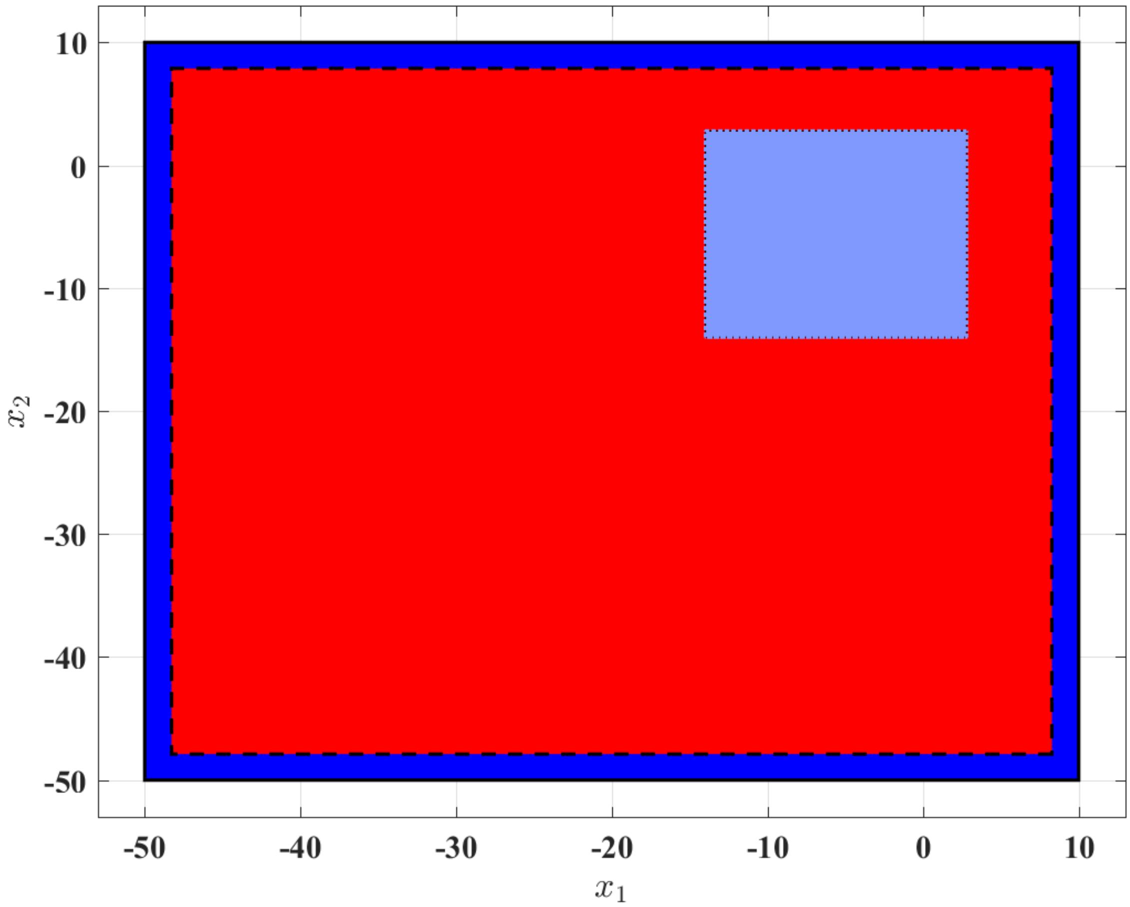


Fig. 4.2 Original constraint on x (blue), tightened constraint on $\bar{x}$ (red), and tightened constraint on $\bar{x}$ using method in [31] (light blue).

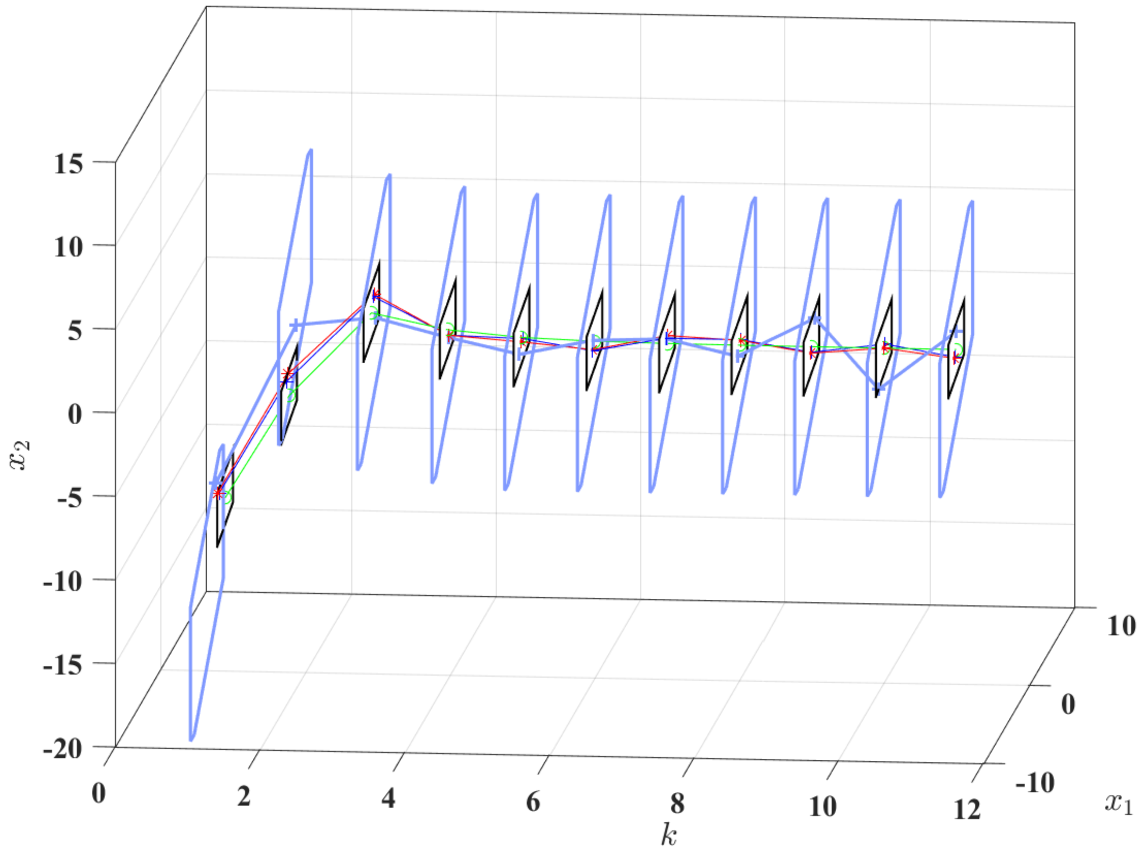


Fig. 4.3 Tube (black), control trajectory of the real (red), estimated (blue) and nominal system (green) states, the tube and the state trajectory of the real system calculated using the method in [31] (light blue).

Numerical tests have shown better control performance by using our optimal RCI sets in the TMPC algorithm. The use of the state feedback and observer gains as variables in the optimization provides extra degrees of freedom and allows the design of tighter tubes and results in better control performance.

Chapter 5

Online MPC with Offline Causal State-feedback Computation

In this chapter, we propose an outline of an online-offline model predictive control (MPC) method for linear discrete-time systems with bounded additive disturbance. We consider a causal state feedback structure on the controller, which comprises a causal state feedback gain and a control perturbation component. In order to improve the efficiency of the online computation, we calculate the state feedback gain offline using a semi-definite program (SDP). Then we propose a novel method to deal with the bounded disturbance as well as the state and input constraints, and compute the control perturbation component online. The online optimization problem is derived using Farkas' Theorem, and then described by a quadratic program (QP) to reduce the online computational burden. A further approximation is made to derive a simplified online optimization problem, which results in a large reduction in the number of variables. The efficiency of this method is demonstrated using numerical examples.

Consider linear discrete-time system (1.1) and the terminal signal (1.5). Although the MPC problem in Theorem 1.2 provides robust solutions using LMIs, the computational burden is heavy. In the following sections, we propose an online-offline method to reduce the computational complexity. The offline and online part of this algorithm are presented separately. The offline algorithm, presented in Section 5.1, is an SDP problem and provides the causal state feedback gains K_0 and K. The online MPC algorithm, presented in Section 5.2 is a QP problem and provides the control perturbation. Another approximation, presented in Section 5.3, simplifies the constraint conditions to further reduce the computational burden.

Two numerical examples are also given to demonstrate the efficiency of these methods in Section 5.5. Through our design methods, the Farkas' Theorem, stated in Theorem 1.1, will be used.

5.1 Offline Calculation

In the objective function (1.13), note that J_0 depends only on K_0 and K while J_1 is quadratic in v for given K_0 and K. Furthermore, since there are a large number of variables in K_0 and K, their online evaluation is generally computationally prohibitive. Hence we propose to determine K_0 and K offline to minimize an upper bound on J_0 for all disturbances and all x_0 satisfying the initial state constraints, and such that the constraints in (1.15) are satisfied for some v. Note, however, that v computed offline will not be used online.

The following theorem is derived from Farkas' Theorem and gives the causal state feedback gain.

Theorem 5.1 (Offline Calculation)**.** *With all definitions as in Section 1.3.2,*

1. *There exist $K_0 \in \mathscr{R}^{N\cdot n_u \times n}$, $K \in \mathscr{R}^{N\cdot n_u \times N\cdot n}$, $v \in \mathscr{R}^{N\cdot n_u}$ such that $J_0 < \bar{\gamma}$ for all $x_0 \in \mathscr{X}_0$ and all $w \in \mathbb{W}$ if there exist matrices $D_x \in \mathscr{D}_+^n$ and $D_w \in \mathscr{D}_+^{(N+1)\cdot n_w}$ such that*

$$\begin{bmatrix} D_w & 0 & 0 & (E_{zw}^K)^T \\ \star & D_x & -\frac{1}{2}D_x(\underline{x}_0+\bar{x}_0) & (E_z^{K_0})^T \\ \star & \star & \bar{\gamma}+\underline{x}_0^T D_x \bar{x}_0 - \bar{w}^T D_w \bar{w} & 0 \\ \star & \star & \star & I \end{bmatrix} \succ 0. \tag{5.1}$$

2. *There exist $K_0 \in \mathscr{R}^{N\cdot n_u \times n}$, $K \in \mathscr{R}^{N\cdot n_u \times N\cdot n}$, $v \in \mathscr{R}^{N\cdot n_u}$ such that $f \in \mathbb{F}$ for all $x_0 \in \mathscr{X}_0$ and all $w \in \mathbb{W}$ if and only if there exist matrices $H_x^i \in \mathscr{D}_+^n$ and $H_w^i \in \mathscr{D}_+^{(N+1)\cdot n_w}$ for all $i \in \mathscr{I}_{(N+1)\cdot m_f}$ such that*

$$\begin{bmatrix} H_w^i & 0 & -(E_{fw}^K)^T e_i \\ \star & H_x^i & -\frac{1}{2}H_x^i(\underline{x}_0+\bar{x}_0) - (E_f^{K_0})^T e_i \\ \star & \star & 2e_i^T(\bar{f}-G_f v) + \underline{x}_0^T H_x^i \bar{x}_0 - \bar{w}^T H_w^i \bar{w} \end{bmatrix} \succ 0. \tag{5.2}$$

Optimal K_0 and K can be obtained by solving the SDPs

$$\min_{\substack{K_0,K \\ (5.1),(5.2)}} \bar{\gamma}. \tag{5.3}$$

Proof.

1. For any matrices $D_x \in \mathscr{D}_+^n$ and $D_w \in \mathscr{D}_+^{(N+1)\cdot n_w}$, we have the identity

$$\begin{aligned} J_0 - \bar{\gamma} = &(E_z^{K_0} x_0 + E_{zw}^K w)^T (E_z^{K_0} x_0 + E_{zw}^K w) - \bar{\gamma} \\ = &-(\bar{w}+w)^T D_w (\bar{w}-w) \\ &-(x_0 - \underline{x}_0)^T D_x (\bar{x}_0 - x_0) \\ &-c^T \mathscr{L}_1 c \end{aligned} \tag{5.4}$$

where $c^T = \begin{bmatrix} w^T & x_0^T & 1 \end{bmatrix}$ and

$$\mathscr{L}_1 = \begin{bmatrix} D_w - (E_{zw}^K)^T E_{zw}^K & -(E_{zw}^K)^T E_z^{K_0} & 0 \\ \star & D_x - (E_z^{K_0})^T E_z^{K_0} & -\frac{1}{2} D_x (\underline{x}_0 + \bar{x}_0) \\ \star & \star & \bar{\gamma} + \underline{x}_0^T D_x \bar{x}_0 - \bar{w}^T D_w \bar{w} \end{bmatrix}$$

Since the first and second terms on the RHS of (5.4) are nonpositive for all $x_0 \in \mathscr{X}_0$ and all $w \in \mathbb{W}$ then $J_0 < \bar{\gamma}$ for all $x_0 \in \mathscr{X}_0$ and all $w \in \mathbb{W}$ if $\mathscr{L}_1 \succ 0$. Effecting a Schur complement for $\mathscr{L}_1 \succ 0$ gives the sufficient condition in (5.1).

2. The constraints (1.15) can be written as

$$\begin{aligned} 2e_i^T (f - \bar{f}) = &2e_i^T (E_f^{K_0} x_0 + G_f v + E_{fw}^K w) - 2e_i^T \bar{f} \\ = &-(\bar{w}+w)^T H_w^i (\bar{w}-w) \\ &-(x_0 - \underline{x}_0)^T H_x^i (\bar{x}_0 - x_0) \\ &-c^T \mathscr{L}_2^i c \end{aligned} \tag{5.5}$$

for any matrices $H_x^i \in \mathscr{D}_+^n$ and $H_w^i \in \mathscr{D}_+^{(N+1)\cdot n_w}$ for all $i \in \mathscr{I}_{(N+1)\cdot m_f}$. Since the first and second terms on the RHS of (5.5) are nonpositive for all $x_0 \in \mathscr{X}_0$ and all $w \in \mathbb{W}$ then $\mathscr{L}_2^i \succ 0$ for all $i \in \mathscr{I}_{(N+1)\cdot m_f}$ implies the constraints are satisfied for all $x_0 \in \mathscr{X}_0$ and all $w \in \mathbb{W}$. This proves sufficiency of (5.2). Necessity follows from Farkas' Theorem.

□

Remark 5.1. *In case there is no feasible K_0 and K satisfying the constraints in (1.15) for all $x_0 \in \mathscr{X}_0$, we may modify the optimization so that $\mathscr{X}_0$ is replaced by $\underline{\mathscr{X}}_0$, the largest volume polytope inside $\mathscr{X}_0$ for which a feasible solution exists. This may be carried out by maximizing the volume of an ellipsoid $\mathscr{Q}$ such that $\mathscr{Q} \subset \underline{\mathscr{X}}_0 \subset \mathscr{X}_0$. However, this is not pursued here.*

5.2 Online Process

The causal state feedback gain of the controller is obtained by solving, offline, the optimization problem in Theorem 5.1. In this section, we propose an online process to compute the control perturbation v to minimize J and satisfy the constraints. Note that J_1 is linear in w. The next result, exploits this fact by using Farkas' Theorem.

Theorem 5.2 (Full QP Method)**.** *At the current time step, the control perturbation sequence that minimizes an upper bound on J can be calculated by solving the following QP problem:*

$$\begin{aligned} \min_{p} \quad & \frac{1}{2} p^T Q p + q^T p \\ & s.t.: \\ & Mp \leq m \end{aligned} \tag{5.6}$$

where

$$p = \begin{bmatrix} v^T & \mu^T & \nu^T \end{bmatrix}^T \in \mathscr{R}^{\left(N \cdot n_u + (N+1)(n_w + n_w \cdot (N+1) \cdot m_f)\right)}$$

$$Q = \begin{bmatrix} 2G_z^T G_z & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$q^T = \begin{bmatrix} 2\left(x_0^T (E_z^{K_0})^T G_z + \bar{w}^T (E_{zw}^K)^T G_z\right) & 4\bar{w}^T & 0 \end{bmatrix},$$

$$M = \begin{bmatrix} -(E_{zw}^K)^T G_z & -I & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \\ 0 & 0 & -I \\ G_f & 0 & I \otimes (2\bar{w}^T) \end{bmatrix}, m = \begin{bmatrix} 0 \\ 0 \\ vec\left((E_{fw}^K)^T\right) \\ 0 \\ \bar{f} - E_f^{K_0} x_0 - E_{fw}^K \bar{w} \end{bmatrix}.$$

Proof. Since J_0 is independent of v and we have $J_0 < \bar{\gamma}$ in Theorem 5.1, $J_0 + J_1 \leq \gamma$ is satisfied if $J_1 \leq \gamma - \bar{\gamma}$. Now, an application of Farkas' Theorem shows that $J_1 \leq \gamma - \bar{\gamma}$ for all $w \in \mathbb{W}$ if an only if there exists $\mu \in \mathscr{R}^{(N+1)\cdot n_w}$ such that

$$\begin{aligned} & v^T G_z^T G_z v + 2(G_z^T E_z^{K_0} x_0 + G_z^T E_{zw}^K \bar{w})^T v + 4\bar{w}^T \mu \leq \gamma - \bar{\gamma} \\ & -\mu \leq 0 \\ & -\mu - (E_{zw}^K)^T G_z v \leq 0. \end{aligned} \tag{5.7}$$

The constraints in (1.15) can be written as:

$$e_i^T \left(E_f^{K_0} x_0 + G_f v + E_{fw}^K w \right) \leq e_i^T \bar{f}, \tag{5.8}$$

for all $i \in \mathscr{I}_{(N+1)\cdot m_f}$ and for all $w \in \mathbb{W}$. Another application of Farkas' Theorem shows that this is equivalent to the system of inequalities

$$\begin{aligned} & -\eta_i \leq 0 \\ & -\eta_i \leq (E_{fw}^K)^T e_i \\ & e_i^T G_f v + 2\bar{w}\eta_i \leq e_i^T (\bar{f} - E_f^{K_0} x_0 - E_{fw}^K \bar{w}) \end{aligned} \tag{5.9}$$

Define $\nu^T = \begin{bmatrix} \eta_1^T & \eta_2^T & \dots & \eta_{(N+1)\cdot m_f}^T \end{bmatrix}$, and combine the constraints in (5.9) with (5.7) to give (5.6). □

5.3 Simplified Online Process

While we have reduced the SDP solution of the MPC problem to a QP problem in Section 5.2, this QP problem nevertheless has $N \cdot n_u + (N+1) \cdot n_w + (N+1) \cdot m_f \cdot (N+1) \cdot n_w$ variables, which is still too large for online computation.

In the following result, we propose a simplified algorithm to reduce the computational burden by expressing the constraints in (5.8) in a more compact way and using an off-line optimization.

Theorem 5.3 (Simplified Algorithm)**.** *At the current time step, the control perturbation sequence can be calculated by solving the following QP problem:*

$$\begin{aligned} \min_{p} \quad & \frac{1}{2}p^T Q p + q^T p \\ & s.t.: \\ & Mp \leq m \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} p &= v \in \mathscr{R}^{N \cdot n_u}, \\ Q &= 2G_z^T G_z, \\ q^T &= 2(G_z^T E_z^{K_0} x_0 + G_z^T E_{zw}^K \bar{w})^T, \\ M &= G_f, \\ m &= \bar{f} - E_f^{K_0} x_0 - |E_{fw}^K| \bar{w}. \end{aligned}$$

Proof. Note from (5.7) that while the variable μ is required to ensure that $J_1 \leq \gamma - \bar{\gamma}$ for all $w \in \mathbb{W}$, this in fact has no effect on the minimizing v and can therefore be dropped.

Next, we can write the constraints (5.8) as

$$e_i^T G_f v \leq e_i^T \bar{f} - e_i^T E_f^{K_0} x_0 - e_i^T E_{fw}^K w$$

for all $w \in \mathbb{W}$. A simple upper bound on $e_i^T E_{fw}^K w$ for all $w \in \mathbb{W}$ is obtained as $e_i^T |E_{fw}^K| \bar{w}$ and so the constraints can be written as

$$G_f v \leq \bar{f} - E_f^{K_0} x_0 - |E_{fw}^K| \bar{w}. \tag{5.11}$$

Combining this constraint with (5.7) gives the optimization problem in (5.10). □

This simplified QP problem (5.10) has only $N \cdot n_u$ variables which reduces the computation burden significantly.

Remark 5.2. *Note that since E_{fw}^K is a function of K, then the quality of the approximation in* (5.11) *can be taken into account in the off-line optimization to compute K, although we do not pursue this here.*

Remark 5.3. *Although our method was derived assuming box-type bounds on the disturbances and constraints, this is only for ease of presentation and the approach is still valid for more general constraints, such as polytopic constraints.*

5.4 RMPC Scheme

In this section, we summarize the overall RMPC scheme from the above analysis.

Algorithm 5.1. *Given system* (1.1) *and* (1.5)*, and set* $\mathscr{W}$.

1. **Offline Computation**

   (a) Calculate K_0 *and* K *by solving the optimization* (5.3).

   (b) Find a triple $(P_t, b_t, K_t) \in \Psi$ *using Algorithm 2.1 and set* $\mathscr{P}(P_t, b_t)$ *as the terminal invariant set,* K_t *as the inner controller.*

2. **Online Computation**

   (a) If the current system state $x \in \mathscr{P}(P_t, b_t)$*, let* $u = K_t x$.

   (b) Else compute v *by solving* (5.6) *or* (5.10) *and set the input* u *as the first element in* $K_0 x + v$.

5.5 Numerical Examples

5.5.1 Example 1

We consider the example in [19], which is originally from [33] and introduce an additive bounded disturbance for system (1.1) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2\sqrt{2} & -4 & 2\sqrt{2} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, B_w = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

C_f, D_f, D_{fw} and $\bar{f}$ are defined according to the input constraints $-4 \leq u(k) \leq 4$ for all $k \in \{0,1,\ldots,N\}$. The disturbance is bounded with $-1 \leq w(k) \leq 1$ for all $k \in \{0,1,\ldots,N\}$. We define the cost signal using

$$C_z = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

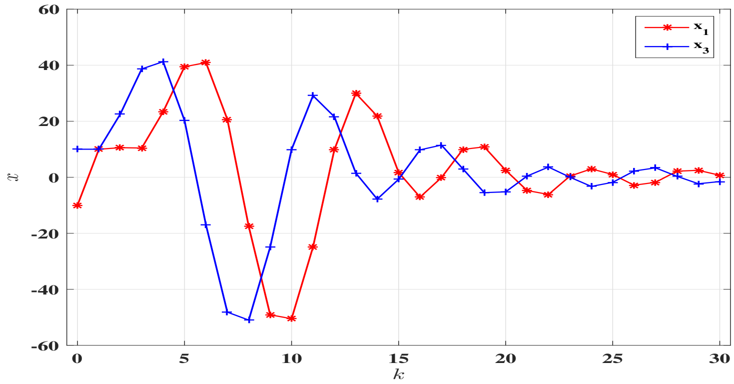
The initial system state condition is given with

$$\bar{x}_0^T = -\underline{x}_0^T = \begin{bmatrix} 10 & 10 & 10 & 10 \end{bmatrix}.$$

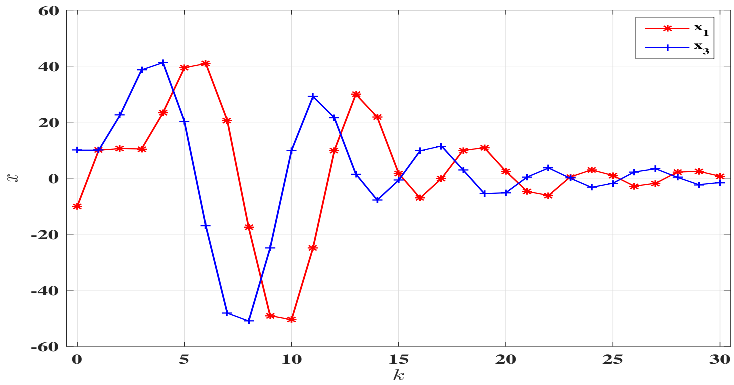
However, there are no constraints for the future system states. K_0 and K are calculated offline using Theorem 5.1 to minimize J_0 and ensure feasibility for all $x_0 \in \mathscr{X}_0$.

Applying our online optimization methods, we have the system state trajectory shown in Fig. 5.1; only the first and the third system state are shown for clarity. Fig. 5.1 (a) shows the response of using the simplified optimization problem in Theorem 5.3, Fig. 5.1 (b) gives the response of using the full QP method in Theorem 5.2, while Fig. 5.1 (c) shows the response of using the SDP method derived from [63] in Theorem 1.2. Although the SDP method gives a less oscillatory response under the disturbance, when using an RCI set as the terminal set and switch the MPC scheme to linear feedback controller within the set, we obtain better control performance as shown in Fig. 5.1 (d).

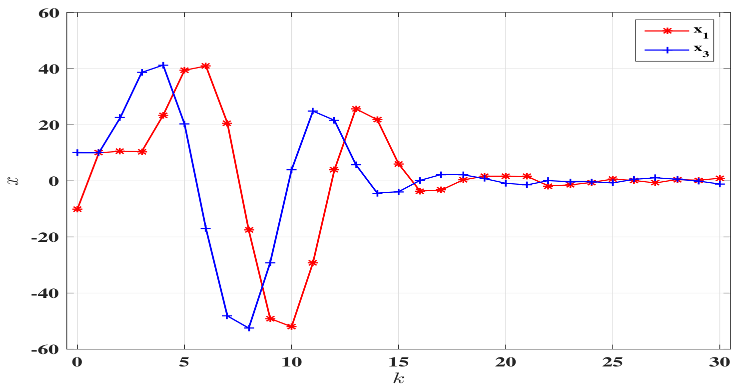
In the simulation, we observe that the simplified method only takes $0.012s$ to complete the optimization on average at each step, the full QP algorithm uses $0.046s$ and the SDP method needs 2.112s. Note that, the average processing speed of our simplified optimization algorithm is more than 170 times faster than the SDP method, we can apply much shorter sampling time for the system by using the simplified method.



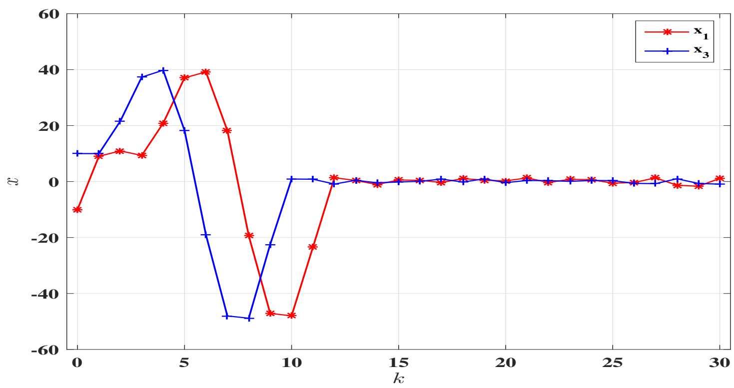
(a) Simplified Algorithm (using Theorem 5.3)



(b) Full QP Method (using Theorem 5.2)



(c) SDP Method (using Theorem 1.2)



(d) Simplified Algorithm and RCI set

Fig. 5.1 State trajectories for Example 1.

5.5.2 Example 2

Taking the example in [63], the discrete time system matrices are given by:

$$A = \begin{bmatrix} 1 & 0.8 \\ 0.5 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

C_f, D_f, D_{fw} and $\bar{f}$ are defined according to the state and input constraints $-0.5 \le u(k) \le 0.5$ and

$$\begin{bmatrix} -3 \\ -3 \end{bmatrix} \le x(k) \le \begin{bmatrix} 3 \\ 3 \end{bmatrix}.$$

In order to demonstrate the robustness of our approach, we increase the bounds on the disturbance:

$$w(k) \in \mathscr{W} := \{w \in \mathscr{R}^2 | \begin{bmatrix} -2 \\ -2 \end{bmatrix} \le w \le \begin{bmatrix} 2 \\ 2 \end{bmatrix}\}.$$

The cost signal is defined using

$$C_z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

which is consistent with the example. The prediction horizon is $N = 9$, and the initial state is $x_0^T = \begin{bmatrix} 3 & -3 \end{bmatrix}$.

Applying our online-offline algorithms with the RCI set as a terminal set for this example gives the system state trajectory shown in Fig. 5.2. The response when using the simplified optimization problem in Theorem 5.3 is shown in red color, and the response when using the SDP method in Theorem 1.2 is shown in blue color. When the system state enters the RCI set, the control scheme is switched to a (pre-computed) linear feedback control law with $K = \begin{bmatrix} -0.4183 & -0.3918 \end{bmatrix}$. The full QP method in Theorem 5.2 gives similar result to the simplified method, hence it is not shown here. In this example, the simplified algorithm only needed $0.009s$ to complete the optimization at each time step, which is almost 380 times faster than the traditional SDP method. The sampling speed can thus be improved greatly without significantly changing the performance. Note that worst case disturbances are used in our examples.

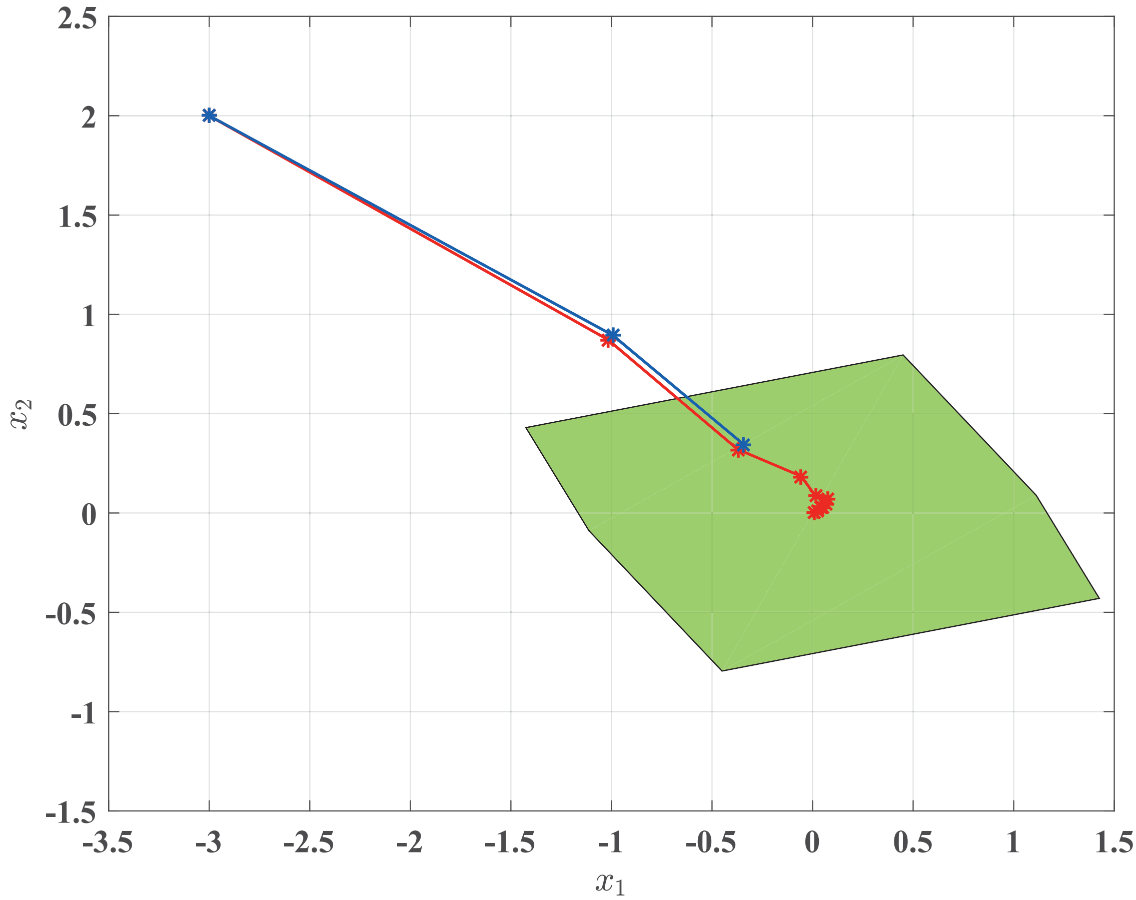


Fig. 5.2 RCI set (green) and state trajectories: using the simplified optimization in Theorem 5.3 (red), using the SDP method in Theorem 1.2 (blue).

5.6 Conclusions

An online-offline method has been proposed for MPC design for state and input constrained linear discrete-time systems subject to additive bounded disturbances in order to reduce the online computational complexity while preserving the robustness against the disturbances. In the offline computation, a causal state feedback controller is designed to minimize part of the cost function while satisfying the constraints over a set of initial states. The online computation involves the solution of a QP problem derived from Farkas' Theorem. The method combines robustness and fast processing speed. A further approximation to this QP problem is made to derive the simplified algorithm, which provides significant improvement in computation efficiency without losing optimality. Our numerical tests indicate that the simplified method can achieve faster processing speed without significant loss in optimality compared with the SDP method.

Chapter 6

Conclusion and Future Work

6.1 Conclusion

RMPC schemes have been well developed in recent years, and the implementation of RCI sets for this scheme are emphasized. This thesis first provides a literature review on recent development of the RCI sets approximations and RMPC algorithms. Considering the computational accuracy and the implemented conveniences, we choose the full-complexity polytopic structure of the RCI set and the online-offline RMPC scheme. The tube based RMPC algorithm is also considered.

In this thesis, linear discrete-time systems subject to additive disturbances and output, initial state and performance constraints, and model uncertainties in some cases, is considered. A novel algorithm for computing the full-complexity inner/outer approximations to polytopic maximal/minimal RCI sets and the corresponding feedback control law (K) is proposed based on convex/LMI optimizations. The necessary and sufficient conditions for the existence of an admissible RCI set and feedback control gain, which are nonlinear and nonconvex in general, are derived. The problem is then relaxed by Farkas' Theorem to obtain sufficient LMI conditions, thus rendering the optimization problem tractable. An initial invariant polytope and the corresponding control law K are first obtained and the set-volume is then iteratively optimized by solving convex/LMI optimizations. These iterations are reminiscent of Newton updates which appears to promote good convergence speed. Both norm-bounded and polytopic model uncertainties can be handled with this algorithm. This algorithm allows arbitrarily large complexity of the invariant polytope and the value of m, coupled with the fact that K is treated as a variable of optimization, results in less conservative inner/outer approximations to the maximal/minimal RCI sets. Numerical examples have shown that our proposed algorithm provides a polytopic RCI set with a substantially improved volume as compared to other schemes from the literature.

Asymmetric output constraints are considered. By considering a symmetric structured polytopic RCI set with a variable center, necessary and sufficient conditions for the existence of an admissible RCI set and feedback control gain are derived. Some corollaries of the Elimination Lemma are employed to relax the problem and obtain sufficient LMI conditions. An optimization algorithm is proposed to yield an initial RCI set and the corresponding control law, then the set-volume is iteratively optimized by solving convex/LMI optimizations. This algorithm results in less conservative and more accurate approximations to polytopic maximal/minimal RCI sets for linear discrete-time systems subject to asymmetric output constraints, which has been illustrated in the numerical examples.

Tube based RMPC schemes provide a novel application of the approximated minimal RCI set. In this thesis, we employ our proposed computation algorithm to calculate the approximated minimal invariant set of the estimation error and the control error for tube based RMPC scheme. Unlike many algorithms in the literature, we reduce the conservative by choosing the full-complexity polytopic set structure and treating the feedback control and the observer gains as variables of optimization, which improved the control performance of tube based RMPC scheme.

Finally, a novel online-offline RMPC scheme is proposed for state and input constrained linear discrete-time systems subject to additive bounded disturbances in order to reduce the online computational complexity while preserving the robustness against the disturbances. A causal state feedback controller is first designed to minimize part of the cost function while satisfying the constraints over a set of initial states. The online procedure then provides the perturbation part of the input by solving a QP problem derived from Farkas' Theorem. This RMPC scheme guarantee both robustness and fast processing speed. The QP problem is simplified further to improve the computation efficiency without loss of optimality. The efficiency of these methods are demonstrated by numerical examples. Furthermore, the RCI sets can be calculated offline and be treated as the terminal target set of the RMPC scheme.

The Matlab CVX toolbox is used to solve our SDP algorithms proposed in this thesis. Numerical examples show that the computational speed of our algorithm of approximating a full-complexity RCI set has no significant increase compared with the low-complexity case. In addition, the proposed online-offline RMPC scheme has achieved very fast processing speeds as illustrated in the examples.

6.2 Future Work

In this section, we highlight possible extensions of the proposed algorithms and control schemes.

- Concerning the computation of the approximated maximal/minimal RCI sets for linear discrete-time systems subject to asymmetric output constraints, we used the full-complexity polytopic structure of the form

$$\mathscr{P}(P,b,x_c) = \{x \in \mathscr{R}^n : -b \leq P(x-x_c) \leq b\},$$

  which is still symmetric around x_c. Future research may concern the form

$$\mathscr{P}(P,\underline{b},\bar{b},x_c) = \{x \in \mathscr{R}^n : \underline{b} \leq P(x-x_c) \leq \bar{b}\},$$

  where $\underline{b} < 0 < \bar{b} \in \mathscr{R}^m$, and with P, $\underline{b}$, $\bar{b}$ and x_c being variable to obtain asymmetric RCI sets.

- In our tube based RMPC scheme, we only considered invariant tubes, that is we use the same RCI set through the control process. Since the computational burden of our approximation algorithm of the RCI sets is not heavy, the online computation of variable tubes can be considered.

- The computation of the tube in the tube based RMPC scheme includes the computation of the RCI set of the estimation error and the corresponding observer gain, and the computation of the RCI set of the control error and the corresponding feedback control gain. These two algorithm are considered separately in our current scheme. Future work will consider computing all the parameters in one algorithm, hence the tube and the corresponding feedback control and observer gains are optimized simultaneously, which may result in a less conservative design.

- The proposed online-offline RMPC scheme is a preliminary work. Further research, particularly on the stability of the proposed algorithm, will be considered.

- The online-offline RMPC scheme can be extended for systems with parameter uncertainties and general convex disturbance bounds and constraints.

References

[1] Allgöwer, F., Badgwell, T. A., Qin, J. S., Rawlings, J. B., and Wright, S. J. (1999). Nonlinear predictive control and moving horizon estimationan introductory overview. In *Advances in control*, pages 391–449. Springer.

[2] Athanasopoulos, N. and Bitsoris, G. (2010). Invariant set computation for constrained uncertain discrete-time linear systems. In *Proceedings of IEEE Conference on Decision and Control*, pages 5227–5232.

[3] Aubin, J.-P. (2009). *Viability theory*. Springer Science & Business Media.

[4] Bemporad, A. and Morari, M. (1999). Robust model predictive control: A survey. In *Robustness in identification and control*, pages 207–226. Springer.

[5] Bemporad, A., Morari, M., Dua, V., and Pistikopoulos, E. N. (2002). The explicit linear quadratic regulator for constrained systems. *Automatica*, 38(1):3–20.

[6] Benzaouia, A. and Burgat, C. (1988). Regulator problem for linear discrete-time systems with non-symmetrical constrained control. *International Journal of Control*, 48(6):2441–2451.

[7] Benzaouia, A. and Mesquine, F. (1994). Regulator problem for uncertain linear discrete-time systems with constrained control. *International Journal of Robust and Nonlinear Control*, 4(3):387–395.

[8] Bitsoris, G. (1988). On the positive invariance of polyhedral sets for discrete-time systems. *Systems & control letters*, 11(3):243–248.

[9] Blanchini, F. (1991a). Constrained control for uncertain linear systems. *Journal of optimization theory and applications*, 71(3):465–484.

[10] Blanchini, F. (1991b). Ultimate boundedness control for uncertain discrete-time systems via set-induced Lyapunov functions. In *Proceedings of IEEE Conference on Decision and Control*, pages 1755–1760.

[11] Blanchini, F. (1994). ultimate boundedness control for uncertain discrete-time systems via set-induced lyapunov functions. *IEEE Transactions on Automatic Control*, 39(2):428–433.

[12] Blanchini, F. (1999). Set invariance in control. *Automatica*, 35(11):1747–1767.

[13] Blanchini, F. and Miani, S. (2007). *Set-theoretic methods in control*. Springer Science & Business Media.

[14] Blanco, T. B., Cannon, M., and De Moor, B. (2010). On efficient computation of low-complexity controlled invariant sets for uncertain linear systems. *International Journal of Control*, 83(7):1339–1346.

[15] Boyd, S. P., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear matrix inequalities in system and control theory*, volume 15. SIAM.

[16] Brezis, H. (1970). On a characterization of flow-invariant sets. *Communications on Pure and Applied Mathematics*, 23(2):261–263.

[17] Cannon, M., Deshmukh, V., and Kouvaritakis, B. (2003). Nonlinear model predictive control with polytopic invariant sets. *Automatica*, 39(8):1487–1494.

[18] Cepeda, A., Limon, D., Alamo, T., and Camacho, E. (2004). Computation of polyhedral h-invariant sets for saturated systems. In *Proceedings of IEEE Conference on Decision and Control*, volume 2, pages 1176–1181.

[19] da Silva Jr, J. G. and Tarbouriech, S. (2001). Local stabilization of discrete-time linear systems with saturating controls: an lmi-based approach. *IEEE Transactions on Automatic Control*, 46(1):119–125.

[20] Dattorro, J. (2008). *Convex optimization and Euclidean distance geometry*.

[21] De Santis, E. (1994). On positively invariant sets for discrete-time linear systems with disturbance: an application of maximal disturbance sets. *IEEE Transactions on Automatic Control*, 39(1):245–249.

[22] Dórea, C. and Hennet, J. (1999). (A, B)-invariant polyhedral sets of linear discrete-time systems. *Journal of optimization theory and applications*, 103(3):521–542.

[23] El Ghaoui, L., Oustry, F., and Lebret, H. (1998). Robust solutions to uncertain semidefinite programs. *SIAM Journal on Optimization*, 9(1):33–52.

[24] Gard, T. C. and Lakshmikantham, V. (1980). Strongly flow-invariant sets. *Applicable Analysis*, 10(4):285–293.

[25] Gilbert, E. G. and Tan, K. T. (1991). Linear systems with state and control constraints: The theory and application of maximal output admissible sets. *IEEE Transactions on Automatic Control*, 36(9):1008–1020.

[26] Glover, J. and Schweppe, F. (1971). Control of linear dynamic systems with set constrained disturbances. *IEEE Transactions on Automatic Control*, 16(5):411–423.

[27] Hennet, J.-C. and Beziat, J.-P. (1991). A class of invariant regulators for the discrete-time linear constrained regulation problem. *Automatica*, 27(3):549–554.

[28] Hu, T., Lin, Z., and Chen, B. M. (2002). Analysis and design for discrete-time linear systems subject to actuator saturation. *Systems & Control Letters*, 45(2):97–112.

[29] Kolmanovsky, I. and Gilbert, E. G. (1998). Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4(4):317–367.

[30] Kothare, M. V., Balakrishnan, V., and Morari, M. (1996). Robust constrained model predictive control using linear matrix inequalities. *Automatica*, 32(10):1361–1379.

[31] Langson, W., Chryssochoos, I., Raković, S., and Mayne, D. Q. (2004). Robust model predictive control using tubes. *Automatica*, 40(1):125–133.

[32] Lee, Y. I. and Kouvaritakis, B. (2000). Robust receding horizon predictive control for systems with uncertain dynamics and input saturation. *Automatica*, 36(10):1497–1504.

[33] Lin, Z. and Saberi, A. (1995). Semi-global exponential stabilization of linear discrete-time systems subject to input saturation via linear feedbacks. *Systems & Control Letters*, 24(2):125–132.

[34] Liu, C. and Jaimoukha, I. M. (2015). The computation of full-complexity polytopic robust control invariant sets. In *Proceedings of IEEE Conference on Decision and Control*, pages 6233–6238.

[35] Luigi Chisci, J. A. R. and Zappa, G. (2001). Systems with persistent disturbances: predictive control with restricted constraints. *Automatica*, 37(7):1019–1028.

[36] Mao, W.-J. (2003). Robust stabilization of uncertain time-varying discrete systems and comments on an improved approach for constrained robust model predictive control. *Automatica*, 39(6):1109–1112.

[37] Mayne, D. and Langson, W. (2001). Robustifying model predictive control of constrained linear systems. *Electronics Letters*, 37(23):1422–1423.

[38] Mayne, D. Q., Rawlings, J. B., Rao, C. V., and Scokaert, P. O. (2000). Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814.

[39] Mayne, D. Q. and Schroeder, W. (1997). Robust time-optimal control of constrained linear systems. *Automatica*, 33(12):2103–2118.

[40] Mayne, D. Q., Seron, M. M., and Raković, S. (2005). Robust model predictive control of constrained linear systems with bounded disturbances. *Automatica*, 41(2):219–224.

[41] Milani, B. E. and Carvalho, A. N. (1995). Robust linear regulator design for discrete-time systems under polyhedral constraints. *Automatica*, 31(10):1489–1493.

[42] Milani, B. E. and Dórea, C. E. (1996). On invariant polyhedra of continuous-time systems subject to additive disturbances. *Automatica*, 32(5):785–789.

[43] Nagumo, M. (1942). Über die lage der integralkurven gewöhnlicher differentialgleichungen. *Proceedings of the Physico-Mathematical Society of Japan*, 24(0):272–559.

[44] Nocedal, J. and Wright, S. (2006). *Numerical optimization*. Springer Science & Business Media.

[45] Olaru, S., De Doná, J., Seron, M., and Stoican, F. (2010). Positive invariant sets for fault tolerant multisensor control schemes. *International Journal of Control*, 83(12):2622–2640.

[46] Ong, C.-J. and Gilbert, E. G. (2006). The minimal disturbance invariant set: Outer approximations via its partial sums. *Automatica*, 42(9):1563–1568.

[47] Ortega, J. M. (1968). The newton-kantorovich theorem. *The American Mathematical Monthly*, 75(6):658–660.

[48] Pluymers, B., Rossiter, J., Suykens, J., and De Moor, B. (2005a). The efficient computation of polyhedral invariant sets for linear systems with polytopic uncertainty. In *Proceedings of American Control Conference*, pages 804–809.

[49] Pluymers, B., Rossiter, J., Suykens, J., and De Moor, B. (2005b). Interpolation based mpc for lpv systems using polyhedral invariant sets. In *Proceedings of American Control Conference.*, pages 810–815.

[50] Pólik, I. and Terlaky, T. (2007). A survey of the s-lemma. *SIAM review*, 49(3):371–418.

[51] Rakovic, S. and Kouramas, K. (2006). The minimal robust positively invariant set for linear discrete time systems: approximation methods and control applications. In *Proceedings of IEEE Conference on Decision and Control*, pages 4562–4567.

[52] Rakovic, S. V., Kerrigan, E. C., Kouramas, K. I., and Mayne, D. Q. (2005). Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control*, 50(3):406–410.

[53] Raković, S. V., Kouvaritakis, B., and Cannon, M. (2013). Equi-normalization and exact scaling dynamics in homothetic tube model predictive control. *Systems & Control Letters*, 62(2):209–217.

[54] Rakovic, S. V., Kouvaritakis, B., Cannon, M., Panos, C., and Findeisen, R. (2011). Fully parameterized tube mpc. In *Proceedings of the 18th IFAC world congress*, pages 197–202.

[55] Raković, S. V., Kouvaritakis, B., Findeisen, R., and Cannon, M. (2012). Homothetic tube model predictive control. *Automatica*, 48(8):1631–1638.

[56] Rawlings, J. B. and Mayne, D. Q. (2009). *Model predictive control: Theory and design*. Nob Hill Pub.

[57] Rotea, M., Corless, M., and Swei, S. (1995). Necessary and sufficient conditions for quadratic controllability of a class of uncertain systems. *Systems & control letters*, 26:195–201.

[58] Scherer C, Gahinet P, C. M. (1997). Multi-objective output-feedback control via LMI optimization. *IEEE Transactions on Automatic Control*, 42(7):896–911.

[59] Seron, M. M., Zhuo, X. W., De Doná, J. A., and Martínez, J. J. (2008). Multisensor switching control strategy with fault tolerance guarantees. *Automatica*, 44(1):88–97.

[60] Suárez, R., Solís-Daun, J., and Álvarez, J. (1994). Stabilization of linear controllable systems by means of bounded continuous nonlinear feedback control. *Systems & control letters*, 23(6):403–410.

[61] Sznaier, M. (1993). A set induced norm approach to the robust control of constrained systems. *SIAM journal on control and optimization*, 31(3):733–746.

[62] Tahir, F. (2010). Efficient computation of robust positively invariant sets with linear state-feedback gain as a variable of optimization. In *Proceedings of IEEE International Conference on Electrical Engineering Computing Science and Automatic Control*, pages 199–204.

[63] Tahir, F. and Jaimoukha, I. M. (2013a). Causal state-feedback parameterizations in robust model predictive control. *Automatica*, 49(9):2675–2682.

[64] Tahir, F. and Jaimoukha, I. M. (2013b). Robust feedback model predictive control of constrained uncertain systems. *Journal of Process Control*, 23(2):189–200.

[65] Tahir, F. and Jaimoukha, I. M. (2015). Low-complexity polytopic invariant sets for linear systems subject to norm-bounded uncertainty. *IEEE Transactions on Automatic Control*, 60(5):1416–1421.

[66] Tarbouriech, S. and Burgat, C. (1991). Positively invariant sets for constrained continuous-time systems with cone properties. In , *Proceedings of the 30th IEEE Conference on Decision and Control*, pages 1748–1754.

[67] Wang, Y. and Boyd, S. (2010). Fast model predictive control using online optimization. *IEEE Transactions on Control Systems Technology*, 18(2):267–278.

[68] Zheng, Z. Q. and Morari, M. (1993). Robust stability of constrained model predictive control. In *Proceedings of American Control Conference*, pages 379–383.

[69] Zhou, B., Duan, G.-R., and Lin, Z. (2010). Approximation and monotonicity of the maximal invariant ellipsoid for discrete-time systems by bounded controls. *IEEE Transactions on Automatic Control*, 55(2):440–446.