Stochastic Maximum Principle and Dynamic Convex Duality in Continuous-time Constrained Portfolio Optimization

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Declaration

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Yusong Li

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Abstract

This thesis seeks to gain further insight into the connection between stochastic optimal control and forward and backward stochastic differential equations and its applications in solving continuous-time constrained portfolio optimization problems. Three topics are studied in this thesis.

- In the first part of the thesis, we focus on stochastic maximum principle, which seeks to establish the connection between stochastic optimal control and backward stochastic differential differential equations coupled with static optimality condition on the Hamiltonian. We prove a weak necessary and sufficient maximum principle for Markovian regime switching stochastic optimal control problems. Instead of insisting on the maximum condition of the Hamiltonian, we show that 0 belongs to the sum of Clarkes generalized gradient of the Hamiltonian and Clarkes normal cone of the control constraint set at the optimal control. Under a joint concavity condition on the Hamiltonian and a convexity condition on the terminal objective function, the necessary condition becomes sufficient. We give four examples to demonstrate the weak stochastic maximum principle.

- In the second part of the thesis, we study a continuous-time stochastic linear quadratic control problem arising from mathematical finance. We model the asset dynamics with random market coefficients and portfolio strategies with convex constraints. Following the convex duality approach,

we show that the necessary and sufficient optimality conditions for both the primal and dual problems can be written in terms of processes satisfying a system of FBSDEs together with other conditions. We characterise explicitly the optimal wealth and portfolio processes as functions of adjoint processes from the dual FBSDEs in a dynamic fashion and vice versa. We apply the results to solve quadratic risk minimization problems with cone-constraints and derive the explicit representations of solutions to the extended stochastic Riccati equations for such problems.

- In the final section of the thesis, we extend the previous result to utility maximization problems. After formulating the primal and dual problems, we construct the necessary and sufficient conditions for both the primal and dual problems in terms of FBSDEs plus additional conditions. Such formulation then allows us to explicitly characterize the primal optimal control as a function of the adjoint processes coming from the dual FBSDEs in a dynamic fashion and vice versa. Moreover, we also find that the optimal primal wealth process coincides with the optimal adjoint process of the dual problem and vice versa. Finally we solve three constrained utility maximization problems and contrasts the simplicity of the duality approach we propose with the technical complexity in solving the primal problem directly.

Keywords: stochastic maximum principle, regime switching, convex duality, forward and backward stochastic differential equation, quadratic risk minimization, stochastic Riccati equation, constrained utility maximization

Contents

Notations

- $(\Omega, \mathcal{F}, \mathbb{P})$ - probability triple consisting of a sample space Ω, the σ-algebra $\mathcal{F}$ which is the set of all measurable events, an the probability measure $\mathbb{P}$;
- $(\mathcal{F}_t)_{t\geq 0}$ - a filtration, an increasing family of sub σ-algebras of $\mathcal{F}; \mathcal{F}_s \subset \mathcal{F}_t, 0 \leq s \leq t$;
- $\mathbb{R}^d$ - the d-dimensional Euclidean space;
- a^+ - $= \max\{a, 0\}$ for any real number a;
- a^- - $= \max\{a, 0\}$ for any real number a;
- 1_A - the indicator function of any set A;
- $M^\intercal$ - the transpose of any vector or matrix M;
- $|M|$ - $= \sqrt{\sum_{i,j} m_{ij}^2}$ for any matrix or vector $M = (m_{ij})$;

In this thesis, by convention, all vectors are column vectors.

Chapter 1

Introduction

1.1 Background

Uncertainty is inherent in most real world systems. The decision maker with certain objectives needs to come up with an optimal strategy to achieve the best expected outcome related to their ultimate goals. The above problem can often be formulated mathematically as a stochastic optimal control problem.

Two most commonly used approaches in solving stochastic optimal control problems are Bellman's dynamic programming principle (DPP) and Pontryagin's maximum principle (MP). The DPP was introduced by R. Bellman in the early 1950s with the following basic idea behind it (see [3, Chapter III.3]):

> An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

The DPP led to the so-called Hamilton-Jacobi-Bellman equation (HJB), which is a non-linear partial differential equation satisfied by the value function. In addition, the notion of viscosity solution introduced by Crandall and Lions in the early 1980s allows us to go beyond the classical verification approach by relaxing the lack of regularity of the value function. Since then, the DPP approach has become a powerful tool to tackle optimal control problems.

Another well known approach to solve optimal control problems is the so called maximum principle (MP) introduced by Pontryagin and his group in the 1950s. It states that an optimal control problem can be decoupled into solving a forward backward differential equation system plus a maximum condition on the Hamiltonian function. The key breakthrough of this theory lies in reducing an infinite-dimensional problem into a static finite-dimensional problem, which is much easier to solve. The original version of deterministic MP was then extended to stochastic case by [53], [36], [11] and [6]. However, prior to 1988, most of the work on around stochastic maximum principle were carried out under the assumption that the control does not enter into the diffusion coefficients and the control space is convex. Following the research seminar study at the Department of Mathematics at Fudan University led by X. Li, the SMP for control dependent diffusion coefficients was introduced in [67] and later simplified in [91].

Stochastic optimal control theory finds numerous applications in many fields that involves decision making under uncertainty. One of the most popular applications of stochastic optimal control lies in the field of portfolio optimization in finance. A classical example of portfolio optimization is as follows. Consider a investor with some initial wealth and wants to invest in financial markets with instruments such as bonds, stocks, currencies and commodities. Portfolio optimization theory studies the optimal investment strategy that maximizes the expected gain and minimizes the risk over the investment horizon.

One of the revolutionary works in modern finance is the portfolio selection theory developed by Harry Markowitz in [61]. He argued that the risk and return characteristics of an investment should be considered together at the portfolio level instead of in isolation. The vehicle to quantify the trade-offs between risk and return inherent in a portfolio was the so-called mean-variance analysis. The portfolio selection work of Markowitz brought mathematics to the art of investment management. Since then, the single period mean-variance framework has initially been extended to multi-period (see [33],[34] and [56]) and then to continuous-time models (see [93], [79] and [87]).

Besides using the return of an investment, a perhaps more realistic way to measure investor satisfaction is the utility function. The concept of utility originates from rational choice theory in Microeconomics and was used as a measure of preferences over certain sets of goods and services. Two landmark papers of Merton [62, 63] studied continuous-time portfolio optimization problems under power and log utility functions with the tools of stochastic optimal control theory. Since then, there has been a great amount of work on continuous-time utility maximization (see [47] and references therein).

1.2 Outline of the thesis

In Chapter 2, we focus on stochastic maximum principle, which seeks to establish the connection between stochastic optimal control and backward stochastic differential differential equations coupled with static optimality condition on the Hamiltonian. We prove a weak necessary and sufficient maximum principle for Markov regime switching stochastic optimal control problems. Instead of insisting on the maximum condition of the Hamiltonian, we show that 0 belongs to the sum of Clarke's generalized gradient of the Hamiltonian and Clarke's normal cone of the control constraint set at the optimal control. Under a joint concavity condition on the Hamiltonian and a convexity condition on the terminal objective function, the necessary condition becomes sufficient. In addition to the theory and proof, we give four examples to demonstrate the weak stochastic maximum principle.

In Chapter 3, we turn our attention to applications of stochastic optimal control theory in mathematical finance. We model the asset dynamics using stochastic differential equations with random market coefficients and portfolio strategies with convex constraints. Following the convex duality approach, we show that the necessary and sufficient optimality conditions for both the primal and dual problems can be written in terms of processes satisfying a system of FBSDEs together with other conditions. We characterise explicitly the optimal wealth and portfolio processes as functions of adjoint processes from the dual FBSDEs in a dynamic fashion and vice versa. We apply the results to solve quadratic risk minimization problems with cone-constraints and derive the explicit representations of solutions to the extended stochastic Riccati equations for such problems.

In Chapter 4, we study constrained utility maximization problems where utility functions are defined on the positive real line. After formulating the primal and dual problems, we construct the necessary and sufficient conditions for both the primal and dual problems in terms of FBSDEs plus additional conditions. Such formulation then allows us to explicitly characterize the primal optimal control as a function of the adjoint processes coming from the dual FBSDEs in a dynamic fashion and vice versa. Moreover, we also find that the optimal primal wealth process coincides with the optimal adjoint process of the dual problem and vice versa. Finally we solve three constrained utility maximization problems and contrasts the simplicity of the duality approach we propose with the technical complexity in solving the primal problem directly.

Finally we conclude the thesis and propose possible areas of future research in the last chapter.

1.3 Published papers and preprints

In this section we list all the published as well as working papers related to this thesis:

[58] Y. Li and H. Zheng. Weak necessary and sufficient stochastic maximum principle for markovian regime-switching diffusion models. *Applied Mathematics and Optimization*, 71:39–77, 2015

[57] Y. Li and H. Zheng. Constrained quadratic risk minimization via forward and backward stochastic differential equations. *Submitted, preprint available at https://arxiv.org/abs/1512.04583*, 2015

[59] Y. Li and H. Zheng. Dynamic convex duality in constrained utility maximization. *Preprint*, 2016

Chapter 2

Weak Necessary and Sufficient Stochastic Maximum Principle for Markovian Regime-Switching Diffusion Models

2.1 Introduction

There has been extensive research in the stochastic control theory. Two principal and most commonly used methods in solving stochastic optimal control problems are the dynamic programming principle (DPP) and the stochastic maximum principle (SMP). The books by Fleming-Rishel [30], Fleming-Soner [31], and Yong-Zhou [89] provide excellent expositions and rigorous treatment of the subject of the dynamic programming principle in the optimal deterministic and stochastic control theory.

Many people have made great contributions in the research of the SMP. Kushner [52, 53] is the first to study the necessary SMP. Haussmann [37], Bensoussan

[5] and Bismut [8, 10, 11] extend Kushner's SMP to more general stochastic control problems with control-free diffusion coefficients. Peng [67] applies the second order spike variation technique to derive the necessary SMP to stochastic control problems with controlled diffusion coefficients. Zhou [91] simplifies Peng's proof. Cadenillas-Karatzas [13] extends Peng's SMP to systems with random coefficients and Tang-Li [80] with jump diffusions. Bismut [11] is the first to investigate the sufficient SMP. Zhou [92] proves that Peng's SMP is also sufficient in the presence of certain convexity condition. Framstad-Øksendal-Sulem [32] extends the sufficient SMP to systems with jump diffusion, Donnelly [26] with Markovian regime-switching diffusion and, most recently, Zhang-Elliott-Siu [90] with Markovian regime-switching jump diffusion.

Briefly speaking, the necessary SMP states that any optimal control along with the optimal state trajectory must solve a system of forward-backward SDEs (stochastic differential equations) plus a maximum condition of the optimal control on the Hamiltonian. The necessary condition together with certain concavity conditions on the Hamiltonian give the sufficient condition of optimality. The major difficulty of generalizing the classical Pontryagin's maximum principle to a stochastic control problem with controlled diffusion term is that, in some cases, the Hamiltonian is a convex function of the control variable and achieves the minimum at the optimal control (see [89, Example 3.3.1]). One of the major contributions of Peng's SMP is the introduction of the generalized Hamiltonian and the second order adjoint stochastic processes. In those cases where the Hamiltonian is convex, it is the second order term that turns the generalized Hamiltonian to a concave function which achieves the maximum at the optimal control. The generalized Hamiltonian and the second order adjoint equation are introduced to preserve the maximum condition of Pontryagin's maximum principle.

However, the second order terms also pose problems. Firstly, one has to assume that all functions involved are twice continuously differentiable in the state variable in order to use the second order variation, which limits the scope of problems applicable to the theorem. Secondly, one has to solve the associated second order adjoint backward stochastic differential equation (BSDE) with the dimensionality equal to the square of that of its first order counterpart, which makes the problem more difficult to solve, at least numerically. Lastly, one can not get the sufficient condition by enhancing the necessary condition with some joint concavity condition to the generalized Hamiltonian and instead one has to add some joint concavity condition to the Hamiltonian (compare [89, Theorem 3.3.2] and [89, Theorem 3.5.2]), which illustrates that the necessary SMP is not completely compatible with the sufficient SMP. This motivates us to relax the requirement of the maximality of the Hamiltonian at the optimal control and to seek a weak but compatible necessary and sufficient SMP.

The main contribution of this chapter is that we prove a weak version of the necessary and sufficient SMP for Markovian regime switching diffusion stochastic optimal control problems. Instead of insisting on the Hamiltonian to achieve the maximum at the optimal control, which is in general impossible, we relax the necessary condition by only requiring the optimal control to be a stationary point of the Hamiltonian. Specifically, we prove that 0 belongs to the sum of Clarke's generalized gradient of the Hamiltonian and Clarke's normal cone of the control constraint set at the optimal control almost surely almost everywhere. Under the joint concavity condition on the Hamiltonian and the convexity condition on the terminal objective function, the necessary condition becomes the sufficient condition.

The advantage of the weak SMP is the following. Firstly, the second order differentiability of the coefficients and the objective functions in the state variable is not required as the weak SMP does not have any second order terms. Secondly, the differentiability of the coefficients and the objective functions in the control variable is not required as the weak SMP uses Clarke's generalized gradients to describe the optimal control. Thirdly, the dimensionality of the BSDE is much reduced as the second order adjoint process is not involved. Lastly, the necessary condition and the sufficient condition are compatible with each other in the sense that the necessary condition provides a stationary point while the sufficient condition confirms its optimality, which is in the same spirit as the necessary and sufficient conditions in the finite dimensional optimization.

In this chapter the control constraint set is assumed to be convex. Under this condition [89] (see case 2, page 120) show that the second order adjoint process can be removed from the SMP but the differentiability condition of the coefficients to the control variable is still required. In the weak SMP, we only assume the differentiability condition in the state variable and the Lipschitz condition in the control variable. This suggests that the Weak SMP is applicable to a more broad range of problems. Example 4.2 shows a non-smooth Hamiltonian in control variable and can not be solved by the existing literature on SMP. The key idea of this chapter is to work on the stationary condition rather than the maximal condition, which opens the possibility of new results when the control constraint set is non-convex.

The rest of the chapter is organized as follows. Section 2 introduces the notations, the formulation of the regime switching stochastic control problem and the basic assumptions. Section 3 states the main theorems of the chapter, the weak necessary SMP (Theorem 2.3.1) and the weak sufficient SMP (Theorem 2.3.2). Section 4 gives four examples to demonstrate the usefulness of the weak SMP in solving regime switching stochastic control problems, including non-smooth non-concave case and regime-switching non-concave case. Section 5 establishes some useful preliminary results on Clarke's generalized gradient and normal cone, Markovian regime switching SDE and BSDE, moment estimates, Lipschitz property, Taylor expansion and duality analysis. Section 6 proves the main theorems. Section 7 concludes. The appendix gives the proof of Theorem 2.5.15 (existence and uniqueness of the solution to a regime switching BSDE) for completeness.

2.2 Problem Formulation

In this section, we formulate the stochastic control problem in a regime switching diffusion model and introduce some assumptions. Here we adopt the model in [26]

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a $\mathbb{P}$ complete right continuous filtration. Let the previsible σ-algebra on $\Omega \times [0, T]$ associated with the filtration $\{\mathcal{F}_t : t \in [0, T]\}$, denoted by $\mathcal{P}^\star$, be the smallest σ-algebra on $\Omega \times [0, T]$ such that every $\{\mathcal{F}_t\}$-adapted stochastic process which is left continuous with right limit is $\mathcal{P}^\star$ measurable. A stochastic process X is previsible, written as $X \in \mathcal{P}^\star$, provided it is $\mathcal{P}^\star$ measurable.

Let $W(\cdot)$ be an m-dimensional standard Brownian motion and $\alpha(\cdot)$ a continuous time finite state observable Markov chain, which are independent of each other. $\{\mathcal{F}_t\}$ is the natural filtration generated by W and α, completed with all $\mathbb{P}$-null sets, denoted by

$$\mathcal{F}_t = \sigma\left[W(s) : 0 \leq s \leq t\right] \bigvee \sigma\left[\alpha(s) : 0 \leq s \leq t\right] \bigvee \mathcal{N},$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets.

Let the Markov chain α take values in the state space $I = \{1, 2, \cdots, d-1, d\}$ and start from initial state $i_0 \in I$ with a $d \times d$ generator matrix $\mathcal{Q} = \{q_{ij}\}_{i,j=1}^d$. For each pair of distinct states (i, j), define the counting process $[Q_{ij}] : \Omega \times [0, T] \to \mathbb{N}$ by

$$[Q_{ij}](\omega, t) := \sum_{0<s\leq t} \mathcal{X}\left[\alpha(s-) = i\right](\omega)\mathcal{X}[\alpha(s) = j](\omega), \forall t \in [0, T],$$

and the compensator process $\langle Q_{ij}\rangle : \Omega \times [0,T] \to [0,+\infty)$ by

$$\langle Q_{ij}\rangle(\omega,t) := q_{ij}\int_0^t \mathcal{X}\left[\alpha(s-)=i\right](\omega)ds, \forall t\in[0,T],$$

where $\mathcal{X}$ is an indicator function. The processes

$$Q_{ij}(\omega,t) := [Q_{ij}](\omega,t) - \langle Q_{ij}\rangle(\omega,t)$$

is a purely discontinuous square-integrable martingale with initial value zero ([74, Lemma IV.21.12]).

Consider a stochastic control model where the state of the system is governed by a controlled Markovian regime-switching SDE:

$$\begin{cases} dx(t) = & b(t,x(t),u(t),\alpha(t-))dt + \sigma(t,x(t),u(t),\alpha(t-))dW(t) \\ x(0) = & x_0\in\mathbb{R}^n, \alpha(0)=i_0\in I, \end{cases} \tag{2.1}$$

where $u(\cdot)$ is a $\mathbb{R}^k$ valued previsible process, $T>0$ is a fixed finite time horizon, $b:[0,T]\times\mathbb{R}^n\times\mathbb{R}^k\times I\to\mathbb{R}^n$ and $\sigma:[0,T]\times\mathbb{R}^n\times\mathbb{R}^k\times I\to\mathbb{R}^{n\times m}$ are given continuous functions satisfying the following assumptions:

(A1) The maps b and σ are measurable, and there exist constant $K>0$ such that for $\varphi=b$ and σ, we have

$$\begin{cases} |\varphi(t,x,u,i)-\varphi(t,\hat{x},\hat{u},i)| \le K\left(|x-\hat{x}|+|u-\hat{u}|\right) \\ \forall t\in[0,T]; i\in I; x,\hat{x}\in\mathbb{R}^n; u,\hat{u}\in\mathbb{R}^k, \\ |\varphi(t,0,0,i)|<K,\ \forall t\in[0,T], \forall i\in I. \end{cases}$$

(A2) The maps b and σ are C^1 in x and there exists a constant $L>0$ and a modulus of continuity $\bar{\omega}:[0,+\infty)\to[0,+\infty)$ such that

$$\begin{cases} |\varphi_x(t,x,u,i)-\varphi_x(t,\hat{x},\hat{u},i)| \le L|x-\hat{x}|+\bar{\omega}(d(u,\bar{u})) \\ \forall t\in[0,T]; i\in I; x,\hat{x}\in\mathbb{R}^n; u,\hat{u}\in\mathbb{R}^k, \end{cases}$$

where $\varphi_x(t,x,u,i)$ is the partial derivative of φ with respect to x at the point (t,x,u,i).

Consider the cost functional

$$J(u) = E\left[\int_0^T f(t, x(t), u(t), \alpha(t))dt + h(x(T), \alpha(T))\right], \tag{2.2}$$

where $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^k \times I \to \mathbb{R}$ and $h : \mathbb{R}^n \times I \to \mathbb{R}$ are given functions satisfying the following assumptions:

(A3) The maps f and h are measurable and there exist constants $K_1, K_2 \geq 0$ such that

$$\begin{cases} |f(t, x, u, i) - f(t, x, \hat{u}, i)| \leq [K_1 + K_2(|x| + |u| + |\hat{u}|)] \, |u - \hat{u}|, \\ |f(t, 0, 0, i)| + |h(0, i)| < K_1, \ \forall t \in [0, T], \forall i \in I. \end{cases}$$

(A4) The maps f and h are C^1 in x and there exist a constant $L > 0$ and a modulus of continuity $\bar{\omega} : [0, +\infty) \to [0, +\infty)$ such that for $\varphi = f$ and h, we have

$$\begin{cases} |\varphi_x(t, x, u, i) - \varphi_x(t, \hat{x}, \hat{u}, i)| \leq L|x - \hat{x}| + \bar{\omega}(d(u, \bar{u})), \\ \forall t \in [0, T]; i \in I; x, \hat{x} \in \mathbb{R}^n; u, \hat{u} \in \mathbb{R}^k, \\ |\varphi_x(t, 0, 0, i)| \leq L, \forall t \in [0, T], i \in I. \end{cases}$$

Remark 2.2.1. Assumptions **(A3)** and **(A4)** together cover many cases, including all quadratic functions in x and u. For instance, if f is Lipschitz in u, then $K_2 = 0$. On the other hand, if f is differentiable with respect to u and f_u satisfies a linear growth condition in u, then K_2 is a positive constant.

Remark 2.2.2. The regime switching component is not critical in this work as our focus is on the stochastic maximum principle approach. One advantage of stochastic maximum principle over dynamic programming is that it does not require Markovian property on the state process. Introducing a jump diffusion process will make the model setup more complicated but will not affect the derivation of the main proof of our result. Moreover, this case has been covered by the paper [90]. Therefore we decided to adopt the simple Brownian setup, which allows read to focus on the main ideas without being distracted by rather complicated setup of the framework.

Consider a measure space $(S, \mathcal{P}^\star, \mu)$, where $S = \Omega \times [0, T]$ and $\mu = \mathbb{P} \times Leb$. Define $L^p(S; \mathbb{R}^q)$ for $p, q \in \mathbb{N}^+$ to be the Banach space of $\mathbb{R}^q$ valued $\mathcal{P}^\star$ measurable functions $f : \Omega \times [0, T] \to \mathbb{R}^q$ such that

$$\|f\| := \left(\int_0^T E|f(t)|^p dt \right)^{\frac{1}{p}} < \infty. \tag{2.3}$$

Similarly, define $L^p_{\mathcal{F}}(S; \mathbb{R}^q)$ for $p, q \in \mathbb{N}^+$ to be the space of $\mathbb{R}^q$ valued $\mathcal{F}_t$ progressively measurable pth order integrable processes.

According to Theorem 2.5.12, under assumption **(A1)**, for any $u \in L^4(S; \mathbb{R}^k)$, the state equation (2.1) admits a unique solution and the cost functional (2.2) is well defined. A control is called admissible if it is valued in U, a non-empty closed convex subset of $\mathbb{R}^k$ and $u \in L^4(S; \mathbb{R}^k)$. Denoted by $\mathcal{U}_{ad}$ the set of admissible controls. In the case that x is a solution of (2.1) corresponding to an admissible control $u \in \mathcal{U}_{ad}$, we call (x, u) an admissible pair and x an admissible state process.

Our optimal control problem can be stated as follows

Problem (S) Minimize (2.2) over $\mathcal{U}_{ad}$.

Any $\bar{u} \in \mathcal{U}_{ad}$ satisfying

$$J(\bar{u}) = \inf_{u \in \mathcal{U}_{ad}} J(u)$$

is called an *optimal control.* The corresponding $\bar{x}$ and $(\bar{x}, \bar{u})$ are called an *optimal state process* and *optimal pair*, respectively.

2.3 Weak Stochastic Maximum Principle

In this section we state the weak necessary and sufficient stochastic maximum principle in the regime-switching diffusion model.

The Hamiltonian $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^k \times I \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}$ for the stochastic control problem (2.1) and (2.2) is defined by:

$$H(t,x,u,i,p,q) := -f(t,x,u,i) + b^{\intercal}(t,x,u,i)p + tr(\sigma^{\intercal}(t,x,u,i)q). \tag{2.4}$$

Given an admissible pair (x,u), the adjoint equation in the unknown adapted processes $p(t-) \in \mathbb{R}^n, q(t) \in \mathbb{R}^{n\times m}$ and $s(t) = (s^{(1)}(t), \cdots, s^{(n)}(t))$, where $s^{(l)}(t) \in \mathbb{R}^{d\times d}$ for $l = 1, \cdots, n$, is the following regime-switching BSDE:

$$\begin{cases} dp(t-) = & -H_x(t,x(t),u(t),\alpha(t-),p(t-),q(t))dt + q(t)dW(t) + s(t) \bullet dQ(t) \\ p(T) = & -h_x(x(T),\alpha(T)), \end{cases} \tag{2.5}$$

where

$$s(t) \bullet dQ(t) \equiv \left(\sum_{j\neq i} s_{ij}^{(1)}(t) dQ_{ij}(t), \cdots, \sum_{j\neq i} s_{ij}^{(n)}(t) dQ_{ij}(t) \right)^{\intercal}.$$

By Theorem 2.5.15, we claim that under assumptions **(A1)-(A4)**, for any $(x,u) \in L^2_{\mathcal{F}}(S;\mathbb{R}^n) \times L^4(S;\mathbb{R}^k)$, (2.5) admits a unique solution $\{(p(t-), q(t), s(t)) | t \in [0,T]\}$ in the sense of Definition 2.5.14. If $(\bar{x},\bar{u})$ is an optimal (resp. admissible) pair and $(\bar{p},\bar{q},\bar{s})$ is the adapted solution of (2.5), then $(\bar{x},\bar{u},\bar{p},\bar{q},\bar{s})$ is called an optimal (resp. admissible) 5-tuple.

We can now state the main results of the chapter.

Theorem 2.3.1. *(Weak Necessary SMP with Regime-Switching) Let assumptions* ***(A1)-(A4)*** *hold. Let $(\bar{x},\bar{u})$ be an optimal pair of* ***Problem (S)****. Then there exists stochastic process $(\bar{p},\bar{q},\bar{s})$ which is an adapted solution to* (2.5)*, such that*

$$0 \in \partial_u(-H)(t,\bar{x}(t),\bar{u}(t),\alpha(t-),\bar{p}(t),\bar{q}(t)) + N_U(\bar{u}(t)), \ a.e.\ t \in [0,T], \mathbb{P}\text{-}a.s., \tag{2.6}$$

where $\partial_u(-H)(t,\bar{x}(t),\bar{u}(t),\alpha(t-),\bar{p}(t),\bar{q}(t))$ *is Clarke's generalized gradient of* $-H$ *with respect to variable* u *at point* $(t,\bar{x}(t),\bar{u}(t),\alpha(t-),\bar{p}(t),\bar{q}(t))$ *and* $N_U(\bar{u}(t))$ *is Clarke's normal cone of* U *at point* $\bar{u}(t)$ *(see Subsection 2.5.1 for details).*

Theorem 2.3.2. *(Weak Sufficient SMP with Regime-Switching) Let assumptions* ***(A1)-(A4)*** *hold and let* $(\bar{x},\bar{u},\bar{p},\bar{q},\bar{s})$ *be an admissible 5-tuple satisfying* (2.6)*. Suppose further that* $h(\cdot,\alpha(T))$ *is convex and the Hamiltonian function* $H(t,\cdot,\cdot,\alpha(t-),\bar{p}(t),\bar{q}(t))$ *is concave for all* $t\in[0,T]$ *a.s. Then* $(\bar{x},\bar{u})$ *is an optimal pair for* ***Problem (S)****.*

Remark 2.3.3. In the special case where $\mathcal{F}_t=\sigma[W(s):0\le s\le t]\bigvee\mathcal{N}$, i.e., the randomness of the system is generated only by the Brownian motion, the Hamiltonian (2.4) and all other functions are free of index i or Markov chain process value $\alpha(t-)$. The adjoint equation (2.5) is a pure Brownian BSDE (no $s(t)\bullet dQ(t)$ term). The weak SMP remains the same as Theorem 2.3.1 and 2.3.2, but only involves the 4-tuple $(\bar{x},\bar{u},\bar{p},\bar{q})$.

Remark 2.3.4. *We call the SMP "weak" because in Peng's necessary stochastic maximum principle it requires the Hamiltonian to reach maximum at the optimal control (see [67]) whereas in our necessary condition we only requires the optimal control to be s stationary point (see Theorem 2.3.1 and Example 2.4.2).*

2.4 Examples

In this section, we present four examples to demonstrate our main theorems.

2.4.1 Examples: Weak SMP without Regime-Switching

In this subsection, we consider two examples from [89] and derive the same results as those in [89] using Theorem 2.3.1 and Theorem 2.3.2. A key property used in our approach is that the adjoint process must be adapted to the filtration $\mathcal{F}_t$.

Example 2.4.1. *(Concave Hamiltonian)* Consider the following stochastic control problem [89, Example 3.5.3]:

$$\begin{cases} dx(t) = u(t)dW(t), t \in [0,1] \\ x(0) = 0 \end{cases} \tag{2.7}$$

with the control constraint set $U = [0,1]$ and the cost functional

$$J(u) = E\left\{ -\int_0^1 u(t)dt + \frac{1}{2}x(1)^2 \right\}.$$

Suppose $(\bar{x}, \bar{u})$ is an optimal pair, then the corresponding adjoint equation is

$$\begin{cases} d\bar{p}(t) = \bar{q}(t)dW(t), t \in [0,1] \\ \bar{p}(1) = -\bar{x}(1). \end{cases} \tag{2.8}$$

Using (2.7) and (2.8) and via a simple calculation we obtain

$$\bar{p}(t) = -\int_0^t \bar{u}(s)dW(s) - \int_t^1 (\bar{u}(s) + \bar{q}(s))\, dW(s).$$

Since the adjoint process $\bar{p}(t)$ is adapted to the filtration $\mathcal{F}_t$, we must have

$$\bar{u}(t) + \bar{q}(t) = 0 \ \textit{for all } t \in [0,1], \mathbb{P}\text{-}a.s. \tag{2.9}$$

The corresponding Hamiltonian is

$$H(t, x, u, \bar{p}(t), \bar{q}(t)) = \bar{q}(t)u + u.$$

Since the problem satisfies **(A1)-(A4)**, by Theorem 2.3.1 and (2.6), we have

$$0 \in -(\bar{q}(t) + 1) + N_{[0,1]}(\bar{u}(t)) \ \textit{for all } t \in [0,1], \mathbb{P}\text{-}a.s.$$

Consequently, on any non-zero measurable set $E \in S = \Omega \times [0,1]$, we can only have the following three cases:

Case 1 : $0 < \bar{u}(t) < 1 \Longrightarrow N_{[0,1]}(\bar{u}(t)) = \{0\}$ *and* $\bar{q}(t) = -1$.

Case 2 : $\bar{u}(t) = 0 \Longrightarrow N_{[0,1]}(\bar{u}(t)) = (-\infty, 0]$ *and* $\bar{q}(t) + 1 \leq 0$.

Case 3 : $\bar{u}(t) = 1 \Longrightarrow N_{[0,1]}(\bar{u}(t)) = [0, +\infty)$ *and* $\bar{q}(t) + 1 \geq 0$.

Suppose Case 1 or Case 2 is true, then $\bar{u}(t) + \bar{q}(t) \leq \bar{u}(t) - 1 < 0$ for some non-zero measurable set $E \in S$, contradiction to (2.9). Hence, we have $\bar{u}(t) = 1$ for every $t \in [0,1], \mathbb{P}$-a.s. and $\bar{x}(t) = W(t)$ and $(\bar{p}(t), \bar{q(t)}) = (-W(t), -1)$ for $t \in [0,1]$. Since $(x,u) \mapsto H(t,x,u,\bar{p}(t),\bar{q}(t)) = -u + u = 0$ is concave and $x \mapsto h(x) = \frac{1}{2}x^2$ is convex, we conclude that $\bar{u}(t) = 1$ is the optimal control using Theorem 2.3.2.

Example 2.4.2. *(Nonconcave nonsmooth Hamiltonian)* Consider the following stochastic control problem

$$\begin{cases} dx(t) = \dfrac{1}{2}|u(t)|dW(t), t \in [0,1] \\ x(0) = 0 \end{cases} \tag{2.10}$$

with the control constraint set $U = [-1,1]$ and the cost functional

$$J(u) = E\left\{\int_0^1 [x(t)^2 - \frac{1}{2}u(t)^2]dt + x(1)^2\right\}.$$

Suppose $(\bar{x}, \bar{u})$ is an optimal pair, then the corresponding adjoint equation is

$$\begin{cases} d\bar{p}(t) = 2\bar{x}(t)dt + \bar{q}(t)dW(t), t \in [0,1] \\ \bar{p}(1) = -2\bar{x}(1). \end{cases} \tag{2.11}$$

Using (2.10), (2.11) and via a simple calculation, we obtain

$$\bar{p}(t) = -\int_0^t (2-t)|\bar{u}(s)|dW(s) - \int_t^1 ((2-s)|\bar{u}(s)| + \bar{q}(s))dW(s).$$

Since the adjoint process $\bar{p}(t)$ is adapted to the filtration $\mathcal{F}_t$, we must have

$$(2-t)|\bar{u}(t)| + \bar{q}(t) = 0 \text{ for all } t \in [0,1], \mathbb{P}\text{-}a.s. \tag{2.12}$$

The corresponding Hamiltonian is

$$H(t,x,u,\bar{p}(t),\bar{q}(t)) = \frac{1}{2}\bar{q}(t)|u| - x^2 + \frac{1}{2}u^2.$$

Since the problem satisfies assumptions **(A1)-(A4)**, by Theorem 2.3.1 and (2.6), we have

$$0 \in \partial_u \left(x(t)^2 - \frac{1}{2}q(t)|u(t)| - \frac{1}{2}u(t)^2 \right) + N_{[-1,1]}(\bar{u}(t)) \text{ for all } t \in [0,1], \mathbb{P}\text{-}a.s. \tag{2.13}$$

Consequently, on any non-zero measurable set $E \in S$, we can only have the following five cases:

Case 1 $\bar{u}(t) = 1 \Longrightarrow 0 \in \left\{ -\frac{1}{2}q(t) - 1 \right\} + [0, +\infty)$ which is compatible with the adaptedness condition (2.12) $\bar{q}(t) = t - 2$.

Case 2 $\bar{u}(t) = -1 \Longrightarrow 0 \in \left\{ \frac{1}{2}q(t) + 1 \right\} + (-\infty, 0]$ which is compatible with (2.12) $\bar{q}(t) = t - 2$.

Case 3 $\bar{u}(t) = 0 \Longrightarrow 0 \in \left[\frac{1}{2}q(t), -\frac{1}{2}q(t) \right] + \{0\}$ which is compatible with (2.12) $\bar{q}(t) = 0$.

Case 4 $\bar{u}(t) \in (0,1) \Longrightarrow 0 \in \left\{ -\frac{1}{2}q(t) - \bar{u}(t) \right\} + \{0\}$ which gives $q(t) = -2\bar{u}(t) < 0$, a contradiction to (2.12) $\bar{q}(t) = (t-2)\bar{u}(t) > 0$.

Case 5 $\bar{u}(t) \in (-1,0) \Longrightarrow 0 \in \left\{ \frac{1}{2}q(t) - \bar{u}(t) \right\} + \{0\}$ which gives $q(t) = 2\bar{u}(t)$, a contradiction to (2.12) $\bar{q}(t) = (2-t)\bar{u}(t)$.

Hence, the set of optimal candidates from Weak Necessary SMP consists of all the progressively measurable processes valued in the set $\{-1, 0, 1\}$. However, since the Hamiltonian is not concave, Theorem 2.3.2 cannot be applied. Substituting $x(t) = \int_0^t \frac{1}{2}|u(s)|dW(s)$ into the cost functional and by simple calculations, we obtain

$$J(u) = -\frac{1}{4}E\int_0^1 t|u(t)|^2 dt.$$

Hence $J(u)$ reaches the minimum at $|\bar{u}(t)| = 1$ a.s. for all t, which implies there are infinitely many optimal controls with any measurable combination of 1 and -1. The optimal state process is $\bar{x}(t) = \frac{1}{2}W(t)$ and the adjoint processes are $\bar{p}(t-) = (t-2)W(t)$ and $\bar{q}(t) = t-2$ for all $t \in [0,1]$.

Remark 2.4.3. Example 2.4.2 shows that the weak necessary SMP can find not only the optimal control for minimization problem (any progressively measurable process taking values -1 or 1) but also the optimal control for maximization problem (the unique progressively measurable process taking value 0), which is in the same spirit of the necessary condition for finite dimensional optimization. The Hamiltonian in Example 2.4.2 is non-smooth in control variable u, which is beyond any known literature on SMP.

Remark 2.4.4. When $dx(t) = u(t)dW(t)$ and $U = [0,1]$ and everything else is kept the same as that in Example 2.4.2, the problem is the same as that of [89, Example 3.3.1]. Theorem 2.3.1 can again be applied to find the optimal control candidate $\bar{u}(t) = 0$. (We leave this to the reader to check.) The Hamiltonian is a convex function of u and $\bar{u}(t) = 0$ is a minimum point. This is the reason that [67] introduces the generalized Hamiltonian $\mathcal{H}$ which makes $\bar{u}(t) = 0$ a maximum point.

2.4.2 Examples: Weak SMP with Regime-Switching

Example 2.4.5. *(Quadratic Loss Minimization)* Here we adopt the setting in [26, Section 6]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$ be a complete probability space on which defined a 1-dimensional standard Brownian motion W and a continuous time Markov chain α valued in a finite state space $I = \{1, \cdots d\}$ with generator matrix $Q = [q_{ij}]_{i,j\in I}$ and initial mode $\alpha(0) = i_0$. Assume that W and α are independent of each other and the filtration is generated jointly by W and α. Consider a market consisting of one risk-free bank account $S_0 = \{S_0(t), t \in [0,T]\}$

and one risky stock $S_1 = \{S_1(t), t \in [0,T]\}$. The risk-free asset's price process satisfies the following equation:

$$\begin{cases} dS_0(t) = r(t,\alpha(t-))S_0(t)dt \; t \in [0,T] \\ S_0(0) = 1, \end{cases}$$

where the risk-free rate of return $r(t,i)$ is a bounded deterministic function for $i \in I$. The price process of the risky stock is given by

$$\begin{cases} dS_1(t) = S_1(t)\left\{b(t,\alpha(t-))dt + \sigma(t,\alpha(t-))dW(t)\right\} \;\; t \in [0,T] \\ S_1(0) = S_1 > 0, \end{cases}$$

where the mean rate of return $b(t,i)$ and the volatility $\sigma(t,i)$ are bounded nonzero deterministic functions for $i \in I$. Define the market price of risk $\theta(t,i) \equiv \sigma^{-1}(t,i)(b(t,i) - r(t,i))$.

Consider an agent with an initial wealth $x_0 > 0$. Let the $\mathcal{F}_t$ previsible real valued process $u(t)$ be the amount allocated to the stock at time t. Then the wealth process x can be written as

$$\begin{cases} dx(t) = \left[r(t,\alpha(t-))x(t) + u(t)\sigma(t,\alpha(t-))\theta(t,\alpha(t-))\right]dt + u(t)\sigma(t,\alpha(t-))dW(t) \\ x(0) = x_0. \end{cases} \tag{2.14}$$

A portfolio $u(\cdot)$ is said to be admissible, written as $u(\cdot) \in \mathcal{U}_{ad}$ if it is $\mathcal{F}_t$-previsible, square integrable and such that the regime switching SDE (2.14) has a unique solution $x(\cdot)$ corresponding to $u(\cdot)$. In this case, we refer to $(x(\cdot), u(\cdot))$ as an admissible pair. The agent's objective is to find an admissible pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ such that

$$E\left(\bar{x}(T) - d\right)^2 = \inf_{u \in \mathcal{U}_{ad}} E(x(T) - d)^2$$

for some fixed constant $d \in \mathbb{R}$.

To solve this problem, first we find potential optimal candidate using Theorem 2.3.1. Suppose that $(\bar{x}(\cdot), \bar{u}(\cdot))$ is an optimal pair. Then the corresponding adjoint equation is

$$\begin{cases} dp(t-) = -r(t,\alpha(t-))p(t-)dt + q(t)dW(t) + s(t) \bullet dQ(t) \ t \in [0,T) \\ -p(T) = 2\bar{x}(T) - 2d. \end{cases} \tag{2.15}$$

To find a solution $(\bar{p}, \bar{q}, \bar{s})$ to (2.15), we try a process

$$\bar{p}(t) = \phi(t,\alpha(t))\bar{x}(t) + \psi(t,\alpha(t)), \tag{2.16}$$

where $\phi(t,i)$ and $\psi(t,i)$ are deterministic smooth functions with terminal conditions

$$\phi(T,i) = 2 \text{ and } \psi(T,i) = -2d \text{ for } \forall i \in I.$$

Applying Ito's formula to (2.16) and comparing coefficients with (2.15) leads to

$$\begin{aligned} -r(t,\alpha(t-))\bar{p}(t) = \textstyle\sum_{i=1}^{d} \mathcal{X}[\alpha(t-) = i] \Big\{ & \bar{x}(t)\left(\phi(t,i)r(t,i) + \Delta\phi(t,i)\right) \\ & +\phi(t,i)\bar{u}(t)\sigma(t,i)\theta(t,i) + \Delta\psi(t,i) \Big\}, \end{aligned} \tag{2.17}$$

$$\bar{q}(t) = \phi(t,\alpha(t-))\sigma(t,\alpha(t-))\bar{u}(t), \tag{2.18}$$

$$\bar{s}_{ij}(t) = \bar{x}(t)(\phi(t,j) - \phi(t,i)) + (\psi(t,j) - \psi(t,i)), \tag{2.19}$$

where for $\varphi = \phi$ and ψ, denote by

$$\Delta\varphi(t,i) \triangleq \varphi_t(t,i) + \sum_{j=1}^{d} q_{ij}(\varphi(t,j) - \varphi(t,i)).$$

The Hamiltonian is given by

$$H(t,x,u,\alpha,p,q) = r(t,\alpha)xp + u\sigma(t,\alpha)q + u\sigma(t,\alpha)\theta(t,\alpha)p. \tag{2.20}$$

By Theorem 2.3.1, we have

$$0 \in \partial_u(-H)(t,\bar{x}(t),\bar{u}(t),\alpha(t-),\bar{p}(t),\bar{q}(t)).$$

Since H is a linear function of $\bar{u}$, we must have

$$\bar{q}(t) = -\theta(t,\alpha(t-))\bar{p}(t). \tag{2.21}$$

Substituting (2.21) and (2.16) into (2.18) we obtain

$$\bar{u}(t) = -\sigma^{-1}(t, \alpha(t-))\theta(t, \alpha(t-))(\bar{x}(t) + \phi^{-1}(t, \alpha(t-))\psi(t, \alpha(t-))). \tag{2.22}$$

Substituting (2.16) and (2.22) into (2.17) leads to the following two differential equations

$$\phi(t, i)(2r(t, i) - |\theta(t, i)|^2) + \Delta\phi(t, i) = 0, \tag{2.23}$$

$$\psi(t, i)(r(t, i) - |\theta(t, i)|^2) + \Delta\psi(t, i) = 0, \tag{2.24}$$

with terminal conditions

$$\phi(T, i) = 2 \text{ and } \psi(T, i) = -2d \text{ for } \forall i \in I.$$

It can be showed that the solutions are

$$\phi(t, i) = 2E\Big\{ \exp\Big[\int_t^T \big(2r(s, \alpha(s)) - |\theta(s, \alpha(s))|^2\big)\, ds \Big] \Big| \alpha(t) = i \Big\}, \tag{2.25}$$

$$\psi(t, i) = -2dE\Big\{ \exp\Big[\int_t^T \big(r(s, \alpha(s)) - |\theta(s, \alpha(s))|^2\big)\, ds \Big] \Big| \alpha(t) = i \Big\}. \tag{2.26}$$

Detailed proofs can be found in [26, Section 6] and [90, Section 5]. Substituting (2.25) and (2.26) back into (2.22) gives the potential optimal portfolio $\bar{u}$ and the corresponding potential optimal wealth process $\bar{x}$.

To verify the optimality of our candidate solution, we apply Theorem 2.3.2. Since **(A1)-(A4)** are satisfied, $h(x(T), \alpha(T)) \equiv (x(T) - d)^2$ is convex and the Hamiltonian (2.20) is concave, we conclude that $(\bar{x}(\cdot), \bar{u}(\cdot))$ is indeed the optimal pair.

Remark 2.4.6. Notice that in this case h is a convex function and the Hamiltonian is concave. Therefore, one can skip the necessary conditions and use a sufficient stochastic maximum principle of Pontryagin's type directly to find the optimal portfolio process. Detailed steps can be found in [26, Section 6] and [90, Section 5]. However, we follow a different approach here. Instead of using the sufficient SMP directly, we first find all admissible portfolios satisfying the necessary conditions stated in Theorem 2.3.1. Combining that with the adjoint equations, we then construct candidate optimal portfolio $\bar{u}$. Finally, an application of Theorem 2.3.2 confirms that $\bar{u}$ is indeed the optimal portfolio. This approach is particularly useful when the conditions for sufficient SMP are not satisfied, e.g. non-concave Hamiltonian.

Example 2.4.7. *(Non-concave Hamiltonian)* Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le 1}, \mathbb{P})$ be a complete probability space. Consider a one-dimensional Brownian motion W and a continuous time finite state Markov chain $\{\alpha(t)|t \in [0,1]\}$ with state space $I := \{1,2\}$ and generator matrix $Q := [q_{ij}]_{i,j=1,2}$. Assume $q_{12} + q_{21} \ge 2$. Consider the following Markovian regime-switching control system

$$\begin{cases} dx(t) = u(t)dW(t), \ t \in [0,1] \\ x(0) = 0 \end{cases}$$

with the control domain $U = [0,1]$ and the cost functional

$$J(u(\cdot)) = E\left[\int_0^1 \big(A(\alpha(t))u(t) + B(\alpha(t))u^2(t) + C(\alpha(t))x^2(t)\big)\,dt + D(\alpha(1))x^2(1)\right],$$

where functions $A, B, C, D : I \to \mathbb{R}$ satisfy

$$\begin{cases} A(1) = -1 \\ A(2) = 0 \end{cases}, \begin{cases} B(1) = 0 \\ B(2) = -\frac{1}{2} \end{cases}, \begin{cases} C(1) = 0 \\ C(2) = 1 \end{cases}, \begin{cases} D(1) = \frac{1}{2} \\ D(2) = 1 \end{cases}.$$

To solve this problem, first we find potential optimal solutions using Theorem 2.3.1. Suppose $(\bar{x}(\cdot), \bar{u}(\cdot))$ is an optimal pair. Then the corresponding adjoint equation is

$$\begin{cases} d\bar{p}(t) = 2C(\alpha(t))\bar{x}(t)dt + \bar{q}(t)dW(t) + \bar{s}(t) \bullet dQ(t) \\ \bar{p}(1) = -2D(\alpha(1))\bar{x}(1) \end{cases}. \tag{2.27}$$

To find a solution $(\bar{p}, \bar{q}, \bar{s})$ to (2.27), we try a process $\bar{p}(t) = \phi(t, \alpha(t))\bar{x}(t)$, where $\phi(t,i), \ i = 1,2$ are deterministic functions satisfying the terminal condition

$\phi(1,i) = -2D(i),\ i = 1,2$. Applying Ito's formula

$$d\bar{p}(t) = \sum_{i=1}^{2} \mathcal{X}[\alpha(t-) = i]\Bigg\{\bar{x}(t)\Bigg(\phi_t(t,i) + \sum_{j=1}^{2} q_{ij}\Big(\phi(t,j) - \phi(t,i)\Big)\Bigg)\Bigg\}dt$$
$$+\phi(t,\alpha(t))\bar{u}(t)dW(t) + \sum_{i\neq j} \bar{x}(t)\left(\phi(t,j) - \phi(t,i)\right)dQ_{ij}. \tag{2.28}$$

Comparing the coefficients of (2.27) and (2.28) leads to

$$2C(\alpha(t))\bar{x}(t) = \sum_{i=1}^{2} \mathcal{X}[\alpha(t-) = i]\Bigg\{\bar{x}(t)\Bigg(\phi_t(t,i) + \sum_{j=1}^{2} q_{ij}(\phi(t,j) - \phi(t,i))\Bigg)\Bigg\} \tag{2.29}$$

$$\bar{q}(t) = \phi(t,\alpha(t))\bar{u}(t) \tag{2.30}$$

$$\bar{s}_{ij}(t) = \bar{x}(t)(\phi(t,j) - \phi(t,i)) \tag{2.31}$$

As (2.29) is a linear equation of $\bar{x}(t)$, we guess that the coefficient of $\bar{x}(t)$ vanishes at optimality and obtain the following two equations

$$\begin{cases} -\phi_t(t,1) - q_{12}(\phi(t,2) - \phi(t,1)) = 0, \\ 2 - \phi_t(t,2) - q_{21}(\phi(t,1) - \phi(t,2)) = 0, \end{cases} \tag{2.32}$$

with terminal conditions

$$\phi(1,1) = -1 \text{ and } \phi(1,2) = -2. \tag{2.33}$$

Solving the system of ordinary differential equations (2.32) with terminal conditions (2.33) gives

$$\begin{cases} \phi(t,1) = \dfrac{q_{12}(q_{12}+q_{21}-2)}{(q_{12}+q_{21})^2}\left(e^{(q_{12}+q_{21}-2)(t-1)} - 1\right) + \dfrac{2q_{12}}{q_{12}+q_{21}}(t-1) - 1 \\ \phi(t,2) = \dfrac{q_{21}(q_{12}+q_{21}-2)}{(q_{12}+q_{21})^2}\left(1 - e^{(q_{12}+q_{21}-2)(t-1)}\right) + \dfrac{2q_{12}}{q_{12}+q_{21}}(t-1) - 2 \end{cases}$$

Moreover, since $q_{12} + q_{21} \geq 2$ and $\frac{q_{21}(q_{12}+q_{21}-2)}{(q_{12}+q_{21})^2} < 1$, we obtain that $\phi(t,i) < -1,\ \forall t \in [0,1), i \in I$. Consider the Hamiltonian

$$\begin{cases} H(t,x,u,1,p,q) = u + qu \\ H(t,x,u,2,p,q) = \frac{1}{2}u^2 - x^2 + uq. \end{cases} \tag{2.34}$$

By Theorem 2.3.1, we have

$$0 \in \partial_u(-H)(t, \bar{x}(t), \bar{u}(t), \alpha(t-), \bar{p}(t), \bar{q}(t)) + N_U(\bar{u}(t))\ \forall t \in [0,1],\ \mathbb{P} - \text{a.s.}$$

Consequently on any non-zero measurable set $E \in S = \Omega \times [0,1)$ such that $\alpha(t-) = 1$, we can only have three cases:

Case 1 : $\bar{u}(t) = 0 \Rightarrow N_{[0,1]}(\bar{u}(t)) = (-\infty, 0]$ and $\bar{q}(t) + 1 \leq 0$.
According to (2.30), $\phi(t,1)\bar{u}(t) \leq -1, \bar{u}(t) \geq -\frac{1}{\phi(t,1)} > 0$, contradiction.

Case 2 : $\bar{u}(t) = 1 \Rightarrow N_{[0,1]}(\bar{u}(t)) = [0, +\infty)$ and $\bar{q}(t) + 1 \geq 0$.
According to (2.30), $\phi(t,1)\bar{u}(t) \geq -1, \bar{u}(t) \leq -\frac{1}{\phi(t,1)} < 1$, contradiction.

Case 3 : $0 < \bar{u}(t) < 1 \Rightarrow N_{[0,1]}(\bar{u}(t)) = \{0\}$ and $\bar{q}(t) = -1$.
According to (2.30), $\bar{u}(t) = -\frac{1}{\phi(t,1)} \in (0,1)$.

Hence we conclude that $\bar{u}(t) = -\frac{1}{\phi(t,1)}$ provided $\alpha(t-) = 1$. Similarly on any non-zero measurable set $E \in S = \Omega \times [0,1)$ such that $\alpha(t-) = 2$, we can only have three cases:

Case 1 : $\bar{u}(t) = 1 \Rightarrow N_{[0,1]}(\bar{u}(t)) = [0, +\infty)$ and $\bar{q}(t) + \bar{u}(t) \geq 0$.
According to (2.30), $(\phi(t,2)+1)\bar{u}(t) \geq 0, \bar{u}(t) \leq \frac{1}{\phi(t,2)+1} < 0$, contradiction.

Case 2 : $\bar{u}(t) \in (0,1) \Rightarrow N_{[0,1]}(\bar{u}(t)) = \{0\}$ and $\bar{q}(t) + \bar{u}(t) = 0$.
According to (2.30), $(\phi(t,2)+1)\bar{u}(t) = 0, \bar{u}(t) = 0$, contradiction.

Case 3 : $\bar{u}(t) = 0 \Rightarrow N_{[0,1]}(\bar{u}(t)) = (-\infty, 0]$ and $\bar{q}(t) + \bar{u}(t) \leq 0$.
According to (2.30), $(\phi(t,2)+1)\bar{u}(t) \leq 0, \bar{u}(t) = 0$.

Hence we must have $\bar{u}(t) = 0$ provided $\alpha(t-) = 2$.
In conclusion, the potential optimal control can be written as

$$\bar{u}(t) = -\frac{1}{\phi(t,1)}\mathcal{X}[\alpha(t-) = 1]. \tag{2.35}$$

Let us now show that $(\bar{x}(\cdot), \bar{u}(\cdot))$ is indeed an optimal pair. Notice that the Hamiltonian (2.34) is not concave function of u, and therefore Theorem 2.3.2 cannot be applied. We have to use other methods to check the optimality of $\bar{u}$. Given any admissible pair $(x(\cdot), u(\cdot))$, apply Ito's formula on $\phi(t, \alpha(t))x^2(t)$ and write it in integral form,

$$E\left[\phi(1, \alpha(1))x^2(1)\right] = E\left[\int_0^1 x^2(t)\left(\phi_t(t, \alpha(t)) + \sum_{j=1}^{2} q_{ij}(\phi(t, j) - \phi(t, \alpha(t)))\right) + \phi(t, \alpha(t))u^2(t)dt\right]. \tag{2.36}$$

Substituting (2.36) into the cost functional and according to (2.29),

$$\begin{aligned} J(u(\cdot)) &= E\left[\int_0^1 \left(A(\alpha(t))u(t) + B(\alpha(t))u^2(t) - \frac{1}{2}\phi(t, \alpha(t))u^2(t)\right) dt\right] \\ &= E\left[\int_{S_1} \left(-u(t) - \frac{1}{2}\phi(t, 1)u^2(t)\right) dt + \int_{S_2} -\frac{1}{2}(1 + \phi(t, 2))u^2(t)dt\right] \\ &= E\left[\int_{S_1} \left(-\frac{1}{2}\phi(t, 1)\left(u(t) + \frac{1}{\phi(t, 1)}\right)^2 + \frac{1}{2\phi(t, 1)}\right) dt + \int_{S_2} -\frac{1}{2}(1 + \phi(t, 2))u^2(t)dt\right], \end{aligned}$$

where $S_1 \equiv \{t | t \in [0, 1] \text{ such that } \alpha(t-) = 1\}$ and $S_2 \equiv \{t | t \in [0, 1] \text{ such that } \alpha(t-) = 2\} = [0, 1] \backslash S_1$. Since $\phi(t, 1) < -1$ and $\phi(t, 2) < -1$ $\forall t \in [0, 1]$, the minimum value of the cost functional is achieved at $\bar{u}$ defined in (2.35).

2.5 Preliminary Results

In this section, we introduce some preliminary results, which will be useful in the sequel. Hereafter, K represents a generic constant.

2.5.1 Clarke's Generalized Gradient and Normal Cone

In this subsection we recall some basic concepts and properties in non-smooth analysis and optimization, which are needed in the statement and proof of the main results (Theorems 2.3.1 and 2.3.2). Clarke's generalized gradient is first introduced to the finite dimensional space in [14] and then extended to the infinite dimensional space in [15, 16] and [1]. Interested readers may refer to [17] for a detailed and complete treatment of the topic.

Definition 2.5.1. *(Generalized directional derivative, see definition in Chapter 3 of [17]) Let C be an open subset of a Banach space X, and let a function $f : C \longrightarrow \mathbb{R}$ be given. We assume that f is Lipschitz on C. The generalized directional derivative of f at x in the direction v, denoted $f^o(x; v)$, is given by*

$$f^o(x; v) = \limsup_{y \to_C x, \lambda \downarrow 0} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

Definition 2.5.2. *(Clarke's generalized gradient) Let X^* denote the dual of X and $\langle \cdot, \cdot \rangle$ be the duality pairing between X and X^*. The generalized gradient of f at x, denoted $\partial f(x)$, is the set of all ζ in X^* satisfying*

$$f^o(x; v) \geq \langle v, \zeta \rangle \text{ for } \forall v \in X.$$

Theorem 2.5.3. *If f attains a local minimum or maximum at x, then $0 \in \partial f(x)$.*

Theorem 2.5.3 is only valid in the case where C is open. When the function is defined on a general non-empty subset of X, we need to introduce the so-called distance function and the concept of Clarke's tangent cone and normal cone.

Definition 2.5.4. *(Distance function) Let X be a Banach space and C be a non-empty subset of X. The distance function $d_C : X \to \mathbb{R}$ is defined as*

$$d_C(x) = \inf\{\|x - c\| : c \in C\}.$$

Theorem 2.5.5. *The function d_C satisfies the following global Lipschitz condition on X*

$$|d_C(x) - d_C(y)| \leq \|x - y\|.$$

Definition 2.5.6. *(Adjacent cone) Let $\bar{C}$ be the closure of C and $x \in \bar{C}$. The adjacent cone to C at x, denoted as $T_C^b(x)$, is defined by*

$$T_C^b(x) := \{v | \lim_{h \to 0^+} d_C(x + hv)/h = 0\}.$$

Definition 2.5.7. *(Tangent cone) Suppose $x \in C$. A vector v in X is a tangent to C at x provided $d_C^o(x; v) = 0$. The tangent cone to C at x, denoted as $T_C(x)$, is the set of all tangents to C at x.*

In addition, when the set C is convex, it can be proved that the adjacent and tangent cones coincide, see [1, Proposition 4.2.1].

Theorem 2.5.8. *Assume that C is convex. Then $T_C(x) = T_C^b(x)$.*

Definition 2.5.9. *(Normal cone) Let $x \in C$. The normal cone to C at x is defined by the polarity with $T_C(x)$:*

$$N_C(x) = \{\xi \in X^* : \langle \xi, v\rangle \leq 0 \text{ for all } v \in T_C(x)\}.$$

For example, take $C = [0, 1]$, we have $N_{[0,1]}(0) = (-\infty, 0]$ and $N_{[0,1]}(1) = [0, +\infty)$. The following necessary optimality condition is proved in [17, page 52 Corollary].

Theorem 2.5.10. *Assume that f is Lipschitz near x and attains a minimum over C at x. Then $0 \in \partial f(x) + N_C(x)$.*

2.5.2 Markovian Regime-Switching SDE and BSDE

In this subsection, we establish the existence and uniqueness theorem of solutions to regime switching SDEs of the form (2.1). First, we give the definition of the solution.

Definition 2.5.11. *[60, Definition 3.11] An $\mathbb{R}^n$ valued stochastic process $\{x(t)\}_{0\leq t\leq T}$ is called a solution of equation* (2.1) *if it has the following properties:*

1. *$\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;*
2. *$\{b(t, x(t), u(t), \alpha(t-))\} \in L^1_{\mathcal{F}}(S; \mathbb{R}^n)$ and $\{\sigma(t, x(t), u(t), \alpha(t-)) \in L^2_{\mathcal{F}}(S; \mathbb{R}^{n\times m})\}$;*

3. for any $t \in [0, T]$*, equation*

$$x(t) = x_0 + \int_0^t b(s, x(s), u(s), \alpha(s-))ds + \int_0^t \sigma(t, x(s), u(s), \alpha(s-))dW(s)$$

holds with probability 1.

A solution $\{x(t)\}$ *is said to be unique if any other solution* $\{\tilde{x}(t)\}$ *is indistinguishable from* $\{x(t)\}$*, that is*

$$\mathbb{P}\{x(t) = \tilde{x}(t) \text{ for all } 0 \leq t \leq T\} = 1$$

Using the same method as in [60, Chapter 3, Theorem 3.13], the existence and uniqueness of solutions to regime-switching SDE of type (2.1) can be proved.

Theorem 2.5.12. *Under assumption* ***(A1)****, given control* $u \in L^4(S; \mathbb{R}^k)$*, there exists a unique solution* $x(t)$ *to equation* (2.1) *and moreover,*

$$E\left(\sup_{0 \leq t \leq T} |x(t)|^2\right) \leq K\left(1 + |x|^2 + \int_0^T E|u(t)|^2 dt\right) \tag{2.37}$$

for some constant $K \geq 0$.

There has been extensive research in BSDE and Markov chain BSDE. Here are some related recent papers on the topic. Cohen [18] studies the existence and uniqueness of solution to Markov chain BSDEs and relates the solution to the nonlinear expectation in [19], in the spirit of [68]. Tao-Wu [81] study the control system driven by 1-dimensional Markov chain FBSDEs and derive the SMP when the coefficients are smooth functions of all variables. Tao-Wu-Zhang [82] study a system of SDEs and BSDEs driven by both Brownian motion and Markov chain with interactions among different groups and with different intensities. Crepey [22] gives a survey on SDEs and their applications, including BSDE, FBSDE and related PDE and Partial Integral Differential Equation (PIDE), etc.

We now develop results for existence and uniqueness of adapted solutions to regime switching BSDEs of type (2.5). Here we use the method of contraction mapping as in [89, Chapter 6, Section 3] and [69, Chapter 6, Section 2] with the help of a martingale representation theorem for the joint filtration of a vector Brownian motion and a finite state Markov chain. Here we introduce the Doléans measure $v_{[Q_{ij}]}$ on the measure space $(\Omega \times [0,T], \mathcal{P}^\star)$:

$$v_{[Q_{ij}]}[A] := E\int_0^T \mathcal{X}_A(\omega,t)d[Q_{ij}](t), \forall A \in \mathcal{P}^\star, \forall i,j \in I, i \neq j.$$

By $G = H$ $v_{[Q]}$-a.e. for $\mathbb{R}^{d\times d}$ mappings G and H on the set $\Omega \times [0,T]$, we mean that

$$\begin{aligned} G_{ij} &= H_{ij}\ v_{[Q_{ij}]}\text{-a.e. } \forall i,j \in I,\ i \neq j \\ \text{and } G_{ii} &= H_{ii}\ (\mathbb{P}\otimes \text{Leb})\text{-a.e. } \forall i \in I. \end{aligned}$$

We start by defining the following spaces for stochastic processes.

$$\begin{aligned} \mathbb{S}^2([0,T]) :=& \Big\{ Y : \Omega \times [0,T] \to \mathbb{R}^n | Y \text{ is } \mathcal{F}_t \text{ progressively measurable} \\ & \text{and } E\left(\sup_{0\le t\le T}|Y(t)|^2\right) < \infty \Big\}, \\ L^2(W,[0,T]) :=& \left\{ \Lambda : \Omega\times[0,T] \to \mathbb{R}^{n\times m} | \Lambda \in \mathcal{P}^\star \text{ and } E\int_0^T \|\Lambda(t)\|^2 dt < \infty \right\}, \\ L^2(Q,[0,T]) :=& \left\{ \Gamma = \left\{ \left(\Gamma_{ij}^{(1)}\right)_{i,j=1}^d, \cdots, \left(\Gamma_{ij}^{(n)}\right)_{i,j=1}^d \right\} \middle| \Gamma_{ii}^{(l)} = 0\ \mathbb{P}\otimes\text{Leb} - a.e. \forall i \in I, \right. \\ & \left. \Gamma_{ij}^{(l)} \in \mathcal{P}^\star \text{ and } \sum_{l=1}^n \sum_{i,j=1}^d E\int_0^T \|\Gamma_{ij}^{(l)}(t)\|^2 d[Q_{ij}](t) < \infty\ \forall i,j \in I, i\neq j \right\}. \end{aligned}$$

It can be proved that $L^2(W,[0,T])$ and $L^2(Q,[0,T])$ are Hilbert spaces (see [25, Lemma A.2.5]). Next we present a martingale representation theorem for square integrable martingales with joint filtration generated by a Brownian motion and a finite state Markov chain. The proof can be found in [25, Theorem B.4.6] and [27, Proposition 3.9].

Theorem 2.5.13. *Suppose the $\mathbb{R}^n$-valued process $\{Y(t), t \in [0,T]\}$ is a square-integrable $\{\mathcal{F}_t\}$-martingale and null at the origin. Then there exists processes*

$\Lambda \in L^2(W,[0,T])$ and $\Gamma \in L^2(Q,[0,T])$ such that Y has the stochastic integral representation

$$Y(t) = Y(0) + \sum_{j=1}^{m} \int_0^t \Lambda_j(s) dW^j(s) + \int_0^t \Gamma(s) \bullet dQ(s) \;\; a.s. \;\; \forall t \in [0,T] \qquad (2.38)$$

with the square-bracket quadratic variation process of Y given by

$$[Y](t) := \sum_{i=1}^{n} \sum_{j=1}^{m} \int_0^t \Lambda_{ij}^2(s) ds + \sum_{l=1}^{n} \sum_{i,j=1}^{d} \int_0^t \left(\Gamma_{ij}^{(l)}(s)\right)^2 d[Q_{ij}](t) \;\; a.s. \;\; \forall t \in [0,T].$$

Moreover, Λ and Γ are unique in the sense that if $\tilde{\Lambda} \in L^2(W,[0,T])$ and $\tilde{\Gamma} \in L^2(Q,[0,T])$ are such that (2.38) holds, then $\Lambda = \tilde{\Lambda}$ $\mathbb{P} \otimes Leb - a.e.$ and $\Gamma = \tilde{\Gamma}$ $v_{[Q]} - a.e.$

Suppose we are given a pair (ξ, f) called the terminal and generator satisfying the following conditions:

(a) $E|\xi|^2 < \infty$,

(b) $f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \to \mathbb{R}^n$ such that

  (i) $f(t,y,z)$ is $\mathcal{F}_t$-progressively measurable for all y, z.

  (ii) $f(t,0,0) \in L_{\mathcal{F}}^2(S;\mathbb{R}^n)$,

  (iii) f satisfies uniform Lipschitz condition in (y,z), i.e $\exists C_f > 0$ such that

$$|f(t,y_1,z_1) - f(t,y_2,z_2)| \leq C_f \left(|y_1 - y_2| + |z_1 - z_2|\right)$$

  $\forall y_1, y_2 \in \mathbb{R}^n, z_1, z_2 \in \mathbb{R}^{n \times m}$ $\mathbb{P} \otimes \text{Leb}$ $a.e.$

Consider the regime switching BSDE

$$-dY(t) = f(t, Y(t), Z(t))dt - Z(t)dW(t) - S(t) \bullet dQ(t), \;\; Y(T) = \xi. \qquad (2.39)$$

Definition 2.5.14. *A solution to the regime switching BSDE (2.39) is a set $(Y,Z,S) \in \mathbb{S}^2([0,T]) \times L^2(W,[0,T]) \times L^2(Q,[0,T])$ satisfying*

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s)))ds - \int_t^T Z(s)dW(s) - \int_t^T S(s) \bullet dQ(t).$$

Now we prove the existence and uniqueness of a solution to the regime switching BSDE of type (2.39).

Theorem 2.5.15. *Given a pair* (ξ, f) *satisfying* (a) *and* (b)*, there exists a unique solution* (Y, Z, S) *to the regime switching BSDE* (2.39).

The proof follows a contraction mapping argument similar to that in [69, Chapter 6, Section 2]. For completeness, we give details in Appendix.

2.5.3 A Moment Estimation

In this subsection, we prove a moment estimation result. A simplified version of the moment estimate can be found in [89, Chapter 3 Lemma 4.2].

Lemma 2.5.16. *Let* $Y(t) \in L^2_{\mathcal{F}}(S; \mathbb{R}^n)$ *be the solution of the following regime switching SDE*

$$\begin{cases} dY(t) = [A(t)Y(t) + \beta(t)]dt + \sum_{j=1}^{m} \left[B^j(t)Y(t) + \gamma^j(t)\right] dW^j(t) \\ Y(0) = y_0 \end{cases} \tag{2.40}$$

where $A, B^j : \Omega \times [0,T] \to \mathbb{R}^{n\times n}$ *and* $\beta, \gamma^j : \Omega \times [0,T] \to \mathbb{R}^n$ *are* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted and*

$$\begin{cases} |A(t)|, |B^j(t)| \leq K \ \ a.e.t \in [0,T], \mathbb{P}\text{-}a.s. \\ \int_0^T E|\beta(s)|^{2k} ds + \int_0^T E|\gamma^j(s)|^{2k} ds < \infty \text{ for some } k \geq 1. \end{cases} \tag{2.41}$$

Then

$$\sup_{t\in[0,T]} E|Y(t)|^{2k} \leq K \left\{ E|y_0|^{2k} + \int_0^T E|\beta(s)|^{2k} ds + \sum_{j=1}^{m} \int_0^T E|\gamma^j(s)|^{2k} ds \right\} \tag{2.42}$$

Proof. For notation simplicity, we prove only the case $m = n = 1$, leaving the case $m, n > 1$ to the interested reader. We first assume that β, γ are bounded.

Let $\epsilon > 0$ and define

$$\langle Y \rangle_\epsilon \triangleq \sqrt{|Y|^2 + \epsilon^2}, \forall Y \in L^2_{\mathcal{F}}(S;\mathbb{R}^n). \tag{2.43}$$

Note that for any $\epsilon > 0$, the map $Y \to \langle Y \rangle_\epsilon$ is smooth and $\langle Y \rangle_\epsilon \to |Y|$ as $\epsilon \to 0$. Applying Ito's formula to $\langle Y(t) \rangle_\epsilon^{2k}$, we have

$$\begin{aligned} d\langle Y(t)\rangle_\epsilon^{2k} &= 2k\langle Y(t)\rangle_\epsilon^{2k-1} \frac{|Y(t)|}{\langle Y(t)\rangle_\epsilon}\Big\{ [A(t)Y(t)+\beta(t)]\,dt + [B(t)Y(t)+\gamma(t)]\,dW(t)\Big\} \\ &+ \left[k(2k-1)\langle Y(t)\rangle_\epsilon^{2k-2}\frac{|Y(t)|^2}{\langle Y(t)\rangle_\epsilon^2} + k\langle Y(t)\rangle_\epsilon^{2k-1}\frac{\epsilon^2}{\langle Y(t)\rangle_\epsilon^3}\right][B(t)Y(t)+\gamma(t)]^2\,dt. \end{aligned}$$

Writing it in integral form and taking expectation. Since $\langle Y(t)\rangle_\epsilon > |Y(t)|$ and $2k-1 \geq 1$, we obtain

$$\begin{aligned} E\langle Y(t)\rangle_\epsilon^{2k} \leq& E\langle Y(0)\rangle_\epsilon^{2k} + 2kE\int_0^t \langle Y(s)\rangle_\epsilon^{2k-1}\{|A(s)|\langle Y(s)\rangle_\epsilon + |\beta(s)|\}\,ds \\ &+ k(2k-1)E\int_0^t \langle Y(s)\rangle_\epsilon^{2k-2}[|B(s)|\langle Y(s)\rangle_\epsilon + |\gamma(s)|]^2\,ds \\ \leq& E\langle Y(0)\rangle_\epsilon^{2k} + KE\int_0^t\Big\{\langle Y(s)\rangle_\epsilon^{2k} + |\beta(s)|\langle Y(s)\rangle_\epsilon^{2k-1} + |\gamma(s)|^2\langle Y(s)\rangle_\epsilon^{2k-2}\Big\}ds, \end{aligned}$$

where K is a constant independent of t. Applying *Young's inequality*, we get

$$E\langle Y(t)\rangle_\epsilon^{2k} \leq E\langle Y(0)\rangle_\epsilon^{2k} + KE\int_0^t\Big\{\langle Y(s)\rangle_\epsilon^{2k} + |\beta(s)|^{2k} + |\gamma(s)|^{2k}\Big\}ds.$$

Finally, *Gronwall's inequality* yields

$$\sup_{t\in[0,T]} E\langle Y(t)\rangle_\epsilon^{2k} \leq K\Big\{E\langle Y(0)\rangle_\epsilon^{2k} + E\int_0^T\left[|\beta(s)|^{2k} + |\gamma(s)|^{2k}\right]ds\Big\}, \tag{2.44}$$

for some constant K. Letting $\epsilon \to 0$ in (2.43), then (2.44) becomes (2.42). $\square$

2.5.4 Lipschitz Property

Lemma 2.5.17. *Let $u_1, u_2 \in L^4(S;\mathbb{R}^k)$ and x_1, x_2 be the associated state processes satisfying* (2.1)*. Then we have the following inequality:*

$$\sup_{t\in[0,T]} E|x_1(t) - x_2(t)|^4 \leq K\|u_1 - u_2\|^4$$

Proof. Let $\xi(t) \triangleq x_1(t) - x_2(t)$. Then we have

$$\begin{aligned} d\xi(t) = &[b(t, x_1(t), u_1(t), \alpha(t-)) - b(t, x_2(t), u_2(t), \alpha(t-))]\, dt \\ &+ [\sigma(t, x_1(t), u_1(t), \alpha(t-)) - \sigma(t, x_2(t), u_2(t), \alpha(t-))]\, dW(t) \end{aligned}$$

For $\varphi = b$ and σ, let

$$\tilde{\varphi}_x(t) = \int_0^1 \varphi_x\left(t, x_2(t) + \theta(x_1(t) - x_2(t)), u_1(t), \alpha(t-)\right) d\theta. \tag{2.45}$$

Substitute (2.45), we obtain

$$\begin{aligned} d\xi(t) = &\left[\tilde{b}_x(t)\xi(t) + b(t, x_2(t), u_1(t), \alpha(t-)) - b(t, x_2(t), u_2(t), \alpha(t-))\right] dt \\ &+ [\tilde{\sigma}_x(t)\xi(t) + \sigma(t, x_2(t), u_1(t), \alpha(t-)) - \sigma(t, x_2(t), u_2(t), \alpha(t-))]\, dW(t). \end{aligned}$$

By Lemma 2.5.16, we obtain

$$\begin{aligned} \sup_{t\in[0,T]} E|\xi(t)|^4 \leq &K \left\{ \int_0^T E|b(t, x_2(t), u_2(t), \alpha(t-)) - b(t, x_2(t), u_1(t), \alpha(t-))|^4 dt \right. \\ &+ \int_0^T E|\sigma(t, x_2(t), u_2(t), \alpha(t-)) - \sigma(t, x_2(t), u_1(t), \alpha(t-))|^4 dt \\ \leq &K \left\{ \int_0^T E|u_1(t) - u_2(t)|^4 dt \right\} \end{aligned}$$

□

Lemma 2.5.18. *The cost functional $J : L^4(S; \mathbb{R}^k) \to \mathbb{R}$ is locally Lipschitz, i.e. for all $\hat{u} \in L^4(S; \mathbb{R}^k)$, there exists a small ball $B_{\hat{u}}^M$ with radius $M > 0$ containing $\hat{u}$ on which, we have*

$$|J(u_1) - J(u_2)| \leq K_{M,\hat{u}} \|u_1 - u_2\|, \tag{2.46}$$

for $\forall u_1, u_2 \in B_{\hat{u}}^M$, where $K_{M,\hat{u}}$ is a constant dependent on M and $\hat{u}$.

Proof. Given $\hat{u} \in L^4(S; \mathbb{R}^k)$ and $M > 0$, define

$$B_{\hat{u}}^M \triangleq \left\{ u \in L^4(S; \mathbb{R}^k) : \|u - \hat{u}\| < M \right\}.$$

For any $u_1, u_2 \in B_{\hat{u}}^M$ with associated state processes x_1, x_2, according to **(A4)**, we have

$$
\begin{aligned}
&E|h(x_1(T),\alpha(T)) - h(x_2(T),\alpha(T))| \\
&\leq E\int_0^1 |\langle h_x(x_1(T)+\theta(x_2(T)-x_1(T)),\alpha(T)), x_2(T)-x_1(T)\rangle| d\theta \\
&\leq K\left\{E\left(1+|x_1(T)|^2+|x_2(T)|^2\right)\right\}^{\frac{1}{2}}\left\{E|x_2(T)-x_1(T)|^2\right\}^{\frac{1}{2}} \\
&\leq K\left\{E(1+|\hat{x}(T)|^2+|\hat{x}(T)-x_1(T)|^2+|\hat{x}(T)-x_2(T)|^2)\right\}^{\frac{1}{2}}\left\{E|x_2(T)-x_1(T)|^2\right\}^{\frac{1}{2}}
\end{aligned}
$$

by Hölder's inequality and Minkowski's inequality. According to Theorem 2.5.12, Jensen's inequality and Lemma 2.5.17,

$$
\begin{aligned}
E|h(x_1(T),\alpha(T)) - h(x_2(T),\alpha(T))| \leq & K_M\left\{1+\left(\int_0^T E|\hat{u}(t)|^2 dt\right)^{\frac{1}{2}}\right\}\left\{E|x_2(T)-x_1(T)|^2\right\}^{\frac{1}{2}} \\
\leq & K_M\left\{1+\left(\int_0^T E|\hat{u}(t)|^4 dt\right)^{\frac{1}{4}}\right\}\left\{E|x_2(T)-x_1(T)|^4\right\}^{\frac{1}{4}} \\
\leq & K_{M,\hat{u}}\|u_1-u_2\|.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&E\int_0^T |f(t,x_1(t),u_1(t),\alpha(t)) - f(t,x_2(t),u_2(t),\alpha(t))| dt \\
\leq & E\int_0^T \left(|f(t,x_1(t),u_1(t),\alpha(t)) - f(t,x_2(t),u_1(t),\alpha(t))|\right. \\
&\left. + |f(t,x_2(t),u_1(t),\alpha(t)) - f(t,x_2(t),u_2(t),\alpha(t))|\right) dt.
\end{aligned}
$$

Following similar arguments, we have

$$
E\int_0^T |f(t,x_1(t),u_1(t),\alpha(t)) - f(t,x_2(t),u_1(t),\alpha(t))| dt \leq K_{M,\hat{u}}\|u_1-u_2\|.
$$

For the second term, by **(A3)** we have

$$
\begin{aligned}
& E\int_0^T |f(t,x_2(t),u_1(t),\alpha(t)) - f(t,x_2(t),u_2(t),\alpha(t))|dt \\
& \le E\int_0^T \{K_1 + K_2(|x_2(t)| + |u_1(t)| + |u_2(t)|)\} |u_1(t) - u_2(t)|dt \\
& \le K\left\{E\int_0^T (1 + |x_2(t)|^2 + |u_1(t)|^2 + |u_2(t)|^2)dt\right\}^{\frac{1}{2}} \|u_1 - u_2\| \\
& \le K_M\left(1 + \left\{E\int_0^T |\hat{u}(t)|^4 dt\right\}^{\frac{1}{4}}\right)\|u_1 - u_2\| \\
& \le K_{M,\hat{u}}\|u_1 - u_2\|,
\end{aligned}
$$

and (2.46) follows by combining the above inequalities. □

2.5.5 Taylor Expansions

Let (x,u) be an admissible pair. Let $v \in \L^4(S;\mathbb{R}^k)$ and $\epsilon > 0$. Define $u^\epsilon(t) \triangleq u(t) + \epsilon v(t)$ for all $t \in [0,T]$. Let (x^ϵ, u^ϵ) satisfy the following stochastic control system:

$$
\begin{cases}
dx^\epsilon(t) = & b(t,x^\epsilon(t),u^\epsilon(t),\alpha(t-))dt + \sigma(t,x^\epsilon(t),u^\epsilon(t),\alpha(t-))dW(t),\ t\in[0,T], \\
x^\epsilon(0) = & x_0 \in \mathbb{R}^L, \alpha(0) = i_0 \in I.
\end{cases}
$$

Next, for $\varphi = b,\ \sigma^j(1 \le j \le M)$ *and* f, we define

$$
\begin{cases}
\varphi_x(t) \triangleq \varphi_x(t,x(t),u(t),\alpha(t-)), \\
\delta\varphi(t) \triangleq \varphi(t,x(t),u^\epsilon(t),\alpha(t-)) - \varphi(t,x(t),u(t),\alpha(t-)).
\end{cases}
$$

Let y^ϵ be the solution of the following regime-switching SDE:

$$
\begin{cases}
dy^\epsilon(t) = & \{b_x(t)y^\epsilon(t) + \delta b(t)\}dt + \sum_{j=1}^m \{\sigma_x^j(t)y^\epsilon(t) + \delta\sigma^j(t)\}dW^j(t),\ t\in[0,T], \\
y^\epsilon(0) = & 0, \alpha(0) = i_0 \in I.
\end{cases}
\tag{2.47}
$$

Remark 2.5.19. The variation in our proof is different from the so-called spike variation technique in the proof of Peng's maximum principle in [67] and [89].

In their proof, where $u^\epsilon(t) = u(t) + 1_{[\tau,\tau+\epsilon]}v(t)$, one first perturbs an optimal control on a small set of size ϵ and then let $\epsilon \to 0$. Whereas, in our proof we perturbs an optimal control over the whole space. The reason behind this is that in the definition of Clarke's generalized directional derivative, $v(t)$ represents a directional vector in $L^4(S;\mathbb{R}^k)$ and must be fixed. One perturbs the control through multiplication of a scalar ϵ and letting $\epsilon \to 0$.

The following lemma gives the Taylor expansion result of the state process and cost functional.

Lemma 2.5.20. *Let assumptions* ***(A1)-(A4)*** *hold. Then, we have*

$$\sup_{t\in[0,T]} E\left|x^\epsilon(t) - x(t)\right|^2 = O(\epsilon^2), \tag{2.48}$$

$$\sup_{t\in[0,T]} E\left|y^\epsilon(t)\right|^2 = O(\epsilon^2), \tag{2.49}$$

$$\sup_{t\in[0,T]} E\left|x^\epsilon(t) - x(t) - y^\epsilon(t)\right|^2 = o(\epsilon^2). \tag{2.50}$$

Moreover, the following expansion holds for the cost functional:

$$J(u^\epsilon) = J(u) + E\langle h_x(x(T),\alpha(T)), y^\epsilon(t)\rangle + E\int_0^T \{\langle f_x(t), y^\epsilon(t)\rangle + \delta f(t)\}\,dt + o(\epsilon). \tag{2.51}$$

Proof. For simplicity, we carry out the proof only for the case $n = m = 1$.

Proof of (2.48). Let $\xi^\epsilon(t) \triangleq x^\epsilon(t) - x(t)$. The we have

$$\begin{cases} d\xi^\epsilon(t) &= \left\{\tilde{b}_x^\epsilon(t)\xi^\epsilon(t) + \delta b(t)\right\}dt + \{\tilde{\sigma}_x^\epsilon(t)\xi^\epsilon(t) + \delta\sigma(t)\}\,dW(t) \\ \xi(0) &= 0, \alpha(0) = i_0. \end{cases} \tag{2.52}$$

where for $\phi = b$ and σ,

$$\tilde{\phi}_x^\epsilon(t) \triangleq \int_0^1 \phi_x(t, x(t) + \theta(x^\epsilon(t) - x(t)), u^\epsilon(t), \alpha(t-))d\theta. \tag{2.53}$$

By Lemma 2.5.16, since $\tilde{b}^\epsilon_x(t)$, and $\tilde{\sigma}^\epsilon_x(t)$ are bounded according to assumption **(A1)**, we obtain

$$\begin{aligned}\sup_{t\in[0,T]} E|\xi^\epsilon(t)|^2 &\leq K\int_0^T E\Big\{|\delta b(s)|^2+|\delta\sigma(s)|^2\Big\}ds\\ &\leq K\epsilon^2\int_0^T E|v(s)|^2 ds\\ &\leq K\epsilon^2.\end{aligned}$$

This proves (2.48).

Proof of (2.49). Similarly, $b_x(t)$ and $\sigma_x(t)$ are bounded according to assumption **(A1)**. Applying Lemma 2.5.16 to (2.47), we obtain

$$\sup_{t\in[0,T]} E|y^\epsilon(t)|^2 \leq K\int_0^T E\Big\{|\delta b(s)|^2+|\delta\sigma(s)|^2\Big\}ds \leq K\epsilon^2.$$

This proves (2.49).

Proof of (2.50). Let $\zeta^\epsilon(t)\triangleq x^\epsilon(t)-x(t)-y^\epsilon(t)\equiv\xi^\epsilon(t)-y^\epsilon(t)$. Then, by (2.52) and (2.47) we have

$$\begin{aligned}d\zeta^\epsilon(t) =& d\xi^\epsilon(t)-dy^\epsilon(t)\\ =&\Big\{\tilde{b}^\epsilon_x(t)\xi^\epsilon(t)-b_x(t)y^\epsilon(t)\Big\}dt+\{\tilde{\sigma}^\epsilon_x(t)\xi^\epsilon(t)-\sigma_x(t)y^\epsilon(t)\}dW(t)\\ =&\Big\{\tilde{b}^\epsilon_x(t)\zeta^\epsilon(t)+\Big[\tilde{b}^\epsilon_x(t)-b_x(t)\Big]y^\epsilon(t)\Big\}dt+\{\tilde{\sigma}^\epsilon_x(t)\zeta^\epsilon(t)+[\tilde{\sigma}^\epsilon_x(t)-\sigma_x(t)]y^\epsilon(t)\}dW(t)\end{aligned}$$

Since $\tilde{b}^\epsilon_x(t)$ and $\tilde{\sigma}^\epsilon_x(t)$ are bounded by assumption **(A1)**, applying Lemma 2.5.16 we obtain

$$\sup_{t\in[0,T]} E|\zeta^\epsilon(t)|^2\leq K\int_0^T E\Big\{\Big|\Big[\tilde{b}^\epsilon_x(t)-b_x(t)\Big]y^\epsilon(t)\Big|^2+|[\tilde{\sigma}^\epsilon_x(t)-\sigma_x(t)]y^\epsilon(t)|^2\Big\}dt. \tag{2.54}$$

Recall that $\bar{\omega}$ appearing in **(A4)** is a modulus of continuity. Thus for any $\rho>0$, there exists a constant $K_\rho>0$ such that

$$\bar{\omega}(r)\leq\rho+rK_\rho,\ \forall r\geq 0. \tag{2.55}$$

By Hölder's inequality, (2.53), (2.49), (2.48) and (2.55), we have

$$
\begin{aligned}
&\int_0^T E\left|\left[\tilde{b}_x^\epsilon(t)-b_x(t)\right]y^\epsilon(t)\right|^2 dt \\
&\le \int_0^T \left(E\left|\tilde{b}_x^\epsilon(t)-b_x(t)\right|^4\right)^{\frac{1}{2}}\left(E\left|y^\epsilon(t)\right|^4\right)^{\frac{1}{2}} dt \\
&\le K\int_0^T \left\{E\int_0^1 \left|b_x\left(t,x(t)+\theta\xi^\epsilon(t),u^\epsilon(t),\alpha(t-)\right)-b_x(t)\right|^4 d\theta\right\}^{\frac{1}{2}} \epsilon^2 dt \\
&\le K\int_0^T \left\{E\left(\xi^\epsilon(t)^4+\bar{\omega}(\epsilon v(t))^4\right)\right\}^{\frac{1}{2}}\epsilon^2 dt \\
&\le K\int_0^T \left\{\epsilon^4+E[\rho+K_\rho\epsilon|v(t)|]^4\right\}^{\frac{1}{2}} dt\epsilon^2.
\end{aligned}
$$

Hence the first term in (2.54) is $o(\epsilon^2)$. Similarly the second and third terms are also $o(\epsilon^2)$, which gives (2.50).

Proof of (2.51). By definition of the cost functional (2.2), we have

$$
\begin{aligned}
&J(u^\epsilon)-J(u) \\
&=E\left\{h(x^\epsilon(T),\alpha(T))-h(x(T),\alpha(T))\right\} \\
&+E\int_0^T \left\{f(t,x^\epsilon(t),u^\epsilon(t),\alpha(t))-f(t,x(t),u(t),\alpha(t))\right\}dt
\end{aligned}
$$

For the first term on the right side of (3.22) we have

$$
\begin{aligned}
&E\left\{h(x^\epsilon(T),\alpha(T))-h(x(T),\alpha(T))\right\} \\
&=E\int_0^1 \langle h_x(x(T)+\theta\xi^\epsilon(T),\alpha(T)),\xi^\epsilon(T)\rangle d\theta \\
&=E\langle h_x(x(T),\alpha(T)),y^\epsilon(T)\rangle+E\langle h_x(x(T),\alpha(T)),\zeta^\epsilon(T)\rangle \\
&+E\int_0^1 \langle h_x(x(T)+\theta\xi^\epsilon(T),\alpha(T))-h_x(x(T),\alpha(T)),\xi^\epsilon(T)\rangle d\theta.
\end{aligned}
$$

Then, by (2.48), (2.50), **(A4)** and applying Hölder's inequality, we have

$$
E\left\{h(x^\epsilon(T),\alpha(T))-h(x(T),\alpha(T))\right\}=E\langle h_x(x(T),\alpha(T)),y^\epsilon(T)\rangle+o(\epsilon). \quad (2.56)
$$

For the second term on the right side of (3.22) we have

$$
\begin{aligned}
&E\int_0^T \{f(t,x^\epsilon(t),u^\epsilon(t),\alpha(t)) - f(t,x(t),u(t),\alpha(t))\}\,dt\\
=&E\int_0^T \left\{\int_0^1 \langle f_x(t,x(t)+\theta\xi^\epsilon(t),u^\epsilon(t),\alpha(t)),\xi^\epsilon(t)\rangle d\theta\right\}\\
&\quad+\{f(t,x(t),u^\epsilon(t),\alpha(t)) - f(t,x(t),u(t),\alpha(t))\}\,dt\\
=&E\int_0^T \{\langle f_x(t),y^\epsilon(t)\rangle + \delta f(t)\}\\
&\quad+\left\{\int_0^1 \langle f_x(t,x(t)+\theta\xi^\epsilon(t),u^\epsilon(t),\alpha(t)) - f_x(t),y^\epsilon(t)\rangle d\theta\right\}\\
&\quad+\left\{\int_0^1 \langle f_x(t,x(t)+\theta\xi^\epsilon(t),u^\epsilon(t),\alpha(t)),\zeta^\epsilon(t)\rangle d\theta\right\}dt
\end{aligned}
$$

Then, using **(A4)** and by a similar argument as in the proof of (2.50), we have

$$
\begin{aligned}
&E\int_0^T \{f(t,x^\epsilon(t),u^\epsilon(t),\alpha(t)) - f(t,x(t),u(t),\alpha(t))\}\,dt\\
=&\ E\int_0^T \{\langle f_x(t),y^\epsilon(t)\rangle + \delta f(t)\} + o(\epsilon).
\end{aligned}
\tag{2.57}
$$

(2.51) follows from (2.56) and (2.57). □

2.5.6 Duality Analysis

Lemma 2.5.21. *Let assumptions **(A1)-(A4)** hold. Let y^ϵ be the solution of* (2.47) *and* (p,q,s) *be the adapted solution of* (2.5). *Then*

$$
E\langle p(T),y^\epsilon(T)\rangle = E\int_0^T \left\{\ \langle p(t-),\delta b(t)\rangle + \langle f_x(t),y^\epsilon(t)\rangle + tr\,(q(t)^\intercal \delta\sigma(t))\ \right\}dt
\tag{2.58}
$$

Proof. Applying Ito's lemma and taking expectation immediately lead to (2.58). □

Now we are able to give the following lemma, which is of great importance.

Lemma 2.5.22. *Let assumptions **(A1)-(A4)** hold. For any $\varepsilon > 0$ and $v \in L^4(S;\mathbb{R}^K)$, define*

$$u^\epsilon(t) \triangleq u(t) + \epsilon v(t) \text{ for } \forall t \in [0,T].$$

Then we have

$$J(u^\epsilon) - J(u)$$
$$= E\int_0^T (-H(t,x(t),u^\epsilon(t),\alpha(t-),p(t-),q(t))) - (-H(t,x(t),u(t),\alpha(t-),p(t-),q(t)))dt + o(\epsilon)$$

Proof. According to Lemma 2.5.20, we have

$$\begin{aligned} &J(u^\epsilon) - J(u) \\ &= E\langle h_x(x(T),\alpha(T)), y^\epsilon(T)\rangle + E\int_0^T \{\langle f_x(t), y^\epsilon(t)\rangle + \delta f(t)\}\,dt + o(\epsilon) \\ &= E\langle -p(T), y^\epsilon(T)\rangle + E\int_0^T \{\langle f_x(t), y^\epsilon(t)\rangle + \delta f(t)\}\,dt + o(\epsilon). \end{aligned}$$

Applying (2.58), we obtain

$$J(u^\epsilon) - J(u) = E\int_0^T -\Big\{\langle p(t-), \delta b(t)\rangle + tr\,(q(t)^\intercal \delta\sigma(t)) - \delta f(t)\Big\}dt + o(\epsilon)$$
$$= E\int_0^T (-H(t,x(t),u^\epsilon(t),\alpha(t-),p(t-),q(t))) - (-H(t,x(t),u(t),\alpha(t-),p(t-),q(t)))dt + o(\epsilon)$$

□

2.6 Proof of the Main Theorems

2.6.1 Proof of Theorem 2.3.1

We follow the technique developed in [15]. Given an optimal 5-tuple $(\bar{x},\bar{u},\bar{p},\bar{q},\bar{s})$, define a functional $\mathcal{H}^{\bar{u}} : L^4(S;\mathbb{R}^k) \to \mathbb{R}$ as following

$$\mathcal{H}^{\bar{u}}(u) = E\int_0^T -H(t,\bar{x}(t),u(t),\alpha(t-),\bar{p}(t),\bar{q}(t))dt.$$

By a similar argument as in Lemma 2.5.18, it can be proved that the functional $\mathcal{H}^{\bar{u}}$ is also locally Lipschitz on $L^4(S;\mathbb{R}^k)$. Next, we define Clarke's generalized gradient of the functionals J and $\mathcal{H}^{\bar{u}}$ at $\bar{u}$ and explore their properties.

Definition 2.6.1. *(This is the same definition as in the previous chapter but applied to J) Let* $L^{\frac{4}{3}}(S;\mathbb{R}^k)$ *denote the dual space of* $L^4(S;\mathbb{R}^k)$ *and* $\langle\cdot,\cdot\rangle$ *denote the duality pairing between* $L^4(S;\mathbb{R}^k)$ *and* $L^{\frac{4}{3}}(S;\mathbb{R}^k)$*. Given an admissible control* $\bar{u}\in L^4(S;\mathbb{R}^k)$*, Clarke's generalized gradient of* J *at* $\bar{u}$*, denoted by* $\partial J(\bar{u})$*, is the set of all* $\zeta\in L^{\frac{4}{3}}(S;\mathbb{R}^k)$ *satisfying*

$$J^o(\bar{u};v)=\limsup_{u\to\bar{u},\epsilon\to 0}\frac{J(u+\epsilon v)-J(u)}{\epsilon}\geq\langle v,\zeta\rangle, \tag{2.59}$$

for all $v\in L^4(S;\mathbb{R}^k)$*. Clarke's generalized gradient of* $\mathcal{H}^{\bar{u}}$ *at* $\bar{u}$ *is defined similarly.*

Then, according to Lemma 2.5.22, given $u\in L^4(S;\mathbb{R}^k)$, for any $\epsilon>0$ and $v\in L^4(S;\mathbb{R}^k)$ such that $u+\epsilon v\in L^4(S;\mathbb{R}^k)$, we have

$$J(u+\epsilon v)-J(u)=\mathcal{H}^{\bar{u}}(u+\epsilon v)-\mathcal{H}^{\bar{u}}(u)+o(\epsilon).$$

Hence, we have

$$J^o(\bar{u};v)=(\mathcal{H}^{\bar{u}})^o(\bar{u};v),\ \textit{for}\ \forall v\in L^4(S;\mathbb{R}^k).$$

Therefore, by Definition 2.6.1, we conclude

$$\partial J(\bar{u})=\partial\mathcal{H}^{\bar{u}}(\bar{u}).$$

Since $\bar{u}$ is an optimal control on $\mathcal{U}_{ad}$, according to Theorem 2.5.10,

$$0\in\partial J(\bar{u})+N_{\mathcal{U}_{ad}}(\bar{u})=\partial\mathcal{H}^{\bar{u}}(\bar{u})+N_{\mathcal{U}_{ad}}(\bar{u}). \tag{2.60}$$

To characterize Clarke's tangent cone in the $L^4(S;\mathbb{R}^k)$ space, we recall [1, Theorem 8.5.1]. Let (Ω,S,μ) be a complete σ-finite measure space and X be a separable Banach space. Consider a measurable set-valued map $K:\Omega\rightsquigarrow X$. We associate with it the subset $\mathcal{K}\subset L^p(\Omega;X,\mu)$ of selections defined by

$$\mathcal{K}:=\{x\in L^p(\Omega;X,\mu)|\ \textit{for almost all}\ \omega\in\Omega, x(\omega)\in K(\omega)\}.$$

Theorem 2.6.2. *Assume that the set-valued map K is measurable and has closed images. Then for every $x \in \mathcal{K}$, the set valued map $\omega \to T^b_{K(\omega)}(x(\omega))$ is measurable. Furthermore*

$$\{v \in L^p(\Omega; X, \mu) | \textit{ for almost all } \omega, v(\omega) \in T^b_{K(\omega)}(x(\omega))\} \subset T^b_{\mathcal{K}}(x).$$

Returning to our proof, since U is convex, by definition, $\mathcal{U}_{ad}$ is also a convex subset of $L^4(S;\mathbb{R}^k)$. Therefore, by Theorem 2.5.8 and Theorem 2.6.2, we obtain

$$T_{\mathcal{U}_{ad}}(\bar{u}) \supset \{v \in L^4(S;\mathbb{R}^k) | v(\omega,t) \in T_U(\bar{u}(\omega,t)) \ \mu\textit{-almost surely}\}. \tag{2.61}$$

The optimality condition (2.60) together with (2.61) implies that $\exists \zeta \in L^{\frac{4}{3}}(S;\mathbb{R}^k)$ such that

$$\begin{cases} E\displaystyle\int_0^T \langle \zeta(t), v(t)\rangle dt \leq 0 \textit{ for } \forall v \in L^4(S;\mathbb{R}^k) \textit{ such that} \\ v(t) \in T_U(\bar{u}(t)) \ \ \textit{for every } t \in [0,T], \mathbb{P}\textit{-almost surely} \\ (\mathcal{H}^{\bar{u}})^o(\bar{u}; v) + E\displaystyle\int_0^T \langle \zeta(t), v(t)\rangle dt \geq 0 \textit{ for } \forall v \in L^4(S;\mathbb{R}^k). \end{cases} \tag{2.62}$$

Now, we recall a version of the measurable selection theorem in [1].

Definition 2.6.3. *[1, Definition 8.1.2] Let $(\Omega, \mathcal{A})$ be a measurable space and X be a complete separable metric space. Consider a set-valued map $F : \Omega \rightsquigarrow X$. A measurable map $f : \Omega \mapsto X$ satisfying*

$$\forall \omega \in \Omega, f(\omega) \in F(\omega)$$

is called a measurable selection of F.

Theorem 2.6.4. *[1, Theorem 8.1.3] Let X be a complete separable metric space, $(\Omega, \mathcal{A})$ a measurable space, F a measurable set-valued map from Ω to closed non-empty subsets of X. Then there exists a measurable selection of F.*

Return to our problem. Fix $u \in \mathcal{U}_{ad}$ and $(\omega, t) \in S$. Let $\mathbb{Q}_+$ denote the set of all strictly positive rationals. Following the argument in [1, Page 325] , we have

$$T_U(u(\omega,t)) = T_U^b(u(\omega,t)) = \bigcap_{n>0} cl\left(\bigcup_{\alpha\in\mathbb{Q}_+}\bigcap_{h\in[0,\alpha]\cap\mathbb{Q}_+}\frac{U-u(\omega,t)}{h}+\frac{1}{n}B\right),$$

where B denotes the unit ball centred at 0. By [1, Theorem 8.2.4], we conclude that the set-valued function $T_U(\bar{u})$ is measurable.

For the first inequality in (2.62), let $M > 0$ and define $\bar{B}_M \triangleq \{v \in \mathbb{R}^k : \|v\| \leq M\}$. For any positive integer n, define a set-valued function Π_n^M as follows

$$\Pi_n^M(\omega,t) = \begin{cases} \{0\}, & \textit{if } \langle \zeta(\omega,t), v\rangle < \frac{1}{n},\ \forall v \in \bar{B}_M \cap T_U(\bar{u}(\omega,t)) \\ \{v \in \bar{B}_M \cap T_U(\bar{u}(\omega,t)) : \langle \zeta(\omega,t), v\rangle \geq \frac{1}{n}\}, & \textit{otherwise.} \end{cases}$$

The map $(\omega, t, v) \to \langle \zeta(\omega, t), v\rangle$ is continuous in v. Moreover, since $\mathbb{R}^k$ is separable, the map can be expressed as the upper limit of a countable family of measurable functions and therefore is measurable. Therefore Π_n^M is measurable since countable intersection of measurable set-valued functions is still measurable. Hence, by Theorem 2.6.4, Π_n^M admits a measurable selection $v_n^M \in L^4(S; \mathbb{R}^k)$. Note that (2.62) implies that the set

$$\{(\omega,t) : \Pi_n^M(\omega,t) \neq \{0\}\}$$

must have μ measure 0. Hence, we conclude that there exists a set, denoted as S_n^M, where

$$S_n^M = \{(\omega,t) : \Pi_n^M(\omega,t) = \{0\}\}$$

and $\mu(S_n^M) = 1$. Consequently, we have

$$\langle \zeta(\omega,t), v\rangle < \frac{1}{n} \quad \forall v \in \bar{B}_M \cap T_U(\bar{u}(\omega,t)) \ \textit{on } S_n^M. \tag{2.63}$$

Define $S^M = \bigcap_{n=1}^{\infty} S_n^M$ with $\mu(S^M) = 1$ since $\mu(S_n^M) = 1\ \forall n \in \mathbb{N}$. Moreover, since (2.63) holds for all n, we have

$$\langle \zeta(\omega,t), v\rangle \leq 0 \ \forall v \in \bar{B}_M \cap T_U(\bar{u}(\omega,t)) \ \textit{on } S^M. \tag{2.64}$$

Since (2.64) holds for arbitrary M, we obtain that

$$\langle \zeta(\omega,t), v\rangle \le 0 \text{ for } \forall v \in T_U(\bar{u}(\omega,t)) \ \mu\text{-almost surely.} \tag{2.65}$$

Next, we consider the second inequality in (2.62). Define the partial generalized directional derivative of the Hamiltonian H at $\bar{u}(t)$ in the direction $v(t)$ as

$$\begin{aligned}-H_u^o(\bar{u}(t); v(t)) = \limsup_{u\to\bar{u},\epsilon\to 0} \frac{1}{\epsilon}\Big\{ &- H(t,\bar{x}(t), u(t)+\epsilon v(t), \alpha(t-), \bar{p}(t), \bar{q}(t), s(t)) \\ &+ H(t,\bar{x}(t), u(t), \alpha(t-), \bar{p}(t), \bar{q}(t), s(t))\Big\}.\end{aligned}$$

Using Fatou's Lemma on the second inequality in (2.62), we have

$$E\left\{\int_0^T -H_u^o(\bar{u}(t); v(t)) + \langle \zeta(t), v(t)\rangle dt\right\} \ge (\mathcal{H}^{\bar{u}})^o(\bar{u}; v) + E\int_0^T \langle \zeta(t), v(t)\rangle dt \ge 0 \tag{2.66}$$

Let $M > 0$ and define $\bar{B}_M \triangleq \{v \in \mathbb{R}^k : \|v\| \le M\}$. For any $n \in \mathbb{N}$, define a set-valued function Γ_n^M as follows

$$\Gamma_n^M(\omega,t) = \begin{cases} \{0\}, \quad \text{if } -H_u^o(\bar{u}(\omega,t); v) + \langle \zeta(\omega,t), v\rangle > -\frac{1}{n} \ \forall v \in \bar{B}_M \\ \{v \in \bar{B}_M : -H_u^o(\bar{u}(\omega,t); v) + \langle \zeta(\omega,t), v\rangle \le -\frac{1}{n}\}, \ \textit{otherwise.} \end{cases}$$

Using a similar argument as above, with the help of Theorem 2.6.4 and (2.66), we can show that the set $\{(\omega,t) : \Gamma_n^M(\omega,t) \ne \{0\}\}$ must have μ measure 0, which implies that

$$-H_u^o(\bar{u}(\omega,t); v)) + \langle \zeta(\omega,t), v\rangle \ge 0 \ \mu\text{-almost surely.} \tag{2.67}$$

Combining (2.65) and (2.67), we conclude

$$0 \in \partial_u(-H)(t,\bar{x}(t),\bar{u}(t),\alpha(t-),\bar{p}(t),\bar{q}(t)) + N_U(\bar{u}(t)), \ a.e. t \in [0,T], \ \mathbb{P}\text{-}a.s.$$

2.6.2 Proof of Theorem 2.3.2

Given admissible pair (x, u), define

$$H(t, x(t), u(t)) \triangleq H(t, x(t), u(t), \alpha(t-), \bar{p}(t), \bar{q}(t)) \textit{ for } \forall t \in [0, T], \ \mathbb{P}\text{-}a.s.$$

Under the convexity condition, Clarke's generalized gradient and normal cone coincide with the subdifferential and normal cone in the sense of convex analysis. Moreover, combining (2.6) and the concavity of $H(t, \bar{x}(t), \cdot)$ for all $t \in [0, T]$ a.s, we conclude that

$$H(t, \bar{x}(t), \bar{u}(t)) = \max_{u \in U} H(t, \bar{x}(t), u), \ \ a.e.\ \ t \in [0, T], \ \mathbb{P}\text{-}a.s.$$

Define $\xi(t) \triangleq x(t) - \bar{x}(t)$ satisfying

$$\begin{cases} d\xi(t) & = \{b(t, x(t), u(t), \alpha(t-)) - b(t, \bar{x}(t), \bar{u}(t), \alpha(t-))\}\, dt \\ & \quad + \sum\limits_{j=1}^{m} \{\sigma^j(t, x(t), u(t), \alpha(t-)) - \sigma^j(t, \bar{x}(t), \bar{u}(t), \alpha(t-))\}\, dW^j(t), \ \ t \in [0, T], \\ \xi(0) & = 0, \alpha(0) = i_0. \end{cases}$$

Following a standard separating hyperplane argument in convex analysis (see [72, Chapter 5]), we obtain

$$\int_0^T \{H(t, x(t), u(t)) - H(t, \bar{x}(t), \bar{u}(t))\} \le \int_0^T \langle H_x(t, \bar{x}(t), \bar{u}(t)), \xi(t)\rangle dt \quad (2.68)$$

for any admissible pair (x, u). Detailed proof of (2.68) can be found in [32].

Applying Ito's formula to $\langle \bar{p}(t), \xi(t)\rangle$, noting the convexity of h, the inequality

(2.68) and the definition of the Hamilitonian (2.4), we have

$$
\begin{aligned}
&E\{h(x(T),\alpha(T)) - h(\bar{x}(T),\alpha(T))\}\\
\geq &E\langle h_x(\bar{x}(T),\alpha(T)),\xi(T)\rangle)\\
= &-E\langle \bar{p}(T),\xi(T)\rangle\\
= &E\int_0^T \Big\{\langle H_x(t,\bar{x}(t),\bar{u}(t)),\xi(t)\rangle\\
&-\langle \bar{p}(t), b(t,x(t),u(t),\alpha(t-)) - b(t,\bar{x}(t),\bar{u}(t),\alpha(t-))\rangle\\
&-\sum_{j=1}^{m}\langle \bar{q}^j(t),\sigma^j(t,x(t),u(t),\alpha(t-)) - \sigma^j(t,\bar{x}(t),\bar{u}(t),\alpha(t-))\rangle\Big\}dt\\
\geq &-E\int_0^T \{f(t,x(t),u(t),\alpha(t-)) - f(t,\bar{x}(t),\bar{u}(t),\alpha(t-))\}dt.
\end{aligned}
$$

Therefore $J(\bar{u}) \leq J(u)$ for all $u \in \mathcal{U}_{ad}$.

2.7 Conclusion

We have proved in the chapter a weak version of the necessary and sufficient stochastic maximum principle in a regime-switching diffusion model. Instead of insisting on the maximum condition of the Hamiltonian, we showed that 0 belongs to the sum of Clarke's generalized gradient of $-H$ and Clarke's normal cone at the optimal control $\bar{u}$, which also removes the requirement of the differentiability of the functions in the control variable. Under certain concavity conditions on the Hamiltonian, the necessary condition becomes sufficient. The theorem does not involve any second order terms, hence the second order differentiability of the functions in the state variable is not required. Moreover, the absence of the second order adjoint equation considerably simplifies the SMP. Further research on this topic includes the extension of the weak SMP to more general stochastic control systems such as non-convex control constraints and locally Lipschitz coefficients. We are currently working on these problems.

Chapter 3

Dynamic Convex Duality in Constrained Quadratic Risk Minimization

3.1 Introduction

In this chapter we study a stochastic control problem arising from mathematical finance. The goal is to minimize a convex cost function that is quadratic in both the wealth process and portfolio strategy in a continuous time complete market with random market parameters and portfolio constraints. Problems of this kind arise naturally in financial applications. We assume that the portfolio must take value in a given closed convex set which is general enough to model short selling, borrowing, and other trading restrictions, see Karatzas-Shreve [47].

There are vast literatures on stochastic linear quadratic (SLQ) optimal control, see Yong-Zhou [89] and references therein. Majority of the existing results are for unconstrained problems. Using the stochastic maximum principle, one can solve the SLQ problem by deriving the optimal control as a linear feedback control of the state and proving the existence and uniqueness of a solution to the resulting stochastic Riccati equation (SRE). When there are no control constraints, the feedback control constructed from the solution of the SRE is automatically admissible, see Zhou-Lim [94] for an example of this method to problems with random coefficients but no portfolio constraints. When there are control constraints, the optimal control is no longer a simple linear feedback control of the state and the SRE method becomes much more difficult and subtle. Hu-Zhou [41] shows the solvability of an extended SRE for constrained SLQ problems with random coefficients.

For convex SLQ problems, it is also natural to use the convex duality method that has been extensively applied to solve utility maximization problems in mathematical finance, see Kramkov-Schachermayer [48, 49] and reference therein. When there are no control constraints and the filtration is generated by driving Brownian motions, one may first convert the original dynamic optimization problem into an equivalent static one, then formulate and solve the static dual problem, and use the dual relation and the martingale property to find the optimal state process for the original problem, finally use the martingale representation theorem to find a replicating portfolio which is the optimal control process. When there are control constraints, the duality method becomes much more complicated. Karatzas-Shreve [47] introduces and solves a family of auxiliary unconstrained problems and shows one of them solves the original constrained problem. Labbé-Heunis [54] applies the convex duality approach, inspired by [9, 73], to solve a mean-variance problem with both random coefficients and portfolio constraints and shows the existence of an optimal solution to the dual problem and constructs the optimal wealth process with the optimal dual solution and the optimal portfolio process with the martingale representation theorem.

Øksendal-Sulem [66] extends the results of [49] to a dynamic setting and proves a close relation between optimal solutions and adjoint processes obtained from forward backward stochastic differential equations (FBSDEs). Specifically, they show that the optimal primal wealth and portfolio processes can be expressed as functions of the optimal adjoint processes of the dual problem and vice versa. This demystifies the opaque relation of the optimal solutions of the primal and dual problems in utility maximization, i.e., given the solution of the dual problem, the optimal control of the primal problem can only be derived from the martingale representation theorem. There are no control constraints in [66] but the asset price process is a general semi-martingale process with some technical conditions.

Inspired by the work of [66], we use the convex duality method to solve the quadratic risk minimization problem with both random coefficients and control constraints. To get a correct formulation of the dual problem, we follow the approach of [54] by first converting the original problem into a static problem in an abstract space, then applying convex analysis to derive its dual problem, and finally getting a specific dual stochastic control problem. It turns out there are three controls in the dual problem, one corresponds to the control constraint set, one to the running cost function, and one to the no-duality-gap relation. Using FBSDEs, we derive the necessary and sufficient conditions for both primal and dual problems, which allows us to explicitly characterise the primal control as a function of the adjoint process coming from the dual FBSDEs in a dynamic fashion and vice versa, similar to those in [66]. Moreover, we also find that the optimal primal wealth process coincides with the optimal adjoint process of the dual problem and vice versa. To the best of our knowledge, this is the first time the dynamic relations of primal and dual problems with control constraints have been explicitly characterized in the literature.

After establishing the optimality conditions for both primal and dual problems, we solve a quadratic risk minimization problem with cone-constraints.

Instead of attacking the primal problem directly, we start from the dual problem and then construct the optimal solution to the primal problem from that of the dual problem. Moreover, we derive the explicit representations of solutions to the extended SREs introduced in [41] in terms of the optimal solutions from the dual problem. The simplicity in solving the dual problem is in good contrast to the technical complexity in solving the extended SREs directly, as discussed in [41]. In addition, we show that when the coefficients are deterministic, the closed form optimal solution to the dual problem can be constructed.

The rest of the chapter is organised as follows. In Section 2 we set up the model and formulate the primal and dual problems following the approach in [54]. In Section 3 we characterise the necessary and sufficient optimality conditions for both the primal and dual problems and establish their connection in a dynamic fashion through FBSDEs. In Section 4 we discuss quadratic risk minimization problems with cone constraints and demonstrate how to construct explicitly the solutions of the extended SREs from those of the dual FBSDEs. In Section 5 we prove the main results. Section 6 concludes.

3.2 Market Model and Primal and Dual Problems

Through out the chapter, we denote by $T > 0$ a fixed terminal time, $\{W(t), t \in [0, T]\}$ a $\mathbb{R}^N$-valued standard Brownian motion with scalar entries $W_m(t)$, $m = 1, \cdots, N$, on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\{\mathcal{F}_t\}$ the $\mathbb{P}$-augmentation of the filtration $\mathcal{F}_t^W = \sigma(W(s), 0 \leq s \leq t)$ generated by W, $\mathcal{P}(0, T; \mathbb{R}^N)$ the set of all $\mathbb{R}^N$-valued progressively measurable processes on $[0, T] \times \Omega$, $\mathcal{H}^2(0, T; \mathbb{R}^N)$ the set of processes x in $\mathcal{P}(0, T; \mathbb{R}^N)$ satisfying $E[\int_0^T |x(t)|^2 dt] < \infty$, and $\mathcal{S}^2(0, T; \mathbb{R}^N)$ the set of processes x in $\mathcal{P}(0, T; \mathbb{R}^N)$ satisfying $E[\sup_{0 \leq t \leq T} |x_t^2|] < \infty$. We write

SDE for stochastic differential equation, BSDE for backward SDE, and FBSDE for forward and backward SDE. We also follow the customary convention that ω is suppressed in SDEs and integrals, except in places where an explicit ω is needed.

Consider a market consisting of a bank account with price $\{S_0(t)\}$ given by

$$dS_0(t) = r(t)S_0(t)dt,\ 0 \le t \le T,\ S_0(0) = 1, \tag{3.1}$$

and N stocks with prices $\{S_n(t)\},\ n = 1, \cdots, N$, given by

$$dS_n(t) = S_n(t)\left[b_n(t)dt + \sum_{m=1}^{N} \sigma_{nm}(t)dW_m(t)\right],\ 0 \le t \le T,\ S_n(0) > 0. \tag{3.2}$$

We assume that $r \in \mathcal{P}(0,T;\mathbb{R})$ (scalar interest rate), $b \in \mathcal{P}(0,T;\mathbb{R}^N)$ (vector of appreciation rates), and $\sigma \in \mathcal{P}(0,T;\mathbb{R}^{N\times N})$ (volatility matrix) are uniformly bounded. We also assume that there exists a positive constant k such that

$$z^\intercal \sigma(t)\sigma^\intercal(t) z \ge k|z|^2$$

for all $(z,\omega,t) \in \mathbb{R}^N \times \Omega \times [0,T]$, where $z^\intercal$ is the transpose of z. Consequently, by [47, Problem 5.8.1, page 372], $\exists k_1 \in (0,\infty)\ s.t\ \max\{\|\sigma^{-1}(t,\omega)z\|, \|[\sigma^\intercal]^{-1}(\omega,t)z\|\}$ $\le k_1\|z\|,\ \forall (z,\omega,t) \in \mathbb{R}^{\mathbb{N}} \times \Omega \times [0,T]$.

Consider a small investor with initial wealth $x_0 > 0$ and a self-financing strategy. Define the set of admissible portfolio strategies by

$$\mathcal{A} := \left\{\pi \in \mathcal{H}^2(0,T;\mathbb{R}^N) : \pi(t) \in K\ for\ t \in [0,T]\ a.e.\right\},$$

where $K \subseteq \mathbb{R}^N$ is a closed convex set and π is a portfolio process with each entry $\pi_n(t)$ defined as the amount invested in the stock n for $n = 1, \ldots, N$. Given any $\pi \in \mathcal{A}$, the investor's total wealth X^π satisfies the SDE

$$\begin{cases} dX^\pi(t) = [r(t)X^\pi(t) + \pi^\intercal(t)\sigma(t)\theta(t)]dt + \pi^\intercal(t)\sigma(t)dW(t), \quad 0 \le t \le T, \\ X^\pi(0) = x_0, \end{cases} \tag{3.3}$$

where $\theta(t) := \sigma^{-1}(t)\,[b(t) - r(t)\mathbf{1}]$ is the market price of risk at time t and is uniformly bounded and $\mathbf{1} \in \mathbb{R}^N$ has all unit entries. A pair (X, π) is *admissible* if $\pi \in \mathcal{A}$ and X is a strong solution to the SDE (3.3) with control process π.

Define a functional $J : \mathcal{A} \to \mathbb{R}$ by

$$J(\pi) := E\left[\int_0^T f(t, X^\pi(t), \pi(t))dt + g(X^\pi(T))\right],$$

where $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ are defined by

$$\begin{cases} f(\omega, t, x, \pi) := \dfrac{1}{2}\left[Q(t)x^2 + 2S^\intercal(t)x\pi + \pi^\intercal R(t)\pi\right], \\ g(\omega, x) := \dfrac{1}{2}\left[ax^2 + 2cx\right]. \end{cases} \tag{3.4}$$

We assume that random variable $a \in L^\infty_{\mathcal{F}_T}(\mathbb{R})$ satisfy

$$\sup_{\omega\in\Omega} a(\omega) < \infty$$

and processes $Q \in \mathcal{P}(0,T;\mathbb{R}), S \in \mathcal{P}(0,T;\mathbb{R}^N), R \in \mathcal{P}(0,T;\mathbb{R}^{N\times N})$ are uniformly bounded, $R(t)$ is a symmetric matrix, and the matrix

$$\begin{pmatrix} Q(t) & S^\intercal(t) \\ S(t) & R(t) \end{pmatrix}$$

is nonnegative definite for all $(\omega, t) \in \Omega \times [0,T]$. Under these assumptions we know J is a convex functional of π.

We consider the following optimization problem:

$$\text{Minimize } J(\pi) \text{ subject to } (X, \pi) \text{ admissible.} \tag{3.5}$$

An admissible control $\hat{\pi}$ is *optimal* if $J(\hat{\pi}) \leq J(\pi)$ for all $\pi \in \mathcal{A}$.

Following the approach introduced in [54], we now set up the dual problem. Denote by

$$\mathbb{B} := \mathbb{R} \times \mathcal{H}^2(0,T;\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R}^N).$$

We write $X \in \mathbb{B}$ if and only if

$$X(t) = x_0 + \int_0^t \dot{X}(\tau)d\tau + \int_0^t \Lambda_X^\intercal(\tau)dW(\tau),\ 0 \leq t \leq T,$$

for some $(x_0, \dot{X}, \Lambda_X) \in \mathbb{B}$. We now reformulate (3.5) as a primal optimization problem over the whole set $\mathbb{B}$. For each $X \equiv (x_0, \dot{X}, \Lambda_X) \in \mathbb{B}$, define

$$\begin{aligned}\mathcal{U}(X) := \{\pi \in \mathcal{A} \ \textit{such that}\ \dot{X}(t) = r(t)X(t) + \pi^\intercal(t)\sigma(t)\theta(t) \\ \textit{and}\ \Lambda_X(t) = \sigma^\intercal(t)\pi(t)\ \textit{for}\ \forall t \in [0,T],\ \mathbb{P} - a.e.\}.\end{aligned}$$

The set $\mathcal{U}(X)$ contains all admissible controls $\pi \in \mathcal{A}$ that make X an admissible wealth process. Note that $\mathcal{U}(X) \neq \varnothing$ if and only if $(\dot{X}(t), \Lambda_X(t)) \in \mathcal{S}(X(t))$ for $(\mathbb{P} \otimes \mathit{Leb})$-a.e. $(\omega, t) \in \Omega \times [0,T]$, where $\mathcal{S}$ is a set valued function defined by

$$\mathcal{S}(\omega, t, x) := \{(v, \xi) : v = r(t)x + \xi^\intercal\theta(t)\ \textit{and}\ \left[\sigma^\intercal\right]^{-1}(t)\xi \in K\}.$$

Define the penalty function $L : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty]$ by

$$L(\omega, t, x, v, \xi) = f\left(\omega, t, x, [\sigma^\intercal]^{-1}(t)\xi\right) + \Psi_{\mathcal{S}(\omega,t,x)}(v, \xi)$$

and the penalty function $l_0 : \mathbb{R} \to [0, \infty]$ by

$$l_0(x) = \Psi_{\{x_0\}}(x),$$

where $\Psi_U(u)$ is a penalty function which equals 0 if u is in set U and $+\infty$ otherwise.

For $X \in \mathbb{B}$, define the cost functional as

$$\Phi(X) := l_0(x_0) + E\left[g(X(T))\right] + E\left[\int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t))dt\right].$$

Note that $\Phi(X) = \infty$ if $X(0) \neq x_0$ or $\mathcal{U}(X) = \varnothing$. Problem (3.5) is equivalent to

Minimize $\Phi(X)$ subject to $X \in \mathbb{B}$.

We now establish the dual problem over the set $\mathbb{B}$. Define the following convex conjugate functions

$$
\begin{aligned}
m_0(y) &:= \sup_{x\in\mathbb{R}}\{xy - l_0(x)\},\\
m_T(\omega, y) &:= \sup_{x\in\mathbb{R}}\{-xy - g(\omega, x)\},\\
M(\omega, t, y, s, \gamma) &:= \sup_{x,v\in\mathbb{R},\xi\in\mathbb{R}^N}\{xs + vy + \xi^{\intercal}\gamma - L(\omega, t, x, v, \xi)\},
\end{aligned}
$$

for all $(\omega, t, y, s, \gamma) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N$. For each $Y \equiv (y, \dot{Y}, \Lambda_Y) \in \mathbb{B}$, define

$$
\Psi(Y) := m_0(y) + E\left[m_T(Y(T))\right] + E\left[\int_0^T M(t, Y(t), \dot{Y}(t), \Lambda_Y(t))dt\right].
$$

Then the dual problem is given by

$$
\text{Minimize } \Psi(Y) \text{ subject to } Y \in \mathbb{B}.
$$

We can write the dual problem equivalently as a stochastic control problem. Some simple calculus gives

$$
\begin{aligned}
m_0(y) &= x_0 y,\\
m_T(\omega, y) &= \frac{(y+c)^2}{2a},\\
M(\omega, t, y, s, \gamma) &= \phi(t, s + r(t)y, \sigma(t)\left[\theta(t)y + \gamma\right]), \qquad (3.6)
\end{aligned}
$$

where ϕ is the conjugate function of $\tilde{f}(\omega, t, x, \pi) = f(\omega, t, x, \pi) + \Psi_K(\pi)$, namely,

$$
\phi(\omega, t, \alpha, \beta) := \sup_{x\in\mathbb{R},\pi\in K}\{x\alpha + \pi^{\intercal}\beta - f(\omega, t, x, \pi)\}.
$$

The dual control problem is therefore given by

$$
\text{Minimize } m_0(y) + E\left[m_T(Y(T))\right] + E\left[\int_0^T \phi(t, \alpha(t), \beta(t))dt\right], \qquad (3.7)
$$

where Y satisfies

$$
\begin{cases}
dY(t) = \left[\alpha(t) - r(t)Y(t)\right]dt + \left[\sigma^{-1}(t)\beta(t) - \theta(t)Y(t)\right]^{\intercal} dW(t)\\
Y(0) = y.
\end{cases} \qquad (3.8)
$$

The dual control process for Y is $(y, \alpha, \beta) \in \mathbb{B}$. From [51, Corollary 2.5.10], we have $Y^{(y,\alpha,\beta)} \in \mathcal{S}^2(0, T; \mathbb{R})$. Note that the control constraint is implicit for the dual problem. For example, if $Q = 0, S = 0, R = 0$, then α must be zero and may be simply dropped in (3.7) and (3.8).

3.3 Main Results

In this section, we derive the necessary and sufficient optimality conditions for primal and dual problems and show the connection between the optimal solutions through their corresponding FBSDEs. To highlight the main results and streamline the discussion, we leave the proofs of all the theorems in Section 5.

Given any admissible control $\pi \in \mathcal{A}$ and solution X^π to the SDE (3.3), the associated adjoint equation in unknown processes $p_1 \in \mathcal{S}^2(0, T; \mathbb{R})$ and $q_1 \in \mathcal{H}^2(0, t; \mathbb{R}^N)$ is the following linear BSDE

$$\begin{cases} dp_1(t) = [-r(t)p_1(t) + Q(t)X(t) + S^\intercal(t)\pi(t)]\, dt + q_1^\intercal(t)dW(t) \\ p_1(T) = -aX^\pi(T) - c. \end{cases} \tag{3.9}$$

From [69, Theorem 6.2.1] we know that there exists a unique solution (p_1, q_1) to the BSDE (3.9). We now state the necessary and sufficient conditions for the primal problem.

Theorem 3.3.1. *(Primal problem and associated FBSDE) Let $\hat{\pi} \in \mathcal{A}$. Then $\hat{\pi}$ is optimal for the primal problem if and only if the solution $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ of FBSDE*

$$\begin{cases} dX^{\hat{\pi}}(t) = \left[r(t)X^{\hat{\pi}}(t) + \hat{\pi}^\intercal(t)\sigma(t)\theta(t)\right] dt + \hat{\pi}^\intercal(t)\sigma(t)dW(t) \\ X^{\hat{\pi}}(0) = x_0 \\ d\hat{p}_1(t) = \left[-r(t)\hat{p}_1(t) + Q(t)X^{\hat{\pi}}(t) + S^\intercal(t)\hat{\pi}(t)\right] dt + \hat{q}_1^\intercal(t)dW(t) \\ \hat{p}_1(T) = -aX^{\hat{\pi}}(T) - c \end{cases} \tag{3.10}$$

satisfies the condition

$$[\hat{\pi}^\intercal - \pi^\intercal]\left[\hat{p}_1(t)\sigma(t)\theta(t) + \sigma(t)\hat{q}_1(t) - S(t)X^{\hat{\pi}}(t) - R(t)\hat{\pi}(t)\right] \geq 0 \tag{3.11}$$

for $(\mathbb{P}\otimes Leb)$*-a.e.* $(\omega, t) \in \Omega \times [0,T]$ *and* $\pi \in K$*.*

Remark 3.3.2. *If* $K = \mathbb{R}^N$*, then condition (3.11) becomes*

$$\hat{p}_1(t)\sigma(t)\theta(t) + \sigma(t)\hat{q}_1(t) - S(t)X^{\hat{\pi}}(t) - R(t)\hat{\pi}(t) = 0.$$

If we further assume $R(t)$ *is positive definite and* $R(t)^{-1}$ *is uniformly bounded, then we can substitute the optimal control* $\hat{\pi}(t)$ *into the FBSDE (3.10) to get a fully-coupled linear FBSDE with random coefficients, see [88] for discussions on the solvability of linear FBSDEs.*

Given any admissible control $(y, \alpha, \beta) \in \mathbb{B}$ and solution $Y^{(y,\alpha,\beta)}$ to the SDE (3.8), the associated adjoint equation in unknown processes $p_2 \in \mathcal{S}^2(0,T;\mathbb{R})$ and $q_2 \in \mathcal{H}^2(0,t;\mathbb{R}^N)$ is the following linear BSDE

$$\begin{cases} dp_2(t) = [r(t)p_2(t) + q_2^\intercal(t)\theta(t)]\,dt + q_2^\intercal(t)dW(t) \\ p_2(T) = -\dfrac{Y^{(y,\alpha,\beta)}(T) + c}{a}. \end{cases} \tag{3.12}$$

From [69, Theorem 6.2.1], we know that there exists a unique solution (p_2, q_2) to the BSDE (4.19). To derive the necessary condition, we need to impose the following assumption on ϕ at the optimal dual control process $(\hat{\alpha}, \hat{\beta})$.

Assumption 3.3.3. *Let* $(\hat{\alpha}, \hat{\beta})$ *be given and* α, β *be any admissible control. Then there exists a* $Z \in \mathcal{P}(0,T;\mathbb{R})$ *satisfying* $E[\int_0^T |Z(t)|dt] < \infty$ *and*

$$Z(t) \geq \frac{\phi(t, \hat{\alpha}(t) + \varepsilon\alpha(t), \hat{\beta}(t) + \varepsilon\beta(t)) - \phi(t, \hat{\alpha}(t), \hat{\beta}(t))}{\varepsilon} \tag{3.13}$$

for $(\mathbb{P}\otimes Leb)$*-a.e.* $(\omega, t) \in \Omega \times [0,T]$ *and* $\varepsilon \in (0,1]$*.*

Remark 3.3.4. *Here are a few comments on Assumption 3.3.3.*

1. *Condition (3.13) is a technical condition that ensures one can apply the monotone convergence theorem and pass the limit under the expectation and integral as $\varepsilon \downarrow 0$, which is used in proving the second and third relations in (3.15), see the proof of Theorem 3.3.5 in Section 3.5. A similar assumption is used in [13, Assumption 1.2] on the data of the primal problem.*

2. *It is difficult to replace this assumption by one on the parameters. I cannot not find counter-examples but this does not justify the assumption itself. In the unconstrained case it is possible to derive more characteristics of the dual function from primal function (such as Lipschitz). However, this is more difficult when it comes to constrained case. I have searched classical books such as [72], but did not find any result that applies to our case*

3. *If $K = \mathbb{R}^N$, $S(t) = 0$ and $Q(t), R(t)$ are positive definite and their inverses are uniformly bounded, then $\phi(t, \alpha, \beta) = \frac{1}{2}Q(t)^{-1}\alpha^2 + \frac{1}{2}\beta^\intercal R(t)^{-1}\beta$. Condition (3.13) holds if Z is chosen to be*

$$Z(t) := Q(t)^{-1}\hat{\alpha}(t)\alpha(t) + \hat{\beta}^\intercal(t)R(t)^{-1}\beta(t) + \frac{1}{2}Q(t)^{-1}\alpha(t)^2 + \frac{1}{2}\beta^\intercal(t)R(t)^{-1}\beta(t).$$

4. *If $Q(t) = 0, S(t) = 0, R(t) = 0$, then $\alpha(t) = 0$ for the dual problem. We may drop α in the expression of ϕ which becomes a support function of K, i.e., ϕ is given by $\phi(t, \beta) = \delta(\beta) := \sup_{\pi \in K} \pi^\intercal \beta$. If we further assume that K is a bounded set, then condition (3.13) holds if Z is chosen to be $Z(t) = \delta(\beta(t))$. However, if K is unbounded, then Assumption 3.3.3 may not hold and we cannot use the monotone convergence theorem to prove (3.15). Other methods may have to be used, see Remark 3.4.1 for further discussions.*

We now state the necessary and sufficient conditions for the dual problem.

Theorem 3.3.5. *(Dual problem and associated FBSDE) Let $(\hat{y}, \hat{\alpha}, \hat{\beta}) \in \mathbb{B}$ satisfy Assumption 3.3.3. Then $(\hat{y}, \hat{\alpha}, \hat{\beta})$ is optimal for the dual problem if and only if* *the solution* $(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}, \hat{p}_2, \hat{q}_2)$ *of FBSDE*

$$\begin{cases} dY^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = \left[\hat{\alpha}(t) - r(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right] dt + \left[\sigma^{-1}(t)\hat{\beta}(t) - \theta(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t)\right]^{\intercal} dW(t) \\ Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(0) = \hat{y} \\ d\hat{p}_2(t) = [r(t)\hat{p}_2(t) + \hat{q}_2^{\intercal}(t)\theta(t)]\, dt + \hat{q}_2^{\intercal}(t)dW(t) \\ \hat{p}_2(T) = -\dfrac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T) + c}{a} \end{cases} \tag{3.14}$$

satisfies the conditions

$$\begin{cases} \hat{p}_2(0) = x_0, \\ \left[\sigma^{\intercal}\right]^{-1}(t)\hat{q}_2(t) \in K, \\ \left(\hat{p}_2(t), \left[\sigma^{\intercal}\right]^{-1}(t)\hat{q}_2(t)\right) \in \partial\phi(\hat{\alpha}(t), \hat{\beta}(t)), \end{cases} \tag{3.15}$$

for $(\mathbb{P} \otimes Leb)$*-a.e.* $(\omega, t) \in \Omega \times [0, T]$.

Remark 3.3.6. *If* $K = \mathbb{R}^N$, $S(t) = 0$ *and* $Q(t), R(t)$ *are positive definite, then from Remark 3.3.4,* ϕ *is a quadratic function of* α *and* β *and we can write optimal controls* $\hat{\alpha}$ *and* $\hat{\beta}$ *in terms of adjoint processes* $\hat{p}_2$ *and* $\hat{q}_2$*. The FBSDE (3.14) becomes a fully coupled linear FBSDE with an additional condition* $\hat{p}_2(0) = x_0$*, which is used to determine the constant control* $\hat{y}$*.*

We can now state the dynamic relations of the optimal portfolio and wealth processes of the primal problem and the adjoint processes of the dual problem and vice versa.

Theorem 3.3.7. *(From dual problem to primal problem) Suppose that* $(\hat{y}, \hat{\alpha}, \hat{\beta})$ *is optimal for the dual problem. Let* $\left(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}, \hat{p}_2, \hat{q}_2\right)$ *be the associated process that satisfies the FBSDE* (3.14) *and condition* (4.26)*. Define*

$$\hat{\pi}(t) := \left[\sigma^{\intercal}\right]^{-1}(t)\hat{q}_2(t),\ \ t \in [0, T]. \tag{3.16}$$

Then $\hat{\pi}$ is the optimal control for the primal problem with initial wealth x_0. The optimal wealth process and associated adjoint processes are given by

$$\begin{cases} X^{\hat{\pi}}(t) = \hat{p}_2(t), \\ \hat{p}_1(t) = Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t), \\ \hat{q}_1(t) = \sigma^{-1}(t)\hat{\beta}(t) - \theta(t)Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) \text{ for } \forall t \in [0,T]. \end{cases} \tag{3.17}$$

Theorem 3.3.8. *(From primal problem to dual problem) Suppose that $\hat{\pi} \in \mathcal{A}$ is optimal for the primal problem with initial wealth x_0. Let $\left(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1\right)$ be the associated process that satisfies the FBSDE* (3.10) *and condition* (3.11). *Define*

$$\begin{cases} \hat{y} = \hat{p}_1(0), \\ \hat{\alpha}(t) = Q(t)X^{\hat{\pi}}(t) + S^{\intercal}(t)\hat{\pi}(t), \\ \hat{\beta}(t) = \sigma(t)\left[\hat{q}_1(t) + \theta(t)\hat{p}_1(t)\right]. \end{cases} \tag{3.18}$$

Then $(\hat{y}, \hat{\alpha}, \hat{\beta})$ is the optimal control for the dual problem. The optimal dual state process and associated adjoint processes are given by

$$\begin{cases} Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) = \hat{p}_1(t), \\ \hat{p}_2(t) = X^{\hat{\pi}}(t), \\ \hat{q}_2(t) = \sigma^{\intercal}(t)\hat{\pi}(t). \end{cases} \tag{3.19}$$

Example 3.3.9. *Assume $Q(t) = 0, S(t) = 0, R(t) = 0$ for $t \in [0,T]$ and $c = 0$ and $K = \mathbb{R}^N$. Then we must have $\alpha(t) = 0, \beta(t) = 0$ for $t \in [0,T]$ a.e. (otherwise $\phi(t, \alpha(t), \beta(t)) = \infty$) and Assumption 3.3.3 is not needed. The solution to the SDE (3.8) is given by $Y(t) = y\Gamma(t)$ for $t \in [0,T]$ a.e., where Γ satisfies the linear SDE*

$$d\Gamma(t) = \Gamma(t)[-r(t)dt - \theta^{\intercal}(t)dW(t)], \ \Gamma(0) = 1.$$

The minimum point $\hat{y}$ for dual problem (3.7) is given by

$$\hat{y} = -\frac{x_0}{E\left[\dfrac{\Gamma(T)^2}{a}\right]}.$$

Let $(\hat{p}_2, \hat{q}_2)$ be the adjoint process associated with the optimal control $(\hat{y}, 0, 0)$, satisfying the BSDE

$$d\hat{p}_2(t) = [r(t)\hat{p}_2(t) + \theta^\intercal(t)\hat{q}_2(t)]dt + \hat{q}_2^\intercal(t)dW(t),\ \hat{p}_2(T) = -\frac{\hat{Y}(T)}{a}.$$

Hence we obtain that

$$\hat{p}_2(t) = \Gamma(t)^{-1}E\left[\Gamma(T)\hat{p}_2(T)\middle|\mathcal{F}_t\right] = -\hat{y}\Gamma(t)^{-1}E\left[\frac{1}{a}\Gamma(T)^2\middle|\mathcal{F}_t\right].$$

By Theorem 3.3.7, we conclude that the optimal wealth and portfolio processes are given by

$$X^{\hat{\pi}}(t) = \hat{p}_2(t),\ \hat{\pi}(t) := [\sigma^\intercal]^{-1}(t)\hat{q}_2(t),\ \forall t \in [0, T].$$

If we define $\hat{X}(t) := X^{\hat{\pi}}(t)\exp(-\int_0^t r(s)ds)$, then $\bar{X}$ satisfies

$$d\bar{X}(t) = e^{-\int_0^t r(s)ds}\hat{q}_2^\intercal(t)(\theta(t)dt + dW(t)),$$

which shows that $\hat{q}_2$ is the martingale representation for the discounted optimal wealth process $\bar{X}$ under the equivalent probability measure Q defined by $dQ/dP = \varepsilon(-\int_0^T \theta(t)dW(t))$.

One can also solve the primal problem directly using the stochastic linear quadratic control theory and stochastic Riccati equation (SRE), see [94]. Since $\hat{p}_2(t) \neq 0$ for $t \in [0, T]$ a.e. if $x_0 \neq 0$, we may define a process P by

$$P(t) := -\frac{\hat{Y}(t)}{\hat{p}_2(t)},\ \forall t \in [0, T].$$

Applying Ito's formula to P, we obtain

$$dP(t) = -2r(t)P(t)dt + \frac{1}{\hat{p}_2(t)^2}P(t)\hat{q}_2^\intercal(t)\hat{q}_2(t)dt - P(t)\left(\theta(t) + \frac{1}{\hat{p}_2(t)}\hat{q}_2(t)\right)^\intercal dW(t).$$

Define a process

$$\Lambda(t) := \frac{\hat{q}_2(t)\hat{Y}(t)}{\hat{p}_2(t)^2} + \frac{\theta(t)\hat{Y}(t)}{\hat{p}_2(t)}.$$

Substituting Λ *into the above equation and rearranging, we have*

$$dP(t) = \left[-2r(t)P(t) + 2\theta^\intercal(t)\Lambda(t) + \theta^\intercal(t)\theta(t)P(t) + \frac{\Lambda^\intercal(t)\Lambda(t)}{P(t)}\right] dt + \Lambda^\intercal(t)dW(t),$$

which is the SRE introduced in [94]. Using the duality approach, we obtain an explicit representation of the unique solution to the SRE.

3.4 Quadratic Risk Minimization with Cone Constraints

In this section we consider the following quadratic risk minimization problem:

$$\begin{cases} \textit{Minimize } J(\pi(\cdot)) = E\left[\frac{1}{2}aX(T)^2\right], \\ \textit{Subject to } \ (X(\cdot), \pi(\cdot)) \ \textit{ is admissible.} \end{cases} \tag{3.20}$$

Assume $K \subset \mathbb{R}^N$ is a closed convex cone. The dual problem is given by

$$\textit{Minimize } x_0 y + E\left[\frac{Y(T)^2}{2a}\right] + E\left[\int_0^T \delta(\beta(t))dt\right] \tag{3.21}$$

over $(y, \beta) \in \mathbb{R} \times \mathcal{H}^2(0, T; \mathbb{R}^N)$, where Y satisfies the SDE (3.8) with $\alpha(t) = 0$ and $\delta(\beta) = \sup_{\pi \in K} \pi^\intercal \beta$, the support function of K. [54, Proposition 5.4] states that there exists an optimal control $(\hat{y}, \hat{\beta})$ to (3.21) with associated optimal state process $\hat{Y}$.

Remark 3.4.1. *Since* K *is unbounded, Assumption 3.3.3 may not hold. Using the subadditivity and positive homogeneity of* δ*, we have (see (3.46))*

$$E\left[\int_0^T \left[\delta(\beta(t)) - \hat{q}_2^\intercal(t)\sigma^{-1}(t)\beta(t)\right] dt\right] \geq 0. \tag{3.22}$$

Let $B := \{(\omega, t) \in \Omega \times [0, T] : [\sigma^\intercal]^{-1}(t)\hat{q}_2(t) \in K\}$. *By [47, Lemma 5.4.2], there exists* $\nu \in \mathcal{P}(0, T; \mathbb{R}^N)$ *such that* $|\nu(t)| \leq 1$ *and* $|\delta(\nu(t))| \leq 1$ *and*

$$[\sigma^\intercal]^{-1}(t)\hat{q}_2(t) \in K \ \Leftrightarrow \ \nu(t) = 0,$$

$$[\sigma^\intercal]^{-1}(t)\hat{q}_2(t) \notin K \ \Leftrightarrow \ \delta(\nu(t)) - \hat{q}_2^\intercal(t)\sigma^{-1}(t)\nu(t) < 0$$

for $(\mathbb{P}\otimes Leb)$*-a.e.* $(\omega,t)\in\Omega\times[0,T]$*. The existence of* ν *ensures that the complement set of* B *has measure zero on* $\Omega\times[0,T]$ *(otherwise there is a contradiction to (3.22)). Hence we conclude* $[\sigma^\intercal]^{-1}(t)\hat{q}_2(t)\in K$ *for* $(\mathbb{P}\otimes Leb)$*-a.e. The third relation in (3.15) can also be proved directly.*

3.4.1 Random coefficient case

We have the following result.

Lemma 3.4.2. *Let* $(\hat{y},\hat{\beta})$ *be the optimal control of the dual problem* (3.21) *and* $\hat{Y}$ *be the corresponding optimal state process. Then* $\hat{\beta}(t)=0$ *if* $\hat{Y}(t)=0$ *for* $(\mathbb{P}\otimes Leb)$*-a.e.* $(\omega,t)\in\Omega\times[0,T]$*.*

Proof. Applying Ito's formula to $\hat{Y}(t)^2$, we get

$$
\begin{aligned}
d\hat{Y}(t)^2 &= \Big[-2r(t)\hat{Y}(t)^2+\Big(\sigma^{-1}(t)\hat{\beta}(t)-\theta(t)\hat{Y}(t)\Big)^\intercal\Big(\sigma^{-1}(t)\hat{\beta}(t)-\theta(t)\hat{Y}(t)\Big)\Big]\,dt\\
&\quad+2\hat{Y}(t)\Big[\sigma^{-1}(t)\hat{\beta}(t)-\theta(t)\hat{Y}(t)\Big]^\intercal dW(t).
\end{aligned}
$$

Define the process

$$
\tilde{S}(t):=\int_0^t 2\hat{Y}(s)[\sigma^{-1}(s)\hat{\beta}(s)-\theta(s)\hat{Y}(s)]^\intercal dW(s).
$$

Following a similar argument as in the proof of Theorem 3.3.1, we know $\tilde{S}$ is a martingale. Taking expectation of $\dfrac{\hat{Y}(T)^2}{2a}$, we have

$$
E\left[\frac{\hat{Y}(T)^2}{2a}\right]:=
$$

$$
E\left[\frac{\hat{y}}{2a}\right]+E\left\{\int_0^T\left[-\frac{r(t)\hat{Y}(t)^2}{a}+\frac{\Big(\sigma^{-1}(t)\hat{\beta}(t)-\theta(t)\hat{Y}(t)\Big)^\intercal\Big(\sigma^{-1}(t)\hat{\beta}(t)-\theta(t)\hat{Y}(t)\Big)}{2a}\right]dt\right\}.
$$

Define the set

$$
\Pi:=\Big\{(\omega,t)\in\Omega\times[0,T]:\hat{Y}(t)=0,\hat{\beta}(t)\neq 0\Big\}.
$$

We must have $(\mathbb{P}\otimes Leb)(\Pi)=0$, otherwise, there is a contradiction. □

Let $\hat{\beta}(t) = \hat{\gamma}(t)\hat{Y}(t)$ for $t \in [0,T]$. Then $\hat{Y}$ follows the SDE

$$\begin{cases} d\hat{Y}(t) = -r(t)\hat{Y}(t)dt + [\sigma^{-1}(t)\hat{\gamma}(t) - \theta(t)]^\intercal \hat{Y}(t)dW(t) \\ \hat{Y}(0) = \hat{y}. \end{cases}$$

Hence, we have $\hat{Y}(t) = \hat{y}\hat{H}(t)$, where

$$\hat{H}(t) := \exp\Bigg(\int_0^t \left[-r(s) - \frac{1}{2}\left(\sigma^{-1}(s)\hat{\gamma}(s) - \theta(s)\right)^\intercal \left(\sigma^{-1}(s)\hat{\gamma}(s) - \theta(s)\right)\right] ds \\ + \left[\sigma^{-1}(s)\hat{\gamma}(s) - \theta(s)\right]^\intercal dW(s) \Bigg).$$

By Theorem 3.3.7, we obtain

$$\hat{p}_2(0) = E\left[\Gamma(T)\hat{p}_2(T)\right] = E\left[-\Gamma(T)\frac{\hat{Y}(T)}{a}\right] = -\hat{y}E\left[\Gamma(T)\frac{\hat{H}(T)}{a}\right] = x_0,$$

which implies

$$\hat{y} = -\frac{x_0}{E\left[\dfrac{\Gamma(T)\hat{H}(T)}{a}\right]}.$$

Moreover, we have

$$\hat{p}_2(t) = \Gamma(t)^{-1}E\left[-\Gamma(T)\frac{\hat{Y}(T)}{a}\bigg|\mathcal{F}_t\right] = -\hat{y}\Gamma(t)^{-1}E\left[\Gamma(T)\frac{\hat{H}(T)}{a}\bigg|\mathcal{F}_t\right],$$

which shows that $\hat{p}_2(t) \neq 0$ $\mathbb{P}$-a.e. for $t \in [0,T]$.

Suppose $x_0 > 0$, then $\hat{Y}(t) < 0$ and $\hat{p}_2(t) > 0$ for $\forall t \in [0,T]$, $\mathbb{P}$-a.e. Define

$$P_+(t) := -\frac{\hat{Y}(t)}{\hat{p}_2(t)} = -\frac{\hat{p}_1(t)}{\hat{X}(t)}, \ \forall t \in [0,T].$$

Applying Ito's formula, we have

$$\begin{aligned} dP_+(t) &= \left[-2r(t)P_+(t) - P_+(t)\frac{\hat{\pi}^\intercal(t)}{\hat{X}(t)}\sigma(t)\theta(t) + \frac{\pi^\intercal(t)\sigma(t)\hat{q}_1(t)}{\hat{X}(t)^2} + \frac{P_+(t)\pi^\intercal(t)\sigma(t)\sigma^\intercal(t)\pi(t)}{\hat{X}(t)^2}\right] dt \\ &\quad + \left[-\frac{\hat{q}_1(t)}{\hat{X}(t)} - P_+(t)\sigma^\intercal(t)\frac{\pi(t)}{\hat{X}(t)}\right]^\intercal dW(t), \\ &= \left[-2r(t)P_+(t) - \hat{\xi}_+^\intercal(t)\left(\sigma(t)\theta(t)P_+(t) + \sigma(t)\Lambda_+(t)\right)\right] dt + \Lambda_+^\intercal(t)dW(t), \end{aligned} \tag{3.23}$$

where

$$\Lambda_+(t) := -\frac{\hat{q}_1(t)}{\hat{X}(t)} - \frac{P_+(t)\sigma^\intercal(t)\pi(t)}{\hat{X}(t)},\ \hat{\xi}_+(t) := \frac{\hat{\pi}(t)}{\hat{X}(t)}.$$

Define

$$\begin{aligned} H_+(t, v, P, \Lambda) :=& v^\intercal P\sigma(t)\sigma^\intercal(t)v + 2v^\intercal\left[\sigma(t)\theta(t)P + \sigma(t)\Lambda\right], \\ H_+^*(t, P, \Lambda) :=& \inf_{v\in K} H_+(t, v, P, \Lambda). \end{aligned}$$

We have

$$\begin{aligned} \partial_v H_+(t, \hat{\xi}_+(t), P_+(t), \Lambda_+(t)) =& 2\left[P_+(t)\sigma(t)\sigma^\intercal(t)\frac{\hat{\pi}(t)}{\hat{X}(t)} + \sigma(t)\theta(t)P_+(t) + \sigma(t)\Lambda_+(t)\right] \\ =& 2\left[-\sigma(t)\frac{\hat{q}_1(t)}{\hat{X}(t)} - \sigma(t)\theta(t)\frac{\hat{p}_1(t)}{\hat{X}(t)}\right]. \end{aligned}$$

Recall that by Theorem 3.3.1, we have

$$[\hat{\pi}(t) - \pi]^\intercal[\hat{p}_1(t)\sigma(t)\theta(t) + \sigma(t)\hat{q}_1(t)] \geq 0 \tag{3.24}$$

for $(\mathbb{P}\otimes \mathit{Leb})$-a.e. $(\omega, t) \in \Omega \times [0, T]$ and $\pi \in K$. According to Theorem 3.3.8, $\hat{X}(t) = \hat{p}_2(t) > 0$. Dividing both sides of (3.24) by $\hat{X}(t)^2$, we obtain that

$$[\hat{\xi}_+(t) - \xi]^\intercal \partial_v H_+(t, \hat{\xi}_+(t), P_+(t), \Lambda_+(t)) \leq 0$$

for $(\mathbb{P}\otimes \mathit{Leb})$-a.e. $(\omega, t) \in \Omega \times [0, T]$ and $\xi \in K$. By [29, Proposition 2.2.1], we conclude that

$$H_+^*(t, P_+(t), \Lambda_+(t)) = H_+(t, \hat{\xi}_+(t), P_+(t), \Lambda_+(t))\ \forall t \in [0, T],\ \mathbb{P} - a.e. \tag{3.25}$$

Moreover, by [17, Page 52, Corollary], we have

$$0 \in P_+(t)\sigma(t)\sigma^\intercal(t)\hat{\xi}_+(t) + \sigma(t)[\theta(t)P_+(t) + \Lambda_+(t)] + N_K(\hat{\xi}_+(t)),\ \forall t \in [0, T]\ \mathbb{P}\text{-}a.e.$$

where $N_K(x) := \{p \in \mathbb{R}^N : p^\intercal(x^* - x) \leq 0, \forall x^* \in K\}$, the normal cone of K at $x \in K$. For all $p \in N_K(x)$, since K is a cone, by choosing $x^* = 2x$ and $x^* = \frac{1}{2}x$, we have $p^\intercal x \leq 0$ and $-\frac{1}{2}p^\intercal x \leq 0$, which gives $p^\intercal x = 0$. Therefore

$$\hat{\xi}_+^\intercal(t)P_+(t)\sigma(t)\sigma^\intercal(t)\hat{\xi}_+(t) + \hat{\xi}_+^\intercal(t)\sigma(t)[\theta(t)P_+(t) + \Lambda_+(t)] = 0. \tag{3.26}$$

Substituting (3.26) into (3.25), we obtain

$$H_+^*(t, P_+(t), \Lambda_+(t)) = \hat{\xi}_+^\intercal(t) \left[\sigma(t)\theta(t)P_+(t) + \sigma(t)\Lambda_+(t)\right] \ \forall t \in [0, T]. \tag{3.27}$$

Substituting (3.27) back into (3.23), we have that P_+ is the solution to the following nonlinear BSDE

$$\begin{cases} dP_+(t) = -\left[2r(t)P_+(t) + H_+^*(t, P_+(t), \Lambda_+(t))\right] dt + \Lambda_+^\intercal(t) dW(t), \\ P_+(T) = a, \\ P_+(t) > 0. \ \forall t \in [0, T]. \end{cases} \tag{3.28}$$

Similarly, if $x_0 < 0$, then $\hat{Y}(t) > 0$ and $\hat{p}_2(t) < 0$ for $t \in [0, T]$, $\mathbb{P}$-a.e. Define

$$P_-(t) := -\frac{\hat{Y}(t)}{\hat{p}_2(t)} = -\frac{\hat{p}_1(t)}{\hat{X}(t)}, \ \forall t \in [0, T].$$

Using a similar approach, it can be shown that P_- is the solution to the following nonlinear BSDE

$$\begin{cases} dP_-(t) = -\left[2r(t)P_-(t) + H_-^*(t, P_-(t), \Lambda_-(t))\right] dt + \Lambda_-^\intercal(t) dW(t), \\ P_-(T) = a, \\ P_-(t) > 0, \ \forall t \in [0, T]. \end{cases} \tag{3.29}$$

where

$$\begin{aligned} H_-(t, v, P, \Lambda) :=& v^\intercal P \sigma(t)\sigma^\intercal(t) v - 2v^\intercal \left[\sigma(t)\theta(t)P + \sigma(t)\Lambda\right], \\ H_-^*(t, P, \Lambda) :=& \inf_{v \in K} H_-(t, v, P, \Lambda). \end{aligned}$$

We find that (3.28) and (3.29) are the extended SRE introduced in [41]. Through the dual approach, we have obtained an explicit representation of the unique solution to the SREs in terms of the optimal state and adjoint processes. Finally, according to Theorem 3.3.7 we conclude that the optimal solution to the primal problem is given by

$$\begin{cases} \hat{\pi}^\intercal(t) = [\sigma^\intercal]^{-1}(t)\hat{q}_2(t), \\ \hat{X}(t) = \hat{p}_2(t) = -\hat{Y}(t)\left[\dfrac{1_{\{x_0>0\}}}{P_+(t)} + \dfrac{1_{\{x_0<0\}}}{P_-(t)}\right]. \end{cases}$$

3.4.2 Deterministic coefficient case

Assume $K \subset \mathbb{R}^N$ is a closed convex cone and r, b, σ are deterministic functions and $a > 0$ is a constant. In this case, the dual problem can be written as

$$\text{Minimize } x_0 y + E\left[\frac{Y(T)^2}{2a}\right]$$

over $(y, \beta) \in \mathbb{R} \times \mathcal{H}^2(0, T; \mathbb{R}^N)$ and Y satisfies the SDE (3.8) with $\alpha(t) = 0$ and $\beta(t) \in K^0$ for $t \in [0, T]$ a.e., where $K^0 := \{\beta : \beta^\intercal \pi \le 0, \forall \pi \in K\}$, the polar cone of K. We solve the above problem in two steps: first, fix y and find the optimal control $\hat{\beta}(y)$; second, find the optimal $\hat{y}$. We can then construct the optimal solution explicitly.

Step 1: Consider the associated HJB equation:

$$\begin{cases} v_t(s,y) - r(s) y v_y(s,y) + \frac{1}{2}\inf_{\beta \in K^0} |\sigma^{-1}(s)\beta - \theta(s) y|^2 v_{yy}(s,y) = 0, \\ v(T,y) = y^2, \end{cases} \tag{3.30}$$

for each $(s, y) \in [t, T] \times \mathbb{R}$. The infimum term in (3.30) can be written explicitly as

1. If $y = 0$, then it is trivial to obtain that

$$\inf_{\beta \in K^0} |\sigma^{-1}(s)\beta - \theta(s)y|^2 = \inf_{\beta \in K^0} |\sigma^{-1}(s)\beta|^2 = 0.$$

2. If $y > 0$, then we have

$$\begin{aligned} \inf_{\beta \in K^0} |\sigma^{-1}(s)\beta - \theta(s)y|^2 &= y^2 \inf_{\beta \in K^0} \left|\sigma^{-1}(s)\left(\frac{\beta}{y}\right) - \theta(s)\right|^2 \\ &= y^2 \inf_{y\bar{\beta} \in K^0} \left|\sigma^{-1}(s)\bar{\beta} - \theta(s)\right|^2 \\ &= y^2 |\sigma^{-1}(s)\beta_+(s) - \theta(s)|^2, \end{aligned}$$

where $\beta_+(s) := \arg\min_{\beta \in K^0} |\sigma^{-1}(s)\beta - \theta(s)|^2$.

3. If $y < 0$, then similarly we have

$$\begin{aligned}\inf_{\beta \in K^0} |\sigma^{-1}(s)\beta - \theta(s)y|^2 =& y^2 \inf_{\beta \in K^0} \left|\sigma^{-1}(s)\frac{\beta}{y} - \theta(s)\right|^2 \\ =& y^2 \inf_{\bar{\beta} \in K^0} \left|\sigma^{-1}(s)\bar{\beta} + \theta(s)\right|^2 \\ =& y^2|\sigma^{-1}(s)\beta_-(s) + \theta(s)|^2,\end{aligned}$$

where $\beta_-(s) := \arg\min_{\beta \in K^0} |\sigma^{-1}(s)\beta + \theta(s)|^2$.

Define

$$\sigma(s,y) := \begin{cases} \sigma^{-1}(s)\beta_+(s) - \theta(s), & \textit{if } y > 0 \\ \sigma^{-1}(s)\beta_-(s) + \theta(s), & \textit{if } y < 0 \\ 0, & \textit{if } y = 0. \end{cases}$$

The HJB equation (3.30) becomes

$$\begin{cases} v_t(s,y) - r(s)yv_y(s,y) + \frac{1}{2}y^2|\sigma(s,y)|^2 v_{yy}(s,y) = 0, \\ v(T,y) = y^2. \end{cases}$$

According to the Feynman-Kac formula, we have

$$v(t,y) = E\left[Y^2(T)|Y(t) = y\right] = y^2 e^{\int_t^T [-2r(s) + |\sigma(s,Y(s))|^2]ds},$$

where the stochastic process Y follows the following geometric Brownian motion

$$dY(s) = -r(s)Y(s)ds + \sigma^\intercal(s,Y(s))Y(s)dW(s),\ Y(t) = y.$$

Moreover, since Y follows a geometric Brownian motion and $\text{sign}(Y(s)) = \text{sign}(y),\ \forall s \in [t,T]$, we have

$$\sigma(s,Y(s)) = \sigma(s,y),\ \forall s \in [t,T].$$

In particular, we have

$$v(0,y) = y^2 e^{\int_0^T [-2r(s) + |\sigma(s,y)|^2]ds}. \tag{3.31}$$

Step 2: Consider the following static optimization problem:

$$\inf_{y\in\mathbb{R}} x_0 y + \frac{1}{2a} v(0,y) \tag{3.32}$$

Substituting (3.31) into the objective function, we obtain that problem (3.32) achieves minimum at

$$\hat{y} = -a x_0 e^{\int_0^T [2r(s) - |\sigma(s,-x_0)|^2] ds}.$$

Hence, we conclude that the optimal control is given by

$$\hat{\beta}(t) = \begin{cases} a x_0 e^{\int_t^T [2r(s)-|\sigma(s,-x_0)|^2] ds} \beta_-(t), & \textit{if } x_0 > 0 \\ -a x_0 e^{\int_t^T [2r(s)-|\sigma(s,-x_0)|^2] ds} \beta_+(t), & \textit{if } x_0 < 0 \\ 0, & \textit{if } x_0 = 0. \end{cases}$$

Moreover, in this case we can construct the solution to the SREs (3.28) and (3.29) explicitly as

$$\hat{P}_+(t) = \hat{P}_-(t) = a e^{\int_t^T [2r(s) + \sigma^\intercal(s,-x_0)\theta(s)] ds}. \tag{3.33}$$

Next, we verify that (3.33) are indeed solutions to the SREs (3.28) and (3.29) with $\Lambda_+(t) = 0$ and $\Lambda_-(t) = 0$, respectively. To this end, we consider the case $x_0 > 0$ and $y < 0$. According to Theorem 3.3.7, we have

$$\hat{X}(t) = \hat{p}_2(t), \forall t \in [0,T], a.e.$$

Hence,

$$\hat{X}(t) = E\left[-\frac{\Gamma(T)Y(T)}{a\Gamma(t)} \middle| \mathcal{F}_t \right] = -\frac{Y(t)}{a} E\left[\frac{\Gamma(T)Y(T)}{\Gamma(t)Y(t)} \middle| \mathcal{F}_t \right], \tag{3.34}$$

where Γ follows the SDE

$$d\Gamma(t) = \Gamma(t)[-r(t)dt - \theta^\intercal(t) dW(t)], \forall t \in [0,T], \Gamma(0) = 1.$$

Applying Ito's lemma, we obtain

$$d\Gamma(t)Y(t) = [-2r(t) - \theta^{\intercal}(t)\sigma(t,y)]Y(t)\Gamma(t)dt - [\sigma^{\intercal}(t,y) + \theta^{\intercal}(t)]Y(t)\Gamma(t)dW(t). \tag{3.35}$$

Combining (3.34) and (3.35), we have

$$\hat{X}(t) = -\frac{Y(t)}{a}e^{\int_t^T [-2r(s)-\theta^{\intercal}(s)\sigma(s,y)]ds}.$$

Applying Ito's lemma again, we have $\hat{X}$ satisfies the SDE

$$d\hat{X}(t) = [r(t)\hat{X}(t) + \theta^{\intercal}(t)\sigma(t,y)\hat{X}(t)]dt + \sigma^{\intercal}(t,y)\hat{X}(t)dW(t). \tag{3.36}$$

Comparing (3.36) with (3.3), we conclude that

$$\hat{\pi}^{\intercal}(t) = \sigma^{\intercal}(t,y)\sigma^{-1}(t)\hat{X}(t),$$

which implies that

$$\hat{\xi}_+^{\intercal}(t) = \frac{\hat{\pi}^{\intercal}(t)}{\hat{X}(t)} = \sigma^{\intercal}(t,y)\sigma^{-1}(t). \tag{3.37}$$

Substituting (3.37) back into (3.27), we have

$$H^*(t, P_+(t), \Lambda_+(t)) = \sigma^{\intercal}(t,y)\theta(t)P_+(t).$$

Taking $x_0 < 0$ and following the same steps, we obtain

$$H^*(t, P_-(t), \Lambda_-(t)) = \sigma^{\intercal}(t,y)\theta(t)P_-(t).$$

Hence, we conclude that $\hat{P}_+(t)$ and $\hat{P}_-(t)$ defined in (3.33) are indeed solutions to SREs (3.28) and (3.29).

Remark 3.4.3. *The example was covered by [54] but the focus was different. The focus of [54] was to apply the duality approach to solve the risk minimisation problem, whereas I would like to demonstrate the dynamic relationship between the primal FBSDE and dual FBSDE. Moreover, from the relationship I was able to recover the explicit construction of SRE that was introduced in [94] and [41]. This is beyond the scope of [54].*

3.5 Proofs of the Main Results

In this section we give proofs of the main results in Section 3.3.

Proof of Theorem 3.3.1. Since the cost functional J is convex, according to [29, Proposition 2.2.1], a necessary and sufficient condition for $\hat{\pi}$ to be optimal is that

$$\langle J'(\hat{\pi}), \hat{\pi} - \pi \rangle \leq 0, \ \forall \pi \in \mathcal{A}, \tag{3.38}$$

where $J'(\hat{\pi})$ is the Gâteaux-derivative of J at $\hat{\pi}$ and can be computed explicitly as (3.3) is a linear SDE and J is a quadratic functional. The optimality condition (3.38) can be written as

$$\begin{aligned} E\Bigg[\int_0^T \Big[&Q(t)X^{\hat{\pi}}(t)\left(X^{\hat{\pi}}(t) - X^{\pi}(t)\right) + S^{\intercal}(t)\left(\hat{\pi}(t)\left(X^{\hat{\pi}}(t) - X^{\pi}(t)\right) + \left(\hat{\pi}(t) - \pi(t)\right)X^{\hat{\pi}}(t)\right) \\ &+ \left(\hat{\pi}^{\intercal}(t) - \pi^{\intercal}(t)\right)R(t)\hat{\pi}(t)\Big]dt + \left[aX^{\hat{\pi}}(T) + c\right]\left(X^{\hat{\pi}}(T) - X^{\pi}(T)\right)\Bigg] \leq 0, \end{aligned} \tag{3.39}$$

for all $\pi \in \mathcal{A}$. Applying Ito's formula to $X^{\hat{\pi}}(t)\hat{p}_1(t)$, we have

$$\begin{aligned} d(X^{\hat{\pi}}(t)\hat{p}_1(t)) = &\left[\hat{p}_1(t)\hat{\pi}^{\intercal}(t)\sigma(t)\theta(t) + \hat{\pi}^{\intercal}(t)\sigma(t)\hat{q}_1(t) + Q(t)X^{\hat{\pi}}(t)^2 + S^{\intercal}(t)X^{\hat{\pi}}(t)\hat{\pi}(t)\right]dt \\ &+ \left[\hat{p}_1(t)\hat{\pi}^{\intercal}(t)\sigma(t) + \hat{q}_1^{\intercal}(t)X^{\hat{\pi}}(t)\right]dW(t). \end{aligned} \tag{3.40}$$

Define the process $\tilde{S}$ as

$$\tilde{S}(t) := \int_0^t \left(\hat{p}_1(s)\hat{\pi}^{\intercal}(s)\sigma(s) + \hat{q}_1^{\intercal}(s)X^{\hat{\pi}}(s)\right)dW(s), \ 0 \leq t \leq T.$$

Obviously, $\tilde{S}$ is a local martingale. To prove that $\tilde{S}$ is a true martingale, it is sufficient to show that $E\left[\sup_{0\leq s\leq T}|\tilde{S}(s)|\right] < \infty$. According to the Burkholder-Davis-Gundy inequality [46, Theorem 3.3.28], it is sufficient to verify that

$$E\left[\left(\int_0^T [|\hat{p}_1(s)\pi^{\intercal}(s)\sigma(s)|^2 + |\hat{q}_1(s)X^{\hat{\pi}}(s)|^2]ds\right)^{\frac{1}{2}}\right] < \infty.$$

Note that from [51, Corollary 2.5.10], we have that $X^{\hat{\pi}} \in \mathcal{S}^2(0,T;\mathbb{R})$. Combining with $p_1 \in \mathcal{S}^2(0,T;\mathbb{R})$ and $q_1 \in \mathcal{H}^2(0,t;\mathbb{R}^N)$ and by Höder's inequality, we have

$$
\begin{aligned}
&E\left[\left(\int_0^T [|\hat{p}_1(s)\pi^\intercal(s)\sigma(s)|^2 + |\hat{q}_1(s)X^{\hat{\pi}}(s)|^2]ds\right)^{\frac{1}{2}}\right] \\
\leq &E\left[\left(\sup_{0\leq s\leq T}|\hat{p}_1(s)|^2\int_0^T|\pi^\intercal(s)\sigma(s)|^2ds + \sup_{0\leq s\leq T}|X^{\hat{\pi}}(s)|^2\int_0^T|q_1(s)|^2ds\right)^{\frac{1}{2}}\right] \\
\leq &\frac{1}{2}E\left[\sup_{0\leq s\leq T}|\hat{p}_1(s)|^2\right] + \frac{1}{2}E\left[\int_0^T|\pi^\intercal(s)\sigma(s)|^2ds\right] + \frac{1}{2}E\left[\sup_{0\leq s\leq T}|X^{\hat{\pi}}(s)|^2\right] \\
&+\frac{1}{2}E\left[\int_0^T|\hat{q}_1(s)|^2ds\right] \\
&<\infty,
\end{aligned}
$$

which implies that $\tilde{S}$ is a true martingale. Taking expectation of $X^{\hat{\pi}}(T)\hat{p}_1(T)$, we have

$$
\begin{aligned}
E\left[X^{\hat{\pi}}(T)\hat{p}_1(T)\right] = x_0\hat{p}_1(0) + E\bigg[\int_0^T\bigg[&\hat{p}_1(t)\hat{\pi}^\intercal(t)\sigma(t)\theta(t) + \hat{\pi}^\intercal(t)\sigma(t)\hat{q}_1(t) \\
&+ Q(t)X^{\hat{\pi}}(t)^2 + S^\intercal(t)X^{\hat{\pi}}(t)\hat{\pi}(t)\bigg]dt\bigg]
\end{aligned}
\tag{3.41}
$$

Similarly, applying Ito's formula to $X^{\pi}(t)\hat{p}_1(t)$ and taking expectation, we obtain that

$$
\begin{aligned}
E\left[X^{\pi}(T)\hat{p}_1(T)\right] = x_0\hat{p}_1(0) + E\bigg[\int_0^T\bigg[&\hat{p}_1(t)\pi^\intercal(t)\sigma(t)\theta(t) + \pi^\intercal(t)\sigma(t)\hat{q}_1(t) \\
&+ Q(t)X^{\hat{\pi}}(t)X^{\pi}(t) + S^\intercal(t)X^{\pi}(t)\hat{\pi}(t)\bigg]dt\bigg]
\end{aligned}
\tag{3.42}
$$

Combining (3.39),(3.41) and (3.42), we obtain that $\hat{\pi} \in \mathcal{A}$ is an optimal control of the primal problem if and only if

$$
E\left[\int_0^T\left[\hat{\pi}^\intercal(t) - \pi^\intercal(t)\right]\left[\hat{p}_1(t)\sigma(t)\theta(t) + \sigma(t)\hat{q}_1(t) + S(t)X^{\hat{\pi}}(t) + R(t)\hat{\pi}(t)\right]dt\right] \geq 0
\tag{3.43}
$$

for all $\pi \in \mathcal{A}$. Define the Hamiltonian function $H : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ as

$$H(\omega, t, x, \pi) \triangleq \pi^\intercal \left[\hat{p_1}(t)\sigma(t)\theta(t) + \sigma(t)\hat{q}_1(t) - S(t)x - \frac{1}{2}R(t)\pi \right] + r(t)\hat{p}_1(t)x - \frac{1}{2}Q(t)x^2$$

and define the set-valued map $F : \Omega \times [0, T] \to K$ as

$$F(\omega, t) := \left\{ \pi \in K : [\hat{\pi}^\intercal(t) - \pi^\intercal]\, H_\pi\left(\omega, t, X^{\hat{\pi}}(t), \hat{\pi}(t)\right) \geq 0 \right\}.$$

Then F is a measurable set-valued map, see [1, Definition 8.1.1]. Given $\pi \in K$, define the set $\mathbb{B}^\pi$ as

$$\mathbb{B}^\pi := \left\{ (\omega, t) \in \Omega \times [0, T] : [\hat{\pi}^\intercal(t) - \pi^\intercal]\, H_\pi(t, X^{\hat{\pi}}(t), \hat{\pi}(t)) < 0 \right\}.$$

According to [1, Theorem 8.14], $\mathbb{B}^\pi_t \in \mathcal{F}_t$ for $t \in [0, T]$. Define an adapted control $\tilde{\pi} : \Omega \times [0, T] \to K$ as

$$\tilde{\pi}(\omega, t) := \begin{cases} \pi & \textit{if } (\omega, t) \in \mathbb{B}^\pi \\ \hat{\pi}(\omega, t), & \textit{otherwise.} \end{cases}$$

Suppose that $(\mathbb{P} \otimes \mathit{Leb})(\mathbb{B}^\pi) > 0$, then

$$E\left[\int_0^T [\hat{\pi}^\intercal(t) - \tilde{\pi}^\intercal(t)] H_\pi(t, X^{\hat{\pi}}(t), \hat{\pi}(t)) dt \right] < 0,$$

contradicting with (3.43). Hence, we conclude that $(\mathbb{P} \otimes \mathit{Leb})(\mathbb{B}^\pi) = 0$ for any fixed $\pi \in K$. Moreover, since K is separable, we conclude that

$$[\hat{\pi}^\intercal(t) - \pi^\intercal] H_\pi(t, X^{\hat{\pi}}(t), \hat{\pi}(t)) \geq 0, \ \forall \pi \in K$$

for $(\mathbb{P} \otimes \mathit{Leb})$-a.e. $(\omega, t) \in \Omega \times [0, T]$. □

Proof of Theorem 3.3.5. Let $(\hat{y}, \hat{\alpha}, \hat{\beta})$ be optimal for the dual problem and $(Y^{(\hat{y}, \hat{\alpha}, \hat{\beta})}, \hat{p}_2, \hat{q}_2)$ satisfy (3.14). Let $(y, \alpha, \beta) \in \mathbb{B}$ and $Y^{(y,\alpha,\beta)}$ satisfy the SDE (3.8). Applying Ito's formula to $\hat{p}_2(t)Y^{(y,\alpha,\beta)}(t)$, we have

$$d(\hat{p}_2(t)Y^{(y,\alpha,\beta)}(t)) = \left[\alpha(t)\hat{p}_2(t) + \hat{q}_2^\intercal(t)\sigma^{-1}(t)\beta(t) \right] dt + \left[\hat{q}_2^\intercal(t)Y^{(y,\alpha,\beta)}(t) + \left(\sigma^{-1}(t)\beta(t) - \theta(t)Y^{(y,\alpha,\beta)}(t) \right)^\intercal \hat{p}_2(t) \right] dW(t).$$

It can be shown, following a similar argument as in the proof of Theorem 3.3.1, that the process

$$\int_0^t \left[\hat{q}_2^\intercal(s)Y^{(y,\alpha,\beta)}(s) + (\sigma^{-1}(s)\beta(s) - \theta(s)Y^{(y,\alpha,\beta)}(s))^\intercal \hat{p}_2(s)\right] dW(s),\ 0 \le t \le T,$$

is a martingale. Taking the expectation of $\hat{p}_2(T)Y^{(y,\alpha,\beta)}(T)$, we obtain

$$E\left[\hat{p}_2(T)Y^{(y,\alpha,\beta)}(T)\right] = \hat{p}_2(0)y + E\left[\int_0^T \left[\alpha(t)\hat{p}_2(t) + \hat{q}_2^\intercal(t)\sigma^{-1}(t)\beta(t)\right] dt\right]. \tag{3.44}$$

For $\varepsilon > 0$ define $(y^\varepsilon, \alpha^\varepsilon, \beta^\varepsilon) \in \mathbb{B}$ by

$$(y^\varepsilon, \alpha^\varepsilon, \beta^\varepsilon) = (\hat{y}, \hat{\alpha}, \hat{\beta}) + \varepsilon(y, \alpha, \beta).$$

Then

$$Y^{(y^\varepsilon,\alpha^\varepsilon,\beta^\varepsilon)}(t) = Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t) + \varepsilon Y^{(y,\alpha,\beta)}(t).$$

Since $(\hat{y}, \hat{\alpha}, \hat{\beta})$ is optimal, we have

$$\frac{1}{\varepsilon}\left[\Psi(y^\varepsilon, \alpha^\varepsilon, \beta^\varepsilon) - \Psi(\hat{y}, \hat{\alpha}, \hat{\beta})\right] \ge 0.$$

Substituting (3.7) into the above inequality, also noting $\hat{p}_2(T) = -\dfrac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T) + c}{a}$, we get

$$\begin{aligned} &yx_0 - E\left[Y^{(y,\alpha,\beta)}(T)\hat{p}_2(T)\right] + \varepsilon E\left[\frac{Y^{(y,\alpha,\beta)}(T)^2}{2a}\right] \\ &+ \frac{1}{\varepsilon}E\left[\int_0^T \left[\phi(\alpha^\varepsilon(t), \beta^\varepsilon(t)) - \phi(\hat{\alpha}(t), \hat{\beta}(t))\right] dt\right] \ge 0. \end{aligned} \tag{3.45}$$

Combining (3.45) with (3.44) and then letting $\varepsilon \downarrow 0$, we have

$$y\left(x_0 - \hat{p}_2(0)\right) + \lim_{\varepsilon\downarrow 0} E\left[\int_0^T [\tilde{g}(t,\varepsilon) - \hat{q}_2^\intercal(t)\sigma^{-1}(t)\beta(t) - \alpha(t)\hat{p}_2(t)]dt\right] \ge 0,$$

where $\tilde{g}(\omega, t, \varepsilon) = \dfrac{1}{\varepsilon}(\phi(t, \alpha^\varepsilon(t), \beta^\varepsilon(t)) - \phi(t, \hat{\alpha}(t), \hat{\beta}(t)))$. Let $\alpha(t) = 0$ and $\beta(t) = 0$ for $t \in [0, T]$, we get

$$y(x_0 - \hat{p}_2(0)) \ge 0,\ \forall y \in \mathbb{R}.$$

Hence, $\hat{p}_2(0) = x_0$. Recall that the function f in (3.4) is convex and the set K is convex, according to [72, Theorem 26.3], ϕ has directional derivative at $(\hat{\alpha}(t), \hat{\beta}(t))$ in any direction $(\mathbb{P} \otimes Leb)$ a.e. on $\Omega \times [0.T]$. Since $\varepsilon \to \tilde{g}(\omega, t, \varepsilon)$ is a nondecreasing function, Assumption 3.3.3 and the monotone convergence theorem imply that

$$E\left[\int_0^T \left[\phi^o\left(t, \hat{\alpha}(t), \hat{\beta}(t); \alpha(t), \beta(t)\right) - \hat{q}_2^\intercal(t)\sigma^{-1}(t)\beta(t) - \alpha(t)\hat{p}_2(t)\right] dt\right] \geq 0 \tag{3.46}$$

where

$$\phi^o\left(\omega, t, \hat{\alpha}, \hat{\beta}; \alpha, \beta\right) := \lim_{\varepsilon \downarrow 0} \frac{\phi(t, \hat{\alpha} + \varepsilon\alpha, \hat{\beta} + \varepsilon\beta) - \phi(t, \hat{\alpha}, \hat{\beta})}{\varepsilon}.$$

For $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^N$, define the set $\mathbb{B}^{(\alpha,\beta)}$ as

$$\mathbb{B}^{(\alpha,\beta)} := \left\{(\omega, t) \in \Omega \times [0, T] : \phi^o\left(\hat{\alpha}(t), \hat{\beta}(t); \alpha, \beta\right) - \hat{q}_2^\intercal(t)\sigma^{-1}(t)\beta - \alpha\hat{p}_2(t) < 0\right\}.$$

Using a similar argument as in the proof of Theorem 3.3.1, we conclude that $\mathbb{B}_t^{(\alpha,\beta)} \in \mathcal{F}_t$ for $t \in [0, T]$ and $(\mathbb{P} \otimes Leb)\left(\mathbb{B}^{(\alpha,\beta)}\right) = 0$ for all $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^N$. Equivalently, given any $(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^N$,

$$\phi^o\left(\hat{\alpha}(t), \hat{\beta}(t); \alpha, \beta\right) - \hat{q}_2^\intercal(t)\sigma^{-1}(t)\beta - \alpha\hat{p}_2(t) \geq 0,$$

for $(\mathbb{P} \otimes Leb)$-a.e. $(\omega, t) \in \Omega \times [0, T]$. In addition, by the separability of the space $\mathbb{R}^{N+1}$, we conclude that

$$\phi^o\left(\hat{\alpha}(t), \hat{\beta}(t); \alpha, \beta\right) - \hat{q}_2^\intercal(t)\sigma^{-1}(t)\beta - \alpha\hat{p}_2(t) \geq 0, \forall(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^N$$

for $(\mathbb{P} \otimes Leb)$-a.e. $(\omega, t) \in \Omega \times [0, T]$. By the definition of Clarke's generalized gradient [17, Chapter 2], the above condition can be written as

$$\left(\hat{p}_2(t), [\sigma^\intercal]^{-1}(t)\hat{q}_2(t)\right) \in \partial\phi\left(\hat{\alpha}(t), \hat{\beta}(t)\right).$$

According to [72, Theorem 23.5] , we conclude that $x\hat{\alpha}(t) + \pi^\intercal\hat{\beta}(t) - \tilde{f}(t, x, \pi)$ achieves the supreme at $(\hat{p}_2(t), [\sigma^\intercal]^{-1}(t)\hat{q}_2(t))$ for $(\mathbb{P} \otimes Leb)$-a.e. $(\omega, t) \in \Omega \times [0, T]$, which implies

$$[\sigma^\intercal]^{-1}(t)\hat{q}_2(t) \in K,$$

for $(\mathbb{P}\otimes Leb)$-a.e. $(\omega,t)\in\Omega\times[0,T]$. We have proved the necessary condition. Let $(\hat{y},\hat{\alpha},\hat{\beta})\in\mathbb{B}$ be an admissible control to the dual problem with processes $\left(Y^{(\hat{y},\hat{\alpha},\hat{\beta})},\hat{p}_2,\hat{q}_2\right)$ satisfying the FBSDE (3.14) and conditions (3.15). Define the Hamiltonian function $H:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ as

$$H(\omega,t,\alpha,\beta)=\hat{q}_2^{\intercal}(t)\sigma^{-1}(t)\beta+\alpha\hat{p}_2(t)-\phi(t,\alpha,\beta).$$

By condition (4.26) and the classical result in duality theorem, we have

$$(0,0)\in\partial H\left(\hat{\alpha}(t),\hat{\beta}(t)\right), \tag{3.47}$$

for $(\mathbb{P}\otimes Leb)$-a.e. $(\omega,t)\in\Omega\times[0,T]$. Given any admissible control $(y,\alpha,\beta)\in\mathbb{B}$, define

$$\tilde{y}=y-\hat{y},\ \ \tilde{\alpha}=\alpha-\hat{\alpha},\ \ \tilde{\beta}=\beta-\hat{\beta}.$$

Let $Y^{(y,\alpha,\beta)}$ and $Y^{(\tilde{y},\tilde{\alpha},\tilde{\beta})}$ be the associated state processes satisfying the SDE (3.8). According to the definition of the dual problem, also noting m_T is a convex function, we have

$$\begin{aligned}\Psi\left(y,\alpha,\beta\right)-\Psi\left(\hat{y},\hat{\alpha},\hat{\beta}\right)\geq&\tilde{y}x_0+E\left[Y^{(\tilde{y},\tilde{\alpha},\tilde{\beta})}(T)\frac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(T)+c}{a}\right]\\&+E\left[\int_0^T\left[\phi(t,\alpha(t),\beta(t))-\phi(t,\hat{\alpha}(t),\hat{\beta}(t))\right]dt\right].\end{aligned}$$

Replacing $\dfrac{Y^{(\hat{y},\hat{\alpha},\hat{\beta})}+c}{a}$ with $-\hat{p}_2(T)$ in the above inequality, we have

$$\begin{aligned}\Psi\left(y,\alpha,\beta\right)-\Psi\left(\hat{y},\hat{\alpha},\hat{\beta}\right)\geq&\tilde{y}(x_0-\hat{p}_2(0))+E\left[\int_0^T\left[\hat{q}_2^{\intercal}(t)\sigma^{-1}(t)\tilde{\beta}(t)-\tilde{\alpha}(t)\hat{q}_2(t)\right]dt\right]\\&+E\Big[\int_0^T\left[\phi(t,\alpha(t),\beta(t))-\phi(t,\hat{\alpha}(t),\hat{\beta}(t))\right]dt\Big]\\=&E\left[\int_0^T\left[-H(t,\alpha(t),\beta(t))+H(t,\hat{\alpha}(t),\hat{\beta}(t))\right]dt\right].\end{aligned}$$

According to condition (3.47) and the concavity of H, we conclude that

$$\Psi\left(\bar{y},\bar{\alpha},\bar{\beta}\right)-\Psi\left(\hat{y},\hat{\alpha},\hat{\beta}\right)\geq 0.$$

Since $(y, \alpha, \beta) \in \mathbb{B}$ is arbitrary, we have proved the sufficient condition. $\square$

Proof of Theorem 3.3.7. Suppose that $(\hat{y}, \hat{\alpha}, \hat{\beta}) \in \mathbb{B}$ is optimal for the dual problem. By Theorem 3.3.5, the process $\left(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t), \hat{p}_2(t), \hat{q}_2(t)\right)$ solves the dual FBSDE (3.14) and satisfies condition (3.15). Define $\hat{\pi}(t)$ and $(X^{\hat{\pi}}(t), \hat{p}_1(t), \hat{q}_1(t))$ as in (3.16) and (3.17), respectively. According to Theorem 3.3.5 and condition (3.15), we have $\hat{\pi}(t) \in K$ $\mathbb{P}$-a.s. and

$$\left(X^{\hat{\pi}}(t), \hat{\pi}(t)\right) \in \partial \phi \left(\hat{\alpha}(t), \hat{\beta}(t)\right).$$

The classical result in conjugate duality theory (see [72, Theorem 23.5]) implies

$$\left(\hat{\alpha}(t), \hat{\beta}(t)\right) \in \partial \tilde{f} \left(X^{\hat{\pi}}(t), \hat{\pi}(t)\right).$$

Recall that $\tilde{f}(\omega, t, x, \pi) = f(\omega, t, x, \pi) + \Psi_K(\pi)$, we can get

$$\hat{\alpha}(t) = Q(t)X^{\hat{\pi}}(t) + S^{\intercal}(t)\hat{\pi}(t), \tag{3.48}$$

$$\hat{\beta}(t) \in S(t)X^{\hat{\pi}}(t) + R(t)\hat{\pi}(t) + \partial \Phi_K(\hat{\pi}(t)) \tag{3.49}$$

for $(\mathbb{P} \otimes \mathit{Leb})$-a.e. $(\omega, t) \in \Omega \times [0, T]$. Combining (3.16), (3.17) and (3.48), we obtain that $\left(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1\right)$ solves the primal FBSDE (3.10). Moreover, combining (3.17) and (3.49) gives condition (3.11). Using the sufficient condition for optimality in Theorem 3.3.1, we conclude that $\hat{\pi}$ is indeed an optimal control for the primal problem. $\square$

Proof of Theorem 3.3.8. Suppose that $\hat{\pi} \in \mathcal{A}$ is an optimal control for the primal problem. By Theorem 3.3.1, the process $\left(X^{\hat{\pi}}(t), \hat{p}_1(t), \hat{q}_1(t)\right)$ solves the FBSDE (3.10) and satisfies condition (3.11). Define $(\hat{y}, \hat{\alpha}(t), \hat{\beta}(t))$ and $(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}(t), \hat{p}_2(t), \hat{q}_2(t))$ as in (3.18) and (3.19), respectively. Substituting them into the primal FBSDE (3.10), we obtain that $\left(Y^{(\hat{y},\hat{\alpha},\hat{\beta})}, \hat{p}_2, \hat{q}_2\right)$ satisfies the dual FBSDE (3.14). By the construction in (3.18) and (3.19), the first two conditions in (3.15) are satisfied. In addition, by condition (3.11) and the concavity of H, we have

$$\hat{\beta}(t) \in \partial_{\pi} \tilde{f} \left(X^{\hat{\pi}}(t), \hat{\pi}(t)\right).$$

Consequently, we have

$$\left(\hat{\alpha}(t), \hat{\beta}(t)\right) \in \partial \tilde{f}\left(X^{\hat{\pi}}(t), \hat{\pi}(t)\right),$$

which is equivalent to the third condition in (3.15). By Theorem 3.3.5, we conclude that $\left(\hat{y}, \hat{\alpha}, \hat{\beta}\right)$ is indeed an optimal control to the dual problem. □

3.6 Conclusion

In this chapter, we discuss a continuous-time constrained quadratic risk minimization problem with random market coefficients. Following a convex duality approach, we derive the necessary and sufficient optimality conditions for primal and dual problems in terms of FBSDEs plus additional conditions. We establish an explicit connection between primal and dual problems in terms of their associated forward backward systems. We prove that the optimal controls of primal and dual problems can be written as functions of adjoint processes of their counterpart. Moreover, we also find that the optimal state processes for both problems coincide with the optimal adjoint processes of their counterpart. We solve both the unconstrained and cone-constrained quadratic risk minimization problems using the dual approach. We recover the solutions to the extended SREs introduced in the literature from the optimal solutions to the dual problem and find the closed-form solutions to the extended SREs when the coefficients are deterministic.

Chapter 4

Dynamic Convex Duality in Constrained Utility Maximization

4.1 Introduction

One of the most commonly studied problems in mathematical economics is the optimal consumption/investment problem. Such problems have as their goal of constructing the investment strategy that maximizes the agent's expected utility of the wealth at the end of the planning horizon. Here we assume that the problem must take value in a closed convex set which is general enough to incorporate the case of short selling, borrowing, and other trading restrictions, see [47].

There has been extensive research in the area of utility maximization. The stochastic control approach was first introduced in the two landmark papers of Merton [62, 63], which was wedded to the Hamilton-Jacobi-Bellman equation and the requirement of an underlying Markov state process. The optimal consumption/investment problem in a non-Markov setting was solved using the martingale method by, among others, Pliska [70], Cox and Huang [20, 21], Karatzas, Lehoczky and Shreve [42]. The stochastic duality theory of Bismut [9] was first employed to study the constrained optimal investment problem in Shreve and Xu [86] where the authors studied the problems of long only constrains with $K = [0, \infty)^N$. The effectiveness of convex duality method was later adopted to tackle the more traditional incomplete market models in the works of, among others, Karatzas, Lehoczky, Shreve and Xu[43], Pearson and He [39, 38], Cvitanić and Karatzas [23]. The spirit of this approach is to suitably embed the constrained problem in an appropriate family of unconstrained ones and find a member of this family for which the corresponding optimal policy obeys the constrains. However, despite the evident power of this approach, it is nevertheless true that obtaining the corresponding dual problem remains a challenge as it often involves clever experimentation and subsequently show to work as desire. To bring some transparency to the dual problem, Labbé and Heunis [55] established a simple synthetic method of arriving at a dual functional, bypassing the need to formulate a fictitious market. It often happens that the dual problem is much nicer than the primal problem in the sense that it is easier to show the existence of a solution and in some cases explicitly obtain a solution to the dual problem than it is to do likewise for the primal problem.

In this chapter, we follow the approach as in Labbé and Heunis [55] by first converting the original problem into a static problem in an abstract space. Then we apply convex analysis to derive its dual problem and get the specific dual stochastic control problem. Subsequently, following the approach in [75] and [83] we progress to a stochastic approach to simultaneously characterise the necessary and sufficient optimality conditions for both the primal and dual problems as systems of Forward and Backward Stochastic Differential Equations (FBSDEs) coupled with static optimality conditions. Such formulation then allows us to characterize the primal optimal control as a function of the adjoint processes coming from the dual FBSDEs in a dynamic fashion and vice versa. Moreover, we also find that the optimal primal wealth process coincides with the optimal adjoint process of the dual problem and vice versa. To the best of our knowledge, this is the first time the dynamic relations of the primal and dual problems have been explicitly established for constrained utility maximization problems under a non-Markov setting. After establishing the optimality conditions and the relations for the primal and dual problems, we solve three constrained utility maximization problems with both Markov and non-Markov setups. Instead of tackling the primal problem directly, we start from the dual problem and then construct the optimal solution to the primal problem from that to the dual problem. In all examples, we contrasts the simplicity of the duality approach we propose with the technical complexity in solving the primal problem directly.

The rest of the chapter is organised as follows. In Section 2, we set up the market model and formulate the primal and dual problems following the approach in [55]. Section 3 is devoted to the main proof of necessary and sufficient optimality conditions for the the primal and dual problems and their connections in a dynamic fashion. To demonstrate the effectiveness of the dynamic duality approach in solving constrained utility maximization problems, we give three examples in Section 4. Section 5 concludes the chapter.

4.2 Market Model and Primal and Dual Problems

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which is defined some $\mathbb{R}^N$-valued standard Brownian motion $\{W(t), t \in [0,T]\}$ with $T > 0$ denoting a fixed terminal time. Let $\{\mathcal{F}_t, t \in [0,T]\}$ be the standard filtration induced by

W, where

$$\mathcal{F}_t \triangleq \sigma\{W(s), s \in [0,t]\} \bigvee \mathcal{N}(P), t \in [0,T],$$

in which $\mathcal{N}(P)$ denotes the collection of all $\mathbb{P}$-null events in $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by $\mathcal{F}^*$ the σ-algebra of $\mathcal{F}_t$ progressively measurable sets on $\Omega \times [0,T]$. For any stochastic process $v : \Omega \times [0,T] \to \mathbb{R}^m,\ m \in \mathbb{Z}^+$, we write $v \in \mathcal{F}^*$ to indicate v is $\mathcal{F}^*$ measurable. We introduce the following notations:

$$\mathcal{H}^1(0,T;\mathbb{R}^m) \triangleq \left\{ v : \Omega \times [0,T] \to \mathbb{R}^m \mid v \in \mathcal{F}^*, E\left[\int_0^T \|v(t)\| dt\right] < \infty \right\},$$

$$\mathcal{H}^2(0,T;\mathbb{R}^m) \triangleq \left\{ \xi : \Omega \times [0,T] \to \mathbb{R}^m \mid \xi \in \mathcal{F}^*, E\left[\int_0^T \|\xi(t)\|^2 dt\right] < \infty \right\},$$

where $m \in \mathbb{Z}^+$.

Consider a market consisting of a bank account with price $\{S_0(t)\}$ given by

$$dS_0(t) = r(t)S_0(t)dt,\ 0 \le t \le T,\ S_0(0) = 1, \tag{4.1}$$

and N stocks with prices $\{S_n(t)\},\ n = 1, \cdots, N$, given by

$$dS_n(t) = S_n(t)\left[b_n(t)dt + \sum_{m=1}^{N} \sigma_{nm}(t)dW_m(t)\right],\ 0 \le t \le T,\ S_n(0) > 0. \tag{4.2}$$

Through out the chapter we assume that the interest rate $\{r(t)\}$, the appreciation rates on stocks denoted by entries of the $\mathbb{R}^N$-valued process $\{b(t)\}$ and the volatility process denoted by entries of the $N \times N$ matrix $\{\sigma(t)\}$ are uniformly bounded $\{\mathcal{F}_t\}$-progressively measurable scalar processes on $\Omega \times [0,T]$. We also assume that there exists a positive constant k such that

$$z^\intercal \sigma(t)\sigma^\intercal(t) z \ge k|z|^2$$

for all $(z, \omega, t) \in \mathbb{R}^N \times \Omega \times [0,T]$, where $z^\intercal$ is the transpose of z. According to [86, p.90 (2.4) and (2.5)], the strong non-degeneracy condition above ensures that matrices $\sigma(t), \sigma^\intercal(t)$ are invertible and uniformly bounded.

Consider a small investor with initial wealth $x_0 > 0$ and a self-financing strategy. Define the set of admissible portfolio strategies by

$$\mathcal{A} := \left\{ \pi \in \mathcal{H}^2(0,T;\mathbb{R}^N) : \pi(t) \in K \text{ } for \text{ } t \in [0,T] \text{ } a.e. \right\},$$

where $K \subseteq \mathbb{R}^N$ is a closed convex set with $0 \in K$ and π is a portfolio process with each entry $\pi_n(t)$ defined as the fraction of the investor's total wealth put into the stock n for $n = 1, \ldots, N$ at time t. Given any $\pi \in \mathcal{A}$, the investor's total wealth X^π satisfies the SDE

$$\begin{cases} dX^\pi(t) = X^\pi(t)\{[r(t) + \pi^\intercal(t)\sigma(t)\theta(t)]dt + \pi^\intercal(t)\sigma(t)dW(t)\}, & 0 \le t \le T, \\ X^\pi(0) = x_0, \end{cases} \tag{4.3}$$

where $\theta(t) := \sigma^{-1}(t)\,[b(t) - r(t)\mathbf{1}]$ is the market price of risk at time t and is uniformly bounded and $\mathbf{1} \in \mathbb{R}^N$ has all unit entries. A pair (X, π) is *admissible* if $\pi \in \mathcal{A}$ and X is a strong solution to the SDE (4.3) with control process π.

Remark 4.2.1. *Here we define the nth entry of $\pi(t)$ as the fraction of small investor's wealth invested in the stock n at time t.Such set-up ensures the positivity of the wealth process X^π, but surrenders the Lipschitz property of the coefficients in both X and π. Hence, the stochastic maximum principle developed in [13] and [67] are not directly applicable in our case.*

Let $U : [0, \infty] \to \mathbb{R}$ of class C^2 be a given utility function assumed to be strictly increasing, strictly concave and satisfies the following conditions:

$$U(0) \triangleq \lim_{x \searrow 0} U(x) > -\infty, \ \lim_{x\to\infty} U(x) = \infty, \lim_{x \searrow 0} U'(x) = \infty, \ and \ \lim_{x\to\infty} U'(x) = 0.$$

Define the value of the expected utility maximization problem as

$$V \triangleq \sup_{\pi \in \mathcal{A}} E\left[U\left(X^\pi(T)\right)\right].$$

To avoid trivialities, we assume that

$$V < +\infty.$$

The constrained utility maximization can be written as the following stochastic optimization problem:

$$\textit{Find optimal } \pi^* \in \mathcal{A} \textit{ such that } E\left[U\left(X^{\pi^*}(T)\right)\right] = \sup_{\pi\in\mathcal{A}} E\left[U\left(X^{\pi}(T)\right)\right] = V.$$

In the rest of this section, we formulate the dual problem following the approach in [55]. Given any continuous $\{\mathcal{F}_t\}$ semimartingale process X, we write $X \in \mathbb{R} \times \mathcal{H}^1(0,T;\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R}^N)$ if

$$X(t) = X_0 + \int_0^T \dot{X}(s)ds + \int_0^T \Lambda_X^{\intercal}(s)dW(s),$$

where $(X_0, \dot{X}, \Lambda_X) \in \mathbb{R} \times \mathcal{H}^1(0,T;\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R}^N)$.
Define the following sets:

$$\mathcal{U}(X) \triangleq \left\{\pi \in \mathcal{A} | \dot{X}(t) = X(t)\left[r(t) + \pi^{\intercal}(t)\sigma(t)\theta(t)\right] \textit{ and } \Lambda_x(t) = X(t)\sigma^{\intercal}(t)\pi(t) \textit{ a.e.}\right\},$$
$$\mathbb{B} \triangleq \left\{X \in \mathbb{R} \times \mathcal{H}^1(0,T;\mathbb{R}) \times \mathcal{H}^2(0,T;\mathbb{R}^N) | X(0) = x_0 \textit{ and } \mathcal{U}(x) = \varnothing\right\}.$$

Moreover, to remove the portfolio constraints, define the penalty functions:

$$l_0(x) = \begin{cases} 0, & \text{if } x = x_0, \\ +\infty, & \text{otherwise,} \end{cases} \qquad l_T(x) = \begin{cases} -U(x), & \text{if } x \in (0,\infty) \\ +\infty, & \text{otherwise} \end{cases} \tag{4.4}$$

$$L(t,x,v,\xi) = \begin{cases} 0, & \text{if } x > 0, v = xr(t) + \xi^{\intercal}\theta(t) \textit{ and } x^{-1}[\sigma^{\intercal}(t)]^{-1}\xi \in K, \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.5}$$

Hence, following [55, Remark 3.4], we obtain

$$-V = \inf_{X\in\mathbb{B}} \Phi_p(X) = \Phi_p(\hat{X}) \textit{ for some } \hat{X} \in \mathbb{B},$$

where $\Phi_p \triangleq l_0(X(0)) + E\left[l_T(X(T))\right] + E\int_0^T L(t, X(t), \dot{X}(t), \Lambda_X(t))dt$. The dual problem is formulated in terms of the following pointwise convex conjugate transforms of the three penalty functions:

$$m_0(y) \triangleq \sup_{x\in\mathbb{R}}[xy - l_0(x)] = x_0 y,$$

$$m_T(y) \triangleq \sup_{x\in\mathbb{R}}[x(-y) - l_T(y)] = \begin{cases} \tilde{U}(y) \triangleq \sup_{x>0}[U(x) - xy], & \text{if } y \in [0,\infty), \\ \infty, & \text{otherwise.} \end{cases}$$

$$M(t,y,s,\gamma) \triangleq \sup_{x,v\in\mathbb{R},\ \xi\in\mathbb{R}^N} \{xs + vy + \xi^\intercal\gamma - L(t,x,v,\xi)\}$$

$$= \begin{cases} 0, & \text{if } s + yr(t) + \delta_K(-\sigma(t)[y\theta(t)+\gamma]) < \infty, \\ \infty, & \text{otherwise,} \end{cases}$$

where $\delta_K(\cdot)$ is the support function of the set $-K$ defined by

$$\delta_K(z) \triangleq \sup_{\pi\in K}\{-\pi^\intercal z\}, z \in \mathbb{R}^N. \tag{4.6}$$

Consequently, the dual objective function Φ_D is given by

$$\Phi_D(Y) \triangleq m_0(Y(0)) + E\left[m_T(Y(T))\right] + E\int_0^T M(t, Y(t), \dot{Y}(t), \Lambda_Y(t))dt,$$

$\forall Y \in \mathbb{R}\times\mathcal{H}^1(0,T;\mathbb{R})\times\mathcal{H}^2(0,T;\mathbb{R}^N)$. Define the set

$$\mathcal{D} \triangleq \left\{ v \triangleq \Omega\times[0,T] \to \mathbb{R}^N | v \in \mathcal{F}^* \text{ and } \int_0^T \left[\delta_K(v(t)) + \|v(t)\|^2\right] dt < \infty \ a.s.\right\}.$$

Given $(y,v) \in (0,\infty)\times\mathcal{D}$, the corresponding state process $Y^{(y,v)}$ satisfies the SDE

$$\begin{cases} dY^{(y,v)}(t) = -Y^{(y,v)}(t)\left\{[r(t) + \delta_K(v(t))]dt + [\theta(t) + \sigma^{-1}(t)v(t)]^\intercal dW(t)\right\}, \ 0 \le t \le T, \\ Y^{(y,v)}(0) = y, \end{cases} \tag{4.7}$$

The optimal value of the dual function is given by

$$\tilde{V} \triangleq \inf_{(y,v)\in(0,\infty)\times\mathcal{D}} \left\{x_0 y + E\left[\tilde{U}(Y^{(y,v)}(T))\right]\right\}.$$

The dual problem can be written as the following stochastic optimization problem:

Find the optimal $(y^*, v^*) \in (0,\infty)\times\mathcal{D}$ *such that* $\tilde{V} = x_0 y^* + E\left[\tilde{U}(Y^{(y^*,v^*)}(T))\right]$.

The duality relation follows from [55, Corollary 4.12]. In this chapter, instead of applying the convex duality method of [9], we use the machinery of stochastic maximum principle and BSDEs to derive the necessary and sufficient conditions of the primal and dual problems separately. After establishing the optimal conditions as two systems of FBSDEs, we explicitly characterise the primal optimal solution as functions of the adjoint process coming from the dual FBSDEs in a dynamic fashion and vice versa.

4.3 Main Results

In this section, we derive the necessary and sufficient optimality conditions for the primal and dual problems and show that the connection between the optimal solutions through their corresponding FBSDEs.

Given an admissible control $\pi \in \mathcal{A}$ and solution X^π to the SDE (4.3), the associated adjoint equation in the unknown processes $p_1 \in \mathcal{H}^2(0,T;\mathbb{R})$ and $q_1 \in \mathcal{H}^2(0,T;\mathbb{R}^N)$ is the following BSDE:

$$\begin{cases} dp_1(t) = -\left\{\left[r(t) + \pi^\intercal(t)\sigma(t)\theta(t)\right]p_1(t) + q_1^\intercal(t)\sigma^\intercal(t)\pi(t)\right\}dt + q_1^\intercal(t)dW(t), \\ p_1(T) = -U'(X^\pi(T)). \end{cases} \tag{4.8}$$

Assumption 4.3.1. *The utility function U satisfies the following conditions*

(i) $x \to xU'(x)$ *is non-decreasing on* $(0,\infty)$.

(ii) There exists $\gamma \in (1,\infty)$ *and* $\beta \in (0,1)$ *such that* $\beta U'(x) \geq U'(\gamma x)$ *for all* $x \in (0,\infty)$.

Moreover, we assume that for $\forall \pi \in \mathcal{A}$ *and corresponding* X^π *satisfying the SDE* (4.3), $E[|U(X^\pi(T))|] < \infty$ *and* $E\left[(U'(X^\pi(T))X^\pi(T))^2\right] < \infty$.

Remark 4.3.2. *The above assumption corresponds to Remark 3.4.4 in [47]. Firstly, under Assumption 4.3.1 (i),*

(i') For a utility function U of class $C^2(0,\infty)$ (which is true in our set-up), the Arrow-Pratt index of relative risk aversion $R(x) = -\frac{xU''(x)}{U'(x)}$ does not exceed 1.

Moreover, set $y = U'(x)$ and we have $xU'(x) = yI(y) = -y\tilde{U}'(y)$. Hence, we conclude:

(ii') $z \to \tilde{U}(e^z)$ is convex in $\mathbb{R}$ when $\tilde{U}$ is the convex dual of U.

Finally, replacing x by $-\tilde{U}'(y)$, we claim that Assumption 4.3.1 (ii) is equivalent to $\tilde{U}'(\beta y) \geq \gamma \tilde{U}'(y)$ for $\forall y \in (0,\infty)$ and some $\beta \in (0,1), \gamma \in (1,\infty)$. Iterating the above inequality we obtain

(iii') $\forall \beta \in (0,1)\ \exists \gamma \in (1,\infty)$ s.t $\tilde{U}'(\beta y) \geq \gamma \tilde{U}'(y)$ for $\forall y \in (0,\infty)$.

Lemma 4.3.3. *Let $\hat{\pi} \in \mathcal{A}$ be an optimal control to the primal problem with corresponding wealth process $X^{\hat{\pi}}$ satisfying the SDE* (4.3)*. The there exists a solution to the adjoint BSDE* (4.8)*.*

Proof. The process defined as

$$\alpha(t) \triangleq E\left[-X^{\hat{\pi}}(T)U'(X^{\hat{\pi}}(T)\big|\mathcal{F}_t\right],\ t \in [0,T] \tag{4.9}$$

is square integrable. In addition, it is the unique solution of the BSDE

$$\alpha(t) = -X^{\hat{\pi}}(T)U'(X^{\hat{\pi}}(T)) - \int_t^T \beta^\intercal(t)dW(t),\ t \in [0,T],$$

where β is a square integrable process with values in $\mathbb{R}^N$. Applying Ito's lemma to $\frac{\alpha(t)}{X^{\hat{\pi}}(t)}$, we have

$$\begin{aligned} d\frac{\alpha(t)}{X^{\hat{\pi}}(t)} =& \frac{\beta(t)}{X^{\hat{\pi}}(t)}dW(t) - \frac{\alpha(t)}{X^{\hat{\pi}}(t)}\left\{[r(t) + \hat{\pi}^\intercal(t)\sigma(t)\theta(t)]dt + \hat{\pi}^\intercal(t)\sigma(t)dW(t) + |\pi^\intercal(t)\sigma(t)|^2 dt\right\} \\ &- \frac{\hat{\pi}^\intercal(t)\sigma(t)\beta(t)}{X^{\hat{\pi}}(t)}dt \\ =& -\left\{[r(t) + \hat{\pi}^\intercal(t)\sigma(t)\theta(t)]\hat{p}_1(t) + \hat{q}_1^\intercal(t)\sigma(t)\hat{\pi}(t)\right\}dt + \hat{q}_1^\intercal(t)dW(t), \end{aligned}$$

where we define

$$\hat{p}_1(t) \triangleq \frac{\alpha(t)}{X^{\hat{\pi}}(t)} \; \textit{and} \; \hat{q}_1(t) \triangleq -\frac{\beta(t)}{X^{\hat{\pi}}(t)} - \frac{\alpha(t)\sigma^{\intercal}(t)\hat{\pi}(t)}{X^{\hat{\pi}}(t)}. \tag{4.10}$$

Hence, we conclude that $(\hat{p}_1, \hat{q}_1)$ solves the adjoint BSDE (4.8). $\square$

We now state the necessary condition for the optimality of primal problem.

Theorem 4.3.4. *(Necessary condition for the primal problem) Let $\hat{\pi} \in \mathcal{A}$ be the optimal control of the primal problem with corresponding wealth process $X^{\hat{\pi}}$. Then there exists a solution $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ to the following FBSDE:*

$$\begin{cases} dX^{\hat{\pi}}(t) = X^{\hat{\pi}}(t)\{[r(t) + \hat{\pi}^{\intercal}(t)\sigma(t)\theta(t)]dt + \hat{\pi}^{\intercal}(t)\sigma(t)dW(t)\}, \\ X^{\hat{\pi}}(0) = x_0, \\ dp_1(t) = -\left\{[r(t) + \hat{\pi}^{\intercal}(t)\sigma(t)\theta(t)]\, p_1(t) + q_1^{\intercal}(t)\sigma^{\intercal}(t)\hat{\pi}(t)\right\} dt + q_1^{\intercal}(t)dW(t), \\ p_1(T) = -U'(X^{\hat{\pi}}(T)). \end{cases} \tag{4.11}$$

Moreover, let $N_K(x)$ be the normal cone to the closed convex set K at $x \in K$, defined as

$$N_K(x) \triangleq \left\{y \in \mathbb{R}^N : \forall x^* \in K, y^{\intercal}(x^* - x) \le 0\right\}.$$

Then the following condition hold

$$-X^{\hat{\pi}}(t)\sigma(t)\left[\hat{p}_1(t)\theta(t) + \hat{q}_1(t)\right] \in N_K(\hat{\pi}(t)) \; \textit{for} \; \forall t \in [0, T], \; \mathbb{P} - a.s. \tag{4.12}$$

Proof. Let $\tilde{\pi} \in \mathcal{A}$ be an admissible control and $\rho \triangleq \tilde{\pi} - \hat{\pi}$. Let $\tau_n \triangleq T \wedge \inf\left\{t \ge 0, \int_0^t \|\rho(s)\sigma(s)\|^2 ds \ge n \; or \; \int_0^t \|\rho^{\intercal}(s)\sigma(s)\sigma^{\intercal}(s)\hat{\pi}(s)\|^2 ds \ge n\right\}$. Hence, $\lim_{n\to\infty} \tau_n = T$ almost surely. Define $\rho_n(t) \triangleq \rho(t)1_{\{t \le \tau_n\}}$. Define the function $\phi_n(\varepsilon) \triangleq U\left(X^{\hat{\pi}+\varepsilon\rho_n}(T)\right)$ where $\varepsilon \in [0, 1]$. Set $G(x) \triangleq U(x_0e^x)$ and taking derivatives, we have

$$\begin{aligned} G'(x) &= U'(x_0e^x)x_0e^x \ge 0, \\ G''(x) &= x_0e^x\left(U'(x_0e^x) + U'(x_0e^x)x_0e^x\right) \ge 0, \end{aligned}$$

by Assumption 4.3.1. Differentiating ϕ on $(0,1)$, we have

$$\begin{aligned}\phi_n'(\varepsilon) =& G'(\cdot)\bigg[\int_0^T \left(\rho_n^\intercal(t)\sigma(t)\theta(t) - \rho_n^\intercal(t)\sigma^\intercal(t)\sigma(t)\left(\hat{\pi}(t)+\varepsilon\rho_n(t)\right)\right)dt \\ &+ \int_0^T \rho_n^\intercal(t)\sigma(t)dW(t)\bigg]. \\ \phi_n''(\varepsilon) =& G''(\cdot)\bigg[\int_0^T \left(\rho_n^\intercal(t)\sigma(t)\theta(t) - \rho_n^\intercal(t)\sigma^\intercal(t)\sigma(t)\left(\hat{\pi}(t)+\varepsilon\rho_n(t)\right)\right)dt \\ &+ \int_0^T \rho_n^\intercal(t)\sigma(t)dW(t)\bigg]^2 - G'(\cdot)\left[\int_0^T \rho_n^\intercal(t)\sigma^\intercal(t)\sigma(t)\rho_n(t)dt\right] \\ \geq& 0.\end{aligned}$$

Hence we conclude that the function $\Phi_n(\varepsilon) \triangleq \frac{\phi_n(\varepsilon)-\phi(0)}{\varepsilon}$ is a decreasing function and we have

$$\lim_{\varepsilon\to 0}\Phi_n(\varepsilon) = U'(X^{\hat{\pi}}(T))X^{\hat{\pi}}(T)H_n^\rho(T), \tag{4.13}$$

where $H_n^\rho(t) \triangleq \int_0^t \left(\rho_n^\intercal(s)\sigma(s)\theta(s) - \rho_n^\intercal(s)\sigma^\intercal(s)\sigma(s)\hat{\pi}(s)\right)ds + \int_0^t \rho_n^\intercal(s)\sigma(s)dW(s)$. Moreover, we obtain

$$E\left[|U'(X^{\hat{\pi}}(T))X^{\hat{\pi}}(T)H_n^\rho(T)|\right] \leq E\left[\left(U'(X^{\hat{\pi}}(T))X^{\hat{\pi}}(T)\right)^2\right]^{\frac{1}{2}} E\left[H_n^\rho(T)^2\right]^{\frac{1}{2}} < \infty.$$

Note that for $\forall\varepsilon \in [0,1], \Phi_n(\varepsilon) \geq \Phi_n(1) = U(X^{\hat{\pi}+\rho_n}(T)) - U(X^{\hat{\pi}}(T))$ with $E\left[\Phi_n(1)\right] < \infty$. Therefore the sequence $\Phi_n(\varepsilon)$ is bounded from below. By Monotonic Convergence Theorem, we have

$$\lim_{\varepsilon\to 0}\frac{E\left[U(X^{\hat{\pi}+\varepsilon\rho_n}(T))\right] - E\left[U(X^{\hat{\pi}}(T))\right]}{\varepsilon} = E\left[U'(X^{\hat{\pi}}(T))X^{\hat{\pi}}(T)H_n^\rho(T)\right].$$

In addition, since $\hat{\pi}$ is optimal, we conclude

$$E\left[U'(X^{\hat{\pi}}(T))X^{\hat{\pi}}(T)H_n^\rho(T)\right] \leq 0. \tag{4.14}$$

To this end, let (α,β) be as defined in Lemma 4.3.3 and $(\hat{p}_2), \hat{q}_2)$ be the adjoint process corresponding to $\hat{\pi}$. Applying Ito's lemma to $-\alpha(t)H_n^\rho(t)$, we have

$$
\begin{aligned}
-d\alpha(t)H_n^\rho(t) =& \beta^\intercal(t)H^\rho(t)dW(t) - \alpha(t)\left(\rho_n^\intercal(t)\sigma(t)\theta(t) - \rho_n^\intercal(t)\sigma(t)\sigma^\intercal(t)\hat{\pi}(t)\right)dt \\
& - \alpha(t)\rho_n^\intercal(t)\sigma(t)dW(t) + \rho_n^\intercal(t)\sigma(t)\beta(t)dt \\
=& \Big[-\hat{p}_1(t)X^{\hat{\pi}}(t)\rho_n^\intercal(t)\sigma(t)\Big(\theta(t) - \sigma^\intercal(t)\hat{\pi}(t)\Big) \\
& -\rho_n^\intercal(t)\sigma(t)\left(X^{\hat{\pi}}(t)\hat{q}_1(t) + X^{\hat{\pi}}(t)\hat{p}_1(t)\sigma^\intercal(t)\hat{\pi}(t)\right)\Big]dt \\
& + \left[\beta^\intercal(t)H_n^\rho(t) - \alpha(t)\rho_n^\intercal(t)\sigma(t)\right]dW(t).
\end{aligned}
$$

Rearranging the above equation, we obtain

$$
-d\alpha(t)H_n^\rho(t) = -X^{\hat{\pi}}(t)\rho_n^\intercal(t)\sigma(t)\Bigg(\hat{p}_1(t)\theta(t)+\hat{q}_1(t)\Bigg)dt+\left[\beta^\intercal(t)H_n^\rho(t) - \alpha(t)\rho_n^\intercal(t)\sigma(t)\right]dW(t). \tag{4.15}
$$

Next, we prove that the local martingale $\int_0^t \beta^\intercal(s)H^\rho(s) - \alpha(s)\rho^\intercal(s)\sigma(s)dW(s)$ is indeed a true martingale.

$$
\begin{aligned}
& E\left[\sup_{t\in[0,T]}|H_n^\rho(t)|^2\right] \\
=& E\left[\sup_{t\in[0,T]}\left|\int_0^t\left(\rho_n^\intercal(s)\sigma(s)\theta(s) - \rho_n^\intercal(s)\sigma(s)\sigma^\intercal(s)\hat{\pi}(s)\right)ds + \int_0^t\rho_n^\intercal(s)\sigma(s)dW(s)\right|^2\right] \\
\leq& C\Bigg\{E\left[\sup_{t\in[0,T]}\left|\int_0^t\rho_n^\intercal(s)\sigma(s)dW(s)\right|^2\right] \\
& + E\left[\sup_{t\in[0,T]}\left|\int_0^t\left(\rho_n^\intercal(s)\sigma(s)\theta(s) - \rho_n^\intercal(s)\sigma(s)\sigma^\intercal(s)\hat{\pi}(s)\right)ds\right|^2\right]\Bigg\} \\
\leq& C\left\{E\left[\int_0^T|\rho_n^\intercal(s)\sigma(s)|^2ds\right] + E\left[\int_0^T|\rho_n^\intercal(s)\sigma(s)|^2ds\right] + E\left[\int_0^T|\rho_n^\intercal(s)\sigma(s)\sigma^\intercal(s)\hat{\pi}(s)|^2ds\right]\right\} \\
<& \infty,
\end{aligned}
$$

by Burkeholder-Davis-Gundy inequality. In addition, we have

$$
E\left[\int_0^t|\alpha(s)\rho_n^\intercal(s)\sigma(s)|^2ds\right] < \infty.
$$

Hence, (4.14) can be reduced to the following

$$
E\left[\int_0^{\tau_n} -X^{\hat{\pi}}(t)\rho_n^\intercal(t)\sigma(t)\Bigg(\hat{p}_1(t)\theta(t) + \hat{q}_1(t)\Bigg)dt\right] \leq 0 \;\forall n \in \mathbb{N}. \tag{4.16}
$$

To this end,we define the following sets:

$$B \triangleq \{(t,\omega) \in [0,T] \times \Omega : (\pi^\intercal - \hat{\pi}^\intercal(t))\, \sigma(t)\, (\hat{p}_1(t)\theta(t) + \hat{q}_1(t)) < 0,\ \textit{for}\ \forall \pi \in K\}\,.$$

Moreover, for any $\pi \in K$

$$B^\pi \triangleq \{(t,\omega) \in [0,T] \times \Omega : (\pi^\intercal - \hat{\pi}^\intercal(t))\, \sigma(t)\, (\hat{p}_1(t)\theta(t) + \hat{q}_1(t)) < 0\}\,.$$

Obviously for each $t \in [0,T]$, $B_t^\pi \in \mathcal{F}_t$. Let us consider the control $\tilde{\pi} : [0,T] \times \Omega \to K$ defined by

$$\tilde{\pi}(t,\omega) \triangleq \begin{cases} \pi, & \textit{if } (t,\omega) \in B^\pi \\ \hat{\pi}(t,\omega), & \textit{otherwise.} \end{cases}$$

Then $\tilde{\pi}$ is adapted and $\exists n^* \in \mathbb{N}$ such that

$$E\left[\int_0^{\tau_n} X^{\tilde{\pi}}(t)\, (\tilde{\pi}^\intercal(t) - \hat{\pi}^\intercal(t))\, \sigma(t)\, (\hat{p}_1(t)\theta(t) + \hat{q}_1(t))\, dt\right] < 0\ \forall n > n^*,$$

contradicting (4.16), unless $(Leb \otimes \mathbb{P})\{B^\pi\} = 0$ for $\forall \pi \in K$. Since $\mathbb{R}^N$ is a separable metric space, we denote $\{\pi_n\}$ to be a countable dense subset of K. Consequently, we have $B = \cup_{n=1}^\infty B^{\pi_n}$ and $(Leb \otimes \mathbb{P})\{B\} = (Leb \otimes \mathbb{P})\{\cup_{n=1}^\infty B^{\pi_n}\} \leq \sum_{n=1}^\infty (Leb \otimes \mathbb{P})\{B^{\pi_n}\} = 0$. Hence, we conclude that

$$-X^{\hat{\pi}}(t)\sigma(t)\, [\hat{p}_1(t)\theta(t) + \hat{q}_1(t)] \in N_K(\hat{\pi}(t))\ \textit{for}\ \forall t \in [0,T],\ \mathbb{P} - a.s.$$

□

We now state the sufficient condition for the optimality of primal problem.

Theorem 4.3.5. *(Sufficient condition for the primal problem) Let $\hat{\pi} \in \mathcal{A}$. Then $\hat{\pi}$ is optimal for the primal problem if there exists a solution $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ to the* *following FBSDE:*

$$\begin{cases} dX^{\hat{\pi}}(t) = X^{\hat{\pi}}(t)\{[r(t) + \hat{\pi}^{\intercal}(t)\sigma(t)\theta(t)]dt + \hat{\pi}^{\intercal}(t)\sigma(t)dW(t)\}, \\ X^{\hat{\pi}}(0) = x_0, \\ dp_1(t) = -\left\{[r(t) + \hat{\pi}^{\intercal}(t)\sigma(t)\theta(t)]\, p_1(t) + q_1^{\intercal}(t)\sigma^{\intercal}(t)\hat{\pi}(t)\right\} dt + q_1^{\intercal}(t)dW(t), \\ p_1(T) = -U'(X^{\hat{\pi}}(T)). \end{cases} \tag{4.17}$$

and satisfies

$$-X^{\hat{\pi}}(t)\sigma(t)\left[\hat{p}_1(t)\theta(t) + \hat{q}_1(t)\right] \in N_K(\hat{\pi}(t)) \ \text{for} \ \forall t \in [0,T], \ \mathbb{P}-a.s. \tag{4.18}$$

where $N_K(x)$ is the normal cone to the closed convex set K at $x \in K$, defined as

$$N_K(x) \triangleq \left\{y \in \mathbb{R}^N : \forall x^* \in K, y^{\intercal}(x^* - x) \leq 0\right\}.$$

Proof. Let$(X^{\hat{\pi}}, \hat{\pi})$ be an admissible pair such that $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ is a solution to the FBSDE (4.17) and satisfied condition (4.18). Applying Ito's formula, we have

$$\begin{aligned} &\left(X^{\hat{\pi}}(t) - X^{\pi}(t)\right)\hat{p}_1(t) \\ =&\int_0^t \left(X^{\hat{\pi}}(s) - X^{\pi}(s)\right)\{-\left[(r(s) + \hat{\pi}^{\intercal}(s)\sigma(s)\theta(s))\,\hat{p}_1(s) + \hat{q}_1^{\intercal}(t)\sigma^{\intercal}(t)\pi(t)\right] dt + \hat{q}_1^{\intercal}(t)dW(t)\} \\ &+ \int_0^t \hat{p}_1^{\intercal}(s)\Big\{ \left[X^{\hat{\pi}}(t)\left(r(s) + \hat{\pi}^{\intercal}(s)\sigma(s)\theta(s)\right) - X^{\pi}(s)\left(r(s) + \pi^{\intercal}(s)\sigma(s)\theta(s)\right)\right] ds \\ &+ \left[X^{\hat{\pi}}(s)\hat{\pi}^{\intercal}(s)\sigma(s) - X^{\hat{\pi}}(s)\pi^{\intercal}(s)\sigma(s)\right] dW(s)\Big\} \\ &+ \int_0^t \left[X^{\hat{\pi}}(s)\hat{\pi}^{\intercal}(s)\sigma(s) - X^{\pi}(s)\pi^{\intercal}(s)\sigma(s)\right]\hat{q}_1(s)ds. \end{aligned}$$

Rearranging the above equation, we have

$$\begin{aligned} &\left(X^{\hat{\pi}}(t) - X^{\pi}(t)\right)\hat{p}_1(t) \\ =&\int_0^t \left(\hat{\pi}^{\intercal}(s) - \pi^{\intercal}(s)\right) X^{\hat{\pi}}(s)\sigma(s)\left[\hat{p}_1(s)\theta(s) + \hat{q}_1(s)\right] ds \\ &+ \int_0^t \left[\left(X^{\hat{\pi}}(s) - X^{\pi}(s)\right) q^{\intercal}(s) + X^{\hat{\pi}}(s)\left(\hat{\pi}^{\intercal}(t) - \pi^{\intercal}(t)\right)\sigma(s)\right] dW(s). \end{aligned}$$

Hence, by Condition (4.18) and the definition of normal cone, taking expectation of the above, we have

$$E\left[\left(X^{\hat{\pi}}(T)-X^{\pi}(T)\right)\hat{p}_1(T)\right]\leq 0.$$

Combining with concavity of U gives us

$$\begin{aligned}&E\left[U\left(X^{\pi}(T)\right)-U\left(X^{\hat{\pi}}(T)\right)\right]\leq E\left[\left(X^{\pi}(T)-X^{\hat{\pi}}(T)\right)U'\left(X^{\hat{\pi}}(T)\right)\right]\\&=E\left[\left(X^{\hat{\pi}}(T)-X^{\pi}(T)\right)\hat{p}_1(T)\right]\leq 0.\end{aligned}$$

Hence $\hat{\pi}$ is indeed an optimal control. □

Next we address the dual problem. To establish the existence of an optimal solution, we impose the following condition:

Assumption 4.3.6. *([55, Condition 4.14]) For any $(y,v)\in(0,\infty)\times\mathcal{D}$, we have $E\left[\tilde{U}\left(Y^{(y,v)}(T)\right)^2\right]<\infty$.*

According to [55, Proposition 4.15], there exists some $(\hat{y},\hat{v})\in(0,\infty)\times\mathcal{D}$ such that $\tilde{V}=x_0\hat{y}+E\left[\tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)\right]$. Given admissible control $(\hat{y},\hat{v})\in(0,\infty)\times\mathcal{D}$ with the state process $Y^{(y,v)}$ that solves the SDE (4.7) and $E\left[\tilde{U}\left(Y^{(y,v)}(T)\right)^2\right]<\infty$, the associated adjoint equation for the dual problem is the following linear BSDE in the unknown processes $p_2\in\mathcal{H}^2(0,T;\mathbb{R})$ and $q_2\in\mathcal{H}^2(0,T;\mathbb{R}^N)$:

$$\begin{cases} dp_2(t)=\left\{\left[r(t)+\delta_K(v(t))\right]^\intercal p_2(t)+q_1^\intercal(t)\left[\theta(t)+\sigma^{-1}(t)v(t)\right]\right\}dt+q_2^\intercal(t)dW(t),\\ p_2(T)=-\tilde{U}'(Y^{(y,v)}(T)).\end{cases}\tag{4.19}$$

Lemma 4.3.7. *Let $(y,v)\in(0,\infty)\times\mathcal{D}$ and $Y^{(y,v)}$ be the corresponding state process satisfying the SDE (4.7) with $E\left[\tilde{U}\left(Y^{(y,v)}(T)\right)^2\right]<\infty$. Then the random variable $Y^{(y,v)}(T)\tilde{U}'(Y^{(y,v)}(T))$ is square integrable and there exists a solution to the adjont BSDE (4.19).*

Proof. According to Assumption 4.3.6 (i), we have $E\left[\tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)^2\right] < \infty$. Following similar arguments as in [47, page 290] we have that since $\tilde{U}$ is a decreasing function

$$\begin{aligned}\tilde{U}(\eta) - \tilde{U}(\infty) &\geq \tilde{U}(\eta) - \tilde{U}(\frac{\eta}{\beta}) \\ &= \int_{\eta}^{\frac{\eta}{\beta}} I(d)du \\ &\geq \left(\frac{\eta}{\beta} - \eta\right) I\left(\frac{\eta}{\beta}\right) \\ &\geq \frac{1-\beta}{\beta\gamma}\eta I(\eta),\end{aligned}$$

for $0 < \eta < \infty$, where $\beta \in (0,1)$ and $\gamma \in (1,\infty)$ are as in Condition 4.3.6. Consequently, we conclude that the random variable $Y^{(\hat{y},\hat{v})}(T)\tilde{U}'(Y^{(\hat{y},\hat{v})}(T))$ is square integrable. To this end, define the process

$$\phi(t) \triangleq E\left[-Y^{(\hat{y},\hat{v})}(T)\tilde{U}'(Y^{(\hat{y},\hat{v})}(T))\middle|\mathcal{F}_t\right],\ t \in [0,T].$$

By the martingale representation theorem, it is the unique solution to the BSDE

$$\phi(t) = -Y^{(\hat{y},\hat{v})}(T)\tilde{U}'(Y^{(\hat{y},\hat{v})}(T)) - \int_t^T \varphi^{\intercal}(s)dW(s),$$

where φ is a square integrable process with values in $\mathbb{R}^N$. Applying Ito's formula to $\frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)}$, we have

$$\begin{aligned}d\frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)} =& \left\{\frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)}\left[r(t) + \delta_K(\hat{v}(t)) + |\theta(t) + \sigma^{-1}(t)\hat{v}(t)|^2\right] + \frac{\varphi(t)}{Y^{(\hat{y},\hat{v})}(t)}[\theta(t) + \sigma(t)^{-1}\hat{v}(t)]\right\}dt \\ &+ \left\{\frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)}\left[\theta(t) + \sigma(t)^{-1}\hat{v}(t)\right]^{\intercal} + \frac{\varphi^{\intercal}(t)}{Y^{(\hat{y},\hat{v})}(t)}\right\}dW(t).\end{aligned}$$

Rearranging the above equation, we have

$$d\hat{p}_2(t) = \left\{[r(t) + \delta_K(\hat{v}(t))]^{\intercal}\hat{p}_2(t) + \hat{q}_2(t)\left[\theta(t) + \sigma(t)^{-1}\hat{v}(t)\right]^{\intercal}\right\}dt + \hat{q}_2^{\intercal}(t)dW(t),$$

where $(\hat{p}_2, \hat{q}_2)$ are defined as

$$\hat{p}_2(t) \triangleq \frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)} \ \textit{and}\ \hat{q}_2(t) \triangleq \hat{p}_2(t)\left[\theta(t) + \sigma(t)^{-1}\hat{v}(t)\right]^{\intercal} + \frac{\varphi^{\intercal}(t)}{Y^{(\hat{y},\hat{v})}(t)}.$$

Hence, we conclude that $(\hat{p}_2, \hat{q}_2)$ solves the BSDE (4.19). □

Next, we prove the necessary and sufficient conditions of optimality of the dual problem.

Theorem 4.3.8. *(Necessary condition for optimality of the dual problem) Let $(\hat{y}, \hat{v}) \in (0, \infty) \times \mathcal{D}$ be the optimal control for the dual problem with corresponding state process $Y^{(\hat{y},\hat{v})}$ satisfying the SDE* (4.7)*. Then there exists a triple process $(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2)$ that solves the FBSDE*

$$\begin{cases} dY^{(\hat{y},\hat{v})}(t) = -Y^{(\hat{y},\hat{v})}(t)\left\{[r(t) + \delta_K(\hat{v}(t))]dt + [\theta(t) + \sigma^{-1}(t)\hat{v}(t)]^\intercal dW(t)\right\}, \\ Y^{(\hat{y},\hat{v})}(0) = \hat{y}, \\ dp_2(t) = \left\{[r(t) + \delta_K(v(t))]^\intercal p_2(t) + q_2^\intercal(t)\left[\theta(t) + \sigma^{-1}(t)v(t)\right]\right\}dt + q_2^\intercal(t)dW(t), \\ p_2(T) = -\tilde{U}'(Y^{(y,v)}(T)). \end{cases} \tag{4.20}$$

and satisfies the following conditions

$$\begin{cases} \hat{p}_2(0) = x_0, \\ \hat{p}_2(t)^{-1}\left[\sigma^\intercal(t)\right]^{-1}\hat{q}_2(t) \in K, \\ \hat{p}_2(t)\delta_K(\hat{v}(t)) + \hat{q}_2'(t)\sigma^{-1}(t)\hat{v}(t) = 0, \ \textit{for } \forall t \in [0,T] \ \mathbb{P} - a.s. \end{cases} \tag{4.21}$$

Proof. Let $(\hat{y}, \hat{v})$ be an optimal control of the dual problem and $Y^{(\hat{y},\hat{v})}$ be the corresponding state process. Define function $h(\xi) \triangleq x_0\xi\hat{y} + E\left[\tilde{U}\left(\xi Y^{(\hat{y},\hat{v})}(T)\right)\right]$, and $\inf_{\xi\in(0,\infty)} h(\xi) = h(1)$. Then following the argument in [43, Lemma 11.7, page 725] by the convexity of $\tilde{U}$, the dominated convergence theorem and Lemma 4.3.7, we conclude that $h(\cdot)$ is continuously differentiable at $\xi = 1$ and the derivative $\partial h(1) = x_0\hat{y} + E\left[Y^{(\hat{y},\hat{v})}(T)\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)\right]$ holds. Hence, we conclude that

$$\hat{p}_2(0) = -\frac{1}{\hat{y}}E\left[Y^{(\hat{y},\hat{v})}(T)\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)\right] = x_0. \tag{4.22}$$

Let $(\hat{y}, \tilde{v})$ be an admissible control and $\eta \triangleq \tilde{v} - \hat{v}$. Similar to the argument in [23, page 781-782], let the stopping time $\tau_n \triangleq T \wedge \inf\{t \in [0,T]; \int_0^t \|\delta_K(\eta(s))\|^2 + \|\theta^\intercal(s)\sigma^{-1}(s)\eta(s)\|^2 + \|\hat{v}^\intercal(s)[\sigma^{-1}(s)]^\intercal\sigma^{-1}(s)\eta(s)\|^2 + \|\phi(s)\eta(s)\|^2 ds \geq n$ *or*

$\left|\int_0^t \eta^\intercal(s)[\sigma^{-1}(s)]^\intercal dW(s)\right| \geq 0\}$. Let $\eta_n(t) \triangleq \eta(t)1_{t\leq\tau_n}$. Define function $\tilde{\phi}_n(\varepsilon) \triangleq \tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) = \tilde{U}\left\{\exp\left[\ln\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right)\right]\right\}$. According to Assumption 4.3.1, $g(z) \triangleq \tilde{U}(e^z)$ is a convex function that is non-increasing. Moreover, since δ_K is convex, $f(\varepsilon) \triangleq \ln\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right)$ is a concave function of ε. Hence $\tilde{\phi}_n(\varepsilon) = g(f(\varepsilon))$ is a convex function. Consequently, $\tilde{\Phi}_n(\varepsilon) \triangleq \frac{\tilde{\phi}_n(\varepsilon)-\tilde{\phi}_n(0)}{\varepsilon}$ is an increasing function. To this end we define $\tilde{H}^{\eta_n}_\varepsilon(t)$ and $\tilde{H}^{\eta_n}(t)$ as

$$
\begin{aligned}
\tilde{H}^{\eta_n}_\varepsilon(t) \triangleq & \int_0^t \delta_K(\hat{v}(s)+\varepsilon\eta_n(s)) - \delta_K(\hat{v}(s)) + \varepsilon\theta^\intercal(s)\sigma^{-1}(s)\eta_n(s) \\
& + \varepsilon\hat{v}^\intercal(s)[\sigma^{-1}(s)]^\intercal\sigma^{-1}(s)\eta_n(s) + \frac{1}{2}\varepsilon^2\eta_n^\intercal(s)[\sigma^{-1}(s)]^\intercal\sigma^{-1}(s)\eta_n(s)ds \\
& + \int_0^t \varepsilon\eta_n^\intercal(s)[\sigma^{-1}(s)]^\intercal dW(s), \\
\tilde{H}^{\eta_n}(t) \triangleq & \int_0^t \delta_K(\eta_n(s)) + \theta^\intercal(s)\sigma^{-1}(s)\eta_n(s) + \hat{v}^\intercal(s)[\sigma^{-1}(s)]^\intercal\sigma^{-1}(s)\eta_n(s)ds \\
& + \int_0^t \eta_n^\intercal(s)[\sigma^{-1}(s)]^\intercal dW(s).
\end{aligned}
$$

Let $\varepsilon \in (0,1)$, we have

$$
\begin{aligned}
\tilde{\Phi}_n(\varepsilon) = & \frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) - \tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{\varepsilon} \\
= & \frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) - \tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T) - Y^{(\hat{y},\hat{v})}(T)} \frac{Y^{(\hat{y},\hat{v})}(T)}{\varepsilon}\left[\frac{Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)}{Y^{(\hat{y},\hat{v})}(T)} - 1\right] \\
= & \frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) - \tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T) - Y^{(\hat{y},\hat{v})}(T)} \frac{Y^{(\hat{y},\hat{v})}(T)}{\varepsilon}\left[\exp\left(-\tilde{H}^{\eta_n}_\varepsilon(T)\right) - 1\right] \\
\leq & \frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right) - \tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T) - Y^{(\hat{y},\hat{v})}(T)} \frac{Y^{(\hat{y},\hat{v})}(T)}{\varepsilon} \\
& \left\{-1 + \exp\left[-\varepsilon\int_0^T \left(\delta_K(\eta_n(t)) + \theta^\intercal(t)\sigma^{-1}(t)\eta_n(t) + \hat{v}^\intercal(t)[\sigma^{-1}(t)]^\intercal\sigma^{-1}(t)\eta_n(t)\right.\right.\right. \\
& \left.\left.\left. + \frac{1}{2}\varepsilon\eta_n^\intercal(t)[\sigma^{-1}(t)]^\intercal\sigma^{-1}(t)\eta_n(t)\right)dt - \varepsilon\int_0^T \eta_n^\intercal(t)[\sigma^{-1}(t)]^\intercal dW(t)\right]\right\}.
\end{aligned}
$$

Hence, taking lim sup on both sides, we have

$$
\limsup_{\varepsilon\to 0} \tilde{\Phi}_n(\varepsilon) \leq -\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right) Y^{(\hat{y},\hat{v})}(T)\tilde{H}^{\eta_n}(T)
$$

with

$$E\left[\left|\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)Y^{(\hat{y},\hat{v})}(T)\tilde{H}^{\eta_n}(T)\right|\right] \le E\left[\left(\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)Y^{(\hat{y},\hat{v})}(T)\right)^2\right]^{\frac{1}{2}} E\left[\tilde{H}^{\eta_n}(T)^2\right]^{\frac{1}{2}} < \infty.$$

Moreover, notice that as $\varepsilon \in (0,1)$ approaches zero, the sequence $\left(\frac{\tilde{U}\left(Y^{(\hat{y},\hat{v}+\varepsilon\eta_n)}(T)\right)-\tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)}{\varepsilon}\right)_{\varepsilon\in(0,1)}$ is bounded from above by $|\tilde{\Phi}_n(1)|$ and $E\left[|\tilde{\Phi}_n(1)|\right] < \infty$. Consequently by the reverse fatou's lemma, we have

$$0 \le \limsup_{\varepsilon\to 0} E\left[\tilde{\Phi}_n(\varepsilon)\right] \le E\left[\limsup_{\varepsilon\to 0}\tilde{\Phi}_n(\varepsilon)\right] \le E\left[-\tilde{U}'\left(Y^{(\hat{y},\hat{v})}(T)\right)Y^{(\hat{y},\hat{v})}(T)\tilde{H}^{\eta_n}(T)\right].$$

To this end, let (ϕ,φ) be defined as in Lemma 4.3.7 and $(\hat{p}_2,\hat{q}_2)$ be the adjoint process corresponding to $(\hat{y},\hat{v})$. Apply Ito's lemma to $\phi(t)\tilde{H}^{\eta}_n(t)$, we obtain

$$\begin{aligned}
&d\phi(t)\tilde{H}^{\eta_n}(t)\\
=&-\varphi^{\intercal}(t)\tilde{H}^{\eta_n}(t)dW(t)+\phi(t)\left(\delta_K(\eta_n(t))+\theta^{\intercal}(t)\sigma^{-1}(t)\eta_n(t)+\hat{v}(t)[\sigma^{-1}(t)]^{\intercal}\sigma^{-1}(t)\eta_n(t)\right)dt\\
&+\phi(t)\eta_n^{\intercal}(t)[\sigma^{-1}(t)]^{\intercal}dW(t)+\eta_n^{\intercal}(t)[\sigma^{-1}(t)]^{\intercal}\varphi(t)dt\\
=&Y^{\hat{y},\hat{v}}(t)\left[\delta_K(\eta_n(t))\hat{p}_2(t)+\hat{q}_2(t)\sigma^{-1}(t)\eta_n(t)\right]dt+\left[\phi(t)\eta_n^{\intercal}(t)[\sigma^{-1}(t)]^{\intercal}-\varphi^{\intercal}(t)\tilde{H}^{\eta_n}(t)\right]dW(t)
\end{aligned}$$

Following similar approach as in the proof of necessary condition for the primal problem, it can be shown that $\int_0^t\left[\phi(s)\eta_n^{\intercal}(s)[\sigma^{-1}(s)]^{\intercal}-\varphi^{\intercal}(s)\tilde{H}^{\eta_n}(s)\right]dW(s)$ is a true martingale. Taking expectation of the above equation, we obtain

$$E\left[\int_0^{\tau_n} Y^{(\hat{y},\hat{v})}(t)\left[\delta_K(\eta(t))\hat{p}_2(t)+\hat{q}_2(t)\sigma^{-1}(t)\eta(t))\right]dt\right] \ge 0. \tag{4.23}$$

To this end, note that $\hat{p}_2(t)=\frac{\phi(t)}{Y^{(\hat{y},\hat{v})}(t)}>0$, define the event $B \triangleq \{(\omega,t):\hat{p}_2(t)^{-1}\sigma(t)^{-1}\hat{q}_2(t)\notin K\}$. According to [47, Lemma 5.4.2 on page 207], there exists some $\mathbb{R}^N$ valued progressively measurable process η such that $\|\eta(t)\|\le 1$ and $\|\delta_K(\eta(t))\|\le 1$ a.e. and

$$\delta_K(\eta(t))+\hat{p}_2(t)^{-1}\hat{q}_2^{\intercal}(t)\sigma(t)^{-1}<0 \;\; a.e. \;\; on \; B,$$

$$\delta_K(\eta(t))+\hat{p}_2(t)^{-1}\hat{q}_2^{\intercal}(t)\sigma(t)^{-1}=0 \;\; a.e. \;\; on \; B^c.$$

Let $\tilde{v} \triangleq \hat{v}+\eta$. We can easily verify that $\tilde{v}$ is progressively measurable and square integrable. Hence, we obtain that

$$E\left\{\int_0^{\tau_n} Y^{(\hat{y},\hat{v})}(t)\left[\hat{p}_2(t)\left(\delta_K(\eta(t))\right)+\hat{q}_2^\intercal(t)\sigma(t)^{-1}\eta(t)\right]dt\right\}<0,$$

contradicting (4.23). Hence, by the $\mathbb{P}$ strict positivity of $Y^{(\tilde{y},\tilde{v})}(t)\hat{p}_2(t)$, we conclude that $\hat{p}_2(t)^{-1}\sigma(t)^{-1}\hat{q}_2(t) \in K$ a.e. (this argument is essentially identical to the analysis in the proof of Proposition 4.17 in [55]). Take $\tilde{v} = 2\hat{v}$, and we have

$$E\left\{\int_0^{\tau_n} Y^{(\tilde{y},\tilde{v})}(t)\left[\hat{p}_2(t)\left(\delta_K(\hat{v}(t))\right)+\hat{q}_2^\intercal(t)\sigma(t)^{-1}\hat{v}(t)\right]dt\right\}\geq 0. \tag{4.24}$$

Lastly, to prove the third condition, simply take $\tilde{v} = 0$ and by the same analysis, we obtain

$$E\left\{\int_0^{\tau_n} Y^{(\tilde{y},\tilde{v})}(t)\left[\hat{p}_2(t)\left(\delta_K(\hat{v}(t))\right)+\hat{q}_2^\intercal(t)\sigma(t)^{-1}\hat{v}(t)\right]dt\right\}\leq 0.$$

On the other hand, by the definition of δ_K, we have $\delta_K(\hat{v}(t))+\hat{p}_2(t)^{-1}\hat{q}_2^\intercal(t)\sigma^{-1}(t)\hat{v}(t) \geq 0$ a.e. Combining with the $\mathbb{P}$ strict positivity of $Y^{(\tilde{y},\tilde{v})}(t)\hat{p}_2(t)$ gives the last condition. □

Theorem 4.3.9. *(Sufficient conditions for optimality of the dual problem) Let $(\hat{y},\hat{v}) \in (0,\infty)\times\mathcal{D}$. Then $(\hat{y},\hat{v})$ is optimal for the dual problem if $(Y^{(\hat{y},\hat{v})},\hat{p}_2,\hat{q}_2)$ solves the FBSDE*

$$\begin{cases} dY^{(\hat{y},\hat{v})}(t) = -Y^{(\hat{y},\hat{v})}(t)\left\{[r(t)+\delta_K(\hat{v}(t))]dt + [\theta(t)+\sigma^{-1}(t)\hat{v}(t)]^\intercal dW(t),\right\} \\ Y^{(\hat{y},\hat{v})}(0) = \hat{y}, \\ dp_2(t) = \left\{[r(t)+\delta_K(v(t))]^\intercal p_2(t) + q_2^\intercal(t)\left[\theta(t)+\sigma^{-1}(t)v(t)\right]\right\}dt + q_2^\intercal(t)dW(t), \\ p_2(T) = -\tilde{U}'(Y^{(y,v)}(T)). \end{cases} \tag{4.25}$$

and satisfies the following conditions

$$\begin{cases} \hat{p}_2(0) = x_0, \\ \hat{p}_2(t)^{-1}\left[\sigma^\intercal(t)\right]^{-1}\hat{q}_2(t) \in K, \\ \hat{p}_2(t)\delta_K(\hat{v}(t)) + \hat{q}_2^\intercal(t)\sigma^{-1}(t)\hat{v}(t) = 0, \ \textit{for } \forall t\in[0,T]\ \mathbb{P}-a.s. \end{cases} \tag{4.26}$$

Proof. Let $(\hat{y}, \hat{v})$ be an admissible control such that $\left(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2\right)$ solves the FBSDE (4.25) and satisfies conditions (4.26). Let the pair $(\tilde{y}, \tilde{v}) \in (0, \infty) \times \mathcal{D}$ be a given admissible control such that $Y^{(\hat{y},\hat{v})}$ solves the SDE (4.7) and $E\left[\tilde{U}(Y^{(\tilde{y},\tilde{v})}(T))^2\right] < \infty$. By Lemma 4.3.7, we claim that there exists adjoint process $(\tilde{p}_2, \tilde{q}_2)$ that solves the BSDE with control $(\tilde{y}, \tilde{v})$. Applying Ito's formula, we have

$$
\begin{aligned}
&\left(Y^{(\hat{y},\hat{v})}(t) - Y^{(\tilde{y},\tilde{v})}(t)\right) \hat{p}_2(t) \\
=&\hat{p}_2(0)y + \int_0^t \left\{Y^{(\tilde{y},\tilde{v})}(s)\left[r(s) + \delta_K(\tilde{v}(s))\right]^\intercal - Y^{(\hat{y},\hat{v})}(s)\left[r(s) + \delta_K(\hat{v}(s))\right]^\intercal\right\} \hat{p}_2(s)ds \\
&+ \int_0^t \left\{Y^{(\tilde{y},\tilde{v})}(s)[\theta(s) + \sigma^{-1}(s)\tilde{v}(s)]^\intercal - Y^{(\hat{y},\hat{v})}(s)[\theta(s) + \sigma^{-1}(s)\hat{v}(s)]^\intercal\right\} \hat{p}_2(s)dW(s) \\
&+ \int_0^t \left(Y^{(\hat{y},\hat{v})}(s) - Y^{(\tilde{y},\tilde{v})}(s)\right) \left\{[r(s) + \delta_K(\tilde{v}(s))]^\intercal \hat{p}_2(s) + \hat{q}_2^\intercal(s)\left[\theta(s) + \sigma^{-1}(s)\hat{v}(s)\right]\right\} ds \\
&+ \int_0^t \left(Y^{(\hat{y},\hat{v})}(s) - Y^{(\tilde{y},\tilde{v})}(s)\right) \hat{q}_2^\intercal(s)dW(s) \\
&+ \int_0^t \left\{Y^{(\tilde{y},\tilde{v})}(s)[\theta(s) + \sigma^{-1}(s)\tilde{v}(s)]^\intercal - Y^{(\hat{y},\hat{v})}(s)[\theta(s) + \sigma^{-1}(s)\hat{v}(s)]^\intercal\right\} \hat{q}_2(s)ds \\
=&\hat{p}_2(0)y + \int_0^t Y^{(\tilde{y},\tilde{v})}(s)\hat{p}_2(s)\left[\delta_K(\tilde{v}(s)) - \delta_K(\hat{v}(s)) + \hat{q}_2^\intercal(s)\sigma^{-1}(s)\left(\tilde{v}(s) - \hat{v}(s)\right)\right] ds \\
&+ \int_0^t \left\{Y^{(\tilde{y},\tilde{v})}(s)[\theta(s) + \sigma^{-1}(s)\tilde{v}(s)]^\intercal - Y^{(\hat{y},\hat{v})}(s)[\theta(s) + \sigma^{-1}(s)\hat{v}(s)]^\intercal\right\} \hat{p}_2(s)dW(s) \\
&+ \int_0^t \left(Y^{(\hat{y},\hat{v})}(s) - Y^{(\tilde{y},\tilde{v})}(s)\right) \hat{q}_2^\intercal(s)dW(s).
\end{aligned}
$$

By (4.26) and taking expectation, we have

$$E\left[\left(Y^{(\hat{y},\hat{v})}(T) - Y^{(\tilde{y},\tilde{v})}(T)\right) \hat{p}_2(T)\right] \geq y\hat{p}_2(0).$$

Consequently, by convexity of $\tilde{U}$ we obtain

$$x_0\tilde{y} + E\left[\tilde{U}(Y^{(\tilde{y},\tilde{v})}(T))\right] - x_0\hat{y} - E\left[\tilde{U}(Y^{(\hat{y},\hat{v})}(T))\right] \geq y(x_0 - \hat{p}_2(0)) = 0.$$

Hence, we conclude that $(\hat{y}, \hat{v})$ is indeed an optimal control of the dual problem.

□

We can now state the dynamic relations of the primal portfolio and wealth processes of the primal problem and the adjoint processes of the dual problem and vice versa.

Theorem 4.3.10. *(From dual problem to primal problem) Suppose that* $(\hat{y}, \hat{v})$ *is optimal for the dual problem. Let* $\left(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2\right)$ *be the associated process that solves the FBSDE* (4.20) *and satisfies condition* (4.21)*. Define*

$$\hat{\pi}(t) \triangleq \frac{\left[\sigma^{\intercal}(t)\right]^{-1} \hat{q}_2(t)}{\hat{p}_2(t)}, \ t \in [0, T]. \tag{4.27}$$

Then $\hat{\pi}$ *is the optimal control for the primal problem with initial wealth* x_0*. The optimal wealth process and associated adjoint process are given by*

$$\begin{cases} X^{\hat{\pi}}(t) = \hat{p}_2(t), \\ \hat{p}_1(t) = -Y^{(\hat{y},\hat{v})}(t), \\ \hat{q}_1(t) = Y^{(\hat{y},\hat{v})}(t)[\sigma^{-1}(t)\hat{v}(t) + \theta(t)]. \end{cases} \tag{4.28}$$

Proof. Suppose that $(\hat{y}, \hat{v}) \in (0.\infty) \times \mathcal{D}$ is optimal for the dual problem. By Theorem 4.3.8, the process $\left(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2\right)$ solves the dual FBSDE (4.20) and satisfies condition (4.21). Construct $\hat{\pi}$ and $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ as in (4.27) and (4.28), respectively. Substituting back into the (4.17), we conclude that $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ solves the FBSDE for the primal problem. Moreover, by (4.21) it can be easily shown that $\hat{\pi} \in \mathcal{A}$ and (4.18) holds. By condition (4.21), it can be easily shown that $\pi \in \mathcal{A}$. Moreover, we have

$$\begin{aligned} & X^{\hat{\pi}}(t)\sigma(t)\left[\hat{p}_1(t)\theta(t) + \hat{q}_1(t)\right] \\ & = \hat{p}_2(t)\sigma(t)\left\{-Y^{(\hat{y},\hat{v})}(t)\theta(t) + Y^{(\hat{y},\hat{v})}(t)\left[\sigma^{-1}(t)\hat{v}(t) + \theta(t)\right]\right\} \\ & = Y^{(\hat{y},\hat{v})}(t)\hat{p}_2(t)\hat{v}(t). \end{aligned}$$

Combining with the third statement of (4.21) and the almost surely positivity of $Y^{(\hat{y},\hat{v})}\hat{p}_2$, we claim that condition (4.18) holds. Consequently, by Theorem 4.3.9 we conclude that $\hat{\pi}$ is indeed an optimal control to the primal problem. □

Theorem 4.3.11. *(From primal problem to dual problem) Suppose that $\hat{\pi} \in \mathcal{A}$ is optimal for the primal problem with initial wealth x_0. Let $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ be the associated process that satisfies the FBSDE* (4.11) *and conditions* (4.12)*. Define*

$$\begin{cases} \hat{y} \triangleq -\hat{p}_1(0), \\ \hat{v}(t) \triangleq -\sigma(t) \left[\dfrac{\hat{q}_1(t)}{\hat{p}_1(t)} + \theta(t) \right], \ \textit{for } \forall t \in [0, T]. \end{cases} \tag{4.29}$$

Then $(\hat{y}, \hat{v})$ is an optimal control for the dual problem. The optimal dual state process and associated adjoint process are given by

$$\begin{cases} Y^{(\hat{y},\hat{v})}(t) = -\hat{p}_1(t), \\ \hat{p}_2(t) = X^{\hat{\pi}}(t), \\ \hat{q}_2(t) = \sigma^{\intercal}(t)\hat{\pi}(t)X^{\hat{\pi}}(t). \end{cases} \tag{4.30}$$

Proof. Suppose that $\hat{\pi} \in \mathcal{A}$ is an optimal control for the primal problem. By Theorem 4.3.4, the process $(X^{\hat{\pi}}, \hat{p}_1, \hat{q}_1)$ solves that FBSDE (4.11) and satisfies conditions (4.12). Define $(\hat{y}, \hat{v})$ and $(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2)$ as in (4.29) and (4.30), respectively. Substituting them back into (4.25), we obtain that $(Y^{(\hat{y},\hat{v})}, \hat{p}_2, \hat{q}_2)$ solves the FBSDE for the dual problem. Moreover, by the construction in (4.29) and (4.30), we have $\hat{p}_2(0) = x_0$ and $[\sigma^{\intercal}(t)]^{-1}\hat{q}_2(t) = \hat{\pi}(t)X^{\hat{\pi}}(t)^{-1} \in K$. Last but not least, substituting $\hat{v}$ into (4.26), it is trivial to prove that the third statement in (4.26) holds. Hence, by Theorem 4.3.9, we conclude that $(\hat{y}, \hat{v})$ is indeed an optimal control to the dual problem. □

4.4 Applications

In this section, we shall use the results introduced in previous sections to address several classical constrained utility maximization problems.

4.4.1 Constrained Power Utility Maximization

In this subsection, we assume U is a power utility function defined by $U(x) \triangleq \frac{1}{\beta}x^{\beta},\ x \in (0,\infty)$, where $\beta \in (0,1)$ is a constant. In addition, we assume that $K \subseteq \mathbb{R}^N$ is a closed convex cone. In this case, the dual problem can be written as

$$Minimize\ x_0 y + E\left[\tilde{U}\left(Y^{(y,v)}(T)\right)\right]$$

over $(y,v) \in (0,\infty)\times\mathcal{D}$, where $\tilde{U}(y) = \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}},\ y \in (0,\infty)$. We solve the above problem in two steps: first fix y and find the optimal control $\hat{v}(y)$; second find the optimal $\hat{y}$. We can then construct the optimal solution explicitly.

Step 1: Consider the associated HJB equation:

$$\begin{cases} v_t(s,y) - r(s)yv_y(s,y) + \frac{1}{2}\inf_{v\in\tilde{K}}\|\sigma^{-1}(s)v + \theta(s)\|^2 y^2 v_{yy}(s,y) = 0 \\ v(T,y) = \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}, \end{cases} \tag{4.31}$$

for each $(s,y) \in [t,T]\times\mathbb{R}$. The infimum term in (4.31) can be written explicitly as $\hat{v}(s) = \sigma(s)proj[-\theta(s)|\sigma^{-1}(s)\tilde{K}]$. Then the HJB equation (4.31) becomes

$$\begin{cases} v_t(s,y) - r(s)yv_y(s,y) + \frac{1}{2}y^2\theta_v^2(s)v_{yy}(s,y) = 0 \\ v(T,y) = \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}, \end{cases}$$

where $\theta_{\hat{v}}(s) = \theta(s) + \sigma^{-1}(s)\hat{v}(s)$.

According to the Feymann-Kac formula, we have

$$\begin{aligned} v(t,y) &= E\left[\frac{1-\beta}{\beta}Y^{\frac{\beta}{\beta-1}}(T)\right] \\ &= \frac{1-\beta}{\beta}y^{\frac{\beta}{\beta-1}}\exp\left\{\int_t^T\left[\frac{1}{2}\frac{\beta}{(\beta-1)^2}\theta_v^2(s) - \frac{\beta}{\beta-1}r(s)\right]ds\right\}, \end{aligned}$$

where the stochastic process Y follows the geometric Brownian motion

$$dY(t) = -Y(t)[r(t)dt + \theta_v(t)dW(t)],\ Y(0) = y.$$

In particular, we have $v(0,y) = y^{\frac{\beta}{\beta-1}}\exp\left\{\int_0^T\left[\frac{1}{2}\frac{\beta}{(\beta-1)^2}\theta_{\hat{v}}^2(s) - \frac{\beta}{\beta-1}r(s)\right]ds\right\}$

Step 2: Solving the following static optimization problem

$$\inf_{y\in\mathbb{R}} x_0 y + y^{\frac{\beta}{\beta-1}} \exp\left\{\int_0^T \left[\frac{1}{2}\frac{\beta}{(\beta-1)^2}\theta_{\hat{v}}^2(s) - \frac{\beta}{\beta-1}r(s)\right] ds\right\},$$

we obtain

$$\hat{y} = x_0^{\beta-1} \exp\left\{(1-\beta)\int_0^T \left[\frac{\beta}{2(\beta-1)^2}\theta_v^2(s) - \frac{\beta}{\beta-1}r(s)\right] ds\right\}. \quad (4.32)$$

Consequently, solving the adjoint BSDE, we have

$$\hat{p}_2(t) = x_0 \exp \int_0^t \left[r(s) + \frac{(1-2\beta)}{2(1-\beta)^2}\theta_{\hat{v}}(s)^2\right] ds + \frac{1}{1-\beta}\int_0^t \theta_{\hat{v}}(s)dW(s), \quad (4.33)$$

$$\hat{q}_2(t) = \frac{\theta_{\hat{v}}(t)}{1-\beta}\hat{p}_2(t). \quad (4.34)$$

To this end, applying Theorem 4.3.10, we can construct the optimal solution to the primal problem using the optimal solutions of the dual problem and hence arrive at the following closed form solutions:

$$\begin{cases} \hat{\pi}(t) = [\sigma(t)^\intercal]^{-1}\dfrac{\theta_{\hat{v}}(t)}{1-\beta}, \\ X^{\hat{\pi}}(t) = x_0 \exp\left\{\displaystyle\int_0^t \left[r(s) + \frac{(1-2\beta)}{2(1-\beta)^2}\theta_{\hat{v}}(s)^2\right] ds + \frac{1}{1-\beta}\int_0^t \theta_{\hat{v}}(s)dW(s)\right\}. \end{cases}$$

4.4.2 Constrained Log Utility Maximization with Random Coefficients

In this section, we assume that U is a log utility function defined by $U(x) = \log x$ for $x > 0$. The dual function of U is defined as $\tilde{U}(y) \triangleq -(1+\log y),\ y \geq 0$. Assume that $K \subseteq \mathbb{R}^N$ is a closed convex set and r, b, σ are uniformly bounded $\{\mathcal{F}_t\}$ progressively measurable processes on $\Omega \times [0,T]$.

Step 1: We fix y and attempt to solve for the optimal control $\hat{v}(y)$. Note that the dynamic programming technique is not appropriate in this case due to the non Markovian nature of the problem. However, following the approach in [23, Section 11, p.790] the problem can be solved explicitly due to the special property of the logarithmic function.

Let $v \in \mathcal{D}$ be any given admissible control and the objective function becomes

$$
\begin{aligned}
&x_0 y + E\left[\tilde{U}\left(Y^{(y,v)}(T)\right)\right] \\
&= x_0 y - 1 - \log y - E\left[\int_0^T r(t) + \delta_K(v(t)) + \frac{1}{2}\|\theta(t) + \sigma(t)v(t)\|^2 dt\right].
\end{aligned}
$$

Consequently, the dual optimization boils down to the following problem of pointwise minimization of a convex function $\delta_K(v) + \frac{1}{2}\|\theta(t) + \sigma(t)v\|^2$ over $v \in \tilde{K}$ for $\forall t \in [0, T]$. Applying classical measurable selection theorem (see [76] and [77]), we conclude that the process defined by

$$
\hat{v}(t) \triangleq \arg\min_{v \in \tilde{K}} \left[\delta_K(v) + \frac{1}{2}\|\theta(t) + \sigma(t)^{-1}v\|^2\right] \tag{4.35}
$$

is $\{\mathcal{F}_t\}$ progressively measurable and therefore is the optimal control given y.

Step 2: Solve the following static optimization problem

$$
\inf_{y \in \mathbb{R}} x_0 y - 1 - \log y - E\left[\int_0^T r(t) + \delta_K(\hat{v}(t)) + \frac{1}{2}\|\theta(t) + \sigma(t)v(t)\|^2 dt\right].
$$

We obtain $\hat{y} = \frac{1}{x_0}$. Hence, the optimal state process for the dual problem is the exponential process satisfying (4.7).

Consequently, solving the adjoint BSDE (4.19), we have

$$
\hat{p}_2(t) Y^{(\hat{y},\hat{v})}(t) = E\left[-Y^{(\hat{y},\hat{v})}(T)\tilde{U}\left(Y^{(\hat{y},\hat{v})}(T)\right)\middle|\mathcal{F}_t\right] = 1. \tag{4.36}
$$

Hence, we have $\hat{p}_2(t) = Y^{(\hat{y},\hat{v})}(T)^{-1}$. Applying Ito's formula on $\hat{p}_2$, we have

$$
\hat{q}_2(t) = Y^{(\hat{y},\hat{v})}(t)^{-1}[\theta(t) + \sigma(t)^{-1}\hat{v}(t)] \; for \; \forall t \in [0, T], \; a.e.
$$

Finally, according to Theorem 4.3.10, we construct the optimal control to the primal problem explicitly form the optimal solution of the dual problem as

$$\hat{\pi}(t) = [\sigma(t)\sigma^{\intercal}(t)]^{-1}\left[\hat{v}(t) + b(t) - r(t)\mathbf{1}\right] \ \textit{for}\ \forall t \in [0,T],\ \textit{a.e.} \tag{4.37}$$

Remark 4.4.1. *In the case where K is a closed convex cone, it is trivial to see that $\delta_K(\hat{v}(t)) = 0$ for $\forall t \in [0,T]$. Then the pointwise minimization problem* (4.35) *becomes an even simpler constrained quadratic minimization problem*

$$\hat{v}(t) \triangleq \arg\min_{v\in\tilde{K}} \|\theta(t) + \sigma(t)^{-1}v\|^2, \forall t \in [0,T].$$

Furthermore, in the case where $K = \mathbb{R}^N$ and $\hat{v} = 0$, the optimal control (4.37) *reduces to $\hat{\pi}(t) = [\sigma(t)\sigma^{\intercal}(t)]^{-1}\left[b(t) - r(t)\mathbf{1}\right]$ for $\forall t \in [0,T]$, and we recover the unconstrained log utility maximization problem discussed in [45].*

Remark 4.4.2. *From the above two examples, we contrast our method to the approach in [23, 43, 47], which rely on the introduction of a family of auxiliary unconstrained problems formulated in auxiliary markets parametrized by money market and stock mean return rates [23, see Section 8]. The existence of a solution to the original problem is then equivalent to finding the fictitious market that provides the correct optimal solution to the primal problem. On the other hand, we explicitly write our the dual problem to the original constrained problem only relying on elementary convex analysis results and characterize its solution in terms of FBSDEs. The dynamic relationship between the primal and dual FBSDEs then allows us to explicitly construct optimal solution to the primal problem from that to the dual problem.*

4.4.3 Constrained Non-HARA Utility Maximization

In this subsection, we assume U is a Non HARA utility function defined by $U(x) = \frac{1}{3}H(x)^{-3} + H(x)^{-1} + xH(x)$ for $x > 0$, where $H(x) = \left(\frac{2}{-1+\sqrt{1+4x}}\right)^{\frac{1}{2}}$. The dual function of U is defined as $\tilde{U} \triangleq \sup_{x>0}[U(x)-xy] = \frac{1}{3}y^{-3}+y^{-1},\ y \in [0,\infty)$. Assume that $K \subseteq \mathbb{R}^N$ is a closed convex cone and r, b, σ are constants. Hence, the dual problem becomes

$$\textit{Minimize } x_0 y + E\left[\frac{1}{3}\left(Y^{(y,v)}(T)\right)^{-3} + \left(Y^{(y,v)}(T)\right)^{-1}\right] \textit{ over } (y,x) \in (0,\infty) \times \mathcal{D}.$$

We solve the above problem in two steps: first, fix y and find the optimal control $\tilde{v}(y)$; second, find the optimal $\hat{y}$. We can then construct the optimal solution explicitly.

Step 1: Consider the associated HJB equation:

$$\begin{cases} v_t(s,y) - ryv_y(s,y) + \frac{1}{2}\inf_{v\in\tilde{K}} \|\sigma^{-1}v + \theta\|^2 y^2 v_{yy}(s,y) = 0, \\ \textit{where } (s,y) \in (0,T) \times [0,\infty), \\ v(T,y) = \frac{1}{3}y^{-3} + y^{-1}, \end{cases} \tag{4.38}$$

for each $(s,y) \in [t,T] \times [0,\infty)$. Let $\hat{v}$ be the minimizer of $\inf_{v\in\tilde{K}} |\theta + \sigma^{-1}v|^2$ and $\hat{\theta} \triangleq \theta + \sigma^{-1}\hat{v}$. To this end, define $w(\tau,y) \triangleq v(s,y)$ with $\tau = T - s$. Consequently, w solves the following PDE:

$$\begin{cases} w_t(\tau,y) + ryw_y(\tau,y) - \frac{1}{2}\hat{\theta}^2 y^2 w_{yy}(\tau,y) = 0, \ \textit{where } (\tau,y) \in (0,T) \times (0,\infty), \\ w(0,y) = \frac{1}{3}y^{-3} + y^{-1}, \end{cases} \tag{4.39}$$

Next, we follow the approach in [7], we solve the above PDE. Let $\alpha = \frac{1}{2} + \frac{r}{\hat{\theta}^2}$, $a = \frac{1}{\sqrt{2}}\hat{\theta}, \beta = -a^2\alpha^2$, and $\hat{w}(s,z) = e^{-az+\beta s}w(t,e^z)$, then $\hat{w}$ solves the heat equation $\hat{w}_t - a^2\hat{w}_{zz} = 0$ and has the initial condition $\hat{w}(0,z) = e^{-az}\left(\frac{e^{-3z}}{3} + e^{-z}\right)$. Using Possion's formula to find $w(s,z)$ and $v(s,y)$, we have

$$v(s,y) = \frac{1}{3}y^{-3}e^{3r(T-s)+6\hat{\theta}^2(T-s)} + \frac{1}{y}e^{r(T-s)+\hat{\theta}^2(T-s)}.$$

Step 2: Considering the following static optimization problem:

$$\inf_{y\in(0,\infty)} x_0 y + \frac{1}{3}y^{-3}e^{3rT+6\hat{\theta}^2 T} + \frac{1}{y}e^{rT+\hat{\theta}^2 T}. \tag{4.40}$$

Solving (4.40), we have

$$-\hat{y}^{-4}e^{3rT+6\hat{\theta}^2 T} - \hat{y}^{-2}e^{rT+\hat{\theta}^2 T} + x_0 = 0. \tag{4.41}$$

Hence, we have $\hat{y} = \frac{1}{\sqrt{2x_0}}\left[e^{(r+\hat{\theta}^2)T} + \sqrt{e^{2(r+\hat{\theta}^2)T} + 4x_0 e^{3(r+2\hat{\theta}^2)T}}\right]^{\frac{1}{2}}$, and the optimal state process for the dual problem is given by

$$\hat{Y}(t) = \frac{1}{\sqrt{2x_0}}\left[e^{(r+\hat{\theta}^2)T} + \sqrt{e^{2(r+\hat{\theta}^2)T} + 4x_0 e^{3(r+2\hat{\theta}^2)T}}\right]^{\frac{1}{2}} e^{(r-\frac{\hat{\theta}^2}{2})t+\hat{\theta}W(t)}. \tag{4.42}$$

Consequently, solving the adjoint BSDE, we have

$$\begin{aligned}\hat{p}_2(t)\hat{Y}(t) &= E\left[\hat{Y}(T)^{-3} + \hat{Y}(T)^{-1}|\mathcal{F}_t\right] \\ &= \hat{y}^{-3}e^{-3(r-\frac{\hat{\theta}^2}{2})T}e^{-3\hat{\theta}W(t)}e^{\frac{9}{2}\hat{\theta}^2(T-t)} + \hat{y}^{-1}e^{-(r-\frac{\hat{\theta}^2}{2})T}e^{-\hat{\theta}W(t)}e^{\frac{1}{2}\hat{\theta}^2(T-t)}\end{aligned}$$

Substituting (4.42) back into the above equation and rearranging, we have

$$\hat{p}_2(t) = \hat{y}^{-4}e^{-3(r+2\hat{\theta}^2)T}e^{-rt-4\hat{\theta}^2 t-4\hat{\theta}W(t)} + \hat{y}^{-1}e^{-rT}e^{-rt-2\hat{\theta}W(t)} \tag{4.43}$$

Applying Ito's formula, we have

$$d\hat{p}_2(t) = \left[-r\hat{p}_2(t) + 4a_1\hat{\theta}^2 S_1(t) + 2a_2\hat{\theta}^2 S_2(t)\right]dt - \left(4a_1\hat{\theta}S_1(t) + 2a_2\hat{\theta}S_2(t)\right)dW(t),$$

where $a_1 = \hat{y}^{-4}e^{-3(r+\hat{\theta}^2)T}$, $a_2 = \hat{y}^{-1}e^{-rT}$, $S_1(t) = e^{-rt-4\hat{\theta}^2 t-4\hat{\theta}W(t)}$ and $S_2(t) = e^{-rt-2\hat{\theta}W(t)}$ for $t \in [0,T]$. Consequently, we have

$$\hat{q}_2(t) = -4a_1\hat{\theta}S_1(t) - 2a_2\hat{\theta}S_2(t),\ t\in[0,T]. \tag{4.44}$$

Finally, according to Theorem 4.3.10, we can construction the optimal solution of the primal problem explicitly from optimal solution to the dual problem as

$$\begin{cases} \hat{\pi}(t) = [\sigma^{\intercal}]^{-1}\hat{q}_2(t)\hat{p}_2^{-1}(t), \\ X^{\hat{\pi}}(t) = -\hat{p}_2(t) = \hat{y}^{-4}e^{-3(r+\hat{\theta}^2)T}e^{-rt-4\hat{\theta}^2 t-4\hat{\theta}W(t)} + \hat{y}^{-1}e^{-rT}e^{-rt-2\hat{\theta}W(t)}. \end{cases}$$

Remark 4.4.3. *Suppose that after attaining $\hat{y}$ and v, we try to recover the optimal solution to the primal problem directly. By the duality relationship between the primal and dual value functions [7, see Theorem 2.6], we have*

$$u(t,x) = v(t,\hat{y}(x)) + v_y(t,\hat{y}(x))\hat{y}(x) = \frac{2}{3}\left(\hat{y}(x)^{-1}e^{(r+\hat{\theta}^2)t} + 2x\hat{y}(x)\right).$$

Hence, to get $(\hat{\pi}, X^{\hat{\pi}})$, we would need to solve the following optimization problem on the Hamiltonian function:

$$\hat{\pi}(t) = \arg\min_{\pi\in K}\left[\left(r(t) + \pi'\sigma(t)\theta(t)\right)u_x(t,x) + \frac{1}{2}tr\left(\sigma\sigma^\intercal u_{xx}(t,x)\right)\right].$$

and substituting the above back to the SDE (4.3)*, which appears to be highly complicated equation to solve. However, in the approach we proposed, the optimal adjoint processes of the dual problem can be written out explicitly as conditional expectations of the dual state process. Consequently, the optimal solution to the primal problem can be constructed explicitly thanks to the dynamic relationship as stated in Theorem 4.3.10.*

4.5 Conclusion

In this chapter, we study constrained utility maximization problem following the convex duality approach. After formulating the primal and dual problems, we construct the necessary and sufficient conditions for both the primal and dual problems in terms of FBSDEs plus additional conditions. Such formulation then allows us to establish an explicit connection between primal and dual optimal solutions in a dynamic fashion. Finally we solve three constrained utility maximization problems using the dynamic convex duality approach we proposed above.

Chapter 5

Conclusions and Future Research

5.1 Conclusion

In this thesis, we study the theoretical aspect of stochastic optimal control theory and its applications in continuous-time portfolio optimization problems.

In the first part of the thesis, we focus on the stochastic maxim principle. and proved a weak version of the necessary and sufficient stochastic maximum principle in a regime-switching diffusion model. Instead of insisting on the maximum condition of the Hamiltonian, we show that the optimal control is a stationary point of the Hamiltonian function. This statement allows us to remove the requirement of the second order differentiability of the functions in the control variable. Under certain concavity conditions on the Hamiltonian, the necessary condition becomes sufficient. The absence of the second order adjoint equation considerably simplifies the SMP.

In the second part of the thesis, we turned our focus to portfolio optimization problems. We first look at a continuous-time constrained quadratic risk minimization problem with random market coefficients. Under the convex duality framework, we derive the necessary and sufficient optimality conditions for primal and dual problems in terms of FBSDEs plus additional conditions.

These results allows us to establish an explicit connection between primal and dual problems in terms of their associated forward backward systems. Following this approach, we solve both the unconstrained and cone-constrained quadratic risk minimization problems. We recover the solutions to the extended SREs introduced in the literature from the optimal solutions to the dual problem and find the closed-form solutions to the extended SREs when the coefficients are deterministic.

Finally, we turned our attention to constrained utility maximization problems. With the tools from convex optimization and stochastic analysis, we construct the necessary and sufficient conditions for both the primal and dual problems in terms of FBSDEs and establish an explicit connection between them. Moreover, using the dynamic convex duality approach we proposed above, we solve three constrained utility maximization problems.

In the rest part of this chapter, we propose two interesting topics for potential future research.

5.2 Numerical methods for stochastic optimization

5.2.1 Introduction

Explicit solutions to stochastic control problems are rare in real world applications, especially when it comes to problems in quantitative finance where one faces the problem of portfolio constrains, transaction costs, partial information, etc. This has led to an important area of research on numerical methods for stochastic optimization. One approach relies on solving directly solving the HJB equation satisfied by the value function using finite difference or finite elements methods. For recent results under this direction, please refer to Barles-Jakobsen

[2] and Krylov [50] and references therein. Alternatively, a probabilistic approach has been introduced by [40]. The essential idea was to approximate the original stochastic control problem by Markov chains on a lattice satisfying local consistency condition. The algorithm was then developed by applying dynamic programming principle to a approximation. However, both of the above approaches are limited to low dimensional problems. Finding an efficient numerical scheme for high dimensional stochastic control problems remains an open problem.

5.2.2 An FBSDE approach

In the first part of the thesis, we study the stochastic maximum principle, which seeks to establish the connection between stochastic optimal control problems and backward stochastic differential equations coupled with static optimality conditions on the Hamiltonian. Suppose that the controlled state process is given by

$$\begin{cases} dX(t) = b(t, X(t), \pi(t))dt + \sigma(t, X(t), \pi(t))dW(t) \\ X(0) = x \end{cases}$$

where the coefficients b, σ are $\mathcal{F}_t$ measurable and Lipschitz in x and π. We seek to maximize the gain functional defined by

$$J(\pi) = E\left[\int_0^T f(t, X(t), \pi(t))dt + g(X(T))\right],$$

where f and g are $\mathcal{F}_t$ and $\mathcal{F}_T$ measurable, respectively. Define the Hamiltonian function $\mathcal{H} : \Omega \times [0, T] \times \mathbb{R}^n \times K \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \to \mathbb{R}$ as

$$\mathcal{H}(t, x, \pi, p, q) \triangleq b(t, x, \pi)p + tr(\sigma^\intercal(t, x, \pi)q) + f(t, x, \pi).$$

The sufficient stochastic maximum principle states that

Theorem 5.2.1. *(Theorem 6.4.6 of [69]) Assume that g is concave in x. Let $(\hat{\pi}, \hat{X})$ be an admissible pair and $(\hat{p}, \hat{q})$ satisfies the BSDE*

$$\begin{cases} d\hat{p}(t) = -\mathcal{H}_x(t, \hat{X}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t))dt + \hat{q}(t)dW(t) \\ \hat{p}(T) = g_x(\hat{X}(T)), \end{cases}$$

such that

$$\mathcal{H}(t, \hat{X}(t), \hat{\pi}(t), \hat{p}(t), \hat{q}(t)) = \max_{\pi \in K} \mathcal{H}(t, \hat{X}(t), \pi, \hat{p}(t), \hat{q}(t)) \ 0 \leq t \leq T \ a.s.$$

and

$$(x, \pi) \to \mathcal{H}(t, x, \pi, \hat{p}(t), \hat{q}(t)) \text{ is a concave function}$$

for all $t \in [0, T]$. Then $\hat{\pi}$ is an optimal control.

Suppose that conditions of Theorem 5.2.1 are satisfied and $\hat{\pi}(t) = \phi(\hat{X}(t), \hat{p}(t), \hat{q}(t))$. Then the decoupled FBSDEs can be written as the following FBSDEs:

$$\begin{cases} d\hat{X}(t) = \tilde{b}(t, \hat{X}(t), \hat{p}(t), \hat{q}(t))dt + \tilde{\sigma}(t, \hat{X}(t), \hat{p}(t), \hat{q}(t))dW(t) \\ X(0) = x \\ d\hat{p}(t) = -\tilde{\mathcal{H}}_x(t, \hat{X}(t), \hat{p}(t), \hat{q}(t))dt + \hat{q}(t)dW(t) \\ \hat{p}(T) = g_x(\hat{X}(T)), \end{cases}$$

where

$$\begin{cases} \tilde{b}(t, \hat{X}(t), \hat{p}(t), \hat{q}(t)) = b(t, \hat{X}(t), \phi(\hat{X}(t), \hat{p}(t), \hat{q}(t))), \\ \tilde{\sigma}(t, \hat{X}(t), \hat{p}(t), \hat{q}(t)) = \sigma(t, \hat{X}(t), \phi(\hat{X}(t), \hat{p}(t), \hat{q}(t))), \\ \tilde{\mathcal{H}}(t, \hat{X}(t), \hat{p}(t), \hat{q}(t)) = \mathcal{H}(t, \hat{X}(t), \phi(\hat{X}(t), \hat{p}(t), \hat{q}(t)), \hat{p}(t), \hat{q}(t)). \end{cases}$$

It is trivial to see that the above FBSDE is coupled since the backward component $(\hat{p}, \hat{q})$ also appears in the forward equation as a consequence of the almost surely optimality condition on the Hamiltonian. Hence, numerically solving the above coupled FBSDE system gives a solution to the corresponding stochastic control problem.

There have been many advances in numerical methods for coupled FBSDEs. One approach is based on the so-called four step scheme initiated in Protter-Ma-Yong [44], which has led recent developments in Milstein-Tretyakov [65] and Delarue-Menozzi [24]. Such methods rely on the relation to quasi-linear parabolic PDEs and provide good convergence results for low dimensional problems. On the other hand, recent paper of Bender-Zhang [4] propose a probabilistic approach to tackle FBSDEs directly for high dimensional problems.

5.3 Portfolio optimization under regime-switching model

5.3.1 Introduction

In the second and third chapters of the thesis,we study constrained portfolio optimizations under standard Brownian motion set-up. Motivated by more realistic models that better reflect random market environment, regime-switching models were introduced by Hamilton [35] to model stock returns under different economic states. Regime-switching models allow for capturing uncertainty coming from two sources:

(i) a standard Brownian motion that models small scale microeconomic fluctuations that affects asset prices,

(ii) a finite state continuous Markov chain independent of the Brownian motion that captures the long-term structural macroeconomic changes that affects long-term outlooks on asset prices.

An example is a two state Markov chain where one state represents bull market with positive outlook and another state represents bear market with generally negative outlooks on future asset prices. Regime-switching models a more accurate representation of financial markets. However, such models are intrinsically incomplete due to the additional source of uncertainty given by the Markov chain.

5.3.2 Potfolio optimization with regime-switching

Some works have been done on extending the traditional portfolio optimization problems under Brownian motion set-up to regime-switching models. In particular Zhou-Yin [95] applied stochastic LQ control to the problem of mean-variance portfolio selection and Sotomayor-Cadenillas [78] followed dynamic programming approach to tackle expected utility maximization problem. However, it is worth noting that both works focused on unconstrained problems and relied on the market parameters being Markov modulated (i.e. at each time the market parameters are completely determined by the state of the Markov chain). Solving constrained portfolio optimization problem with random parameters under regime-switching model remains an open and interesting future research area.

Appendix A

Proof of Theorem 2.5.15

Proof. Consider the function Φ on $\mathbb{S}^2([0,T]) \times L^2(W,[0,T]) \times L^2(Q,[0,T])$ mapping $(Y,Z,S) \in \mathbb{S}^2([0,T]) \times L^2(W,[0,T]) \times L^2(Q,[0,T])$ to $\left(\hat{Y}, \hat{Z}, \hat{S}\right) = \Phi(Y,Z,S)$ defined by

$$\hat{Y}(t) = \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s) - \int_t^T S(s) \bullet dQ(s).$$

Consider the square-integrable martingale

$$M(t) = E\left[\xi + \int_0^T f(s, Y(s), Z(s))ds \middle| \mathcal{F}_t\right].$$

According to Theorem 2.5.13, there exists unique $\left(\hat{Z}, \hat{S}\right) \in L^2(W,[0,T]) \times L^2(Q,[0,T])$ such that

$$M(t) = M(0) + \int_0^t \hat{Z}(s)dW(s) + \int_0^t \hat{S}(s) \bullet dQ(s).$$

We then define the process $\hat{Y}(t)$ by

$$\begin{aligned}
\hat{Y}(t) &= E\left[\xi + \int_t^T f(s, Y(s), Z(s))ds \middle| \mathcal{F}_t\right] \\
&= M(t) - \int_0^t f(s, \alpha(s), Y(s), Z(s))ds \\
&= M(0) + \int_0^t \hat{Z}(s)dW(s) + \int_0^t \hat{S}(s) \bullet dQ(s) - \int_0^t f(s, Y(s), Z(s))ds \\
&= \xi + \int_t^T f(s, Y(s), Z(s))ds - \int_t^T \hat{Z}(s)dW(s) - \int_t^T \hat{S}(s) \bullet dQ(s).
\end{aligned}$$

By Doob's L^2 inequality, we have

$$E\left[\sup_{0\le t\le T}\left|\int_t^T \hat{Z}(s)dW(s)\right|\right] \le 4E\left[\int_0^T |\hat{Z}(s)|^2 ds\right] < \infty,$$

$$E\left[\sup_{0\le t\le T}\left|\int_t^T \hat{S}(s)\bullet dQ(s)\right|\right] \le 4E\left[\sum_{l=1}^n\sum_{i,j=1}^d\int_0^T |\hat{S}_{ij}^{(l)}(s)|^2 d[Q_{ij}](s)\right] < \infty.$$

Under the assumptions on (ξ, f), we conclude that $\hat{Y}\in S^2([0,T])$. Hence Φ is a well defined function from $S^2([0,T])\times L^2(W,[0,T])\times L^2(Q,[0,T])$ into itself. Next, we show that $(\hat{Y},\hat{Z},\hat{S})$ is a solution to the regime switching BSDE (2.39) if and only if it is a fixed point of Φ.

Let (U,V,Γ), $(U',V',\Gamma')\in S^2([0,T])\times L^2(W,[0,T])\times L^2(Q,[0,T])$. Apply function Φ and obtain $(Y,Z,S)=\Phi(U,V,\Gamma)$, $(Y',Z',S')=\Phi(U',V',\Gamma')$. Set $(\bar{U},\bar{V},\bar{\Gamma})=(U-U',V-V',\Gamma-\Gamma')$, $(\bar{Y},\bar{Z},\bar{S})=(Y-Y',Z-Z',S-S')$ and $\bar{f}(t)=f(t,U(t),V(t))-f(t,U'(t),V'(t))$. Take $\beta>0$ to be chosen later and apply Ito's formula to $e^{\beta s}|\bar{Y}|^2$ on $[0,T]$,

$$\begin{aligned}|\bar{Y}(0)|^2 = \;& -\int_0^T e^{\beta t}\left(\beta|\bar{Y}(t)|^2-2\bar{Y}(t)^\intercal\bar{f}(t)\right)dt-\int_0^T e^{\beta t}|\bar{Z}(t)|^2dt\\ &-\int_0^T e^{\beta t}\sum_{l=1}^n\sum_{i,j=1}^d|\bar{S}_{ij}^{(l)}|^2 d\left[Q_{ij}\right](t)-2\int_0^T e^{\beta t}\bar{Y}(t)^\intercal\bar{Z}(t)dW(t)\\ &-2\int_0^T e^{\beta t}\sum_{l=1}^n\sum_{i,j=1}^d\bar{Y}^{(l)}(t)\bar{S}_{ij}^{(l)}(t)dQ_{ij}(t).\end{aligned} \tag{A.1}$$

Observe that, according to Young's inequality

$$E\left[\left(\int_0^T e^{2\beta t}|\bar{Y}(t)|^2|\bar{Z}(t)|^2dt\right)^{\frac{1}{2}}\right]\le\frac{e^{\beta T}}{2}E\left[\sup_{0\le t\le T}|\bar{Y}(t)|^2+\int_0^T|\bar{Z}(t)|dt\right]<\infty,$$

$$\begin{aligned}&E\left[\left(\int_0^T e^{2\beta t}|\bar{Y}^{(l)}(t)|^2|\bar{S}_{ij}^{(l)}(t)|^2 d\left[Q_{ij}\right](t)\right)^{\frac{1}{2}}\right]\\ &\le\frac{e^{\beta T}}{2}E\left[\sup_{0\le t\le T}|\bar{Y}^{(l)}(t)|^2+\int_0^T|\bar{S}_{ij}^{(l)}(t)|^2 d\left[Q_{ij}\right](t)\right]<\infty.\end{aligned}$$

Hence $\int_0^t e^{\beta s}\bar{Y}(s)^\intercal\bar{Z}(s)dW(s)$ and $\int_0^t e^{\beta s}\sum_{l=1}^n\sum_{i,j=1}^d\bar{Y}^{(l)}(s)\bar{S}_{ij}^{(l)}(s)dQ_{ij}(s)$ are true martingales by the Burkholder-Davis-Gundy inequality. Taking expectation in

(A.1), we get

$$
\begin{aligned}
&E|\bar{Y}(0)|^2 + E\Big\{ \int_0^T e^{\beta t}\Big[\left(\beta|\bar{Y}(t)|^2 + |\bar{Z}(t)|^2\right) dt + \sum_{l=1}^{n}\sum_{i,j=1}^{d} |\bar{S}_{ij}^{(l)}(t)|^2 d\left[Q_{ij}\right](t) \Big]\Big\} \\
&= 2E\left[\int_0^T e^{\beta t}\bar{Y}(t)^\intercal \bar{f}(t)dt\right] \le 2C_f E\left[\int_0^T e^{\beta t}|\bar{Y}(t)|\left(|\bar{U}(t)| + |\bar{V}(t)|\right) dt\right] \\
&\le 4C_f^2 E\Big[\int_0^T e^{\beta t}|\bar{Y}(t)|^2 dt\Big] + \frac{1}{2}E\Big[\int_0^T e^{\beta t}\left(|\bar{U}(t)|^2 + |\bar{V}(t)|^2\right) dt\Big].
\end{aligned}
\tag{A.2}
$$

Take $\beta = 1 + 4C_f^2$ and substitute into (A.2), we have

$$
\begin{aligned}
&E\Big[\int_0^T e^{\beta t}\left(|\bar{Y}(t)|^2 + |\bar{Z}(t)|^2\right) dt + \int_0^T e^{\beta t}\sum_{l=1}^{n}\sum_{i,j=1}^{d} |\bar{S}_{ij}^{(l)}(t)|^2 d[Q_{ij}](t)\Big] \\
&\le \frac{1}{2}E\Big[\int_0^T e^{\beta t}\left(|\bar{U}(t)|^2 + |\bar{V}(t)|^2\right) dt\Big] \\
&\le \frac{1}{2}E\Big[\int_0^T e^{\beta t}\left(|\bar{U}(t)|^2 + |\bar{V}(t)|^2\right) dt\Big] + \frac{1}{2}E\Big[\int_0^T e^{\beta t}\sum_{l=1}^{n}\sum_{i,j=1}^{d} |\bar{\Gamma}_{ij}^{(l)}(t)|^2 d[Q_{ij}](t)\Big].
\end{aligned}
$$

Notice that $L^2(W,[0,T])$ and $L^2(Q,[0,T])$ are Hilbert spaces and therefore the space $\mathbb{S}^2([0,T]) \times L^2(W,[0,T]) \times L^2(Q,[0,T])$ endowed with the norm

$$
\|(Y,Z,S)\|_\beta = \left\{ E\Big[\int_0^T e^{\beta t}\left(|\bar{Y}(t)|^2 + |\bar{Z}(t)|^2\right) dt + \int_0^T e^{\beta t}\sum_{l=1}^{n}\sum_{i,j=1}^{d} |\bar{S}_{ij}^{(l)}(t)|^2 d[Q_{ij}](t)\Big] \right\}^{\frac{1}{2}}
$$

is a Banach space. We conclude that Φ admits a unique fixed point which is the solution to the BSDE (2.39). $\square$

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