# Zero-Delay Source-Channel Coding

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### Abstract

In this thesis, we investigate the zero-delay transmission of source samples over three different types of communication channel models. First, we consider the zero-delay transmission of a Gaussian source sample over an additive white Gaussian noise (AWGN) channel in the presence of an additive white Gaussian (AWG) interference, which is fully known by the transmitter. We propose three parameterized linear and non-linear transmission schemes for this scenario, and compare the corresponding mean square error (MSE) performances with that of a numerically optimized encoder, obtained using the necessary optimality conditions. Next, we consider the zero-delay transmission of a Gaussian source sample over an AWGN channel with a one-bit analog-to-digital (ADC) front end. We study this problem under two different performance criteria, namely the MSE distortion and the distortion outage probability (DOP), and obtain the optimal encoder and the decoder for both criteria. As generalizations of this scenario, we consider the performance with a K-level ADC front end as well as with multiple one-bit ADC front ends. We derive necessary conditions for the optimal encoder and decoder, which are then used to obtain numerically optimized encoder and decoder mappings. Finally, we consider the transmission of a Gaussian source sample over an AWGN channel with a one-bit ADC front end in the presence of correlated side information at the receiver. Again, we derive the necessary optimality conditions, and using these conditions obtain numerically optimized encoder and decoder mappings. We also consider the scenario in which the side information is available also at the encoder, and obtain the optimal encoder and decoder mappings. The performance of the latter scenario serves as a lower bound on the performance of the case in which the side information is available only at the decoder.

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## Contents

A	bstra	ct		i
A	cknov	wledge	ments	ii
C	opyri	$\mathbf{ght}$		iii
St	atem	ent of	Originality	$\mathbf{iv}$
C	onter	nts		$\mathbf{v}$
Li	st of	Figure	es	viii
A	bbrev	viation	S	x
Sy	<b>mbo</b>	ls and	Notations	xii
1	Intr	oducti	on	1
	1.1	Overvi	ew	1
	1.2	Backgi	ound	3
		1.2.1	Optimal linear encoding/ decoding	7
		1.2.2	Hybrid Digital Analog Systems	8
		1.2.3	Power Constrained Channel Quantized Vector Quantization (PC-	
			CQVQ)	10
		1.2.4	Shannon-Kotelnikov Mapping	11
		1.2.5	Optimality Conditions for Analog Mappings	12
	1.3	Thesis	Overview and Summary of the Main Contributions	13
2	Zere	o-Delay	y Source-Channel Coding in the Presence of Interference	
	Kno	own at	the Encoder	16
	2.1	Overvi	ew	16
	2.2	Introd	uction	17
	2.3	system	n model	20
	2.4	Param	eterized zero-delay transmission schemes	21
		2.4.1	Interference Cancellation	22
		2.4.2	Interference Concentration	23
		2.4.3	Comparison of ICA and ICO in the Asymptotic Zero-Noise Regime	28
		2.4.4	One Dimensional Lattice	29
		2.4.5	ICO with Non-Uniform Quantizer	30

В	Pro	of of Proposition 3.4.1	105
A	Pre	liminaries: Calculus of Variations	97
5	Con	clusions and Future Work	94
	4.6	Numerical Results	88
		4.5.1 Optimal Encoder and Decoder Design	82 86
	4.5	The DOP Criterion	81
		4.4.5 Shannon Lower Bound	81
		4.4.4 Side Information Available at Both the Encoder and Decoder	79
		4.4.3 Periodic BPSK Transmission	78
		4.4.2 Periodic Linear Transmission	77
	1.1	4.4.1 Optimal Encoder and Decoder Design	75
	4.3 4.4	The MSE distortion criterion	75
	4.2	Introduction	71
	4.1		70
	Info	rmation at the Receiver	70
4	Zer	o-Delay Source-Coding with a One-Bit ADC Front End and Side	
		3.1.2 DOP Criterion	00
		3.7.2 DOP Criterion	02 66
	3.7	Numerical Results	62 62
	- <b>-</b>	3.6.2 DOP Criterion	61
		3.6.1 MSE Criterion	60
	3.6	Multiple Observations	59
		3.5.2 K-Level Front End $\ldots$	58
		3.5.1 Optimal Encoder and Decoder for a One-Bit Front End	55
	3.5	Single Observation: Distortion Outage Probability	55
		3.4.4 <i>K</i> -Level Front End	54
		3.4.3 Digital Transmission for One-bit Front End	53
		3.4.2 Linear Transmission for One-Bit Front End	52
	0.4	3.4.1 Optimal Encoder and Decoder for a Ope-Bit Front End	50 51
	3.3 2.4	System Model	49 50
	3.2 9.9	Introduction	40
	3.1	Overview	45
	End		45
3	3 Zero-Delay Source-Channel Coding with a Low-Resolution ADC Fro		
	2.0		00
	2.6	2.3.1 Numerically Optimized Encoder	30 30
	2.5	Necessary Condition for Optimality and NOE Design	36
	~ ~	2.4.6 1DL with Non-Uniform Quantizer	34

С	Proof of Proposition 3.4.2	113
D	Proof of Proposition 3.5.1	115
E	Proof of Proposition 3.5.2	117
F	Proof of Proposition 3.5.3	118
G	Proof of Proposition 3.6.1	120
н	Proof of Proposition 3.6.2	122
I	Proof of Proposition 4.4.1	124
J	Proof of Remark 4.4.2	127
K	Proof of Proposition 4.5.1	129
$\mathbf{L}$	Proof of Proposition 4.5.2	133
$\mathbf{M}$	Proof of Remark 4.5.2	134
N	Proof of Proposition 4.5.3	135

Bib	liography	
	in graphy	

# List of Figures

1.1	Block diagram of a generic point-to-point communication system	4
1.2	Block diagram of the point-to-point communication system with separate	۲
	source and channel encoders and with additive noise	Э
1.3	Shannon's original example of a 1 : 2 mapping [1].	11
2.1	Zero-delay transmission of a Gaussian source over an AWGN channel in	
	the presence of AWG interference known at the transmitter	20
2.2	Source clipping and mapping at the transmitter.	23
2.3	Source is clipped to the region $\left[-\frac{\kappa}{2}, \frac{\kappa}{2}\right]$ , and the channel input is limited to the interval $\left[-\frac{\kappa+\Delta}{2}, \frac{\kappa+\Delta}{2}\right]$ . Dots are interference concentration points and dashed lines are the decision thresholds for interference concentration at the transmitter	94
0.4	$E_{\text{rescalar}} = \frac{1}{2} \left( \frac{1}{2} \right) \left( \frac{1}{2} + \frac{1}{2} \right)$	24
2.4	Encoder mapping $n(v,s)$ for NOE ( $\sigma_s^2 = 4, \sigma_n^2 = 1$ )	38
2.5	NOE mapping for different interference values ( $\sigma_s^2 = 4, \sigma_n^2 = 1$ )	39
2.6	NOE mapping for source outputs $v = 0.9$ and $v = -1.1$ ( $\sigma_s^2 = 4$ , $\sigma_n^2 = 1$ ).	40
2.7	Average MSE distortion (dB) vs. the average SNR (dB) for the proposed	10
	schemes for $\sigma_s^2 = 4$ , $\sigma_n^2 = 1$	40
2.8	Average MSE distortion (dB) vs. the average SNR (dB) for the proposed	
	schemes for $\sigma_s^2 = 25$ , $\sigma_n^2 = 1$	41
2.9	$\Delta$ vs. power constraint for different values of $\Delta_v$ for ICO ( $\sigma_s^2 = 25$ , $\sigma_n^2 = 1$ )	41
2.10	$\Delta$ vs. channel power constraint for ICO and 1DL for different values of $\Delta = (-2^2 - 2^2 - 1)$ . Bad dotted lines are accessible with ICO and blue	
	$\Delta_v$ ( $\sigma_s = 25, \sigma_n = 1$ ). Red dotted lines are associated with ICO and blue	
	lines are associated with IDL. $\Delta$ for IDL is always greater than that for ICO (for fixed value of power constraint and $\Lambda_{-}$ )	49
9.11	Average MSE distortion us $A$ for the ICO for $\tau^2 = 25$ $\tau^2 = 1$	42
2.11	Average MSE distortion vs. $\Delta_v$ for the ICO and 1DL for $P = 5$ (dP) $\sigma^2 = 4$	40
2.12	Average MSE distortion vs. $\overline{\Delta}$ for 1CO and 1DL for $T = 5$ (dB), $\theta_s = 4$ , $\sigma^2 = 1$	11
	$o_n - 1$	44
3.1	System model for the transmission of a single Gaussian source sample	
	over a quantized vector AWGN channel with $N$ ADC front ends	48
3.2	Illustration of the intervals $I_1$ , $I_2$ and $(I_1 \cap I_2)$ that characterize the opti-	
	mal encoder for the DOP criterion, for two different cases depending on	
	the $(\hat{v}_{(1)}, \hat{v}_{(2)})$ values.	56
3.3	Optimal encoder mappings for the MSE distortion criterion under differ-	
	ent power constraints $P$ (dB) ( $\sigma_w^2 = \sigma_v^2 = 1$ )	63
3.4	Complement of the MSE $1 - \overline{D}$ (dB), versus the SNR $\gamma$ (dB) for the	
	optimal encoder obtained in Proposition 3.4.1 as well as for the linear	
	and digital schemes studied in Sections 3.4.1, 3.4.2 and 3.4.3 ( $\sigma_w^2 = \sigma_v^2 = 1$ ).	64

3.5	NOE mappings for different power constraints (dB) and under the MSE distortion criterion ( $\sigma_w^2 = \sigma_v^2 = 1$ ): (a) $K = 4$ ; (b) $K = 8$ . The curves are labelled by the pair (P,d), where P is the average power constraint (dB)	0.5
3.6	and <i>d</i> is the quantization step size of the ADC front end Complement of the MSE $(1 - \overline{D})$ vs. SNR $(\gamma)$ (linear scale) for different	. 65
	ADC architectures with <i>B</i> -bit outputs ( $\sigma_v^2 = 1$ ). Variance of the AWGN in all scenarios and for all observations is one ( $\sigma_v = 1$ )	66
3.7	Optimal encoder mappings for $N = 1$ , $K = 2$ under the DOP criterion, for different values of the SNR ( $\gamma$ ). The dots are the reconstruction points	
3.8	$\hat{v}_{(1)}$ and $\hat{v}_{(2)}$ ( $\sigma_w^2 = \sigma_v^2 = 1$ and $D = 0.3$ )	. 68
3.9	Complement of the DOP $1 - \epsilon(D)$ versus SNR $\gamma$ (linear scale) for different ADC architectures with <i>B</i> -bit outputs ( $\sigma_w^2 = \sigma_v^2 = 1$ ).	. 69
4.1	System model for the zero-delay transmission of a Gaussian source over an AWGN channel with a one-bit AD front end and correlated side infor-	
	mation at the receiver.	. 73
4.2	Illustration of the PLT encoder mapping for $\alpha = 2, \beta = 2.5$	. 78
4.3	Illustration of the PBT encoder mapping for $\gamma = 0.2, \ \delta = 3. \ldots$	. 79
4.4	Illustration of the optimal encoder mapping when there is no side infor- mation at the receiver ( $\sigma_t^2 = \sigma_w^2 = 1$ ). The application of this mapping to the residual error (4.22) is optimal if the side information U is also known	
	at the encoder.	. 80
4.5	MSE encoder mappings $f(v)$ with different average power values and $r = 0.85$ , ( $\sigma_v = \sigma_w = 1$ ). Increasing the power constraint P has no impact on the period of the NOE mapping	88
4.6	NOE mappings that satisfy the necessary optimality condition for differ- ent correlation coefficients $r$ and an average power constraint of $P = 5$	
	$(\sigma_v^2 = \sigma_w^2 = 1)$	. 89
4.7	Complementary MSE distortion vs. SNR for $r = 0.6$ ( $\sigma_v^2 = \sigma_w^2 = 1$ ).	. 90
4.8	Complementary MSE distortion versus correlation coefficient $r$ under the average power constraint $P = 5$ ( $\sigma_v^2 = \sigma_w^2 = 1$ )	. 91
4.9	DOP encoder mappings $f(v)$ for different correlation coefficients ( $r = 0.1, 0.6, 0.8$ ), ( $\sigma_v^2 = \sigma_w^2 = 1$ ). The power constraint P of the mappings	
	increases in the direction of the arrow.	. 92
4.10	Complementary DOP vs. SNR for $r = 0$ , 0.6, 0.8 ( $\sigma_v^2 = \sigma_w^2 = 1$ )	. 93
B.1	The function $f_o(v)$ in (B.9) and the function $f_{a,b}(v)$ in (B.8) with $a = 4, b = 10 \dots $	. 107
B.2	Plot of the function $\tilde{g}(b)$ in (B.19)	. 109

# Abbreviations

1DL	One-dimensional lattice
1DL-NU	One-dimensional lattice with non-uniform quantizer
ADC	Analog to digital converter
APC	Average power constraint
AWG	Additive white gaussian
AWGN	Additive white gaussian noise
$\operatorname{CCDF}$	Complementary cumulative distribution function
$\operatorname{cdf}$	Cumulative distribution function
CSI	Channel state information
DOP	Distortion outage probability
DM	Discrete memoryless
HDA	Hybrid digital analog
ICA	Interference cancellation
ICO	Interference concentration
ICO-NU	Interference concentration with non-uniform quantizer
iid	Independent and identically distributed
ISI	Inter-symbol interference
JSCC	Joint source channel coding
MAC	Multiple access channel
MAP	Maximum a posteriori
MIMO	Multiple input multiple output
ML	Maximum likelihood
MMSE	Minimum mean square error
MSE	Mean square error
NOE	Numerically optimized encoder

OPTA	Optimal performance theoretically attainable
PBT	Periodic BPSK transmission
PCCOVQ	Power constraint channel optimized vector quantization
PLT	Periodic linear transmission
$\operatorname{pmf}$	Probability mass function
pdf	Probability density function
PPC	Peak power constraint
SSCC	Separate source channel coding
SNR	Signal to noise ratio
SINR	Signal to interference and noise ratio
THP	Tomlinson-Harashima Precoding

# Symbols and Notations

- $\mathbb{R}$  the set of real numbers
- $\mathbb{E}[\cdot]$  statistical expectation
- $\Pr(\cdot)$  probability
- $N(0,\sigma)$  Gaussian distribution with mean zero and variance  $\sigma$
- $Q(\cdot) \qquad \text{CCDF of normal distribution, } Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{u^2}{2}} du$
- $\mathcal{Q}(\cdot)$  quantization operator
- $\lfloor \cdot \rfloor$  floor operation (the maximum integer less than the argument)

## Chapter 1

## Introduction

### 1.1 Overview

Communication technology is an essential part of our modern lives. One of the challenges in designing communication systems is to enable reliable and efficient transmission of information bearing signals, such as an image, audio, text or a video, over a noisy communication medium. Many practical applications require the transmission of a continuous amplitude source over a noisy channel, to be reconstructed at the destination with some performance criteria. For example, transmission of multimedia signals over cellular networks, wireless transmission of sensor measurements to a fusion center, or transmission of received signals at remote radioheads to the baseband processing unit in a cloud radio access network (CRAN).

In almost all communication systems we have today, data compression and transmission over the channels are handled separately. This provides modularity for the practical system design. Moreover, it is motivated by Shannon's source-channel separation theorem, which states that, in a point-to-point scenario, there is no loss of optimality if the source is first compressed, independently of the channel statistics, and then, the compressed bits are transmitted over the channel with a capacity achieving channel code [2].

During the last decades, significant amount of research have been dedicated to improve the performance of source compression and data transmission over a noisy channel. The tremendous progress in both research directions have enabled today's wireless networks that can deliver high data rate content reliably to a large number of users. However, in certain emerging applications, it is impossible to achieve the desired level of reliability and energy efficiency under complexity constraints, due to device limitations, and latency constraints, imposed by the underlying applications. It is known that in those scenarios jointly optimizing the source and channel coding may improve the end-to-end performance. This has led to a growing research interest in JSCC in recent years.

While Shannon's separation theorem proves the optimality of separate design of sufficiently long and unbounded-complexity source and channel codes, many emerging applications today require transmission of measured parameters under extreme latency constraints. For example, state measurements in a smart grid are to be transmitted as fast as possible to the control centre, which then has to estimate the system state and interfere in the case of a black out. Similarly, in a body sensor network the transmission of vital signals to the control centre has to be extremely fast as delays may lead to significant health problems. In these examples, it is not possible for the transmitter to wait to collect a large number of source samples in order to apply vector quantization, or to exploit capacity-achieving channel codes with efficiently large blocklengths. In this thesis, we consider the other extreme of "zero-delay" transmission, that is, each source sample is transmitted over a single use of the channel

A well-known approach to zero-delay transmission is linear encoding and decoding. Both the complexity of encoding/ decoding, and the delay can be significantly reduced by linearly mapping the source samples to channel inputs (linear transmission). Despite its simplicity, linear transmission achieves the MMSE distortion in a point-to-point setting when a Gaussian source is transmitted over a static AWGN channel and the source and the channel are with equal bandwidth; that is, when an average of one source sample is to be transmitted over one channel use [3]. However, linear transmission is in general not capable of exploiting additional degrees-of-freedom available in the system, and its optimality breaks down in many other setups.

The optimality of linear transmission is very sensitive to the matching between the source/ channel distributions, input cost constraint, the distortion measure and the source and channel bandwidths [4]. In the point-to-point setting, linear transmission is an alternative for optimal SSCC, and its main advantages are simplicity and zero-delay. Surprisingly, it has been shown that linear transmission achieves the optimal performance in various other scenarios, such as Gaussian MAC with correlated Gaussian

sources [5, 6], or broadcasting a common source to multiple receivers over Gaussian channels. Specifically, for the latter scenario linear transmission is the only known optimal transmission scheme. Moreover, in some other scenarios, such as transmission with bandwidth mismatch [7], or broadcasting with correlated side information [8], linear transmission is shown to improve the performance when combined with digital coding schemes in the form of HDA transmission.

In general, the optimal zero-delay scheme is not necessarily linear. Indeed, in many other scenarios, there is no explicit way to derive such analog mappings, nor is the optimal performance known. In [1] and [9], Shannon and Kotelnikov utilized space-filling curves for transmission with bandwidth compression. Later, this approach was extended to the transmission of a Gaussian source over an AWGN channel with bandwidth compression and expansion in the work by Fuldseth and Ramstad [10], Chung [11], Vaishampayan and Costa [12], Ramstad [13] and Hekland et.al. [14]. For numerically optimizing the analog mappings two approaches is usually utilized. In the first one, the analog mapping has a structure which is defined by some parameters. The goal of the optimization is to find the optimum values of the parameter set of the structured mapping. It is noticed that in this approach, the performance is limited to the parameters considered for that specific structure. In the second approach, the design of the analog scheme is based on PCCQVQ. In this approach, a discrete form of the problem is tackled utilizing tools developed for vector quantization [15], [16]. Recently, in [17] a novel approach for finding optimal analog mappings has been studied. The main difference between the approach used in [17] and PCCQVQ based approaches is that in [17] the authors derive the necessary conditions of optimality in the original analog domain without any discretization. This in turn provides a theoretic analysis of the problem. Besides, a completely different numerical method, which iteratively imposes the optimality conditions of the original problem, is used to obtain NOE mapping functions.

### **1.2** Background

The two major results in information theory are Shannon's source coding theorem, which characterizes the ultimate limit of compressing a source sequence; and Shannon's channel coding theorem, which characterizes the maximum rate of reliable communication over



FIGURE 1.1: Block diagram of a generic point-to-point communication system.

noisy channels. By using appropriate encoding and decoding techniques for both source and channel coding we can approach these limits arbitrarily closely.

Although the information theoretic performance characterization of many communication scenarios (specially multiuser settings) remain open, there are some special cases in which the ultimate bounds are well established, e.g., the point-to-point communications and lossy compression of a single source. These information theoretic results are built upon unlimited delay and complexity assumptions. Therefore, although the fundamental performance bounds for these special cases have been established, and there exists certain practical coding techniques with high delay and complexity that can approach these bounds, there is still a significant amount of work to do regarding the practical implementations under certain complexity and delay constraints to achieve the optimal performance. On the other hand, the optimal performance under complexity and delay constraints is unknown other than some special scenarios and in the high SNR regime [18], [19].

The block diagram of a generic point-to-point communication system is shown in Figure 1.1. It consists of 1) realizations of a random source V that need to be transmitted to the destination, 2) an encoder, which transforms the source sequence into the channel input, 3) a noisy channel over which the encoded symbols are transmitted, and 4) a decoder that reconstructs the original source sequence from the received noisy signals in either lossless or lossy fashion. In general this setup is a JSCC problem in which the encoder structure and channel input depend on the source and channel statistics. Shannon showed that for the point-to-point setup, by carrying out source encoding/ decoding and channel encoding/ decoding separately, we can reach the theoretical optimal bound. As a result of Shannon's separation theorem the transmitter and the receiver can be decomposed into two separate components (see Figure 1.2):

- Source encoder: compresses the source sequence into bits,
- **Channel encoder**: operates on the compressed bit sequence, and exploits channel coding techniques to protect these bits from channel errors over the noisy channel,



source and channel encoders and with additive noise.

- **Channel decoder**: operates on the received signal from the channel output and extracts the message index,
- Source decoder: based on the channel decoded message index, reconstructs (decompresses) the transmitted source sequence.

Throughout this thesis, we will consider a special case of the general model in Figure 1.1 in which the source sequence consists of independent samples of a Gaussian distribution, and the channel is stationary memoryless with AWGN. In the following, we present Shannon's source coding, channel coding, and source-channel separation theorems.

Shannon's source coding theorem [20]: When compressing a source sequence  $V^m$ , the goal is to represent the source sequence with the lowest possible rate, such that the average distortion  $1/m \sum_{i=1}^{m} \mathbb{E}[d(V_i, \hat{V}_i)], d(v, \hat{v}) : \mathbb{R}^2 \to \mathbb{R}$  of the recovered sequence  $\hat{V}^m$ at the receiver is below a predefined value  $\bar{D}$ . Shannon characterized the minimum rate required to achieve an average distortion of  $\bar{D}$ , called *rate-distortion function*  $R(\bar{D})$  as

$$R(\bar{D}) \triangleq \min_{p(\hat{v}|v): \mathbb{E}[d(V,\hat{V})] \le \bar{D}} I(V;\hat{V}).$$
(1.1)

Dual form of (1.1) can also be considered called the *distortion rate* function  $\overline{D}(R)$ , which is the minimum average achievable distortion  $\overline{D}$  for a source compression rate of R; and it is defined as below

$$D(R) \triangleq \min_{p(\hat{v}|v): I(V;\hat{V}) \le R} \mathbb{E}[d(V;\hat{V})].$$
(1.2)

For an iid Gaussian source, we have  $\overline{D}(R) = \sigma_v^2 2^{-2R}$ , where  $\sigma_v^2$  is the source variance and R is in bits per source sample.

Shannon's channel coding theorem: Assume that we want to transmit a message index M (uniformly distributed over the set  $\mathcal{M}$ ) over a DM noisy channel with channel input X, channel output Y and conditional probability p(y|x). Channel encoder maps each message  $M \in \mathcal{M}$  to a channel input vector  $X^n$  and the channel decoder finds an estimate  $\hat{M}$  of the transmitted message based on receiving  $Y^n$ . The performance metric is given as the probability of decoding error  $P_e = P(M \neq \hat{M})$ . Shannon formulated the channel coding problem as finding the maximum communication rate  $C \triangleq \log \mathcal{M}/n$ , called the channel capacity, such that  $P_e$  can be made arbitrarily small while letting the block length be sufficiently large [21]. He characterized the capacity C as the maximum mutual information between the channel input X and the channel output Y over possible input distributions:

$$C \triangleq \max_{p(x)} I(X;Y). \tag{1.3}$$

Although some numerical algorithms are proposed in [22] to numerically evaluate (1.3), in general it is hard to find closed form expressions for channel capacity. For the special case of an AWGN channel under an APC of P, i.e.,  $1/n \sum_{i=1}^{n} \mathbb{E}\left[|X_i|^2\right] \leq P$ , the capacity is shown to be [2]:

$$C(P) = \frac{1}{2} \log\left(1 + \frac{P}{\sigma_n^2}\right), \qquad (1.4)$$

where  $\sigma_n^2$  is the channel noise variance, and C is in bits per channel use.

Shannon's source-channel separation theorem: Consider the point-to-point communication system in Figure 1.2, and let C be the capacity of the DM channel and R(D)be the rate distortion function of the DM source. Shannon showed that the necessary and sufficient condition for communicating the DM source over the DM channel with predefined distortion level  $\bar{D}$ , is  $R(\bar{D}) \leq C$ .

This theorem states that as long as this inequality holds, a source can be transmitted over the channel optimally with SSCC. This is the main theoretical motivation behind the fact that the communication systems today are designed based on SSCC. It is worth noting that this statement holds when the code length is of infinite length, which introduces infinite delay and complexity. However, infinite delay and complexity in practice is impossible, and enforce the designers to use suboptimal finite length/ complexity codes.

**Optimal Performance Theoretically Attainable:** OPTA is obtained from evaluating the rate distortion function at channel capacity. By expanding the inequality  $R(D) \leq C$  for a Gaussian source and an AWGN channel, OPTA in the Gaussian bandwidth matching (m = n) setting can be found to be

$$D_{\text{OPTA}} = \frac{\sigma_v^2}{1 + \frac{P}{\sigma_n^2}}.$$
(1.5)

In addition to infinite delay and complexity, separate coding over Gaussian channels is also not robust against variations in the channel conditions. In many applications the exact value of the channel SNR is not known at the transmitter. Although systems using digital transmission can take advantages of advanced quantization, compression and error control algorithms, since their structure is dependent on the target SNR, their performance does not improve with increased SNR (leveling-off effect) and breaks down completely when the true SNR is lower than the target SNR (threshold effect). Threshold effect becomes even more drastic when these systems operate close to their optimal performance limits.

Joint Source Channel Coding: For the reasons mentioned in the previous section, JSCC can be an appropriate alternative to SSCC to transmit analog sources, such as video, image, audio, over noisy channels. It has been proven in [3] that for transmitting an iid Gaussian source over an AWGN channel, in the case of bandwidth matching, OPTA is attainable by a direct mapping of the source symbols to the channel inputs. That is, no coding is required, and OPTA is achieved by low complexity and delay free transmission.

Another motivation to study JSCC is their robustness against varying conditions of the channel compared to digital systems. To mention a few of the studies on JSCC, in [7] using nearly robust HDA coding techniques for transmission of a Gaussian source over a broadcast channel, the authors show that the optimal performance in terms of Shannon limit can be approached and the threshold effect can be mitigated. In [23], HDA coding techniques are used to derive lower bounds on the distortion exponent for fading MIMO channels when the channel state is available at the receiver. In [24], it is shown that linear transmission is optimal when both the channel and the side information are time varying. In [25] using HDA codes and optimal power allocation, it is shown that in the case of bandwidth mismatching, the interference (known at the transmitter) over the channel can be cancelled at the receiver. In the following some known JSCC schemes are described.

#### 1.2.1 Optimal linear encoding/ decoding

In general there is no guarantee that OPTA can be achieved by linear encoding/ decoding. Assume we want to send m samples of a source V through n channel uses. The transmitter uses a matrix  $\mathbf{T} \in \mathbb{R}^{n \times m}$  to encode  $v^m$  and produces the channel input signal  $x^n$ . After the transmission over the AWGN channel, at the decoder, the received vector  $y^n$  is decoded by the use of a matrix  $\mathbf{R} \in \mathbb{R}^{m \times n}$ , and  $\hat{v}^n$  is produced as the source estimation. We would like to minimize the MSE distortion

$$\bar{D} = \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}\left[ (V_i - \hat{V}_i)^2 \right], \qquad (1.6)$$

where

$$x^n = \mathbf{T}v^m, \quad \hat{v}^m = \mathbf{R}y^n, \tag{1.7}$$

and  $y^n = x^n + w^n$ , where  $w^n$  is an *n*-dimensional AWGN vector with covariance matrix  $\sigma_n^2 \mathbf{I}_{n \times n}$ . It is shown in [26] that for the bandwidth compression case(m > n) the optimal matrices are in the form of

$$\mathbf{T} = \frac{\sqrt{P}}{\sigma_v} \mathbf{I}_{n \times m}, \quad \mathbf{R} = \frac{\sigma_v \sqrt{P}}{P + \sigma_n^2} \mathbf{I}_{m \times n}, \tag{1.8}$$

where  $\mathbf{I}_{n \times m}$  is a matrix with  $I_{i,i} = 1, i = 1, ..., n$  and zero otherwise. For the case of bandwidth expansion (m < n), we have

$$\mathbf{T} = \sqrt{\frac{n}{m}} \frac{\sqrt{P}}{\sigma_v} \mathbf{I}_{n \times m}, \quad \mathbf{R} = \sqrt{\frac{m}{n}} \frac{\sigma_v \sqrt{P}}{P + \sigma_n^2} \mathbf{I}_{m \times n}.$$
 (1.9)

In [26], it is shown that the performance of linear encoding/ decoding is very close to OPTA for low channel SNR. However, at high SNR, it is far from the OPTA (except bandwidth matching scenario in which BPAM is optimal). To get closer to OPTA we have to investigate nonlinear systems.

#### 1.2.2 Hybrid Digital Analog Systems

Digital systems suffer from two drawbacks known as *threshold effect* and *leveling-off effect*. The threshold effect means that once the channel SNR is less than the SNR at which the digital system has been designed, the performance of the system degrades drastically. On the other hand, the leveling-off effect refers to the fact that due to the finite precision of the quantizers in the system, once the channel SNR is higher than the designed SNR, the performance of the system does not improve and it remains constant. Since linear transmission does not suffer from these two effects, it can be combined with

9

digital coding techniques to improve the performance of the system in case there is SNR mismatch.

Recently, some nonlinear HDA systems have been proposed in the literature. In [27] transmission of a uniform source sample by two channel uses (bandwidth expansion) of an AWGN channel has been studied under the MSE distortion criterion. The encoder quantizes the source sample and transmits the quantization index and the quantization error. The quantizer is optimized with respect to the MSE distortion criterion. It is shown that utilizing a uniform quantizer is nearly optimal for a uniformly distributes source. In [28] the authors propose a scheme which is composed of two parts: digital and analog. In the digital part, the source is quantized with a low delay source optimized vector quantizer (SOVQ) and later the quantization indices are encoded via a high delay channel encoder. In the analog part, the error of the quantization is mapped via a nonlinear analog mapping and scaled. The output of the analog part and the digital part are superimposed and transmitted over an AWGN channel. Employing this structure, it is shown that under both bandwidth compression and expansion, robust and graceful performance can be achieved for a wide range of channel SNR. In [29] the authors study the transmission of two correlated Gaussian memoryless sources over a Gaussian Multiple Access Channel (GMAC). They propose two zero-delay JSCC schemes. The first scheme quantized the sources, such that, one of the source quantization indices are nested in the other source quantization indices. The quantized value of each source output is scaled to an acceptable power level and transmitted over the channel. In the second scheme, one of the sources is mapped using a nonlinear mapping and the second source is quantized. The output of the encoders are added (superimposed) over the channel. It is shown that the proposed schemes are robust against channel variations and have a constant gap with the performance upper bound in the high SNR regime. In [30] zerodelay JSCC transmission of a Gaussian source over an AWGN channel in the presence of a correlated Gaussian side information at the receiver is studied. It is noted that when block coding is allowed, the corresponding source coding problem of this scenario is known as Wyner-Ziv problem [31]. In [30], authors employ a HDA coding scheme based on superimposing the scaled quantization of the source and linear mapping of the quantization error. Having shown the robustness of the scheme for bandwidth matching scenario, it is shown that application of this scheme in the bandwidth expansion scenario performs better than the well known space-filling spiral mapping [1, 9].

### 1.2.3 Power Constrained Channel Quantized Vector Quantization (PC-CQVQ)

In [10, 15] the authors propose an algorithm in order to minimize the MSE distortion

$$\bar{D} = \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}\left[ |V_i - \hat{V}_i|^2 \right],$$
(1.10)

subject to an average power constraint given as

$$\mathcal{P} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\left[ |X_i|^2 \right] \le P, \tag{1.11}$$

where  $X_i$  is the channel input at time index *i*. Using Lagrangian method, authors in [10, 15], consider the unconstraint optimization problem

$$\min_{f,g} \bar{D} + \lambda \mathcal{P}, \tag{1.12}$$

where f, g are the encoder and decoder mappings, respectively, and  $\lambda \geq 0$  is the Lagrangian multiplier. From optimization theory [32, 33], it is well known that the optimal solution to the problem (1.12), in general satisfies the necessary condition of optimality for the constrained optimization problem

$$\begin{array}{ll} \underset{f,g}{\text{minimize}} & \bar{D} \\ \text{subject to} & \mathcal{P} < P. \end{array} \tag{1.13}$$

To solve the problem (1.12), the authors in [10, 15] utilize a vector quantizer followed by a mapping from an m-dimensional source space to an n-dimensional channel space. They consider the channel space as a finite signal set composed of K centroids in the codebook  $C = \{c_i\}_{i=0}^{K-1}$ . The source space is partitioned into K partitions as  $\mathcal{P} = \{\Omega_i\}_{i=0}^{K-1}$ . The vector quantizer maps the source samples to indices. Later each index is mapped to a channel symbol. On the receiver side, the decoder recovers the source based on nearest neighbour detection. The optimization of this algorithm is based on the generalized Lloyd algorithm [34].



FIGURE 1.3: Shannon's original example of a 1 : 2 mapping [1].

#### 1.2.4 Shannon-Kotelnikov Mapping

Unlike quantizing the source in digital systems, Shannon-Kotelnikov mapping [1, 9] is an approach where the source is directly (source encoder and channel encoder are merged together) mapped into the channel. In this approach, the source is represented using a point in the source space  $\mathbb{R}^m$ , and similarly the channel is represented by a point in the channel space  $\mathbb{R}^n$ . The main idea is to use the geometrical interpretation of the communication problem. This geometrical approach first introduced by Shannon in [1], and later in [9] Kotelnikov extended the tools for the bandwidth expansion problem by using a similar signal mapping approach. Shannon-Kotelnikov mappings are used either for bandwidth expansion or bandwidth compression. The former uses the redundant dimension for error control and the latter represents a lossy compression. For the case of the bandwidth matching, it is well known that linear mapping is optimal for a memoryless source and a Gaussian channel [3].

As an example of 1 : 2 bandwidth expansion mapping, Shannon proposed the curve shown in Figure 1.3. In this mapping, the one dimensional source is given by the line space V (e.g., the length along the curve), which is mapped to a two dimensional channel input  $(X_1, X_2)$ . This approach performs better than the one that we send the source sample twice over the channel (repetition coding). By interchanging the source and the channel, Shannon also suggested that the same mapping can be used for bandwidth reduction (every source vector  $(V_1; V_2)$  is projected onto the nearest point on the mapping curve, which will be represented using a one dimensional channel space, e.g., the distance from some reference origin to the projected point on the curve).

One important question regarding Shannon-Kotelnikov mappings is that what is the optimal geometrical structure of such mapping? One possible answer is to look at the discrete mappings obtained from PCCQVQ and connecting the adjacent points of the codebook. As another answer, in [13], it is shown that a good mapping should satisfy some certain requirements: 1) The mapping curve should cover well the entire source space to reduce overload distortion; 2) source symbols with high probability should be mapped to low power channel symbols so that the transmission power is minimized; 3) points in the channel space that are close to each other should be mapped to source symbols that are also close in the source space in order to minimize the distortion when errors occur. Furthermore, when choosing a mapping, all channel representations should have low correlation so that no redundant information is transmitted on different channel symbols.

### 1.2.5 Optimality Conditions for Analog Mappings

In [17], a new approach has been studied to dealing with the source-channel coding problems. The authors in [17] study the functional properties of the MSE distortion under average power constraint for the zero-delay source-channel coding problem, and the problem of obtaining vector transformations that optimally map between the m-dimensional source and n-dimensional channel space. They study the functional properties of the point-to-point problem. In particular, it is shown that the MSE distortion for the zerodelay source-channel coding problem is a concave functional of the source density, given a fixed noise density, and of the noise density given a fixed source. It is also shown that the MSE distortion is a convex functional of the channel input density. This result guarantees that there is a unique encoding mapping function that minimizes the MSE distortion. Using gradient-descent based optimization algorithms such as noisy channel relaxation (NCR) [35–37], a design algorithm has been proposed, which iteratively imposes the optimality conditions to obtain NOE mappings. This approach has also been utilized in [38?, 39] for finding the NOE mapping functions.

An interesting result in [17], regarding the zero-delay source-channel coding problem (point-to-point with infinite resolution front end) is related to the linearity conditions of optimal mappings in terms of the source and channel densities and the channel power constraint. It is shown that given a Gaussian source, optimal mapping are linear in the high SNR regime regardless of the channel density. Likewise, for a Gaussian channel, optimal mappings are linear in the low SNR regime regardless of the source density.

## 1.3 Thesis Overview and Summary of the Main Contributions

In this thesis, we focus on zero-delay source-channel coding techniques under three different scenarios. This work is motivated by many recently emerging applications such as wireless sensor networks (WSN) or Internet-of-Things (IoT), where there are many problems which must be dealt with. For instance, since the real time communication in such applications matters, it is not possible to utilize very long block coding techniques. On the other hand, due to the increasing number of simultaneous users, interference is another problem in providing qualified data at the end user. In addition to these issues, in almost all of the contemporary applications the signals are being digitized before being processed. Therefore, investigating the performance of different scenarios under these circumstances are of interest. In the following, we review the topics and the main contributions presented in the next chapters.

In Chapter 2, we consider zero-delay transmission of a Gaussian source over an AWGN channel in the presence of an independent AWG interference signal. We study the minimization of the MSE distortion under an average power constraint assuming that the interference signal is known at the transmitter. Optimality of simple linear transmission does not hold in this setting due to the presence of the known interference signal. We propose various non-linear transmission schemes. The nonlinear schemes utilize a scalar quantizer to produce the channel input. Based on the numerical observations, it is seen that, in contrast to typical uniform quantization of Gaussian sources, a non-uniform quantizer, whose quantization intervals become smaller as we go further from zero, improves the performance. Given that the optimal decoder is the MMSE estimator, we derive a necessary condition for the optimality of the encoder. Using the optimality condition, we obtain NOE mappings with the help of gradient-descent based algorithm used in [17, 39]. Based on the numerical results, it is shown that one of the schemes with non-uniform quantization performs closer (compared to the other schemes) to the numerically optimized encoder while requiring significantly lower complexity.

In Chapter 3, motivated by practical constraints arising in sensor networks and Internetof-Things applications, we consider the zero-delay transmission of a Gaussian measurement over a vector Gaussian noise channel with a low-resolution ADC front end. We study the optimization of the encoder and the decoder under both the MSE distortion and the DOP criteria with an average power constraint on the channel input. The motivation for considering the DOP criterion, is that, since the MSE distortion criterion is not a good performance metric for applications involving transmission of image or speech [2, Chapter 10], the DOP criterion could be a reasonable metric compared to the MSE distortion criterion. We derive the optimal solutions for the case of a one-bit ADC front end for both criteria. In particular we show that, the optimal encoder mapping tends to a linear encoder only in the low SNR regime. Instead, antipodal digital transmission is optimal in the high SNR regime. For the DOP criterion, we show that the optimal encoder mapping is piecewise constant and can take only two opposite values when it is non-zero. For both the MSE distortion and the DOP criteria, we derive necessary optimality conditions for a K-level ADC front end and for multiple one-bit ADC front ends. We later use these conditions to derive and evaluate numerically optimized solutions. To gain insights into optimal encoding and decoding functions, extensive simulations have been carried out.

In Chapter 4, we study the zero-delay transmission of a Gaussian source over an AWGN channel with a one-bit ADC front end in the presence of correlated side information at the receiver. We focus on the design of the optimal encoder and the decoder under the two performance criteria as in Chapter 3, namely, the MSE distortion and the DOP under an average power constraint on the channel input. For both the MSE distortion and the decoder. For the MSE distortion we observe that the NOE is periodic, and its period increases with the correlation between the source and the receiver side information. For the DOP,

we observe that the NOE mappings are bounded and fluctuating between positive and negative values.

Finally in Chapter 5, we provide the conclusions from our research presented in this thesis, and discuss about the potential research directions that can be considered in the future.

## Chapter 2

# Zero-Delay Source-Channel Coding in the Presence of Interference Known at the Encoder

### 2.1 Overview

Zero-delay transmission of a Gaussian source over an AWGN channel is considered in the presence of an additive Gaussian interference signal. The MSE distortion is minimized under an average power constraint assuming that the interference signal is known at the transmitter. Optimality of simple linear transmission does not hold in this setting due to the presence of the known interference signal. While the optimal encoder-decoder pair remains an open problem, various non-linear transmission schemes are proposed in this chapter. In particular, ICO and 1DL strategies, using both uniform and non-uniform quantization of the interference signal, are studied. It is shown that, in contrast to typical scalar quantization of Gaussian sources, a non-uniform quantizer, whose quantization intervals become smaller as we go further from zero, improves the performance. Given that the optimal decoder is the MMSE estimator, a necessary condition for the optimality of the encoder is derived, and the NOE satisfying this condition is obtained. Based on the numerical results, it is shown that 1DL with non-uniform quantization performs closer

(compared to the other schemes) to the numerically optimized encoder while requiring significantly lower complexity.

#### 2.2Introduction

While the spectral efficiency of communication systems has improved significantly within the last decade, latency remains as the bottleneck for many applications. In many emerging applications, such as those involving *cyber-physical systems* (CPS) or *wireless sensor* networks (WSN), real-time interaction among distributed autonomous agents is crucial. A communication link is called real-time when the communication time is lower than the time constants of the application. Such applications impose significantly lower roundtrip latency requirements compared to what is achievable today. For example, in many applications involving CPSs, local system measurements are reported by sensor nodes via noisy links to other network agents. The need to have near real-time monitoring and control of the underlying physical system imposes strict delay constraints on the communication links. In such a scenario, utilizing long block codes for source compression or channel coding is not viable due to the stringent delay constraint. Similarly, when tactile control of an object and hearing/ seeing its reaction through a wireless connection is desired, a reaction latency on the order of milliseconds will be imposed on the communication link [40]. For example, for a typical 1 m/s speed of a finger on a touch screen, the reaction time for the screen is expected to be approximately 1ms in order to achieve an unnoticeable displacement of 1mm between the object to be moved and the finger [41].

We consider zero-delay transmission of system parameters over wireless channels, that is, a single source sample needs to be transmitted over a single use of the channel. It is well-known that zero-delay linear encoding (uncoded transmission) of a Gaussian source over an AWGN channel does not result in any performance loss in terms of the end-to-end MSE distortion [3]. However, this is not the case if there is bandwidth mismatch between the source and channel [9, 14, 42], if there is correlated source side information at the receiver [43], or if there is a peak power constraint at the transmitter [44]. Characterization of the optimal transmission strategy is challenging in general, and remains an open problem in most cases.

In this chapter we consider zero-delay transmission of a Gaussian source over an AWGN channel in the presence of an AWG interference signal causally known at the transmitter. This is known as the *dirty-tape channel*. Known interference at the transmitter can be used to model communication systems which use superposition coding to transmit multiple data streams simultaneously [23, 25, 45, 46]. For the superposed data streams, the codewords corresponding to lower layers act as known interference. The capacity of the dirty-tape channel was first studied by Shannon [47], who characterized the capacity using the so-called *Shannon strategies*. The channel model when the interference is known non-causally at the transmitter is known as the *dirty-paper channel*. The capacity of the dirty-paper channel was characterized by Gelfand and Pinsker in [48], and it was later shown in [49] that, in the Gaussian setting, the capacity of the dirty-paper channel is equal to the one without interference.

Despite Shannon's single-letter characterization, there is no closed-form capacity expression for the dirty-tape channel even in the Gaussian setting. Willems in [50] proposed the ICO strategy for the Gaussian dirty-tape channel. The basic idea of this scheme is cancelling the interference by giving a structure to it. Willems showed that, it is possible to partially cancel the interference at the receiver by quantizing it at the encoder, and by proper power allocation between the interference quantization error and the channel input signal, which is uniformly distributed over the quantization region. More recently, Erez *et al.* [51] proposed inflated lattice strategies for the Gaussian dirty-tape channel in [50]. They show that the rate loss of their coding scheme with respect to no interference, which is shown to be zero in the case of dirty-paper channel [49], is not more than 0.254 bits per channel use in the asymptotic high SNR regime. On the other hand, it is shown in [52] that the ICO scheme of Willems performs better than the inflated lattice based coding scheme in the low SNR regime. In [53], optimal mappings based on an iterative algorithm are proposed. Based on the numerical results in [53], it is shown that the numerically obtained encoder performs well compared to the scheme proposed in [51].

All of the above mentioned work study the channel coding problem whereas we are interested in zero-delay JSCC over the dirty-tape channel. Note that Shannon's sourcechannel separation theorem [2] does not apply to zero-delay JSCC problems; and hence, we can not directly use the above channel coding results to evaluate the MSE performance. A generalization of this problem is studied in [39], which further allows correlation between the source and the interference signals, and bandwidth mismatch between the source and the channel. While [39] focuses on deriving a numerically optimized encoder and decoder pair, our goal here is to develop low-complexity joint source-channel transmission techniques motivated by the channel coding strategies proposed in [50] and [51]. The problem we consider is intrinsically connected to problems in stochastic control where the controllers must operate at zero delay. A control problem, similar to the zero-delay source channel coding problem here, is the Witsenhausen's well known counterexample [54] (see e.g. [55] for a comprehensive review) where a similar functional optimization problem is studied and it is shown that nonlinear controllers can outperform linear ones in decentralized control settings even under Gaussianity and MSE assumptions. Expanding upon our previous work in [56], we consider ICO and 1DL schemes combined with nonlinear companders. While characterizing the optimal performance is elusive for this problem, we present numerical results comparing the performance of the proposed strategies, and provide some heuristics to improve them. In particular, we propose a counter-intuitive non-uniform quantization scheme in conjunction with the ICO and 1DL schemes, which increases the average quantization error, and hence, the power used for interference concentration, but leads to a lower MSE since the transmitter can then use a compander with a larger dynamical range for the more likely interference states.

Similarly to [39], we also characterize the necessary condition for the optimality of an encoder mapping, and obtain a NOE using steepest descent to search for an encoder that satisfies the derived necessary condition. While the MSE achieved by NOE outperforms the other proposed schemes, it is demanding computationally. It is shown that non-uniform quantization in conjunction with 1DL performs closer (compared to the other schemes) to the NOE, while the number of parameters to be optimized for the 1DL scheme (with non-uniform quantizer) is significantly less than NOE; and hence, it has significantly less computational complexity. We also note that, although we considered Gaussian distributed interference, the results obtained in this chapter can be easily extended to any distribution of the interference.

The rest of the chapter is organized as follows: In Section 2.3 we introduce the system model. In Section 2.4 zero-delay transmission schemes under average power constraint are introduced. In Section 2.5, we characterize the necessary condition for the optimal encoder, and introduce NOE. In Section 2.6, we compare all the proposed transmission schemes numerically.



FIGURE 2.1: Zero-delay transmission of a Gaussian source over an AWGN channel in the presence of AWG interference known at the transmitter.

#### 2.3system model

We consider the transmission of a Gaussian source over an AWGN channel in the presence of an AWG interference signal, which is known at the transmitter. The setup is illustrated in Figure 2.1. Without loss of generality, we assume, that the memoryless Gaussian source sample, V, has zero mean and unit variance, i.e.,  $V \sim \mathcal{N}(0,1)$ . The interference signal is independent of the source, and also follows a Gaussian distribution,  $S \sim \mathcal{N}(0, \sigma_s^2)$ . The discrete memoryless channel output Y, is given by Y = X + S + W, where X is the channel input, S is the known Gaussian interference signal, and W is the additive Gaussian noise,  $W \sim \mathcal{N}(0, \sigma_n^2)$ , independent of the source and interference signals.

We denote the zero-delay encoding function as X = h(V, S), where  $h(\cdot, \cdot)$  is assumed to be Borel measurable, square integrable function. An average power constraint is imposed on the channel input:

$$\mathbb{E}\left[X^2\right] \le P,\tag{2.1}$$

where the expectation is over all realizations of the source and interference signal. We are interested in transmitting the source samples, V, over the channel under the MSE distortion criterion. We denote the MMSE estimation function at the receiver by  $\hat{V} =$  $g(Y) \triangleq \mathbb{E}[V|Y]$ . Our goal is to characterize the minimum MSE  $\mathbb{E}\left[|V - \hat{V}|^2\right]$ , for given  $P, \sigma_s^2$  and  $\sigma_n^2$  values.

We note that, in our setting, due to the zero-delay constraint, causal and non-causal knowledge of the interference are equivalent. In other words, non-causal knowledge of the interference is useless, and the transmitter only uses the knowledge of the current value of the interference.

We define the functions below, which will be used throughout the chapter.

$$\mathcal{I}_{0}(m_{1}, m_{2}, a, b) \triangleq \frac{\sqrt{m_{1}}e^{-\frac{m_{2}^{2}}{4m_{1}}}}{\sqrt{\pi}} \int_{a}^{b} e^{-m_{1}u^{2}+m_{2}u} du 
= Q\left(\frac{m_{2}-2bm_{1}}{\sqrt{2m_{1}}}\right) - Q\left(\frac{m_{2}-2am_{1}}{\sqrt{2m_{1}}}\right), \quad (2.2)$$

$$\mathcal{I}_{1}(m_{1}, m_{2}, a, b) \triangleq \int_{a}^{b} ue^{-m_{1}u^{2}+m_{2}u} du 
= \frac{1}{2m_{1}} \left(e^{-m_{1}a^{2}+m_{2}a} - e^{-m_{1}b^{2}+m_{2}b}\right) 
+ \frac{\sqrt{\pi}m_{2}e^{\frac{m_{2}^{2}}{4m_{1}}}}{2m_{1}\sqrt{m_{1}}} \cdot \mathcal{I}_{0}(m_{1}, m_{2}, a, b), \quad (2.3)$$

$$\mathcal{I}_{2}(m_{1}, m_{2}, a, b) \triangleq \int_{a}^{b} u^{2} e^{-m_{1}u^{2} + m_{2}u} du$$

$$= \frac{m_{2}}{4m_{1}^{2}} \left( e^{-m_{1}a^{2} + m_{2}a} - e^{-m_{1}b^{2} + m_{2}b} \right)$$

$$+ \frac{1}{2m_{1}} \left( ae^{-m_{1}a^{2} + m_{2}a} - be^{-m_{1}b^{2} + m_{2}b} \right)$$
(2.4)

$$+\frac{\sqrt{\pi}e^{\frac{m_2}{4m_1}}}{2m_1\sqrt{m_1}}\cdot\left(1+\frac{m_2^2}{2m_1}\right)\cdot\mathcal{I}_0(m_1,m_2,a,b),$$
(2.5)

where  $Q(\cdot)$  is the complementary cumulative function defined as  $Q(t) = \int_{t}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$ . Since we deal with definite integrals throughout the chapter, we will avoid writing the boundaries of the integrals explicitly when they are from  $-\infty$  to  $\infty$ . Also, if no limits are specified, the summations are over all integers  $\mathbb{Z}$ . We also define the rectangle function R(t) as

$$R(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$
(2.6)

#### 2.4Parameterized zero-delay transmission schemes

In this section we introduce five different transmission schemes for the setup introduced in Section 2.3 with increasing complexity. Later on, in Section 2.6, we will compare and comment on the performances of these schemes.

#### 2.4.1**Interference Cancellation**

The simplest way to communicate in the presence of a known interference signal is to cancel the interference. In the ICA scheme, the transmitted signal X is a simple linear combination of the source realization V and the interference S. The transmitter decides how much of the interference will be cancelled depending on the system parameters. We have

$$X = aV + bS, (2.7)$$

where a and b are the coefficients to be determined. The channel input has to satisfy

$$\mathbb{E}\left[X^2\right] = a^2 + b^2 \sigma_s^2 \le P. \tag{2.8}$$

With MMSE estimation at the receiver, the achievable average distortion is found as

$$D_{ICA} = \frac{1}{1 + \frac{P - b^2 \sigma_s^2}{(b+1)^2 \sigma_s^2 + \sigma_n^2}}.$$
(2.9)

The optimal b value that minimizes (2.9) is given by

$$b^* = -\frac{P + \sigma_s^2 + \sigma_n^2 - \sqrt{(P - \sigma_s^2)^2 + \sigma_n^2 + 2\sigma_n^2(P + \sigma_s^2)}}{2\sigma_s^2}.$$
 (2.10)

The optimal value for a can be obtained from (2.8) and (2.10). The ICA scheme consumes part of the transmission power for interference cancellation; and thus, is expected to perform poorly especially in the low power regime, when the interference power is relatively high compared to the input power.

*Remark* 2.4.1. We note that in the high SINR regime  $(P \gg \sigma_s^2)$  the average achievable distortion is  $D_{\text{ICA}} = 1/\left(1 + \frac{P - \sigma_s^2}{\sigma_n^2}\right)$ . That is because, by rewriting  $b^*$  we have

$$b^{*} = -\frac{P + \sigma_{s}^{2} + \sigma_{n}^{2} - \sqrt{P^{2} \left(1 - \frac{2\sigma_{s}^{2}}{P} + \frac{2\sigma_{n}^{2}}{P} + \frac{\sigma_{s}^{4} + \sigma_{n}^{2} + 2\sigma_{n}^{2}\sigma_{s}^{2}}{P^{2}}\right)}{2\sigma_{s}^{2}}$$

$$\stackrel{(a)}{\simeq} -\frac{P + \sigma_{s}^{2} + \sigma_{n}^{2} - P\left(1 - \frac{\sigma_{s}^{2}}{P} + \frac{\sigma_{n}^{2}}{P}\right)}{2\sigma_{s}^{2}}$$

$$= -1, \qquad (2.11)$$



FIGURE 2.2: Source clipping and mapping at the transmitter.

where (a) is due to the approximation  $\sqrt{1-x} \simeq 1-x/2$  for small x. This is as though the signal X = aV - S,  $(a = P - \sigma_s^2)$  is transmitted over the channel.

The analysis of the performance of the ICA scheme is relegated to Section 2.6. Next we will introduce alternative non-linear transmission strategies.

#### 2.4.2**Interference Concentration**

This scheme is motivated by Willems' ICO scheme for channel coding [50]. We combine the interference concentration idea with JSCC of a Gaussian signal under a PPC [44]. The interference signal S is concentrated to one of the pre-determined discrete points on the real line; that is, the interference is quantized, and the corresponding quantization noise is cancelled, rather than cancelling the whole interference. Only the signal corresponding to the quantization index of the interference is received at the receiver. The transmitter superposes a companded version of the source signal such that it is compressed into one quantization interval of the quantizer.

The signal transmitted over the channel is given by

$$X = T(V) - (S \mod \Delta), \tag{2.12}$$

where  $(S \mod \Delta) \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$  corresponds to the quantization error, and is defined as

$$S \mod \Delta \triangleq S - \mathcal{Q}(S),$$
 (2.13)


FIGURE 2.3: Source is clipped to the region  $\left[-\frac{\kappa}{2}, \frac{\kappa}{2}\right]$ , and the channel input is limited to the interval  $\left[-\frac{\kappa+\Delta}{2}, \frac{\kappa+\Delta}{2}\right]$ . Dots are interference concentration points and dashed lines are the decision thresholds for interference concentration at the transmitter.

where  $\mathcal{Q}(S)$  is the nearest neighbour quantizer defined as below

$$Q(S) \triangleq \Delta \cdot \left\lfloor \frac{S}{\Delta} + \frac{1}{2} \right\rfloor.$$
 (2.14)

The source is clipped and mapped as in Figure 2.2 to the interval  $[-\kappa/2, \kappa/2]$  as below

$$T(v) = \begin{cases} \frac{\kappa}{2} & v \ge \frac{\Delta_v}{2}, \\ \frac{\kappa}{\Delta_v}v & -\frac{\Delta_v}{2} \le v < \frac{\Delta_v}{2}, \\ -\frac{\kappa}{2} & v < -\frac{\Delta_v}{2}, \end{cases}$$
(2.15)

where  $\kappa = \Delta - 2d$ . Notice that  $\kappa + \Delta$  is the variation range of the channel input X (since T(v) and  $(S \mod \Delta)$  are varying in the intervals  $[-\kappa/2, \kappa/2]$ , and  $[-\Delta/2, \Delta/2]$ ), respectively. Parameter d can be considered as a guard interval between the source mappings into different intervals. Parameters d,  $\kappa$  and  $\Delta$  are illustrated in Figure 2.3.

It can be seen from (2.12) that, in the ICO scheme, s is concentrated to one of the quantization indices in  $\{i\Delta : i \in \mathbb{Z}\}$ , which corresponds to a uniform quantizer with quantization interval size of  $\Delta$ . Power consumed by the transmitter for interference concentration is equivalent to the average quantization noise variance for the interference signal. While the power allocated to interference concentration,  $\sigma_{\text{Smod}\Delta}^2$ , depends only on the value of  $\Delta$ , the power of the compander component,  $\sigma_T^2$ , depends on  $\kappa$  and  $\Delta_v$ parameters.  $\sigma_T^2$  and  $\sigma_{Smod\Delta}^2$  are to be chosen such that the channel power constraint is satisfied. We have

$$\mathbb{E}\left[X^2\right] = \sigma_T^2 + \sigma_{\mathrm{Smod}\Delta}^2 \le P,\tag{2.16}$$

where the expectation is taken over the pdf of the channel input  $f_X(x) = f_T(t) \star$  $f_{S \mod \Delta}(t)$ , where  $\star$  denotes the convolution operation, and we have

$$f_T(t) = \frac{1}{\sqrt{2\pi\alpha}} e^{-\frac{t^2}{2\alpha^2}} R\left(\frac{t}{\kappa}\right) + Q\left(\frac{\Delta_v}{2}\right) \cdot \left(\delta\left(t - \frac{\kappa}{2}\right) + \delta\left(t + \frac{\kappa}{2}\right)\right), \quad (2.17)$$

$$f_{S \text{mod}\Delta}(t) = \sum_{i} \frac{1}{\sqrt{2\pi\sigma_s}} e^{-\left(\frac{i\Delta+t}{\sqrt{2\sigma_s}}\right)^2} \cdot R\left(\frac{t}{\Delta}\right).$$
(2.18)

Therefore, for  $\sigma_T^2$  and  $\sigma_{S \mod \Delta}^2$  we have

$$\sigma_T^2 = \kappa^2 \left[ \frac{1}{\Delta_v^2} + Q\left(\frac{\Delta_v}{2}\right) \left(\frac{1}{2} - \frac{2}{\Delta_v^2}\right) - \frac{e^{-\frac{\Delta_v^2}{8}}}{\Delta_v \sqrt{2\pi}} \right],\tag{2.19}$$

$$\sigma_{\text{Smod}\Delta}^2 = \frac{1}{\sqrt{2\pi}\sigma_s} \sum_i e^{-\frac{(i\Delta)^2}{2\sigma_s^2}} \mathcal{I}_2\left(\frac{1}{2\sigma_s^2}, -\frac{i\Delta}{\sigma_s^2}, -\frac{\Delta}{2}, \frac{\Delta}{2}\right).$$
(2.20)

Since solving (2.16) for equality with respect to d,  $\Delta$  and  $\Delta_v$  is cumbersome, we resort to numerical techniques to find the  $\Delta$  and  $\Delta_v$  parameters that satisfy the average power constraint. The received signal is given by

$$Y = X + S + W$$
  
=  $T(V) - (S \mod \Delta) + S + W$   
=  $T(V) + Q(S) + W.$  (2.21)

MMSE estimation is directly applied on the received signal to reconstruct the transmitted source sample:

$$g(y) = \frac{\iint v f_V(v) f_S(s) f_{Y|V,S}(y|v,s) dv ds}{\iint f_V(v) f_S(s) f_{Y|V,S}(y|v,s) dv ds}$$
  
$$= \frac{\sum_i p(q_i) \int v f_V(v) f_W(y - T(v) - q_i) dv}{\sum_i p(q_i) \int f_V(v) f_W(y - T(v) - q_i) dv}$$
  
$$= \frac{\sum_i p(q_i) \cdot \left( \mathbf{F}_{y,q_i,\frac{\kappa}{\Delta v}} + \frac{e^{-\frac{\Delta^2 v}{8}}}{\sqrt{2\pi}} \cdot \left( f_W \left( y - \frac{\kappa}{2} - q_i \right) - f_W \left( y + \frac{\kappa}{2} - q_i \right) \right) \right)}{\sum_i p(q_i) \cdot \left( \mathbf{G}_{y,q_i,\frac{\kappa}{\Delta v}} + Q(\frac{\Delta v}{2}) \cdot \left( f_W \left( y - \frac{\kappa}{2} - q_i \right) + f_W \left( y + \frac{\kappa}{2} - q_i \right) \right) \right)}, \quad (2.22)$$

where  $q_i \triangleq i \cdot \Delta$ ,  $i \in \mathbb{Z}$ , are the points to which the interference is concentrated when we have  $s \in \omega_i$ ,  $\omega_i = \left[q_i - \frac{\Delta}{2}, q_i + \frac{\Delta}{2}\right)$ , and we have

$$p(q_i) = \int_{\omega_i} f_S(s) ds = \mathcal{I}_0\left(\frac{\sigma_s^2}{2}, q_i, -\frac{\Delta}{2\sigma_s^2}, \frac{\Delta}{2\sigma_s^2}\right)$$
(2.23)

$$\mathbf{F}_{y,q_i,\frac{\kappa}{\Delta_v}} \triangleq \frac{e^{-\frac{(y-q_i)^2}{2\sigma_n^2}}}{2\pi\sigma_n} \cdot \mathcal{I}_1\left(\frac{\Delta_v^2 \sigma_n^2 + \kappa^2}{2\Delta_v^2 \sigma_n^2}, \frac{\kappa(y-q_i)}{\Delta_v \sigma_n^2}, -\frac{\Delta_v}{2}, \frac{\Delta_v}{2}\right),$$
(2.24)

$$\mathbf{G}_{y,q_i,\frac{\kappa}{\Delta_v}} \triangleq \frac{\Delta_v e^{-\frac{-v(s-q_{11})}{2(\Delta_v^2 \sigma_n^2 + \kappa^2)}}}{\sqrt{2\pi \left(\Delta_v^2 \sigma_n^2 + \kappa^2\right)}} \cdot \mathcal{I}_0\left(\frac{\Delta_v^2 \sigma_n^2 + \kappa^2}{2\Delta_v^2 \sigma_n^2}, \frac{\kappa(y-q_i)}{\Delta_v \sigma_n^2}, -\frac{\Delta_v}{2}, \frac{\Delta_v}{2}\right).$$
(2.25)

Finally, the corresponding average distortion is evaluated as below

$$D \stackrel{(a)}{=} 1 - \mathbb{E}[V\hat{V}]$$

$$= 1 - \sum_{i} p(q_{i}) \int \int vg(T(v) + q_{i} + w)f_{W}(w)f_{V}(v)dwdv$$

$$= 1 - \sum_{i} p(q_{i}) \cdot \left(e^{-\frac{\Delta_{v}^{2}}{8}} \cdot \int \left(g\left(\frac{\kappa}{2} + w + q_{i}\right) - g\left(-\frac{\kappa}{2} + w + q_{i}\right)\right)f_{W}(w)dw$$

$$- \int_{-\frac{\Delta_{v}}{2}}^{\frac{\Delta_{v}}{2}} \int vf_{V}(v)g\left(\frac{\kappa}{\Delta_{v}}v + w + q_{i}\right)f_{W}(w)dwdv\right), \quad (2.26)$$

where (a) is due to the MMSE estimation.

Remark 2.4.2. In this remark, we consider a suboptimal decoding algorithm and study its complexity with respect to the optimal MMSE estimation. For the ICO scheme (as well as the other non-linear encoding schemes introduced later in this chapter) it is also possible to use an alternative suboptimal decoding scheme called MAP-MMSE. In MAP-MMSE, we first decode the interference concentration index  $\hat{q}_i$  using MAP decoding at the receiver, and then cancel the decoded interference from the received signal  $y = T(v) + q_i + w$ . Finally MMSE estimation is applied on the remaining signal  $y = T(v) + q_i - \hat{q}_i + w$  to estimate the source sample.

For MAP decoding of the transmitted interference concentrated index  $q_i$ , the term Z = T(V) + W is assumed as noise with pdf  $f_Z$ . We define  $a_{ri}$ ,  $a_{li}$  as the decision threshold distances out of which the incorrect decoding for  $q_i$  is occurred. We have

$$p(q_i)f_Z(a_{ri}) = p(q_{i+1})f_Z(\Delta - a_{ri}),$$
  

$$p(q_{i-1})f_Z(\Delta - a_{li}) = p(q_i)f_Z(-a_{li}),$$
(2.27)

where

$$f_Z(z) = \frac{e^{-\frac{z^2}{2(\sigma_n^2 + \alpha^2 \sigma_v^2)}}}{\sqrt{2\pi(\sigma_n^2 + \alpha^2 \sigma_v^2)}} \cdot \mathcal{I}_0\left(\frac{\sigma_n^2 + \alpha^2 \sigma_v^2}{2\alpha^2 \sigma_v^2 \sigma_n^2}, \frac{z}{\sigma_n^2}, -\frac{\kappa}{2}, \frac{\kappa}{2}\right) + \frac{Q(\frac{\Delta v}{2\sigma v})}{\sqrt{2\pi}\sigma_n} \cdot \left[e^{-\frac{(z-\frac{\kappa}{2})^2}{2\sigma_n^2}} + e^{-\frac{(z+\frac{\kappa}{2})^2}{2\sigma_n^2}}\right].$$
(2.28)

The final distortion can be written as

$$D_{\text{map}} = \sum_{i} D_{\text{map}}^{i} (S \in \omega_{i}) p(q_{i})$$
$$= \sum_{i} \sum_{j} p(q_{i}) D_{\text{map}}^{ij} (q_{i} + Z \in \zeta_{j}) p(q_{i} + Z \in \zeta_{j}), \qquad (2.29)$$

where  $\zeta_i = [q_i - a_{li}, q_i + a_{ri})$  is the MAP decoder decision region for declaring  $q_i$  as output and  $p(q_i + Z \in \zeta_j)$  and  $D_{map}^{ij}(q_i + Z \in \zeta_j)$  are obtained as

$$p(q_i + Z \in \zeta_j) = \int_{q_j - a_{lj} - q_i}^{q_j + a_{rj} - q_i} f_Z(z) dz,$$
(2.30)

$$D_{\text{map}}^{ij}(q_{i} + Z \in \zeta_{j}) = \sigma_{v}^{2} - \int \int_{q_{j}-a_{lj}-q_{i}-T(v)}^{q_{j}+a_{rj}-q_{i}-T(v)} vg_{\text{map}}(T(v) + w + q_{i} - q_{j})f_{V}(v)f_{W}(w)dwdv$$

$$= \sigma_{v}^{2} - \sigma_{v}e^{-\frac{\Delta_{v}^{2}}{8\sigma_{v}^{2}}} \cdot \left(\int_{q_{j}-a_{lj}-q_{i}-\frac{\kappa}{2}}^{q_{j}+a_{rj}-q_{i}-\frac{\kappa}{2}}g_{\text{map}}\left(\frac{\kappa}{2} + w + q_{i} - q_{j}\right)(f_{W}(w) - f_{W}(w + \kappa))dw\right)$$

$$- \int_{-\frac{\Delta_{v}}{2}}^{\frac{\Delta_{v}}{2}} \int_{q_{j}-a_{lj}-q_{i}-\alpha v}^{q_{j}+a_{rj}-q_{i}-\alpha v} vf_{V}(v)g_{\text{map}}(\alpha v + w + q_{i} - q_{j})f_{W}(w)dwdv, \qquad (2.31)$$

where

$$g_{\rm map}(y_{\rm new}) = E[V|y_{\rm new}] = \frac{\int v f_{Y_{\rm new}|V}(y_{\rm new}|v)dv}{\int f_{Y_{\rm new}|V}(y_{\rm new}|v)dv},$$
(2.32)

where  $y_{\text{new}} = T(v) + w + q_i - \hat{q}_j$ . By expanding  $f_{Y_{\text{new}}|V}(y_{\text{new}}|v)$ , we have

$$\begin{split} f_{Y_{\text{new}}|V}(y_{\text{new}}|v) &= \sum_{i} p(q_i) f_{Y_{\text{new}}|V,\mathcal{Q}}(y_{\text{new}}|v,q_i) \\ &= \sum_{i} \sum_{j} p(q_i) f_{W}(y_{\text{new}}|v,q_i,\hat{q}=q_j) p(\hat{q}=q_j|v,q_i) \end{split}$$

$$= \sum_{i} \sum_{j} p(q_{i}) f_{W}(y_{\text{new}} | v, q_{i}, \hat{q} = q_{j}) \cdot \int f_{W}(w) p(\hat{q} = q_{j} | v, q_{i}, w) dw$$
  
$$= \frac{1}{\sqrt{2\pi\sigma_{n}^{2}}} \cdot \sum_{i} \sum_{j} p(q_{i})$$
  
$$\cdot e^{-\frac{(y_{\text{new}} - v - q_{i} - q_{j})^{2}}{2\sigma_{n}^{2}}} \cdot \int_{a_{lj} - T(v) - q_{i}}^{a_{rj} - T(v) - q_{i}} f_{W}(w) dw.$$
(2.33)

This algorithm, in addition to being suboptimal, is also computationally more demanding due to the increased computational complexity in the MMSE stage (2.32). Even considering ML decoder instead of MAP decoder does not improve the computation time significantly. This is because of the dependence of the noise with the signal produced by MAP decoder namely  $y_{\text{new}}$ , which increases the complexity of computing  $f_{Y_{\text{new}}|V}(y_{\text{new}}|v)$ in (2.33). Hence, we restrict our numerical analysis to MMSE estimation as it provides the optimal performance in a shorter time.

*Remark* 2.4.3. In this remark we note that ICO scheme is very similar to what is known as THP [57, 58] studied in the context of the ISI channel. In THP, the massage is discrete (replicas of M-PAM constellation points on the real line), and what is transmitted over the channel is the error of the interference with respect to the nearest intended constellation point, whereas in ICO the intended massage is continuous and what is transmitted over the channel is the error of the massage with respect to the quantized version of the interference. Indeed, ICO can be interpreted as the JSCC version of THP scheme.

#### 2.4.3 Comparison of ICA and ICO in the Asymptotic Zero-Noise Regime

In order to illustrate the benefits of ICO over ICA we consider the asymptotic zeronoise regime, i.e., we assume that  $\sigma_n^2 \to 0$ . For ICA, one can see that, if  $P \ge \sigma_s^2$  then the interference can be completely removed using part of the available power, and zero distortion is achieved in the limit as the noise disappears. On the other hand, when  $P < \sigma_s^2$ , the best achievable distortion is  $(\sigma_s^2 - P)/\sigma_s^2$  (this can be easily verified from (2.10) and (2.9)); that is, there is always residual distortion in the estimation even if there is no noise in the system. On the other hand, one can show that in the asymptotic zero-noise regime, independent of the input power constraint, zero distortion can be achieved by the ICO scheme. In the absence of noise, since the received signal is always within the quantization region of the interference signal, the quantization index can always be detected correctly. Once the quantization index is known, the effect of interference can be completely removed.

With MMSE estimation applied on the received noiseless signal T(V) + Q(S), the reconstructed source samples can be written as below

$$g\left(T(v) + \mathcal{Q}(S)\right) = \begin{cases} \frac{e^{-\frac{\Delta_v^2}{8}}}{\sqrt{2\pi}} & v \ge \frac{\Delta_v}{2}, \\ v & -\frac{\Delta_v}{2} \le v < \frac{\Delta_v}{2}, \\ -\frac{e^{-\frac{\Delta_v^2}{8}}}{\sqrt{2\pi}} & v < -\frac{\Delta_v}{2}. \end{cases}$$
(2.34)

The remaining distortion is only due to the companding of the source samples to squeeze them into the quantization region. By letting  $\Delta_v$  go to infinity we can reconstruct the source perfectly, and zero distortion can be achieved asymptotically. Note that, letting  $\Delta_v \to \infty$  also means that the average input power depends only on  $\Delta$  in the limit.

These arguments show that the ICO scheme can provide significant improvements compared to ICA, particularly when the interference is strong and the noise in the system is low. In the following, we provide other techniques based on the idea of providing a structure to the interference. We will observe that these techniques will further improve the performance of the ICO scheme.

#### **One Dimensional Lattice** 2.4.4

The idea of using a lattice structure for communication in the presence of known interference has been considered in [51] for the channel coding problem. Here we consider using a similar lattice structure for JSCC. The channel input for the 1DL scheme is given by

$$X = (T(V) - S) \mod \Delta, \tag{2.35}$$

where  $T(\cdot)$  is as defined in (2.15). In the 1DL scheme, the term T(v) - s is concentrated to one of the quantization points in  $\{i \cdot \Delta\}_{i=-\infty}^{\infty}$ .

In order to satisfy the average power constraint, we need to characterize the pdf of X, which can be obtained as follows:

0 0

$$f_X(x) = \begin{cases} \sum_i f_{T*S}(i\Delta + x) & -\frac{\Delta}{2} \le x < \frac{\Delta}{2}, \\ 0 & \text{otherwise,} \end{cases}$$
(2.36)

where  $f_{T*S}(u)$  is defined as

$$f_{T*S}(u) = \frac{\Delta_v e^{-\frac{\Delta_v^2 u^2}{2(\Delta_v^2 \sigma_s^2 + \kappa^2)}}}{\sqrt{2\pi (\Delta_v^2 \sigma_s^2 + \kappa^2)}} \mathcal{I}_0\left(\frac{\sigma_s^2 \Delta_v^2 + \kappa^2}{2\kappa^2 \sigma_s^2}, \frac{u}{\sigma_s^2}, -\frac{\kappa}{2}, \frac{\kappa}{2}\right) + \frac{Q\left(\frac{\Delta_v}{2}\right)}{\sqrt{2\pi}\sigma_s}\left(e^{-\frac{(u-\frac{\kappa}{2})^2}{2\sigma_s^2}} + e^{-\frac{(u+\frac{\kappa}{2})^2}{2\sigma_s^2}}\right).$$
(2.37)

Notice that in 1DL, the channel input X is limited to  $[-\Delta/2, \Delta/2)$ . The quantization step size,  $\Delta$ , must be chosen such that the channel input power constraint is satisfied. We recall here that due to the fact that finding a closed form expression for channel power constraint with respect to  $\Delta$  and  $\Delta_v$  is cumbersome, we resort to numerical calculations to evaluate the value of  $\Delta$  that satisfies the power constraint.

The received signal can be written as

$$Y = X + S + W$$
  
=  $(T(V) - S) \mod \Delta + S + W$   
=  $-[T(V) - S - (T(V) - S) \mod \Delta] + T(V) + W$   
=  $T(V) - Q(T(V) - S) + W.$  (2.38)

The numerical results for the 1DL scheme are presented in Section 2.6. Here we just state that the 1DL scheme achieves lower MSE distortion compared to ICO, since 1DL supports a larger  $\Delta$  value for an equal power constraint. This is mainly due to the bounded channel input which leads to a more efficient use of the available power.

#### 2.4.5 ICO with Non-Uniform Quantizer

Note that both the ICO and 1DL schemes give some shape to the interference, rather than simply reducing its variance as in ICA. Both schemes use uniform quantization for this. In this section we consider using a non-uniform quantizer for the ICO scheme, and different companders for the source sample depending on the interference signal.

In classical scalar quantization, non-uniform quantization is employed in order to reduce the quantization noise for the more likely values of the underlying signal at the expense of the less likely values. With such a quantizer, in our setting, we would have smaller intervals around zero, and the interval size would increase as we go further away from the origin. Note that, this would reduce the transmission power allocated for interference concentration, since it achieves a lower quantization noise variance. However, this would also mean that we have to compress the source signal even further when the interference realization is close to zero. We observe that the final distortion benefits more from increasing  $\Delta$ ; that is, having quantization points with larger separation. Hence, we apply the opposite of classical non-uniform scalar quantization, and use a lower resolution quantization for more likely values of the interference, and decrease the quantization interval size as we go further away from zero.

As before, the interference signal is concentrated to the middle point of the quantization interval into which it falls. Since the length of the quantization interval depends on the realization of the interference, a different compander function will be used for each interval. We denote by  $\mathcal{Q}^{N}(\cdot)$  the non-uniform quantizer with decision intervals  $\omega_{i}$ defined as

$$\omega_0 \triangleq \left\{ s : -\frac{\Delta_0}{2} \le s < \frac{\Delta_0}{2} \right\}, \quad i = 0,$$
  

$$\omega_i \triangleq \left\{ s : B_i \le s < B_i + \Delta_i \right\}, \quad i = 1, 2, ...,$$
  

$$\omega_i \triangleq \left\{ s : -B_i - \Delta_i \le s < -B_i \right\}, \quad i = -1, -2, ...,$$
(2.39)

and quantizations indices  $q_i^N$  corresponds to the middle point of each interval. We have

$$q_i^N = \operatorname{sgn}(i) \cdot \left(B_i + \frac{\Delta_i}{2}\right), \qquad (2.40)$$

where  $sgn(\cdot)$  is the sign function<sup>1</sup>, and

$$B_{i} \triangleq \begin{cases} |i| \cdot \Delta_{0}/2, & \text{if } i = \{1, 0, -1\}, \\ \frac{\Delta_{0}}{2} + \sum_{j=1}^{|i|-1} \Delta_{j}, & \text{otherwise.} \end{cases}$$
(2.41)

<sup>1</sup>We have sgn(x) = 1 if x > 0, -1 if x < 0, and 0 if x = 0.

We define the function  $\bar{f}_S(s)$  as follows

$$\bar{f}_S(s) \triangleq \frac{1}{f_S(s)^a},\tag{2.42}$$

where  $f_S(s)$  is the pdf of the interference S, and  $a \ge 0$  is a parameter to be optimized. The length of the i-th quantization interval,  $\Delta_i$ , is chosen such that

$$2\int_{s\in\omega_0} \bar{f}_S(s)ds = \int_{s\in\omega_i} \bar{f}_S(s)ds, \quad i = 1, 2, \dots$$
(2.43)

For Gaussian interference and  $a \ge 0$  it can be shown that

$$|i| > |j| \Rightarrow \Delta_i \le \Delta_j,$$
  
$$|i| = |j| \Rightarrow \Delta_i = \Delta_j.$$
 (2.44)

At the transmitter, if S falls into the quantization interval  $\omega_i$ , we have  $\mathcal{Q}^N(S) = q_i^N$ . Therefore, source V is transformed as follows

$$T(v, q_i^N) \triangleq \begin{cases} \frac{\Delta_i}{2} & v \ge \frac{\Delta_v}{2}, \\ \frac{\Delta_i}{\Delta_v}v & -\frac{\Delta_v}{2} \le v < \frac{\Delta_v}{2}, \\ -\frac{\Delta_i}{2} & v < -\frac{\Delta_v}{2}, \end{cases}$$
(2.45)

where we have defined  $T(v, q_i^N)$  to denote the companding function, in order to highlight its dependence on the realization of the interference quantization  $q_i^N$ . The transmitted signal is generated as below

$$X = T(V, \mathcal{Q}^N(S)) - (S \mod \Delta^N), \qquad (2.46)$$

where  $(S \mod \Delta^N)$  denotes the quantization noise for the non-uniform scalar quantizer with  $\mathcal{Q}^N(\cdot)$ .

To satisfy the average power constraint we follow the same approach as in Section 2.4.2. For brevity we define  $U \triangleq (S \mod \Delta^N)$ . We have

$$\mathbb{E}[X^2] = \sum_{i} p\left(q_i^N\right) \cdot \int_{-\Delta_i}^{\Delta_i} x^2 f_{X|\mathcal{Q}^N}\left(x|\mathcal{Q}^N(S) = q_i^N\right) dx$$

$$= \sum_{i} p\left(q_{i}^{N}\right) \cdot \left(\sigma_{T\left(V,q_{i}^{N}\right)}^{2} + \sigma_{U|\mathcal{Q}^{N}(S)=q_{i}^{N}}^{2}\right),$$
  
$$= \sum_{i} p\left(q_{i}^{N}\right) \cdot \sigma_{T\left(V,q_{i}^{N}\right)}^{2} + \sigma_{U}^{2},$$
 (2.47)

where

$$p\left(q_i^N\right) = \mathcal{I}_0\left(\frac{\sigma_s^2}{2}, q_i^N, -\frac{\Delta_i}{2\sigma_s^2}, \frac{\Delta_i}{2\sigma_s^2}\right)$$
(2.48)

$$\sigma_{T\left(V,q_{i}^{N}\right)}^{2} = \Delta_{i}^{2} \left[ \frac{1}{\Delta_{v}^{2}} + Q\left(\frac{\Delta_{v}}{2}\right) \left(\frac{1}{2} - \frac{2}{\Delta_{v}^{2}}\right) - \frac{e^{-\frac{\Delta_{v}^{2}}{8}}}{\Delta_{v}\sqrt{2\pi}} \right].$$
 (2.49)

To evaluate  $\sigma_U^2$  in (2.47), we need the distribution of U for the non-uniform quantizer. Since  $\max_i \{\Delta_i\} = \Delta_0$ , we have  $U \in [-\Delta_0, \Delta_0)$ . The cdf of U can be written as

$$F_U(u) = \sum_i F_{U|\mathcal{Q}^N} \left( u | \mathcal{Q}^N(S) = q_i^N \right) p\left(q_i^N\right), \qquad (2.50)$$

where  $F_{U|Q^N}\left(u|Q^N=q_i^N\right)$  for different *i*'s can be expanded as below

$$F_{U|\mathcal{Q}^N}\left(u|\mathcal{Q}^N(S)=q_i^N\right) = \frac{1}{p\left(q_i^N\right)} \int_{q_i^N - \frac{\Delta_i}{2}}^{q_i^N + u} f_S(s)ds, \quad -\frac{\Delta_i}{2} \le u < \frac{\Delta_i}{2}.$$
 (2.51)

By differentiating (2.50) with respect to u, and recalling that  $q_{-i}^N = -q_i^N$  and  $p\left(q_{-j}^N\right) = p\left(q_j^N\right)$ , we obtain

$$f_U(u) = \frac{f_S(u)}{p(q_0^N)} + \sum_{i=1}^{\infty} \frac{f_S(-q_i^N + u) \cdot R\left(\frac{2u + B_i + B_{i+1}}{2(B_{i+1} - B_i)}\right) + f_S(q_i^N + u) \cdot R\left(\frac{2u - B_i - B_{i+1}}{2(B_{i+1} - B_i)}\right)}{p(q_i^N)}, \quad -\frac{\Delta_0}{2} \le u < \frac{\Delta_0}{2}.$$

$$(2.52)$$

Using conventional tools in probability theory,  $\sigma_U^2$  in (2.47) can be evaluated as

$$\sigma_U^2 = \sum_i \frac{e^{-\frac{(q_i^N)^2}{2\sigma_s^2}}}{\sqrt{2\pi}\sigma_s} \mathcal{I}_2\left(\frac{1}{2\sigma_s^2}, \frac{q_i^N}{\sigma_s^2}, -\frac{\Delta_i}{2}, \frac{\Delta_i}{2}\right).$$
(2.53)

The received signal for the ICO-NU scheme is given by

$$Y = X + S + W$$
  
=  $T(V, \mathcal{Q}^N(S)) - (S \mod \Delta^N) + S + W$   
=  $T(V, \mathcal{Q}^N(S)) + \mathcal{Q}^N(S) + W.$  (2.54)

At the receiver we use MMSE estimation as introduced in Section 2.4.2. The source reconstruction and final distortion is obtained as follows.

$$g^{N}(y) = \frac{\sum_{i} p(q_{i}^{N}) \cdot \left(\mathbf{F}_{y,q_{i}^{N},\frac{\Delta_{j}}{\Delta_{v}}} + \frac{e^{-\frac{\Delta_{v}^{2}}{8}}}{\sqrt{2\pi}} \cdot \left(f_{W}(y - \frac{\Delta_{i}}{2} - q_{i}^{N}) - f_{W}(y + \frac{\Delta_{i}}{2} - q_{i}^{N})\right)\right)}{\sum_{i} p(q_{i}^{N}) \cdot \left(\mathbf{G}_{y,q_{i}^{N},\frac{\Delta_{j}}{\Delta_{v}}} + Q(\frac{\Delta_{v}}{2}) \cdot \left(f_{W}(y - \frac{\Delta_{i}}{2} - q_{i}^{N}) + f_{W}(y + \frac{\Delta_{i}}{2} - q_{i}^{N})\right)\right)},$$
(2.55)

$$D = 1 - \sum_{i} p(q_i^N) \iint v g^N(T(v, q_i^N) + q_i^N + w) f_W(w) f_V(v) dw dv.$$
(2.56)

#### 2.4.61DL with Non-Uniform Quantizer

In this section we consider the 1DL scheme combined with a non-uniform quantizer similarly to the ICO scheme in Section 2.4.5. The transmitted signal is given as below

$$X_{1\mathrm{DL}} = \left( [T(V, \mathcal{Q}^N(S)) - S] \mod \Delta^N \right).$$
(2.57)

To satisfy the average power constraint we follow the same approach as in Section 2.4.2. We have

$$\mathbb{E}\left[X^2\right] = \sum_i p(q_i^N) \cdot \int_{-\frac{\Delta_i}{2}}^{\frac{\Delta_i}{2}} x^2 f_{X|\mathcal{Q}^N}\left(x|\mathcal{Q}^N(S) = q_i^N\right) dx.$$
(2.58)

where

$$f_{X|\mathcal{Q}^N}\left(x|\mathcal{Q}^N(S) = q_i^N\right) = f_c^{q_i^N}(q_i^N + x) + f_c^{q_i^N}(q_i^N + x - \operatorname{sgn}(x)\Delta_i), \quad -\frac{\Delta_i}{2} \le x < \frac{\Delta_i}{2} \quad \text{for all } i,$$
(2.59)

where  $f_c^{q_i^N}(u) = f_{T(V,q_i^N)}(v) * f_{S|\mathcal{Q}^N}(s|\mathcal{Q}^N(S) = q_i^N)$ , and we have

$$f_c^{q_i^N}(u) = \begin{cases} \alpha_i \mathcal{I}_0\left(\beta_i, \frac{u}{\sigma_s^2}, -\frac{\Delta_i}{2}, u + \frac{\Delta_i}{2}\right) + \frac{Q\left(\frac{\Delta_v}{2}\right)e^{-\frac{(u+\Delta_i/2)^2}{2\sigma_s^2}}}{p(q_i^N)\sqrt{2\pi\sigma_s}} & -\Delta_i \le u < 0, \\ \alpha_i \mathcal{I}_0\left(\beta_i, \frac{u}{\sigma_s^2}, u - \frac{\Delta_i}{2}, \frac{\Delta_i}{2}\right) + \frac{Q\left(\frac{\Delta_v}{2}\right)e^{-\frac{(u-\Delta_i/2)^2}{2\sigma_s^2}}}{p(q_i^N)\sqrt{2\pi\sigma_s}} & 0 \le u < \Delta_i, \end{cases}$$
(2.60)

where  $\alpha_i$  and  $\beta_i$  are given as

$$\alpha_i = \frac{\Delta_v e^{-\frac{\Delta_v^2 u^2}{2(\Delta_v^2 \sigma_s^2 + \Delta_i^2)}}}{p(q_i^N) \sqrt{2\pi (\Delta_v^2 \sigma_s^2 + \Delta_i^2)}},\tag{2.61}$$

$$\beta_i = \frac{\Delta_v^2 \sigma_s^2 + \Delta_i^2}{2\sigma_s^2 \Delta_i^2}.$$
(2.62)

The received signal for the 1DL-NU scheme is given by

$$Y = X + S + W$$
  
=  $([T(V, Q^N(S)) - S] \mod \Delta^N) + S + W$   
=  $T(V, Q^N(S)) - Q^N(S) + W.$  (2.63)

At the receiver we use MMSE estimation as introduced in Section 2.4.2. To reconstruct the source samples at the receiver we use (2.55) and (2.56).

Remark 2.4.4. The intuition behind choosing the function in 2.42 is the following. We know that clipping the source sample injects a distortion at the encoder side. Therefore, given an average power constraint, the larger the  $\Delta_v$  value, the smaller the distortion introduced by clipping. Since the interference has a Gaussian distribution, we know that realizations of the interference around the origin are more likely than those towards the tails of the distribution. Note that the quantization noise variance of the quantization scheme we use, corresponds to the power spent for concentrating the interference at the transmitter. Therefore, Uniformly quantizing the interference S, is equivalent to assigning the power budget uniformly across realizations of the interference, whereas, assigning larger intervals around the origin distributes the power budget non-uniformly among interference realizations, such that, more likely interference realizations are quantized with larger quantization intervals; and hence, they require more power, but allow smaller distortion due to clipping. The heuristic interference quantization scheme used here can be further improved by devising a numerical technique similar to the classical Lloyd-Max algorithm [34]. We leave the optimization of the interference quantizer for the ICO and 1DL schemes as a future work.

#### 2.5Necessary Condition for Optimality and NOE Design

As stated in the Introduction, the optimal zero or low-delay joint source-channel coding scheme is an open problem in most communication scenarios, with few exceptions [3]. A common approach for these problems in the literature [17], [15] is to formulate the optimal encoder mapping as an unconstrained optimization problem through the Lagrangian, then to apply calculus of variations techniques to obtain a necessary condition for the optimal mapping, and finally numerically obtain an encoder mapping, typically using an iterative steepest descent algorithm, that satisfies this necessary condition. Due to lack of convexity, this solution does not guarantee global optimality, and the final solution is highly sensitive to the initial mapping. Despite these drawbacks, with carefully chosen initial mappings, and sufficiently high-grained quantization of the continuous source and channel alphabets, these NOEs achieve the best known performance in most scenarios.

In this section, we will follow the same approach as in [17], [39]. We briefly include the derivations for completeness, and then, we numerically obtain the encoder that satisfies this condition. Numerical techniques have been previously used for joint source-channel mappings in various scenarios in [15-17, 38]. By writing the Lagrangian cost function for this system model we have

$$J(h,g) = \mathbb{E}[(V-\hat{V})^2] + \lambda \cdot \mathbb{E}[h(V,S)^2]$$
  
=  $1 - \iint \left( \int vg(h(v,s) + w + s)f_W(w)dw - \lambda h(v,s)^2 \right) f_V(v)f_S(s)dvds, \quad (2.64)$ 

where  $\lambda$  is the Lagrangian multiplier, and h(V,S) is the encoder mapping function. By writing Euler-Lagrange equations [32, Section 7.5] we have

$$\nabla_h J(h,g) = \left(2\lambda h(s,v) - \int v \dot{g}(h(v,s) + w + s) f_W(w) dw\right) \cdot f_V(v) f_S(s), \quad (2.65)$$

where  $\dot{g}(\cdot)$  is the derivative of  $g(\cdot)$ . From calculus of variations [32], it is well-known that for the optimal encoder mapping, (2.65) must be zero. This yields the necessary condition for optimality as below

$$h(v,s) = \frac{v}{2\lambda} \int \dot{g}(h(v,s) + w + s) f_W(w) dw.$$
(2.66)

The optimal decoder is the MMSE estimator, and the corresponding distortion is obtained as below

$$D = 1 - \iiint vg(h(v,s) + s + w)f_S(s)f_V(v)f_W(w)dvdsdw.$$
 (2.67)

Remark 2.5.1. We note here that the uncoded transmission satisfies the necessary condition in (2.66). Considering the transmitted signal as h(v,s) = av, where  $a = \sqrt{P}$ , the MMSE estimation can be simplified to g(y) = cy, where  $c = \frac{a}{a^2 + \sigma_s^2 + \sigma_n^2}$  and y is the received signal. Substituting  $h(\cdot), g(\cdot)$  in (2.66) we have

$$av = \frac{cv}{2\lambda} \int f_W(w)dw \Rightarrow \lambda = \frac{c}{2a}.$$
 (2.68)

Substituting  $\lambda = \frac{c}{2a}$  in (2.65) makes the gradient  $\nabla_h J(h,g)$  zero. The final resulting distortion in this case is  $D = 1/\left(1 + \frac{P}{\sigma_s^2 + \sigma_n^2}\right)$ . It is also noticed that for the ICA scheme, since  $\lambda = \frac{a}{c} + \frac{bs}{c}$  is not a constant, that does not satisfy (2.66).

#### 2.5.1Numerically Optimized Encoder

Since the optimality condition for the encoder derived above does not have a closed form expression, we use the iterative steepest decent algorithm to obtain the encoder numerically. During the iterations the encoder is updated as below

$$h_{i+1}(v,s) = h_i(v,s) - \mu \nabla_h(h,g), \qquad (2.69)$$

where i is the iteration index,  $\mu$  is the step size, and  $\nabla_h(h,g)$  is obtained as in (2.65). At each iteration the initial cost (2.64) is decreasing. Iterations are performed until  $\nabla_h(h,g)$ reaches a predefined threshold value. In order to calculate the integrals in (2.65) at each iteration we use discretization. It is worth mentioning that, since discretization injects



FIGURE 2.4: Encoder mapping h(v,s) for NOE ( $\sigma_s^2 = 4, \sigma_n^2 = 1$ ).

some residual error into the algorithm, it is essential to increase the accuracy in order to make the residual distortion (due to discretization) negligible compared to the final achievable distortion. Hence, the simulation takes considerably longer time to converge at high SNR values.

In our simulations we start from the low power constraints. The algorithm is initiated with a vector whose elements are the different values assigned to each discretized pair of (v, s). For the lowest power constraint, the initial values are chosen close to zero to make sure that they satisfy the average power constraint. The final solution obtained for a low power constraint is used as the initial guess for the higher power constraint, and so on. It should be remarked, that there is no guarantee that this iterative optimization scheme converges to the global optimal solution. We have also tried to initiate the NOE from the encoder mappings obtained for ICO and 1-DL schemes, as well as their non-uniform quantized counterparts; but in all cases we obtained the exact same final encoder mapping.

In Figure 2.4, the encoder structure for numerically optimized encoder with P = 4 dB is shown. The plot shows, in a colour-coded fashion, the channel input value (here in range [-40, 40]) corresponding to each discretized pair of source v and interference s values. To



FIGURE 2.5: NOE mapping for different interference values ( $\sigma_s^2 = 4, \sigma_n^2 = 1$ ).

elaborate the details of Figure 2.4, the encoder mapping for different values of the source and the interference outputs are shown in Figures 2.5 and 2.6, respectively. From Figures 2.4-2.6, we observe that, i) the source is clipped similarly to the parameterized nonlinear schemes considered in Section 2.4 (see Figure 2.5 and Figure 2.2 for comparison); ii) Depending on which interval the interference falls into, the transmitted signal resembles a shifted version of a linear mapping (see Figure 2.6). This is similar to the transmission of quantization noise in ICO and 1DL schemes.

#### 2.6Numerical results

We remark here that obtaining closed-form expressions for the optimal performance of JSCC under strict delay constraints is extremely difficult if not impossible. Instead, in this section, we provide numerical results comparing the performances of the proposed transmission schemes. We will also include the Shannon theoretic lower bound (SLB) obtained by evaluating the rate-distortion function of the Gaussian source at the capacity of the underlying channel when the interference is completely removed. Not surprisingly this lower bound is quite loose in general.



FIGURE 2.6: NOE mapping for source outputs v = 0.9 and v = -1.1 ( $\sigma_s^2 = 4, \sigma_n^2 = 1$ ).



FIGURE 2.7: Average MSE distortion (dB) vs. the average SNR (dB) for the proposed schemes for  $\sigma_s^2=4,\,\sigma_n^2=1$  .



FIGURE 2.8: Average MSE distortion (dB) vs. the average SNR (dB) for the proposed schemes for  $\sigma_s^2=25,\,\sigma_n^2=1$  .



FIGURE 2.9:  $\Delta$  vs. power constraint for different values of  $\Delta_v$  for ICO ( $\sigma_s^2 = 25$ ,  $\sigma_n^2 = 1$ ).



FIGURE 2.10:  $\Delta$  vs. channel power constraint for ICO and 1DL for different values of  $\Delta_v$  ( $\sigma_s^2 = 25$ ,  $\sigma_n^2 = 1$ ). Red dotted lines are associated with ICO and blue lines are associated with 1DL.  $\Delta$  for 1DL is always greater than that for ICO (for fixed value of power constraint and  $\Delta_v$ )

In Figures 2.7 and 2.8, performances of the proposed transmission schemes are illustrated and compared with SLB for different SNR levels. For ICO-NU and 1DL-NU we optimize (2.22) and (2.26) over  $\Delta$  and  $\Delta_v$  as well as a. As it can be seen in Figures 2.7 and 2.8, non-uniform quantization improves the performance of both the ICO and 1DL schemes. 1DL-NU outperforms all the other schemes proposed in Section 2.4 in both the low and high SNR regimes. We expect that SLB is loose in general (especially in the high interference regime), and identifying a tighter lower bound will be instrumental in characterizing the performance limits in this problem. We see that, as expected, NOE outperforms all other encoding schemes, but this is at the expense of a much longer computation time. We also observe that the proposed low-complexity parameterized encoding schemes perform close to NOE, particularly in the high SNR regime.

We also observe that, 1DL outperforms ICO, even though the performances of the two schemes have relatively similar behaviour. Also, in the low interference regime ICA outperforms both ICO and ICO-NU. In Figure 2.9, the size of optimal  $\Delta$  versus different channel power constraints for ICO is shown for different values of  $\Delta_v$ . It can be seen from the figure that by increasing either P or  $\Delta_v$ , size of  $\Delta$  grows non-linearly. This



FIGURE 2.11: Average MSE distortion vs.  $\Delta_v$  for the ICO for  $\sigma_s^2=25,\,\sigma_n^2=1$  .

can be easily verified from (2.16), (2.19) and (2.20). Note that (2.20) tends to  $\sigma_s^2$  as  $\Delta$  increases. On the other hand, in (2.19) it can be shown that as  $\Delta$  increases,  $\Delta_v$ increases too (for a fixed value of  $\sigma_T^2$ ). For high values of  $\Delta$  and  $\Delta_v$  (2.16) simplifies to  $\Delta = \Delta_v \sqrt{P - \sigma_s^2}$ . This linear relation is also observed in the figure. In Figure 2.10 the size of the quantization interval  $\Delta$  for both ICO and 1DL is plotted against the power constraint. As it is seen in this figure, for all power constraint values, 1DL uses a larger  $\Delta$ than ICO for quantization, which explains the improved performance of 1DL compared to ICO (larger  $\Delta$  means that the source is mapped into a larger interval, and hence, can be reconstructed with a smaller average distortion). A similar observation applies also to the ICO-NU and 1DL-NU schemes, and the latter outperforms the former.

In Figure 2.11 the average distortion versus  $\Delta_v$  is plotted for ICO, and for different input power constraints. We observe from the figure that the average distortion is a convex function of  $\Delta_v$ . For high power constraints, distortion is almost constant beyond a certain value for  $\Delta_v$  (the bottom curve in Figure 2.11). As it can be seen, the higher the input power constraint (since  $\sigma_n^2 = 1$ , increasing P is equivalent to increasing SNR) the higher the optimal value for  $\Delta_v$ , which achieves the minimum average distortion.

In Figure 2.12 the average distortion with respect to normalized  $\left(\frac{\kappa}{\Delta} = \frac{\Delta - 2d}{\Delta}\right)$  is plotted



Average MSE distortion vs.  $\frac{\kappa}{\Delta}$  for ICO and 1DL for P=5 (dB),  $\sigma_s^2=4,\,\sigma_n^2=1~.$ Figure 2.12:

for both ICO and 1DL. It is observed that the average distortion has a minima with regard to the noise gap, d. It is seen from the figure that there is space to improve the achievable average distortion by optimizing over d. Since optimizing the achievable distortion over d,  $\Delta_v$ ,  $\Delta$  is demanding, we have obtained the noise gap effect d on the final distortion only for P = 5 dB in Figure 2.12 (for the remainder of the simulations we have assumed d = 0).

## Chapter 3

# Zero-Delay Source-Channel Coding with a Low-Resolution ADC Front End

## 3.1 Overview

Motivated by the practical constraints arising in sensor networks and Internet-of-Things applications, the zero-delay transmission of a Gaussian measurement over a vector additive white Gaussian noise channel is studied with a low-resolution ADC front end. The optimization of the encoder and the decoder is tackled under both the MSE distortion and the DOP criteria, with an average power constraint on the channel input. Optimal encoder and decoder mappings are identified for the case of a one-bit ADC front end for both criteria. For the MSE distortion, the optimal encoder mapping is shown to be non-linear in general, and it tends to a linear encoder, in the low SNR regime, and to an antipodal digital encoder in the high SNR regime. This is in contrary to the optimality of linear encoding at all SNR values in the presence of a full precision front end. For the DOP criterion, it is shown that the optimal encoder mapping is piecewise constant, and can take only two opposite values when it is non-zero. For both the MSE distortion and the DOP criteria, necessary optimality conditions are derived for a K-level ADC front end and for multiple one-bit ADC front ends. These conditions then are used to obtain numerically optimized solutions. Extensive numerical results are provided to gain insights into the optimal encoding and decoding functions.

## 3.2 Introduction

Power consumed by ADCs grows exponentially with the number of bits, and linearly with the sampling rate [59], [60]. This limits the resolution of the ADCs when the available power is limited, for example for sensor nodes or mobile devices that operate on limited battery power. As an extreme case, one-bit ADCs are of particular interest due to their low hardware complexity, since they can be realized using a simple threshold comparator, and without the need for automatic gain control [61], [62].

Motivated by these considerations the impact of a one-bit ADC front end on the performance of a communication system has been studied in the literature for various models. In [63], it is shown that antipodal signalling, or BPSK, is capacity achieving for a real-valued AWGN channel with a one-bit ADC front end, whereas, for the complex counterpart, QPSK is optimal. While, these results hold under the assumption that the one-bit ADC is symmetric (that it, a zero-threshold comparator), in [64] it is shown that, in the low SNR regime, a symmetric quantizer is not optimal, and the optimal performance is achieved by flash signalling [65, Def. 2] together with an optimized asymmetric quantizer. In [66], it is shown that, for a point-to-point Rayleigh fading channel with a one-bit ADC front end, under the assumption of perfect CSI at the receiver, QPSK is capacity achieving. When the CSI is not available at the receiver, it is shown in [67]that, QPSK is optimal above an SNR threshold that depends on the coherence time of the channel, while, for lower SNRs, on-off QPSK achieves the capacity. For the pointto-point MIMO scenarios with a one-bit ADC front end at each receive antenna, the capacity is unknown. In [68], it is shown that, with perfect CSI at the receiver, QPSK is optimal at very low SNRs. In [69], upper and lower bounds on the capacity with perfect receiver CSI are presented.

While the reviewed works focus on reliable transmission of digital information over long blocks, in many applications, such as the Internet of Things, cyber-physical systems, and wireless sensor networks, the low-delay transfer of analog measurements is crucial. Motivated by this observation, this work considers the zero-delay transmission of an analog Gaussian source over a vector AWGN channel followed by a low-resolution ADC front end. With an infinite resolution front end and equal bandwidth of a Gaussian source and an AWGN channel [3], it is well known that under an average power constraint and MSE distortion measure, linear transmission and MMSE estimation are, respectively, the optimal encoder and decoder. Instead, the optimal encoder and decoder are unknown in the finite resolution scenario studied here.

For the general case of bandwidth mismatch between the source and the channel, there is no explicit method to obtain the optimal encoding and decoding functions, except for the special case of the matched source-channel pairs [4], [3]. Various solutions have been proposed for specific source-channel pairs. Notable examples are the space-filling curves, originally proposed by Shannon [1] and Kotelnikov [9], and later extended in [10–14] for the delay-limited transmission of a Gaussian source over an AWGN channel. In [10, 15] and [16], design of an optimal encoder is studied based on PCCOVQ, where a discretized version of the problem is tackled using tools developed for vector quantization. In [17], the authors studied optimal vector transformations from the m-dimensional source space to the k-dimensional channel under a given transmission power constraint and for the MSE distortion criterion. Having obtained the necessary conditions of optimality for the encoder and the decoder, [17] shows that, in point-to-point zero-delay transmission, i.e., each source sample transmitted over a single use of the channel, the optimal solution is unique.

In this chapter, we study the optimization of encoding and decoding functions for the transmission of a Gaussian source sample over the single use of a vector AWGN channel with a low-resolution ADC front end. We tackle the optimization problem of the encoder and the decoder under two different criteria; namely the MSE distortion, and the DOP with an average power constraint on the channel input. We derive the optimal solutions for the case of a one-bit ADC front end for both criteria. For the MSE distortion, we show that the optimal encoder mapping tends to a linear encoder, which is optimal with a full-precision front end, only in the low SNR regime. For the DOP criterion, we derive the optimal encoder mapping, showing that it is piecewise constant and can take only two opposite values when it is non-zero. For both the MSE distortion and the DOP criteria, we study necessary optimality conditions for a K-level ADC front end and for multiple one-bit ADC front ends. These conditions are used to obtain numerically optimized solutions.



FIGURE 3.1: System model for the transmission of a single Gaussian source sample over a quantized vector AWGN channel with N ADC front ends.

The rest of the chapter is organized as follows. In Section 3.3, we introduce the system model. In Section 3.4, we consider the design of the optimal transceiver under the MSE distortion criterion when the receiver has a single observation of the source. In Section 3.5, we study the same design problem under the DOP criterion. In Section 3.6, we study a more general case in which the receiver makes multiple one-bit observations under both the MSE distortion and the DOP criteria. In Section 3.7, numerical results are provided.

Notations: Throughout the chapter,  $\mathbb{R}$  denotes the set of real numbers; uppercase and lowercase letters denote random variables and realizations, respectively. We use  $b_j^N$  to denote the bit-wise representation of the number  $2^N - j$ ,  $j = 1, \ldots, 2^N$  with length N.  $\mathbb{E}[\cdot]$  and  $\Pr(\cdot)$  denote the expectation and probability operators, respectively. Let  $f'(x) = \frac{df(x)}{dx}$ ,  $f''(x) = \frac{d^2f(x)}{dx^2}$  denote the first and second order derivatives of the continuously differentiable function f with respect to the argument. The standard normal distribution is denoted by  $\mathcal{N}(0,1)$ , with distribution function  $\Phi(\cdot)$ , and the CCDF by  $Q(\cdot)$ , which is given by

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\frac{t^2}{2}} dt$$

Unless stated otherwise, boundaries of the integrals are from  $-\infty$  to  $\infty$ .

## 3.3 System Model

We consider the system model in Figure 3.1, in which a single sample of a Gaussian source  $V \sim \mathcal{N}(0, \sigma_v^2)$  is transmitted over a single use of a quantized vector AWGN channel. The encoded signal is given as X = f(V), where  $f : \mathbb{R} \to \mathbb{R}$  is a mapping from the source sample to the channel input, with average transmission power  $P = \mathbb{E}[f(V)^2]$ . The receiver makes N noisy measurements of the encoded signal, which are digitized by means of a low-resolution ADC front end. Mathematically, each noisy received signal is modelled as

$$Z_i = f(V) + W_i, \ i = 1, ..., N, \tag{3.1}$$

where the noise  $W_i \sim \mathcal{N}(0, \sigma_{w_i}^2)$ , i = 1, ..., N, is independent over index *i*. Each received signal  $Z_i$  is quantized with a scalar *K*-level ADC producing quantized level

$$Y_i = \mathcal{Q}(Z_i), \ i = 1, \dots, N.$$

$$(3.2)$$

The scalar K-level ADC is characterized by fixed quantization intervals and corresponding quantized level, namely

$$Q(z) = y_{(j)}, \text{ for } z \in [z_{(j-1)}, z_{(j)}), \ j = 1, \dots, K,$$
(3.3)

where  $z_{(j-1)}$  and  $z_{(j)}$  are the lower and upper bounds of the interval corresponding to the quantized signal  $y_{(j)}$ , respectively, for j = 1, ..., K, and we have  $z_{(0)} = -\infty$  and  $z_{(K)} = \infty$ . Note that the ADCs employed to quantize different channel outputs all have the same structure, that is, the quantization intervals and reconstruction levels.

For most of the chapter we will consider a symmetric one-bit ADC, with threshold  $z_{(1)} = 0$  and reconstruction levels  $y_{(1)} = 1$  and  $y_{(2)} = 0$ :

$$Q(z) = \begin{cases} 0 & z \ge 0, \\ 1 & z < 0. \end{cases}$$
(3.4)

We define the SNR as

$$\gamma = \frac{NP}{\sum\limits_{i=1}^{N} \sigma_{w_i}^2}.$$
(3.5)

Based on the quantized signals  $(Y_1, ..., Y_N) \triangleq Y^N$ , the decoder produces an estimate  $\hat{V}$  of V using the decoding function  $g : \{y_{(1)}, ..., y_{(K)}\}^N \to \mathbb{R}$ , i.e.,  $\hat{V} = g(Y^N)$ .

Two performance criteria are considered, namely, the MSE distortion defined as

$$\bar{D} = \mathbb{E}\left[ (V - \hat{V})^2 \right], \qquad (3.6)$$

and the DOP defined as

$$\epsilon(D) = \Pr\left((V - \hat{V})^2 \ge D\right).$$
(3.7)

In both cases, we aim at studying the optimal encoder mapping function f, along with the corresponding optimal estimator g at the decoder, such that,  $\overline{D}$  and  $\epsilon(D)$  are minimized subject to the average power constraint. More specifically, as it is common in related works (see, e.g., [17]), we consider the unconstrained minimization

$$\min_{\substack{f,g\\ f,g}} L(f,g,\lambda),$$
(3.8)

where

$$L(f,g,\lambda) = \begin{cases} \bar{D} + \lambda \mathbb{E}[f(V)^2] & \text{for the MSE criterion,} \\ \epsilon(D) + \lambda \mathbb{E}[f(V)^2] & \text{for the DOP criterion,} \end{cases}$$
(3.9)

with  $\lambda \geq 0$  being a Lagrange multiplier that defines the relative weight given to the average transmission power  $\mathbb{E}[f(V)^2]$  as compared to the distortion criterion.

## 3.4 Single Observation: MSE Distortion

In this section, we study the design of the encoder and the decoder under the MSE criterion by focusing on the case of a single observation (N = 1). For the one-bit ADC in (3.4), we obtain the optimal encoder and decoder in Section 3.4.1. Furthermore, we consider the conventional linear transmission and digital modulation schemes for reference in Section 3.4.2 and Section 3.4.3, respectively. Finally, we consider the extensions to a K-level front end, and obtain a necessary condition on the optimal mapping in Section 3.4.4. For brevity, throughout this section, we drop the subscript i = 1 identifying the observation index.

#### 3.4.1 Optimal Encoder and Decoder for a One-Bit Front End

To elaborate on the optimal encoder and decoder for the one-bit ADC in (3.4), without loss of generality, we write the receiver mapping function as

$$g(Y) = \hat{V} = \begin{cases} \hat{v}_{(1)} & Y = 0, \\ \hat{v}_{(2)} & \text{if } Y = 1, \end{cases}$$
(3.10)

which is defined by the pair of parameters  $(\hat{v}_{(1)}, \hat{v}_{(2)})$ . In (3.10), since, for any encoder mapping f, the MMSE estimator is optimal under the MSE criterion, and hence also for problem (3.8), we have  $\hat{v}_{(1)} = \mathbb{E}[V|Y=0]$  and  $\hat{v}_{(2)} = \mathbb{E}[V|Y=1]$ . The next proposition provides the optimal encoder mapping function.

Proposition 3.4.1. The optimal mapping f for problem (3.8) under the MSE criterion is unique up to a sign, is an odd function of v, and is defined by the implicit equation

$$f(v)e^{\frac{f(v)^2}{2\sigma_w^2}} = \frac{v}{\sqrt{2\pi}\sigma_w\lambda}.$$
(3.11)

*Proof*: See Appendix B.

Illustrations of the optimal mappings satisfying (3.11) will be given in Section 3.7. Here, we observe that, by expanding the Taylor series of the exponential function in (3.11), it can be easily verified that, in the low SNR regime, that is, as  $\sigma_w^2 \to \infty$ , the optimal mapping satisfies the condition  $f(v) \propto v$ , that is, it approaches a linear mapping.

Furthermore, given that the optimal mapping function f(v) is odd, we can write

$$\hat{v}_{(1)} = \mathbb{E}[V|Y=0] = \frac{\frac{1}{\sigma_v} \int v\Phi\left(\frac{v}{\sigma_v}\right) \Pr(Y=0|V=v) dv}{\Pr(Y=0)}$$
(3.12a)

$$= \frac{-2}{\sigma_v} \int v \Phi\left(\frac{v}{\sigma_v}\right) Q\left(\frac{f(v)}{\sigma_w}\right) dv$$
(3.12b)

$$= \frac{2}{\sigma_v} - \frac{4}{\sigma_v} \int_0^\infty v \Phi\left(\frac{v}{\sigma_v}\right) Q\left(\frac{f(v)}{\sigma_w}\right) dv, \qquad (3.12c)$$

and hence the average distortion can be simplified as

$$\bar{D} = \sigma_v^2 - \mathbb{E}[V\hat{V}] \tag{3.13a}$$

$$= \sigma_v^2 - \frac{1}{2} \left( \hat{v}_{(1)} \mathbb{E}[V | \hat{V} = \hat{v}_{(1)}] + \hat{v}_{(2)} \mathbb{E}[V | \hat{V} = \hat{v}_{(2)}] \right)$$
(3.13b)

$$=\sigma_v^2 - \hat{v}_{(1)}^2, \tag{3.13c}$$

where (3.13a) is due to the orthogonality property of MMSE estimation; (3.13b) follows from the fact that the optimal encoder is odd; and (3.13c) is due to the chain of equalities  $\mathbb{E}[V|\hat{V} = \hat{v}_{(1)}] = \mathbb{E}[V|Y = 0] = \hat{v}_{(1)} = -\hat{v}_{(2)} = -\mathbb{E}[V|\hat{V} = \hat{v}_{(2)}].$ 

#### 3.4.2 Linear Transmission for One-Bit Front End

Here we consider the performance of linear transmission in the presence of a one-bit ADC front end. The encoder mapping for linear transmission is given by

$$f(v) = \sqrt{\frac{P}{\sigma_v^2}}v. \tag{3.14}$$

As seen in Section 3.4.1, linear transmission is optimal in the low-SNR asymptotic. In the following we elaborate on its performance for any given channel SNR  $\gamma$ . The MSE distortion achieved by linear transmission,  $\bar{D}_l$ , can be found by substituting (3.14) in (3.13c). Since the resulting expression is not in closed form, we derive analytical upper and lower bounds that can be useful to obtain additional insights. Using the lower bound in [70] for the Q function, namely  $Q(x) \geq \beta e^{-\frac{kx^2}{2}}$ , where  $\beta = \frac{e^{(\pi(k-1)+2)^{-1}}}{2k} \sqrt{\frac{1}{\pi}(k-1)(\pi(k-1)+2)}$  for any  $k \geq 1$ , the distortion of linear transmission can be lower bounded as

$$\bar{D}_l \ge \sigma_v^2 \left( 1 - \frac{2}{\pi} \left( 1 - \frac{2\beta}{1 + k \cdot \gamma} \right)^2 \right). \tag{3.15}$$

On the other hand, using the inequality  $Q(x) \leq \frac{1}{4}(e^{-x^2} + e^{-\frac{x^2}{2}})$  in [71] we have the upper bound as

$$\bar{D}_l \le \sigma_v^2 \left( 1 - \frac{1}{2\pi} \left( \frac{\gamma(3+8\gamma)}{(\gamma+1)(2\gamma+1)} \right) \right). \tag{3.16}$$

Using these bounds we observe that, in the asymptotic limit of low SNR, i.e., as  $\gamma \to 0$ , we have the average distortion  $\bar{D}_l = \sigma_v^2$ , while, in the high SNR asymptotic, i.e., as  $\gamma \to \infty$ , we obtain  $\bar{D}_l = \sigma_v^2(1 - 2/\pi)$ . Both distortions can be argued to be optimal asymptotically. In fact, for zero SNR, the MMSE estimate even with an infiniteresolution front end is given by  $\hat{V} = 0$ , which yields  $\bar{D} = \sigma_v^2$ . Instead, for infinite SNR, the best mapping is given by the optimal binary quantizer, which yields  $\overline{D} = \sigma_v^2 (1-2/\pi)$  (see, e.g., [2, Section 10.1]).

#### 3.4.3 Digital Transmission for One-bit Front End

Here we consider a conventional digital transmission scheme, which is based on quantizing and mapping the source to a discrete constellation for transmission over the channel. Accordingly, the source is quantized to one of the M levels, each characterized by the interval  $[v_{(l-1)}, v_{(l)})$ , l = 1, ..., M, where  $v_{(M)} = \infty$ ,  $v_{(0)} = -\infty$ , and  $v_{(l)} \ge v_{(l-1)}$  for all l = 1, ..., M. Each interval  $[v_{(l-1)}, v_{(l)})$  is mapped to the corresponding channel input  $X = x_{(l)}$ . We take the constellation of possible transmission points to be  $\{X = A(2l - 1 - M), l = 1, ..., M\}$ , for some parameter  $A \ge 0$ , such that, the average power constraint is satisfied. Note that, when M is even, this corresponds to the M-PAM modulation, while if M is odd, the constellation includes the zero-power signal, i.e.,  $x_{(\frac{M+1}{2})} = 0$ . The average transmission power can be written as  $\mathbb{E}[X^2] = \sum_{l=1}^{M} x_{(l)}^2 \cdot \Pr(X = x_{(l)})$ , where  $\Pr(X = x_{(l)}) = \frac{1}{\sigma_v} \int_{v_{(l-1)}}^{v_{(l)}} \Phi(v/\sigma_v) dv$ .

The average achievable distortion for M levels of symmetric digital transmission, i.e.,  $v_{(l)} = -v_{(M-l)}$ , can be easily obtained as

$$\bar{D}_{d,M} = \sigma_v^2 - \left(\frac{2}{\sigma_v} \sum_{l=1}^M Q\left(\frac{(2l-1-M)\sqrt{\gamma}}{S}\right) \int_{v_{(l-1)}}^{v_{(l)}} v\Phi\left(\frac{v}{\sigma_v}\right) dv\right)^2,\tag{3.17}$$

where  $S^2 = \sum_{l=1}^{M} (2l - 1 - M)^2 \Pr(x_{(l)}).$ 

As a special case, when M = 2, setting the quantization threshold as  $v_{(1)} = 0$ , we obtain BPSK transmission. The resulting achievable distortion can be computed from (3.17) as

$$\bar{D}_{d,2} = \sigma_v^2 \left( 1 - \frac{2}{\pi} \left( 1 - 2Q \left( \sqrt{\gamma} \right) \right)^2 \right).$$
(3.18)

We observe that, as for linear transmission, when  $\gamma \to \infty$ , we have  $\bar{D}_{d,2} = \sigma_v^2(1-2/\pi)$ , and when  $\gamma \to 0$ , we have  $\bar{D}_{d,2} = \sigma_v^2$ . Also, from (3.18), one can check that the slope of the average distortion for BPSK transmission as  $\gamma \to 0$  is  $-4\sigma_v^2/\pi^2$ , which is smaller than the slope for linear transmission that can be obtained from (3.13c) as  $-2\sigma_v^2/\pi$ . This shows that in the low SNR regime linear transmission is preferable to BPSK transmission.

As another example, for M = 3, we set the quantization thresholds as  $v_{(1)} = -c$  and  $v_{(2)} = c$ , so that [-c, c] is the interval of source values for which the transmission symbol is  $x_{(2)} = 0$ . The MSE distortion can be computed by solving the following optimization problem with line search:

$$\bar{D}_{d,3} = \min_{c \ge 0} \sigma_v^2 \left( 1 - \frac{2e^{-\frac{c^2}{\sigma_v^2}}}{\pi} \cdot \left( 1 - 2Q\left(\sqrt{\frac{\gamma}{2Q\left(\frac{c}{\sigma_v}\right)}}\right) \right)^2 \right).$$
(3.19)

#### 3.4.4 K-Level Front End

In this section, we consider the system model in Figure 3.1 with a single observation, i.e., N = 1, but with a K-level ADC front end as in (3.3). As in the case of single one-bit observation, without loss of generality, we write the receiver mapping function as

$$g(Y) = \hat{V} = \hat{v}_{(j)}, \text{ if } Y = y_{(j)},$$
(3.20)

for j = 1, ..., K. In the next proposition, we obtain a necessary optimality condition for the encoding and decoding functions f and g.

Proposition 3.4.2. The optimal encoder and decoder mappings f and g for the K-level ADC front end in (3.3) satisfy the necessary conditions

$$f(v) = \frac{1}{2\sqrt{2\pi}\sigma_w\lambda} \sum_{j=1}^{K} \left( e^{-\frac{\left(z_{(j-1)}-f(v)\right)^2}{2\sigma_w^2}} - e^{-\frac{\left(z_{(j)}-f(v)\right)^2}{2\sigma_w^2}} \right) \hat{v}_{(j)} \left(2v - \hat{v}_{(j)}\right), \quad (3.21)$$

and (3.20) with

$$\hat{v}_{(j)} = \frac{\int v\Phi\left(\frac{v}{\sigma_v}\right) \left(Q\left(\frac{z_{(j-1)}-f(v)}{\sigma_w}\right) - Q\left(\frac{z_{(j)}-f(v)}{\sigma_w}\right)\right) dv}{\int \Phi\left(\frac{v}{\sigma_v}\right) \left(Q\left(\frac{z_{(j-1)}-f(v)}{\sigma_w}\right) - Q\left(\frac{z_{(j)}-f(v)}{\sigma_w}\right)\right) dv}, \quad j = 1, ..., K.$$
(3.22)

Furthermore, the gradient of the Lagrangian function  $L(f, g, \lambda)$  over f for g in (3.20) is given as

$$\nabla L = 2\lambda f(v) - \frac{1}{\sqrt{2\pi}\sigma_w} \sum_{j=1}^K \left( e^{-\frac{\left(z_{(j-1)} - f(v)\right)^2}{2\sigma_w^2}} - e^{-\frac{\left(z_{(j)} - f(v)\right)^2}{2\sigma_w^2}} \right) \hat{v}_{(j)} \left( 2v - \hat{v}_{(j)} \right). \quad (3.23)$$

*Proof*: See Appendix C.

We note that the gradient in (3.23), along with (3.22), will be used in Section 3.7 to obtain numerically optimized encoders and decoders.

### 3.5 Single Observation: Distortion Outage Probability

In this section, we study the optimal encoder and decoder under the DOP criterion defined in (3.7) for the case of a single observation (N = 1). We first study the case of a one-bit front end in Section 3.5.1, and then we extend the results to a K-level front end in Section 3.5.2.

#### 3.5.1 Optimal Encoder and Decoder for a One-Bit Front End

With no loss of generality, the decoder is given as in (3.10) for some reconstruction points  $(\hat{v}_{(1)}, \hat{v}_{(2)})$ . To proceed, we first focus on the optimization of the encoder mapping f for a given decoder in (3.10). We then tackle the problem of minimizing the DOP over the reconstruction points  $(\hat{v}_{(1)}, \hat{v}_{(2)})$ .

To elaborate, we define the intervals

$$I_{j} \triangleq \{ v : (v - \hat{v}_{(j)})^{2} < D \},$$
(3.24)

for j = 1, 2, which are depicted in Figure 3.2. Each interval  $I_j$ , corresponds to the set of source values that are within the allowed distortion D of the reconstruction point  $\hat{v}_{(j)}$ . The following claims hold: (i) For all source outputs v in the set  $(I_1 \cup I_2)^C = \{v : \min_{j=1,2}(v - \hat{v}_{(j)})^2 > D\}$ , outage occurs (superscript C denotes the complement set). We refer to this event as *source outage*. (ii) For all source values in the interval  $I_1 \cap I_2$ , either of the reconstruction points yield a distortion no more than the target value D. Therefore, regardless of which of the two reconstruction levels,  $\hat{v}_{(1)}$  and  $\hat{v}_{(2)}$ ), is selected by the receiver, no outage occurs. From observations (i) and (ii), it easily follows that, for all source values v inside the intervals  $(I_1 \cup I_2)^C$  and  $(I_1 \cap I_2)$ , the optimal mapping is f(v) = 0, since, for both intervals, the occurrence of an outage event is independent of the transmitted signal.



FIGURE 3.2: Illustration of the intervals  $I_1$ ,  $I_2$  and  $(I_1 \cap I_2)$  that characterize the optimal encoder for the DOP criterion, for two different cases depending on the  $(\hat{v}_{(1)}, \hat{v}_{(2)})$ values.

From the discussion above, we only need to specify the optimal mapping for the intervals  $I_1 \setminus I_2$  and  $I_2 \setminus I_1$ . This should be done by accounting not only for the *source outage* event mentioned above, but also for the *channel outage* events. In particular, the distortion outage probability  $\epsilon(D)$  can be written as

$$\epsilon(D) = \Pr\left(V \in (I_1 \cup I_2)^C\right) + \Pr\left(V \in (I_1 \setminus I_2), \hat{V} = \hat{v}_{(2)}\right) + \Pr\left(V \in (I_2 \setminus I_1), \hat{V} = \hat{v}_{(1)}\right), \qquad (3.25)$$

where the first term accounts for the source outage event, while the second and third terms are the probabilities of outage due to channel transmission errors. For instance, the second term is the probability that the decoder selects  $\hat{V} = \hat{v}_{(2)}$  while V is in the interval  $I_1 \setminus I_2$  (see Figure 3.2). The next proposition characterizes the optimal encoder mapping.

Proposition 3.5.1. Given a target distortion D, and arbitrary reconstruction points  $\hat{v}_{(1)}$ and  $\hat{v}_{(2)}$ , the optimal mapping f for the problem (3.8) is given by

$$f(v) = \begin{cases} 0 & v \in (I_1 \cup I_2)^C \cup (I_1 \cap I_2), \\ -u & v \in (I_2 \setminus I_1), \\ u & v \in (I_1 \setminus I_2), \end{cases}$$
(3.26)

where u is the unique solution of

$$ue^{\frac{u^2}{2\sigma_w^2}} = \frac{1}{2\sqrt{2\pi}\sigma_w\lambda}.$$
(3.27)

*Proof*: See Appendix D.

We note here that, for given  $\lambda \geq 0$ , the optimal u is independent of the values of  $\hat{v}_{(1)}$ and  $\hat{v}_{(2)}$ . Examples of optimal encoders will be provided in Section 3.7. In the next proposition, we turn to the optimization of the reconstruction levels  $(\hat{v}_{(1)}, \hat{v}_{(2)})$ .

Proposition 3.5.2. The optimal reconstruction points  $(\hat{v}_{(1)}, \hat{v}_{(2)})$ , are given by

$$\hat{v}_{(1)} = \sqrt{D} - a^*,$$
(3.28a)

$$\hat{v}_{(2)} = -\hat{v}_{(1)},\tag{3.28b}$$

where  $a^*$  is obtained from

$$a^{*} = \underset{a \in [0, \sqrt{D}]}{\operatorname{arg min}} 2Q\left(\frac{2\sqrt{D}-a}{\sigma_{v}}\right) + 2\left(Q\left(\frac{u}{\sigma_{w}}\right) + \lambda u^{2}\right)$$
$$\cdot \left(Q\left(\frac{a}{\sigma_{v}}\right) - Q\left(\frac{2\sqrt{D}-a}{\sigma_{v}}\right)\right), \quad (3.29)$$

where u is obtained by solving (3.27).

#### *Proof*: See Appendix $\mathbf{E}$ .

To summarize, the optimal encoder and decoder are obtained as follows. First, given the Lagrange multiplier  $\lambda \geq 0$ , the value of u is obtained by solving (3.27). Then the decoder's reconstruction points  $(\hat{v}_{(1)}, \hat{v}_{(2)})$  are computed from (3.28)-(3.29). Finally, Finally, the optimal encoder mapping is given by (3.26). The next remark elaborates on the optimal encoder and decoder in two asymptotic SNR regimes.

*Remark* 3.5.1. If  $\lambda$  is large, i.e., in the low-SNR regime, from (3.27) we have  $u \approx 0$ . Also from (3.27) it can be verified that  $\lambda u^2 = \frac{e^{-\frac{u^2}{\sigma_w^2}}}{8\pi\sigma_w^2\lambda} \approx 0$ . Hence, from (3.29) we obtain

$$a^* \approx \underset{a \in [0,\sqrt{D}]}{\operatorname{arg min}} Q\left(\frac{a}{\sigma_v}\right) + Q\left(\frac{2\sqrt{D}-a}{\sigma_v}\right),$$
 (3.30)

yielding  $\hat{v}_{(1)} = \hat{v}_{(2)} = 0$ , that is,  $I_1 = I_2$  and  $\epsilon(D) = 2Q\left(\frac{\sqrt{D}}{\sigma_v}\right)$ . On the other hand, for small values of  $\lambda$ , corresponding to the high-SNR regime, u becomes large. Also, from (3.27) we have  $\lambda = 1/\left(2\sqrt{2\pi}\sigma_w u e^{\frac{u^2}{2\sigma_w^2}}\right)$ . Hence, we have  $\lambda u^2 = u/\left(2\sqrt{2\pi}\sigma_w e^{\frac{u^2}{2\sigma_w^2}}\right) \approx 0$ , and

$$a^* \approx \underset{a \in [0,\sqrt{D}]}{\operatorname{arg min}} 2Q\left(\frac{2\sqrt{D}-a}{\sigma_v}\right),$$
(3.31)

which yields distinct intervals  $I_1$  and  $I_2$  with  $\hat{v}_{(1)} = -\hat{v}_{(2)} = \sqrt{D}$ , and  $\epsilon(D) = 2Q\left(\frac{2\sqrt{D}}{\sigma_v}\right)$ .

#### **3.5.2** *K*-Level Front End

Here we turn to the case of K-level front end under the DOP criterion. With no loss of optimality, the decoder is given as in (3.20) for some reconstruction levels  $\{\hat{v}_{(1)}, \ldots, \hat{v}_{(K)}\}$  to be optimized. For a subset  $\mathcal{V} \subset \{\hat{v}_{(1)}, \ldots, \hat{v}_{(K)}\}$ , let  $I_{\mathcal{V}}$  be the set of source outputs v for which the quadratic distance between v and the reconstruction points in set  $\mathcal{V}$  is less than D, while the quadratic distance with respect to the reconstruction points in  $\mathcal{V}^C$  is larger than D. This set is obtained as

$$I_{\mathcal{V}} = \bigcap_{\hat{v}_{(j)} \in \mathcal{V}} I_j \setminus \bigcup_{\hat{v}_{(j)} \in \mathcal{V}^C} I_j, \ |\mathcal{V}| = 1, ..., K,$$
(3.32)

where  $I_i$  is defined as

$$I_j \triangleq \left\{ v : |v - \hat{v}_{(j)}|^2 < D \right\}.$$
(3.33)

We also define the set  $I_{\emptyset}$  corresponding to  $\mathcal{V} = \emptyset$  as

$$I_{\emptyset} \triangleq \left(\bigcup_{\mathcal{V}: |\mathcal{V}| \neq 0} I_{\mathcal{V}}\right)^C, \qquad (3.34)$$

that is, the set of values  $v \in \mathbb{R}$  that do not have a reconstruction value  $\hat{v}_{(j)}$  within a distance  $\sqrt{D}$ . Note that the sets  $\{I_{\mathcal{V}}: \mathcal{V} \subset \{\hat{v}_{(1)}, ..., \hat{v}_{(K)}\}\}$ , is a partition of the whole real line.

As for the single one-bit front end (3.25), the DOP depends on both source and channel outage events. Note that the source outage occurs if  $V \in I_{\emptyset}$ , whereas the channel outage occurs when  $V \notin I_{\emptyset}$ , with no outage occurring when  $V \in I_{\{\hat{v}_{(1)},...,\hat{v}_{(K)}\}}$ . In the following proposition, we present necessary optimality conditions for the encoder and decoder mappings.

Proposition 3.5.3. For a K-level ADC front end, the optimal encoder and decoder mappings f and g satisfy the necessary conditions

$$f(v) = \frac{1}{2\lambda}G(f, v), \qquad (3.35)$$

where G(f, v) is defined as

$$G(f, v) \triangleq \begin{cases} 0 & v \in I_{\mathcal{V}} : |\mathcal{V}| = 0, K, \\ \frac{1}{\sqrt{2\pi\sigma_{w}}} \sum_{j: \ \hat{v}_{(j)} \in \mathcal{V}} \left( e^{-\frac{\left(z_{(j-1)} - f(v)\right)^{2}}{\sigma_{w}^{2}}} - e^{-\frac{\left(z_{(j)} - f(v)\right)^{2}}{\sigma_{w}^{2}}} \right) & v \in I_{\mathcal{V}} : |\mathcal{V}| = 1, \dots, K-1, \end{cases}$$

$$(3.36)$$

and (3.20) with

$$\hat{v}_{(j)} = \arg\max_{t} \int_{t-\sqrt{D}}^{t+\sqrt{D}} \Phi\left(\frac{v}{\sigma_{v}}\right) \left(Q\left(\frac{z_{(j-1)} - f(v)}{\sigma_{w}}\right) - Q\left(\frac{z_{(j)} - f(v)}{\sigma_{w}}\right)\right) dv. \quad (3.37)$$

Furthermore, the gradient of the Lagrangian function  $L(f, g, \lambda)$  over f for g in (3.20) is given as

$$\nabla L = 2\lambda f(v) - \frac{1}{2}G(f, v).$$
(3.38)

*Proof*: See Appendix  $\mathbf{F}$ .

As for Proposition 3.4.2, in Section 3.7, we will use (3.37) and (3.38) to obtain numerically optimized encoders and decoders, that satisfy the necessary optimality conditions.

### 3.6 Multiple Observations

In this section, we study the more general case in which the receiver has N > 1 noisy one-bit quantized observations,  $Y^N$ , of the transmitted source sample. For both the
MSE distortion and the DOP criteria, without loss of generality, we write the receiver mapping function as

$$g(Y^N) = \hat{V} = \hat{v}_{(j)}, \text{ if } Y^N = b_j^N,$$
 (3.39)

where  $b_j^N$  is the N-length binary representation of the number  $2^N - j$  for  $j = 1, ..., 2^N$ . Note that there are  $2^N$  reconstruction levels  $\hat{v}_{(j)}$ , each of them corresponding to a different configuration of the received signal  $Y^N \in \{0,1\}^N$ . We will denote by  $b_j^N(k)$  the k-th element of the vector  $b_j^N$ .

# 3.6.1 MSE Criterion

Recalling that the optimal decoding function under the MSE criterion is the MMSE estimator, the optimal decoder satisfies  $\hat{v}_{(j)} = \mathbb{E}[V|Y^N = b_j^N]$ . In the next proposition, we provide necessary conditions for the optimal encoder and decoder mappings along with an expression for the gradient of the Lagrangian in (3.9) over the encoder mapping f.

Proposition 3.6.1. Given a front end with N one-bit ADCs, the optimal encoder and decoder mappings f and g satisfy the necessary conditions

$$f(v) = \frac{1}{2\lambda} \sum_{j=1}^{2^N} \Theta\left(N, f(v), j\right) \hat{v}_{(j)} \left(2v - \hat{v}_{(j)}\right), \qquad (3.40)$$

and (3.39) with

$$\hat{v}_{(j)} \triangleq \frac{\int v\Phi\left(\frac{v}{\sigma_v}\right) \prod_{i=1}^N Q\left(\frac{(-1)^{b_j^N(i)+1}f(v)}{\sigma_{w_i}}\right) dv}{\int \Phi\left(\frac{v}{\sigma_v}\right) \prod_{i=1}^N Q\left(\frac{(-1)^{b_j^N(i)+1}f(v)}{\sigma_{w_i}}\right) dv},\tag{3.41}$$

where  $\Theta(N, f(v), j)$  is defined as

$$\Theta(N, f(v), j) \triangleq \sum_{k=1}^{N} \left( \frac{(-1)^{b_j^N(k)} e^{-\frac{f(v)^2}{2\sigma_{w_k}^2}}}{\sqrt{2\pi}\sigma_{w_k}} \prod_{l=1, l \neq k}^{N} Q\left(\frac{(-1)^{b_j^N(l)+1} f(v)}{\sigma_{w_l}}\right) \right).$$
(3.42)

Furthermore, the gradient of the Lagrangian function  $L(f, g, \lambda)$  over f for g in (3.39) is given as

$$\nabla L = 2\lambda f(v) - \sum_{j=1}^{2^N} \hat{v}_{(j)} \Theta(N, f(v), j) \left(2v - \hat{v}_{(j)}\right).$$
(3.43)

*Proof*: See Appendix G.

## 3.6.2 DOP Criterion

In this subsection, we consider the DOP criterion. The analysis follows the same steps as in Section 3.5.2. In particular, we define the different intervals  $I_{\mathcal{V}}$ , where  $\mathcal{V}$  is a subset of  $\{\hat{v}_{(1)}, \ldots, \hat{v}_{(2^N)}\}$  as in (3.32). Based on this definition, in the next proposition, we derive necessary optimality conditions for the encoder and decoder mappings.

Proposition 3.6.2. Given a front end with N one-bit ADCs, the optimal encoder and decoder mappings f and g satisfy the necessary conditions

$$f(v) = \frac{1}{2\lambda} \tilde{G}(f, v), \qquad (3.44)$$

where  $\tilde{G}(f, v)$  is defined as

$$\tilde{G}(f,v) \triangleq \begin{cases}
0 & v \in I_{\mathcal{V}} : |\mathcal{V}| = 0, 2^{N}, \\
\sum_{k: \ \hat{v}_{(k)} \in \mathcal{V}} \left( \sum_{i=1}^{N} \frac{(-1)^{b_{k}^{N}(i)} e^{-\frac{f(v)^{2}}{2\sigma_{w_{i}}^{2}}}}{\sqrt{2\pi\sigma_{w_{i}}}} \prod_{\substack{l=1\\l \neq i}}^{N} Q\left( \frac{(-1)^{b_{k}^{N}(l)+1} f(v)}{\sigma_{w_{l}}} \right) \right) \quad v \in I_{\mathcal{V}} : |\mathcal{V}| = 1, \dots, 2^{N} - 1, \end{cases}$$
(3.45)

and (3.39) with

$$\hat{v}_{(j)} = \arg\max_{t} \int_{t-\sqrt{D}}^{t+\sqrt{D}} \Phi\left(\frac{v}{\sigma_{v}}\right) \prod_{i=1}^{N} Q\left(\frac{(-1)^{b_{j}^{N}(i)+1}f(v)}{\sigma_{w_{i}}}\right) dv, \ j = 1, \dots, 2^{N}.$$
(3.46)

Furthermore, the gradient of the Lagrangian function  $L(f, g, \lambda)$  over f for g in (3.39) is given as

$$\nabla L = 2\lambda f(v) - \frac{1}{2}\tilde{G}(f, v).$$
(3.47)

*Proof*: See Appendix H.

Remark 3.6.1. From the optimal decoder in (3.46) and (3.41), it can be easily verified that with equal noise variances for the N observations, i.e.,  $\sigma_{w_i}^2 = \sigma_w^2$  for i = 1, ..., N, the reconstruction points  $\hat{v}_{(j)}$  corresponding to those vectors  $b_j^N$  that have the same number of ones are equal. As a consequence, there are only N + 1 efficient reconstruction points instead of  $2^N$ .

# 3.7 Numerical Results

In this section, we provide some illustrations of the results derived above by means of numerical examples. We consider the MSE distortion and the DOP in separate subsections. Throughout this section, we set  $\sigma_v^2 = \sigma_w^2 = 1$ , so that the SNR in (3.5) is proportional to the power constraint P.

#### 3.7.1 MSE Criterion

We start by considering the MSE distortion in a single measurement system (N = 1) with a one-bit ADC front end (K = 2). Figure 3.3 shows the optimal mapping functions obtained from Proposition 3.4.1 for different values of P. The value of the Lagrange multiplier  $\lambda$  in (3.11) is obtained by means of bisection so as to satisfy the power constraint  $\mathbb{E}[f(V)^2] = P$ . We can observe from Figure 3.3 that, as discussed in Section 3.4.1, for low SNR, the optimal mapping function tends to linear transmission, while, for high SNR, the mapping mapping function tends to resemble a step function, which corresponds to a digital transmission with M = 2.

To further elaborate on the case N = 1 and K = 2, in Figure 3.4, the MSE distortion of the optimal, linear and digital transmission schemes are plotted versus the SNR  $\gamma$ . For clarity of illustration, we plot the complementary MSE distortion  $1 - \overline{D}$ , where we note



FIGURE 3.3: Optimal encoder mappings for the MSE distortion criterion under different power constraints P (dB) ( $\sigma_w^2 = \sigma_v^2 = 1$ ).

that  $\overline{D} = \sigma_v^2 = 1$  is achievable by setting  $\hat{V} = 0$  irrespective of the received signal. The figure confirms that, linear transmission approaches optimality at low SNR, whereas, for high SNR, digital schemes outperform linear transmission. Also, it is seen that, for digital transmission, increasing the number of constellation points generally improves the performance, although, in the high SNR regime, binary transmission is sufficient to achieve the optimal performance.

We now investigate other scenarios with a K-level ADC (N = 1, K > 2) studied in Section 3.4.4, and multiple one-bit ADCs (N > 1, K = 2) studied in Section 3.6.1. For these cases, for which Proposition 3.4.2 and Proposition 3.6.1 provide respective necessary optimality conditions, we resort to a gradient-descent approach, as in, e.g., [17]. The gradient-descent algorithm updates the current iterate  $f^{(i)}(v)$  as

$$f^{(i+1)}(v) = f^{(i)}(v) - \mu \nabla L, \qquad (3.48)$$

where *i* is the iteration index and  $\mu > 0$  is the step size.  $\nabla L$  is the derivative of the Lagrangian (3.8) with respect to the mapping function *f* at  $f^{(i)}$ , which can be found in (3.23) and (3.43) for *K*-level ADC and multiple one-bit ADCs, respectively. The



FIGURE 3.4: Complement of the MSE  $1 - \overline{D}$  (dB), versus the SNR  $\gamma$  (dB) for the optimal encoder obtained in Proposition 3.4.1 as well as for the linear and digital schemes studied in Sections 3.4.1, 3.4.2 and 3.4.3 ( $\sigma_w^2 = \sigma_v^2 = 1$ ).

algorithm is initialized with an arbitrary mapping, here we use linear mapping specified in (3.14). It is noted that the algorithm is not guaranteed to converge to a global optimal solution. We refer to mappings obtained from (3.48) as numerically optimized encoder (NOE) mappings.

In Figure 3.5, obtained NOE mappings are illustrated for a K=4 level and a K=8 level ADC front end, respectively, under the MSE distortion criterion. The decision thresholds for K=4 and K=8 are chosen as  $[\infty, -d, 0, d, \infty]$  and as  $[-\infty, -3d, -2d, -d, 0, d, 2d, 3d, \infty]$ , respectively, where the parameter d is optimized by means of a line search. From the results obtained in Figure 3.5, it is noticed that, in the low SNR regime, the NOE mappings for both K = 4 and K = 8 approach linear transmission as discussed in Section 3.4.1. Instead, as the SNR  $\gamma$  increases, the NOE encoding functions resemble piecewise-constant digital mappings.

We now compare two front-end architectures, both of which receive B output bits per received sample. The first architecture uses all bits to quantize a single observation, i.e., N = 1 and  $K = 2^B$ , while the second one receives B one-bit measurements, i.e., N = B, K = 2. In Figure 3.6, the achievable MSE distortion is shown for the two



FIGURE 3.5: NOE mappings for different power constraints (dB) and under the MSE distortion criterion ( $\sigma_w^2 = \sigma_v^2 = 1$ ): (a) K = 4; (b) K = 8. The curves are labelled by the pair (P, d), where P is the average power constraint (dB) and d is the quantization step size of the ADC front end.



FIGURE 3.6: Complement of the MSE  $(1 - \overline{D})$  vs. SNR  $(\gamma)$  (linear scale) for different ADC architectures with *B*-bit outputs ( $\sigma_v^2 = 1$ ). Variance of the AWGN in all scenarios and for all observations is one ( $\sigma_w = 1$ )

architectures for B = 1, 2, 3, along with the Shannon lower bound [3, Equation 21] as the optimal performance regarding N = 1 and  $K = \infty$ . It is seen that, for the same number of bits per sample, B, as the SNR  $\gamma$  increases, the 2<sup>B</sup>-level ADC architecture outperforms the receiver with B one-bit ADCs, whereas, the opposite is true for low SNR. This shows that, for high SNR, it is more beneficial to invest additional output bits in improving the ADC resolution. In contrast, for low SNR, it is preferable to increase the number of observations in order to improve the effective SNR by collecting independent measurements of the transmitted signal.

### 3.7.2 DOP Criterion

Here, we turn to the optimal mapping and performance under the DOP criterion. We start by considering the case of a one-bit ADC front end with a single observation, i.e., N = 1, K = 2. The optimal mapping along with the corresponding optimal reconstruction points  $(\hat{v}_{(1)}, \hat{v}_{(2)})$  for three different values of the power constraint P are shown in Figure 3.7, as obtained in Propositions 3.5.1 and 3.5.2. It is seen that, as the SNR decreases, the optimal reconstruction points,  $\hat{v}_{(1)}$  and  $\hat{v}_{(2)}$ , tend to zero, while, for high SNR, they tend to  $\hat{v}_{(1)} = -\hat{v}_{(2)} = \sqrt{D}$ , as per Remark 3.5.1. This observation can be explained as follows: for low SNR, the DOP is generally dominated by the probability of channel outage, i.e., by the last two terms in (3.25), which are zero for  $\hat{v}_{(1)} = \hat{v}_{(2)} = 0$ . In contrast, for high SNR, the optimal solution aims at minimizing the probability of source outage events, i.e., the first term in (3.25), which requires  $\hat{v}_{(1)} = -\hat{v}_{(2)} = \sqrt{D}$ .

Continuing the analysis of the N = 1, K = 2 case, in Figure 3.8, the complement of the DOP,  $1 - \epsilon(D)$ , is plotted with respect to the SNR for different values of D. As SNR decreases, based on the discussion above and Remark 3.5.1, DOP tends to the first term in (3.25) when  $\hat{v}_{(1)} = \hat{v}_{(2)} = 0$ , which can be computed as  $Q(\sqrt{D}/\sigma_v)$ . Furthermore, as the SNR increases, DOP tends to the first term in (3.25) but with  $\hat{v}_{(1)} = -\hat{v}_{(2)} = \sqrt{D}$ , resulting in  $\epsilon(D) = Q(2\sqrt{D}/\sigma_v)$ , indicated by the dashed lines in Figure 3.8.

In order to derive NOE mapping functions under the DOP criterion for K > 2 or N > 1, we apply the following iterative gradient-descent based algorithm based on the results obtained in Propositions 3.5.3 and 3.6.2:

- 1. Initialize the set of reconstruction points  $\{\hat{v}_{(j)}\}, j = 1, \dots, 2^B;$
- 2. Find the NOE mapping corresponding to the decoder (3.20) and (3.39) for K-level ADC and multiple one-bit ADCs, respectively, with  $\{\hat{v}_{(j)}\}, j = 1, ..., 2^B$ , selected at step 1 using the gradient-descent algorithm (3.48), with  $\nabla L$  defined as in (3.38) for K > 2, and as in (3.47) for N > 1;
- 3. Find the optimal reconstruction points corresponding to the obtained NOE mapping using (3.37) for K > 2, and (3.46) for N > 1;
- 4. If convergence is not obtained, go back to step 2.

For low SNR, based on the results in Section 3.5.1 (see also Figure 3.7) the reconstruction points are close to zero; and hence, we can initialize the algorithm with all-zero values when  $\lambda$  is very large. Therefore, we first set a large value for  $\lambda$  and consider all-zero vector as the initial mapping. Then, we consider successively smaller values of  $\lambda$ , i.e., increase the SNR. We use the reconstruction points obtained for the previous value of the Lagrange multiplier  $\lambda$  to initialize the algorithm for the current value of  $\lambda$ .

In Figure 3.9, the complement of the DOP,  $1 - \epsilon(D)$ , is shown for two architectures with B = 2 output bits, namely one observation with 4-level ADC (N = 1, K = 4), and two



FIGURE 3.7: Optimal encoder mappings for N = 1, K = 2 under the DOP criterion, for different values of the SNR ( $\gamma$ ). The dots are the reconstruction points  $\hat{v}_{(1)}$  and  $\hat{v}_{(2)}$ ( $\sigma_w^2 = \sigma_v^2 = 1$  and D = 0.3).



FIGURE 3.8: Complement of the DOP  $(1-\epsilon(D))$  versus SNR  $(\gamma)$  (linear scale). Dashed lines represent the DOP in the high SNR regime for the corresponding distortion target  $(\sigma_w^2 = \sigma_v^2 = 1).$ 



FIGURE 3.9: Complement of the DOP  $1 - \epsilon(D)$  versus SNR  $\gamma$  (linear scale) for different ADC architectures with *B*-bit outputs ( $\sigma_w^2 = \sigma_v^2 = 1$ ).

observations with one-bit ADCs (N = 2, K = 2), as well as for the architecture with a one-bit ADC (N = 1, K = 2), for a target distortion of D = 0.09. In accordance with the discussion above, all architectures have the same performance for low SNR, namely  $\epsilon(D) = 2Q(\sqrt{D}/\sigma_v) = 2Q(\sqrt{0.09})$ . Furthermore, in a manner similar to the discussion on Figure 3.6 for the MSE criterion, in the low SNR regime, it is beneficial to increase the number of observations, whereas at high SNR, it is preferable to increase the ADC resolution. In this regard, we note that, in the high SNR the optimal K = 4 levels are selected to minimize the probability of source outage, that is, the probability  $\Pr(V \in I_{\emptyset})$ , yielding the reconstruction points  $\hat{V} = \{-3\sqrt{D}, -\sqrt{D}, \sqrt{D}, 3\sqrt{D}\}$  and a DOP of  $2Q(4\sqrt{D}/\sigma_v) = 2Q(1.2) = 0.2301$ . Instead, for N = 2 and K = 1, the three effective reconstruction points (see Remark 3.6.1) at high SNR tend to  $\hat{V} = \{-2\sqrt{D}, 0, 2\sqrt{D}\}$ and the minimum DOP is lower bounded by  $Q(3\sqrt{D}/\sigma_v) = 2Q(.9) = 0.3681$ .

# Chapter 4

# Zero-Delay Source-Coding with a One-Bit ADC Front End and Side Information at the Receiver

# 4.1 Overview

Zero-delay transmission of a Gaussian source over an AWGN channel with a one-bit ADC front end is considered in the presence of a correlated side information at the receiver. The design of the optimal encoder and the decoder is studied for two performance criteria, namely, the MSE distortion and the DOP, under an average power constraint on the channel input. For both the MSE distortion and the DOP, conditions of optimality for the encoder and the decoder are derived, and it is observed that the presence of receiver side information has a significant impact on the structure of the optimal encoder mapping. For the MSE distortion it is observed that the NOE is periodic, and its period increases with the correlation between the source and the receiver side information. For the DOP, it is observed that the NOE mappings are bounded and fluctuating between positive and negative values. The fluctuation of the encoder mapping is damped and approaches zero with the increasing magnitude of the source values.

# 4.2 Introduction

A key component of the front end of any digital receiver is the ADC that is typically connected to each receiving antenna. The energy consumption of an ADC (in Joules/sample) increases exponentially with its resolution (in bits/sample) [60]. This leads to a growing concern regarding the energy consumption of digital receivers, either due to the increasing number of receiving antennas, e.g., for massive MIMO transceivers [72], or due to the limited availability of energy, e.g., in energy harvesting terminals [73]. Energy-efficient operation of digital receivers may hence impose constraints on the resolution of the ADCs that can be employed for each receiving antenna.

Motivated by communication among energy- and complexity-limited sensor nodes, we study zero-delay transmission of analog sensor measurements to a receiving sensor equipped with a 1-bit ADC front end. In keeping with the scenario of a network of sensors, we further assume that the receiving sensor nodes has its own correlated measurement of the transmitted source sample. Focusing on the MSE distortion and DOP criteria, our goal is to gain insights into the structure and the performance of optimal encoder and decoder functions when the source sample and the side information are jointly Gaussian.

This work contributes to a line of research that endeavors to understand the impact of front end ADC limitations on the performance limits of communication systems. The capacity analysis of a real discrete-time AWGN channel with a K-level ADC front end is studied in [63], proving the sufficiency of K + 1 constellation points at the encoder. Furthermore, it is shown in [63] that BPSK modulation achieves the capacity when the receiver front end is limited to a 1-bit ADC. In [64], the authors show that, in the low SNR regime, the symmetric threshold 1-bit ADC is suboptimal, while asymmetric threshold quantizers and asymmetric signalling constellations are needed to obtain the optimal performance. Generalization of the analysis from single-antenna AWGN channels to MIMO fading systems are put forth in [68], and, more recently, to massive MIMO systems in [72] and [74]. In [75] some of the authors of this work considered the set-up analysed here, but in the absence of correlated side information at the receiver. It is noted that the zero-delay constraint prevents the application of the mentioned channel capacity results to this set-up, and that, as it will be seen, the presence of correlated side information at the receiver significantly modifies the optimal design problem. In this work, we derive necessary optimality conditions of an encoder mapping for two performance criteria, namely, the MSE distortion and the DOP. For the MSE criterion, we observe that, similarly to the case with an infinite resolution front end studied in [17, 38, 43], the optimal encoder mapping is periodic. Furthermore, the period of this function depends on the correlation coefficient between the source and the side information, and is independent of the input power constraint, or equivalently the channel SNR. Motivated by the structure of NOE mappings, we also propose two simple parameterized mappings, which, although being suboptimal, approach the performance of NOE mappings in low and high SNR regimes. For the DOP criterion, we observe that the NOE mappings are fluctuating between positive and negative values. Additionally, the NOE mappings for the DOP are damped as the source magnitude increases, and they become zero when the absolute value of the source output is greater than some threshold value.

The rest of the chapter is organized as follows. In Section 4.3, the system model is explained. Considering the MSE criterion in Section 4.4, we study the optimal design of the encoder and the decoder in Section 4.4.1. In Sections 4.4.2 and 4.4.3, we propose two parameterized encoding schemes. In Section 4.4.4, we consider the scenario in which the side information is also available at the encoder. We obtain the optimal performance for this scenario, which can be considered as a lower bound on the performance of the original problem with decoder-only side information studied in this chapter. In Section 4.4.5, we study the asymptotic performance of the problem when the block length of the transmission tends to infinity. Focusing on the DOP criterion in Section 4.5.1. Next in Section 4.5.2, as for the MSE counterpart, we consider the case in which the side information is also available at the encoder and decoder in Section 4.5.1. Next in Section 4.5.2, as for the MSE counterpart, we obtain the optimal performance of the side information is also available at the encoder. We obtain the optimal performance of the scenario, which accordingly can be considered as a lower bound for the performance of the decoder-only side information problem. In Section 4.6, numerical results are provided.

Notations: Throughout the chapter, upper case and lowercase letters denote random variables and realizations, respectively. The standard normal distribution is denoted by  $\mathcal{N}(0,1)$ , and its pdf by  $\Phi(\cdot)$ . Operators  $\mathbb{E}[\cdot]$  and  $\Pr(\cdot)$  stand for the expectation and probability, respectively.  $Q(\cdot)$  denotes the CCDF of the standard normal distribution,



FIGURE 4.1: System model for the zero-delay transmission of a Gaussian source over an AWGN channel with a one-bit AD front end and correlated side information at the receiver.

defined as

$$Q(z) \triangleq \frac{1}{\sqrt{2\pi}} \int_{z}^{\infty} e^{-\frac{x^2}{2}} dx.$$
(4.1)

The boundaries of integrals are from  $-\infty$  to  $\infty$  unless otherwise stated. We denote the pdf of a standard bivariate normal distribution with correlation r as

$$\Phi(v,u) = \frac{1}{2\pi\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left(v^2 + u^2 - 2rvu\right)},\tag{4.2}$$

and the conditional distribution for these variables as

$$\Phi(v|u) = \frac{1}{\sqrt{2\pi(1-r^2)}} e^{-\frac{(v-ru)^2}{2(1-r^2)}}.$$
(4.3)

# 4.3 System Model

We consider the system model in Figure 4.1, in which a single source sample  $V \sim \mathcal{N}(0, \sigma_v^2)$  is transmitted over a single use of a channel characterized by AWGN followed by a one-bit ADC front end. The receiver has access to side information  $U \sim \mathcal{N}(0, \sigma_u^2)$ , which is correlated with the source V. The correlation matrix of the source and the side information is given by

$$\Lambda = \begin{bmatrix} \sigma_v^2 & r\sigma_v\sigma_u \\ r\sigma_v\sigma_u & \sigma_u^2 \end{bmatrix}.$$
 (4.4)

Accordingly, the encoded signal is given as X = f(V), where  $f : \mathbb{R} \to \mathbb{R}$  is a mapping that is constrained to satisfy an average power constraint  $\mathbb{E}[f(V)^2] \leq P$ . At the receiver, the received noisy signal is modelled as

$$Z = f(V) + W, \tag{4.5}$$

where  $W \sim \mathcal{N}(0, \sigma_w^2)$  is independent of the source and side information. The noisy signal Z is quantized with a one-bit ADC producing the received signal as

$$Y = Q(Z) = \begin{cases} 0 & Z \ge 0, \\ 1 & Z < 0. \end{cases}$$
(4.6)

We define the signal-to-noise ratio (SNR) as  $\gamma = \frac{P}{\sigma_w^2}$ . Based on Y and U, the decoder produces an estimate  $\hat{V}$  of V using a decoding function  $g : \{0, 1\} \times \mathbb{R} \to \mathbb{R}$ , i.e.,  $\hat{V}_Y(U) = g(Y, U)$ .

Two performance criteria are considered in this chapter, namely, the MSE distortion defined as

$$\bar{D} = \mathbb{E}\left[ (V - \hat{V})^2 \right], \tag{4.7}$$

and the DOP defined as

$$\epsilon(D) = \Pr\left((V - \hat{V})^2 \ge D\right). \tag{4.8}$$

In both cases, we aim at studying the optimal mapping function f, along with the corresponding optimal estimator g at the receiver, such that,  $\overline{D}$  and  $\epsilon(D)$  are minimized subject to the average power constraint. More specifically, as it is common in related works (see, e.g., [17]), we consider the unconstrained minimization

$$\underset{f,g}{\text{minimize}} \quad L(f,g,\lambda), \tag{4.9}$$

where

$$L(f,g,\lambda) = \begin{cases} \bar{D} + \lambda E[f(V)^2] & \text{for the MSE criterion,} \\ \epsilon(D) + \lambda E[f(V)^2] & \text{for the DOP criterion,} \end{cases}$$
(4.10)

with  $\lambda \ge 0$  being a Lagrange multiplier that defines the relative weight given to the average transmission power  $\mathbb{E}[f(V)^2]$  as compared to the distortion criterion.

# 4.4 The MSE distortion criterion

In this section, we study the performance of the system model in Figure 4.1 under the MSE distortion criterion. In the following, we first consider the optimal design of the encoder and the decoder, and obtain a necessary condition for the optimality of the encoder. For reference we also study two parameterized encoding schemes, namely PLT and PBT. Then, as a lower bound, we consider the optimal design when the side information is available at both the encoder and the decoder. We also consider the Shannon lower bound as the optimal performance of the system in the asymptotically long block length.

#### 4.4.1 Optimal Encoder and Decoder Design

The design goal is to minimize the Lagrangian in (4.9) for the MSE distortion criterion. Note that, for any encoding function, the optimal decoder is always the MMSE estimator; therefore, we focus on the design of the encoder mapping f. With an MMSE estimator at the receiver, the optimal reconstruction function for any encoder mapping is given by

$$\hat{v}_y(u) = g(y, u) = \mathbb{E}[V|Y = y, U = u]$$
(4.11a)

$$= \frac{\int v\Phi\left(\frac{v}{\sigma_v} \middle| \frac{u}{\sigma_u}\right) Q\left(\frac{(-1)^{y+1}f(v)}{\sigma_w}\right) dv}{\int \Phi\left(\frac{v}{\sigma_v} \middle| \frac{u}{\sigma_u}\right) Q\left(\frac{(-1)^{y+1}f(v)}{\sigma_w}\right) dv}.$$
(4.11b)

The following proposition provides a necessary condition for the optimal encoder mapping.

Proposition 4.4.1. The optimal encoder mapping f for problem (4.9) under the MSE distortion criterion must satisfy the implicit equation

$$2\sqrt{2\pi}\sigma_w\sigma_u\lambda f(v)e^{\frac{f(v)^2}{2\sigma_w^2}} = 2vA(v) - B(v), \qquad (4.12)$$

where  $\lambda \geq 0$  and is given. The functions A(v) and B(v) are defined as

$$A(v) \triangleq \int \Phi\left(\frac{u}{\sigma_u} \left| \frac{v}{\sigma_v} \right) \left( g(0, u) - g(1, u) \right) du, \tag{4.13a}$$

$$B(v) \triangleq \int \Phi\left(\frac{u}{\sigma_u} \middle| \frac{v}{\sigma_v}\right) \left(g(0, u)^2 - g(1, u)^2\right) du, \qquad (4.13b)$$

and g(y, u), y = 0, 1, is the optimal MMSE estimation defined in (4.11). Furthermore, the gradient of the Lagrangian function  $L(f, g, \lambda)$  over f, for g given as in (4.11), is given by

$$\nabla L = 2\lambda f(v) - \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w\sigma_u}(2vA(v) - B(v)).$$

$$(4.14)$$

*Proof*: See Appendix I.

Remark 4.4.1. To elaborate further on the necessary condition obtained in (4.12), we consider two extreme values of the correlation coefficient r.

• independent side information (r = 0): When the correlation coefficient is zero, the necessary condition (4.12) reduces to

$$2\sqrt{2\pi}\sigma_w\lambda f(v)e^{\frac{f(v)^2}{2\sigma_w^2}} = 2v\left(\hat{v}_0 - \hat{v}_1\right) - \left(\hat{v}_0^2 - \hat{v}_1^2\right),\tag{4.15}$$

where  $\hat{v}_y$ , for y = 1, 0, can be obtained from (4.11) as

$$\hat{v}_{y} \triangleq \mathbb{E}[V|Y=y] = \frac{\int v\Phi\left(\frac{v}{\sigma_{v}}\right) Q\left(\frac{(-1)^{y+1}f(v)}{\sigma_{w}}\right) dv}{\int \Phi\left(\frac{v}{\sigma_{v}}\right) Q\left(\frac{(-1)^{y+1}f(v)}{\sigma_{w}}\right) dv}.$$
(4.16)

In the absence of side information at the receiver, it is shown in [75] that the optimal mapping is an odd function. Therefore, with no loss of optimality, it can be easily verified that  $\hat{v}_0 = -\hat{v}_1$ . Hence, the condition in (4.15) can be further simplified as

$$f(v)e^{\frac{f(v)^2}{2\sigma_w^2}} = \frac{2v\hat{v}_0}{\sqrt{2\pi}\sigma_w\lambda},\tag{4.17}$$

which is the result obtained in [75].

• Perfect side information (r = 1): In this case, we have  $\Phi(u/\sigma_u|v/\sigma_v) = \frac{\sigma_u}{\sigma_v}\delta(u - v)$ , where  $\delta(\cdot)$  is the Dirac delta function. Therefore, it can be easily verified from (4.12) that the optimal mapping is f(v) = 0, as expected. Note that, the

result is valid also for the negative correlation, i.e., r = -1. In this case we have  $\Phi(u/\sigma_u|v/\sigma_v) = -\frac{\sigma_u}{\sigma_v}\delta(u-v).$ 

Remark 4.4.2. Due to the symmetry of the zero-threshold ADC at the receiver and the noise distribution and motivated by the optimality of an odd function in the absence of receiver side information [75], we conjecture that the optimal encoder in the presence of correlated side information is also an odd function of v. While this argument is strengthened by our numerical observations (see Section 4.6), we leave the proof of the validity of this conjecture as an open problem for future work. Here, we show that an odd function can indeed satisfy the necessary condition in (4.12). Assuming that the optimal mapping is odd, i.e., f(-v) = -f(v), it is shown that A(v) and B(v) are even and odd functions of v, respectively (see Appendix J). Therefore we can write

$$2\sqrt{2\pi}\sigma_w\sigma_u\lambda f(-v)e^{\frac{f(-v)^2}{2\sigma_w^2}} = -2vA(-v) - B(-v)$$
(4.18a)

$$= -2vA(v) + B(v)$$
 (4.18b)

$$= -2\sqrt{2\pi}\sigma_w\sigma_u\lambda f(v)e^{\frac{f(v)^2}{2\sigma_w^2}},\qquad(4.18c)$$

concluding that an odd mapping can satisfy the necessary condition. Also, following similar steps as in Appendix J, it can also be shown that for an odd encoder mapping we have g(1, -u) = -g(0, u).

In Section 4.6, we will present NOE mappings obtained by using a gradient descent approach. We observe that, due to the correlated receiver side information, the resulting encoder mappings are periodic, with a period that depends on the correlation coefficient r. Motivated by this observation, and results for the case with an infinite-resolution front end in [17], we propose two simple parameterized encoder mappings. Their performance will be compared with that of the NOE mapping in Section 4.6.

#### 4.4.2 Periodic Linear Transmission

The first proposed encoder mapping is a periodic linear function with period  $2\beta$ , and slope  $\alpha$  within each period. The encoder function is defined as

$$f_{\rm PLT}(v) = \alpha(-1)^{\left\lfloor \frac{v}{\beta} + \frac{1}{2} \right\rfloor} \cdot \left( \beta \left\lfloor \frac{v}{\beta} + \frac{1}{2} \right\rfloor - v \right), \tag{4.19}$$



FIGURE 4.2: Illustration of the PLT encoder mapping for  $\alpha = 2, \ \beta = 2.5$ .

where  $\lfloor x \rfloor$  is the largest integer less than or equal to x. In Figure 4.2, an illustration of this mapping for  $\alpha = 2$ ,  $\beta = 2.5$  is shown. To satisfy the average power constraint we have

$$\mathbb{E}[f(V)^{2}] = \alpha^{2} \left( \sigma_{v}^{2} + \beta^{2} \sum_{i=-\infty}^{\infty} i^{2} \left( Q \left( \frac{-\frac{\beta}{2} + i\beta}{\sigma_{v}} \right) - Q \left( \frac{\frac{\beta}{2} + i\beta}{\sigma_{v}} \right) \right) - \frac{2\beta\sigma_{v}}{\sqrt{2\pi}} \sum_{i=-\infty}^{\infty} i \left( e^{-\frac{\left(-\frac{\beta}{2} + i\beta\right)^{2}}{2\sigma_{v}^{2}}} - e^{-\frac{\left(\frac{\beta}{2} + i\beta\right)^{2}}{2\sigma_{v}^{2}}} \right) \right) \leq P.$$

$$(4.20)$$

The parameters  $\alpha$  and  $\beta$  are optimized under a given average power constraint P in order to minimize the MSE distortion.

# 4.4.3 Periodic BPSK Transmission

The second proposed encoder mapping, unlike PLT, adopts digital modulation with two levels, namely,  $\gamma$  and  $-\gamma$ , with a period of  $\delta$ . The mapping is defined as

$$f_{\rm PBT}(v) = \gamma \left( 1 + 2\mathcal{Q}(v) \cdot \operatorname{mod}\left( \left\lfloor \frac{2v}{\delta} \right\rfloor \right)_2 \right),$$
(4.21)



FIGURE 4.3: Illustration of the PBT encoder mapping for  $\gamma = 0.2$ ,  $\delta = 3$ .

where  $\text{mod}(\cdot)_2$  is the argument in modulo 2. In Figure 4.3, an illustration of this mapping for  $\gamma = 0.2$ , and  $\delta = 2.5$  is shown. Due to the average power constraint, we set  $\gamma = \sqrt{P}$ , and parameter  $\delta$  is optimized to minimize the MSE.

#### 4.4.4 Side Information Available at Both the Encoder and Decoder

In this section, we consider the scenario in which both the encoder and the decoder have access to the side information U. In this case, without loss of optimality, the encoder can encode the error

$$T = V - \frac{\sigma_v}{\sigma_u} r U, \tag{4.22}$$

where the random variable  $\sigma_v r U/\sigma_u$  is the MMSE estimate of V given U, which can be computed at both the encoder and the decoder. Since the random variable T, which is distributed as  $\mathcal{N}(0, \sigma_t^2)$ , with  $\sigma_t^2 = \sigma_v^2(1 - r^2)$ , is independent of the side information U, the encoder can directly encode the error T via a mapping function  $\tilde{f}(t)$  of the error T by neglecting the presence of the side information U at the receiver. Therefore, the problem reduces to the one studied in [75].



FIGURE 4.4: Illustration of the optimal encoder mapping when there is no side information at the receiver ( $\sigma_t^2 = \sigma_w^2 = 1$ ). The application of this mapping to the residual error (4.22) is optimal if the side information U is also known at the encoder.

In [75, Proposition. I], it is shown that the optimal zero-delay encoder mapping in the absence of side information is obtained from the implicit equation

$$\tilde{f}(t)e^{\frac{\tilde{f}(t)^2}{2\sigma_w^2}} = \frac{t}{\sqrt{2\pi}\sigma_w\lambda},\tag{4.23}$$

where  $\lambda \geq 0$  is chosen such that the power constraint is satisfied. Examples of the optimal mapping are shown in Figure 4.4. It is observed that, in the high SNR regime, the optimal mapping tends to digital 2-level antipodal signalling, whereas, in the low SNR regime it tends to linear mapping.

In Section 4.6, we will use the resulting optimal performance in the presence of side information at both the encoder and decoder as a lower bound on the performance of the set-up under study in which the side information is available solely at the receiver.

Remark 4.4.3. In Section 4.6, it will be seen that the application of a gradient descent based optimization procedure yields periodic NOE mappings, whose periods are dependent on the correlation coefficient r. The periodic behaviour of the NOE mappings can be explained with reference to the optimal solution discussed above, for the scenario in which U is also known at the encoder. In fact, in that case, a mapping  $f(v) = \tilde{f}(v - \sigma_v r u / \sigma_u)$  is optimal, where  $\tilde{f}(\cdot)$  is shown in Figure 4.4. Therefore, the optimal mapping is centred on the MMSE estimate  $\sigma_v r u / \sigma_u$ . When the latter is not available at the encoder, the NOE consists of periodic replicas of a basic mapping that behaves in a manner similar to  $\tilde{f}(\cdot)$  in Figure 4.4. As further discussed in Section 4.6, the period increases as the variance of the MMSE estimate of V given U, namely  $\sigma_v^2(1-r^2)$ , decreases.

#### 4.4.5 Shannon Lower Bound

A lower bound on the MSE distortion can be obtained by relaxing the zero-delay constraint, and using the Shannon source-channel separation theorem. In [63], it is shown that the capacity of the AWGN channel with a 1-bit ADC in (4.6) is given by

$$C = 1 - h\left(Q\left(\sqrt{\mathrm{SNR}}\right)\right),\tag{4.24}$$

where  $h(\cdot)$  is the binary entropy function defined as  $h(p) \triangleq -p \log_2 p - (1-p) \log_2 (1-p)$ . Furthermore, the rate-distortion function of a Gaussian source with correlated Gaussian side information at the receiver is given by the Wyner-Ziv rate-distortion function [31]

$$R(\bar{D}) = \frac{1}{2} \left[ \log_2 \frac{\sigma_v^2 (1 - r^2)}{\bar{D}} \right]^+, \tag{4.25}$$

where  $[x]^+ = \max(0, x)$ . Combining (4.24) and (4.25) a lower bound on the MSE distortion  $\overline{D}$  is obtained as

$$\bar{D}_{\text{lower}} = (1 - r^2)\sigma_v^2 2^{-2\left(1 - h\left(Q\left(\sqrt{\text{SNR}}\right)\right)\right)}.$$
(4.26)

# 4.5 The DOP Criterion

In this section, we consider the performance of the system in Figure 4.1 in terms of the DOP. In the following, we first study the necessary optimality conditions of the encoder and the decoder when the side information is solely available at the decoder. Next, as a lower bound, we study the DOP when the side information is available at both the encoder and the decoder.

#### 4.5.1 Optimal Encoder and Decoder Design

In this section, we derive the necessary conditions for an optimal encoder and decoder pair by considering the Lagrangian function in (4.10) under the DOP criterion. In the following, we first obtain the necessary optimality condition of an encoder mapping function f for a given decoder g. Then, we obtain the optimal decoder g for a given encoder mapping f.

• Optimal encoder: Here we consider the optimal design of the encoder mapping f. To elaborate, for a given side information realization u, we define the intervals

$$I_{y}(u) = \left\{ v : (v - g(y, u))^{2} < D \right\}, \ y = 0, 1,$$
(4.27)

where  $g(\cdot, \cdot)$  is the decoder reconstruction function, which is assumed to be fixed and given. Note that, for a given side information realization u, the decoder output is two separate points (depending on the output of the one-bit ADC). Therefore, for different side information realizations u, we have different reconstruction points, i.e., g(0, u) and g(1, u), and accordingly, different intervals  $I_0(u)$  and  $I_1(u)$ corresponding to g(0, u) and g(1, u), respectively.

Given u, each interval  $I_0(u)$  and  $I_1(u)$  in (4.27), corresponds to the set of source values that are within the allowed distortion target D of the reconstruction points g(0, u) and g(1, u), respectively. Hence, the following claims hold: (i) For all source outputs v in the set  $(I_0(u) \cup I_1(u))^C = \{v : \min_{y=0,1}(v - g(y, u))^2 \ge D\}$ , outage occurs (superscript C denotes the complement set). We refer to this event as *source outage*. (ii) For all source values in the interval  $I_0(u) \cap I_1(u)$ , either of the reconstruction points yield a distortion no more than the target value D. Therefore, regardless of which output  $(g_0(u), g_1(u))$  is selected by the receiver, no outage occurs.

With these observations in mind, the next proposition characterizes the optimal encoder mapping for a given decoder g.

Proposition 4.5.1. Given a target distortion D, and a decoder with reconstruction function g(Y, U), the optimal mapping  $f(\cdot)$  for the problem (4.10) under the DOP

criterion satisfies

$$f(v) = \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{2\lambda\sqrt{2\pi}} \left( \Pr\left(U \in S_{0\backslash 1}(v)\right) - \Pr\left(U \in S_{1\backslash 0}(v)\right) \right).$$
(4.28)

where  $S_{0\backslash 1}(v)$  and  $S_{1\backslash 0}(v)$  are defined as

$$S_{0\setminus 1}(v) \triangleq \{u: \ b_{0r}(u) \ge v \text{ and } b_{0l}(u) \le v\},\$$
  
$$S_{1\setminus 0}(v) \triangleq \{u: \ b_{1r}(u) \ge v \text{ and } b_{1l}(u) \le v\},\$$
(4.29)

and  $b_{1r}(u)$ ,  $b_{1l}(u)$ ,  $b_{0r}(u)$  and  $b_{0l}(u)$  are defined as below

$$b_{0r}(u) \triangleq \begin{cases} g(0,u) + \sqrt{D} & g(0,u) \ge g(1,u), \\ \min\left\{g(1,u) - \sqrt{D}, g(0,u) + \sqrt{D}\right\} & g(0,u) < g(1,u), \end{cases}$$
(4.30a)

$$b_{0l}(u) \triangleq \begin{cases} \max\left\{g(1,u) + \sqrt{D}, g(0,u) - \sqrt{D}\right\} & g(0,u) \ge g(1,u), \\ g(0,u) - \sqrt{D} & g(0,u) < g(1,u), \end{cases}$$
(4.30b)

$$b_{1r}(u) \triangleq \begin{cases} \min\left\{g(1,u) + \sqrt{D}, g(0,u) - \sqrt{D}\right\} & g(0,u) \ge g(1,u), \\ g(1,u) + \sqrt{D} & g(0,u) < g(1,u), \end{cases}$$
(4.30c)

$$b_{1l}(u) \triangleq \begin{cases} g(1,u) - \sqrt{D} & g(0,u) \ge g(1,u), \\ \max\left\{g(1,u) - \sqrt{D}, g(0,u) + \sqrt{D}\right\} & g(0,u) < g(1,u). \end{cases}$$
(4.30d)

Furthermore, the gradient of the Lagrangian function  $L(f, g, \lambda)$  over f, for a given g, is found as

$$\nabla L = 2\lambda f(v) - \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}} \left( \Pr\left(U \in S_{0\backslash 1}(v)\right) - \Pr\left(U \in S_{1\backslash 0}(v)\right) \right).$$
(4.31)

*Proof*: See Appendix K.

• Optimal decoder: Assuming that the encoder mapping f is given, we aim to minimize the Lagrangian function in (4.10) for the DOP criterion over the decoding function g. The next proposition characterizes the optimal decoder mapping for a given encoder f. Proposition 4.5.2. Given a target distortion D and an encoder mapping f, the optimal decoder  $g(\cdot, \cdot)$  for the problem (4.10) under the DOP criterion is obtained as

$$g(y,u) \in \arg\max_{\hat{v}} \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} \Phi\left(\frac{v}{\sigma_v} \Big| \frac{u}{\sigma_u}\right) Q\left(\frac{(-1)^{y+1}f(v)}{\sigma_w}\right) dv.$$
(4.32)

*Proof*: See Appendix L.

*Remark* 4.5.1. In the following, as for the MSE distortion criterion, we consider two asymptotic values for the correlation between the source and the side information.

Independent side information(r = 0): We note that when r = 0, the decoder outputs are two separate points, namely g(0, u) = v̂<sub>0</sub> and g(1, u) = v̂<sub>1</sub>, depending only on the one-bit ADC output. Therefore, it can be verified from (4.30) that b<sub>1r</sub>(u), b<sub>1l</sub>(u), b<sub>0r</sub>(u) and b<sub>0l</sub>(u) are also independent of u. With no loss of generality, assuming v̂<sub>0</sub> ≥ v̂<sub>1</sub>, we have<sup>1</sup>

$$b_{1l} = \hat{v}_1 - \sqrt{D},$$
  

$$b_{1r} = \min\{\hat{v}_1 + \sqrt{D}, \hat{v}_0 - \sqrt{D}\},$$
  

$$b_{0l} = \max\{\hat{v}_1 + \sqrt{D}, \hat{v}_0 - \sqrt{D}\},$$
  

$$b_{0r} = \hat{v}_0 + \sqrt{D}.$$
(4.33)

Hence, from (4.29) it can be verified that

$$S_{0\setminus 1}(v) = \begin{cases} \mathbb{R} & b_{0l} \le v \le b_{0r}, \\ \emptyset & \text{otherwise}, \end{cases}$$
(4.34a)

$$S_{1\setminus 0}(v) = \begin{cases} \mathbb{R} & b_{1l} \le v \le b_{1r}, \\ \emptyset & \text{otherwise.} \end{cases}$$
(4.34b)

Substituting in (4.28), we obtain

$$2\lambda\sqrt{2\pi}f(v)e^{\frac{f(v)^2}{2\sigma_w^2}} = \begin{cases} 1 & b_{0l} \le v \le b_{0r}, \\ -1 & b_{1l} \le v \le b_{1r}, \\ 0 & \text{otherwise,} \end{cases}$$
(4.35)

<sup>&</sup>lt;sup>1</sup>Due to the independence of the intervals from u, we drop the argument u

which is the result obtained for the optimal mapping when there is no side information at the receiver [75].

• Perfect side information (r = 1): When the receiver has access to a perfect side information, i.e., r = 1, from (4.32), the optimal decoder is obtained as follows

$$g(y,u) \in \arg\max_{\hat{v}} \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} \delta\left(v - \frac{\sigma_v u}{\sigma_u}\right) Q\left(\frac{(-1)^{y+1}f(v)}{\sigma_w}\right) dv$$
(4.36a)

$$= \arg \max_{\hat{v}} Q\left(\frac{(-1)^{y+1}f\left(\frac{\sigma_{v}u}{\sigma_{u}}\right)}{\sigma_{w}}\right) \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} \delta\left(v-\frac{\sigma_{v}u}{\sigma_{u}}\right) dv \qquad (4.36b)$$

$$= \left\{ \hat{v} : \hat{v} \in \left( \frac{\sigma_v u}{\sigma_u} - \sqrt{D}, \frac{\sigma_v u}{\sigma_u} + \sqrt{D} \right) \right\}$$
(4.36c)

we can choose the decoder to be  $g(Y,U) = g(U) = \frac{\sigma_v U}{\sigma_u}$ . Therefore, we have

$$b_{0r}(u) = \frac{\sigma_v}{\sigma_u} u + \sqrt{D}, \qquad (4.37a)$$

$$b_{0l}(u) = \frac{\sigma_v}{\sigma_u} u + \sqrt{D}, \qquad (4.37b)$$

$$b_{1r}(u) = \frac{\sigma_v}{\sigma_u} u - \sqrt{D}, \qquad (4.37c)$$

$$b_{1l}(u) = \frac{\sigma_v}{\sigma_u} u - \sqrt{D}.$$
(4.37d)

It can be easily verified that

$$\Pr\left(u \in S_{0\setminus 1}(v)\right) = \Pr\left(u \in S_{1\setminus 0}(v)\right) = 0.$$

$$(4.38)$$

Hence, from (4.28), we get

$$f(v) = 0.$$

This result is quite intuitive: when r = 1 the source can be recovered directly from the side information by linear scaling, and hence, no transmission is needed. Note that, as for the MSE distortion criterion, when the correlation is negative, i.e., r = -1, it can be easily checked that the decoder is  $g(Y, U) = g(U) = -\frac{\sigma_v U}{\sigma_u}$ .

Remark 4.5.2. In the low SNR regime, from (4.32), we have  $g(y, u) \simeq \frac{r\sigma_v}{\sigma_u} u$ , y = 0, 1. Therefore, in the low SNR regime, the DOP at the receiver can be approximated as (see Appendix M)

$$\epsilon(D) = 2Q\left(\frac{\sqrt{D}}{\sigma_v\sqrt{1-r^2}}\right). \tag{4.39}$$

# 4.5.2 Side Information Available at Both the Encoder and the Decoder

Now, we consider the case in which the side information U is also available to the encoder. Based on the available side information, both the encoder and the decoder can reconstruct the source under the DOP criterion as  $\sigma_v ru/\sigma_u$ . We have

$$\arg\min_{\hat{v}} \Pr(|V - \hat{v}|^2 \ge D|U = u)$$
(4.40)

$$= \arg \max_{\hat{v}} \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} \Phi\left(\frac{v}{\sigma_v} \middle| \frac{u}{\sigma_u}\right) dv$$
(4.41)

$$=\frac{\sigma_v}{\sigma_u}ru.\tag{4.42}$$

At the transmitter, similarly to the MSE distortion criterion, the encoder uses the optimal mapping for the scenario without side information to transmit the error signal

$$T = V - \frac{r\sigma_v}{\sigma_u} U. \tag{4.43}$$

In [75], it is shown that the optimal encoder mapping has the following structure

$$\tilde{f}(t) = \begin{cases} 0 & t \in (I_0 \cup I_1)^C \cup (I_0 \cap I_1), \\ -\tau & t \in (I_1 \setminus I_0), \\ \tau & t \in (I_0 \setminus I_1), \end{cases}$$
(4.44)

where for  $\lambda \geq 0$ , the value of  $\tau$  is the unique solution of

$$\tau e^{\frac{\tau^2}{2\sigma_w^2}} = \frac{1}{2\sqrt{2\pi}\sigma_w\lambda}.$$
(4.45)

The intervals  $I_0$  and  $I_1$  are defined as

$$I_y = \{t : (t - \hat{t}_y)^2 \le D\},\tag{4.46}$$

where  $\hat{t}_y$ , y = 0, 1, are the optimal reconstruction points given by

$$\hat{t}_0 = \sqrt{D} - a^*, \tag{4.47a}$$

$$\hat{t}_1 = -\hat{v}_0,$$
 (4.47b)

where  $a^*$  is obtained from

$$a^* = \underset{a \in [0, \sqrt{D}]}{\operatorname{arg min}} 2Q\left(\frac{2\sqrt{D} - a}{\sigma_v}\right) + 2\left(Q\left(\frac{\tau}{\sigma_w}\right) + \lambda\tau^2\right) \cdot \left(Q\left(\frac{a}{\sigma_v}\right) - Q\left(\frac{2\sqrt{D} - a}{\sigma_v}\right)\right)$$
(4.48)

Note that  $\lambda \geq 0$  is chosen so that for a given pair of reconstruction points  $(\hat{t}_1, \hat{t}_0)$  the power constraint is satisfied.

In the following proposition we obtain the optimal decoder for this scenario.

Proposition 4.5.3. Given a target distortion D, and an encoder mapping  $\tilde{f}$  in (4.44), the optimal decoder  $g(\cdot, \cdot)$  for the problem (4.10) under the DOP criterion is obtained as

$$g(y,u) = \frac{r\sigma_v}{\sigma_u} u + \hat{t}_y, \ y = 0, 1,$$
(4.49)

where  $\hat{t}_y$  is the optimal reconstruction points corresponding to the source of variance  $(1-r^2)\sigma_v^2$  defined as in (4.47). The DOP is obtained as

$$\begin{aligned} \epsilon(D) &= 2Q\left(\frac{2\sqrt{D}-a}{\sqrt{1-r^2}\sigma_v}\right) \\ &+ 2Q\left(\frac{t}{\sigma_w}\right)\left(Q\left(\frac{a}{\sqrt{1-r^2}\sigma_v}\right) - Q\left(\frac{2\sqrt{D}-a}{\sqrt{1-r^2}\sigma_v}\right)\right), \end{aligned}$$

where t is the solution of the equation  $te^{\frac{t^2}{2\sigma_w^2}} = \frac{1}{2\sqrt{2\pi}\sigma_w\lambda}$ .

*Proof*: See Appendix N.

In Section 4.6, we will use the resulting DOP in the presence of side information at both encoder and decoder as a lower bound on the performance of the original decoder-only side information set-up under study.



FIGURE 4.5: MSE encoder mappings f(v) with different average power values and r = 0.85, ( $\sigma_v = \sigma_w = 1$ ). Increasing the power constraint P has no impact on the period of the NOE mapping.

# 4.6 Numerical Results

In this section, we present numerical results with the aim of assessing the performance of the encoder/ decoder pairs proposed in the previous sections. In order to derive the NOE mappings we apply a gradient descent-based algorithm. The algorithm performs a gradient descent search in the opposite direction of the derivative of the Lagrangian (4.10) with respect to the encoder mapping  $f(\cdot)$ . The update is done as

$$f_{i+1}(v) = f_i(v) - \mu \nabla_f L,$$
 (4.50)

where *i* is the iteration index,  $\nabla_f L$  is defined in (4.14) and (4.31) for the MSE distortion and DOP criterion, respectively.  $\mu > 0$  is the step size. The algorithm is initialized with an arbitrary mapping, e.g., linear mapping. It is noted that the algorithm is not guaranteed to converge to a global optimal solution. We also remark that different power constraints are imposed by means of a linear search over the Lagrange multiplier  $\lambda$ . In the following, we first discuss the numerical results for the MSE distortion criterion, followed by the numerical results for the DOP criterion.



FIGURE 4.6: NOE mappings that satisfy the necessary optimality condition for different correlation coefficients r and an average power constraint of P = 5 ( $\sigma_v^2 = \sigma_w^2 = 1$ ).

In Figure 4.5, NOE mappings for the MSE distortion criterion are plotted for different average power constraints, for a correlation coefficient of r = 0.85. We note the periodic structure of the mapping, which is due to the available side information at the receiver as discussed in Remark 4.4.3. In contrast, the optimal mapping obtained in [75] when r = 0 is a monotonically increasing function (see Figure 4.4). We also observe that the average power constraint does not affect the period of the mapping. In Figure 4.6, NOE mappings for an average power of P = 5 are plotted for different correlation coefficients. We see that the period of the mapping indeed depends on r: the higher the correlation coefficient r the smaller the period of the mapping.

In Figure 4.7, we plot the complementary MSE distortion  $(1 - \overline{D})$  versus SNR for NOE, as well as for the PLT and PBT schemes, for the correlation coefficient of r = 0.6. The SLB and the MSE distortion achieved when both the encoder and the decoder have access to the side information U, which is referred to as the encoder side information lower bound (ESLB), are also included for comparison. We observe that the performance of PBT is close to that of NOE at high SNR values. On the other hand, for low SNRs, PLT outperforms PBT and approaches the NOE performance. The results are aligned with the shapes of the NOE mappings illustrated in Figure 4.5. We observe that the



FIGURE 4.7: Complementary MSE distortion vs. SNR for r = 0.6 ( $\sigma_v^2 = \sigma_w^2 = 1$ ).

NOE resembles PLT whereas it approaches the shape of PBT as SNR increases (see Section 4.4.4).

In Figure 4.8, the complementary MSE distortion  $(1-\overline{D})$  is plotted versus the correlation coefficient r for a fixed average power constraint of P = 5. We observe that, for this SNR value, PBT performs very close to NOE for a wide range of r values. However, as r approaches 1, PLT outperforms PBT, and approaches the performance of NOE. This can be explained based on the observation in Figure 4.5 that, as average power constraint decreases, the NOE mappings resembles the PLT mapping.

We observe from the comparison of the SLB and ESLB bounds in both Figure 4.7 and Figure 4.8, that the zero-delay constraint entails a significant loss with respect to the case in which block processing is allowed. Also observed from Figure 4.8 is that the ESLB is tight only for low and high correlation regime, and in general there is a loss in the MSE distortion by not having the side information at the encoder. We note that this is not the case for the SLB, and the same MSE distortion performance is achieved when the side information is available, or not, at the encoder. This is because of the no rate-loss property of the Gaussian source in Wyner-Ziv compression [31].



FIGURE 4.8: Complementary MSE distortion versus correlation coefficient r under the average power constraint P = 5 ( $\sigma_v^2 = \sigma_w^2 = 1$ ).

Considering the DOP criterion, in Figure 4.9, NOE mappings for different power constraints and for three different correlation coefficients are shown. For the lower correlation coefficient r = 0.1 (the NOE mappings on the top of Figure 4.9) the NOE mappings resemble the optimal mappings in the absence of receiver side information obtained in [75]. As the correlation between the source and the side information increases, the domain of the mapping on which it is non-zero expands. This can be explained as follows. With a higher correlation coefficient r, receiver's information on the range of the source output becomes more accurate. This allows the encoder to allocate its power budget over a larger set of source output values, as the distortion target can be reached for a larger set of source values. It is also observed that unlike the encoder mappings for the MSE distortion criterion, here, NOE mappings become almost zero when the absolute value of the source output is greater than some positive value. We give an example to explain this behaviour. Assume a source with unit variance  $\sigma_v^2 = 1$ , that outputs v = 10. Also assume that the distortion target at the receiver is D = 0.09. Based on the side information (we assume unit variance  $\sigma_u^2 = 1$  with correlation coefficient r = 0.5) at the receiver, we can estimate the source to be  $\hat{V} = rU$ . Roughly speaking, to be able to utilize the side information at the receiver we need to have  $|v - rU|^2 < D$ , i.e.,  $|10 - 0.5U|^2 < 0.09$ .



FIGURE 4.9: DOP encoder mappings f(v) for different correlation coefficients (r = 0.1, 0.6, 0.8), ( $\sigma_v^2 = \sigma_w^2 = 1$ ). The power constraint P of the mappings increases in the direction of the arrow.

This results in  $Pr(19.4 < u < 20.6) \simeq 0$ . Although joint decoding based on the channel output and the side information can improve this probability, it would be far less than what is needed to increase the probability to a reasonable value. Therefore, the optimal mapping does not allocate much power to very large source outputs.

In Figure 4.10, we plot the complementary DOP,  $1-\epsilon(D)$ , versus SNR for NOE mappings and for correlation coefficients r = 0, 0.6, 0.8. In addition to that, the DOP when both the encoder and the decoder have access to the side information U, which is referred to as encoder side information lower bound (ESLB), is also included for comparison. We observe that in the low SNR regime the DOP is close to the ESLB. This is because in



FIGURE 4.10: Complementary DOP vs. SNR for r = 0, 0.6, 0.8  $(\sigma_v^2 = \sigma_w^2 = 1)$ .

the low SNR regime the channel output is not reliable and the decoding is done almost based on the side information. We also observe that the DOP saturated as the SNR increases. In the high SNR regime there is practically no channel outage and the overall DOP is dominated by the source outage probability, which is independent of the SNR.

# Chapter 5

# **Conclusions and Future Work**

In this thesis, we addressed the communication limits under the zero-delay constraint. We focused on the point-to-point transmission with the MSE distortion and the DOP criteria under the average power constraint on the channel input. In the following, we conclude the results of the previous chapters respectively and discuss about the future possible directions as continuation of this research.

In chapter two, we have studied the problem of zero-delay transmission of a Gaussian source over an AWGN channel in the presence of known interference at the transmitter. Due to the zero-delay constraint and the memoryless nature of the source samples and the interference signals over time, causal and non-causal availability of the interference information are equivalent in this setting. We have proposed one linear and five nonlinear zero-delay JSCC schemes. The linear scheme is based on interference cancellation, whereas the non-linear schemes shape the interference and convert it into structured interference, and use companding for the transmission of the source samples.

We have shown that the proposed non-linear coding schemes can achieve zero-distortion in the limit of zero noise, whereas this is not possible through the linear ICA scheme when the interference is strong. We have also introduced the novel idea of non-uniform interference quantization for this problem, and have shown that the corresponding 1DL-NU scheme achieves the best performance among the proposed parametric transmission techniques.

We have also studied the necessary condition for optimality, and obtained a NOE using this optimality condition. While NOE outperforms other proposed encoders, it has a significantly higher computational complexity compared to the parameterized schemes. Based on the numerical results it is shown that 1DL-NU performs closer (among the proposed parameterized schemes) to NOE. We have also observed that the structure of the encoder mapping of the proposed parameterized transmission schemes, resemble that of the encoder mapping obtained numerically in the NOE scheme. Based on our numerical performance results and the latter observation, we argue that the proposed low-complexity parameterized transmission schemes can be instrumental in practical systems to achieve reasonably good performance with limited computational resources.

In chapter three, we considered the zero-delay transmission of a single sample of a Gaussian source over a quantized vector AWGN channel. We first studied a system with a one-bit ADC front end at the receiver for two performance criteria, namely the MSE distortion, and the DOP. For the MSE distortion, we have shown that the optimal encoder mapping is odd, and that, in the low SNR regime, linear transmission approaches the optimal performance, whereas digital transmission becomes optimal in the high SNR regime. For the DOP criterion, we have obtained the optimal structure of the encoder and the decoder, demonstrating that the optimal encoder function is symmetric and piecewise constant. For both the MSE distortion and the DOP, we also derived necessary optimality conditions of the encoder and decoder mappings for a K-level ADC front end and for multiple one-bit ADC observations.

Among open problems that are left for future research, we mention here the joint optimization of encoding and decoding functions as well as of the quantizers, possibly dithered, used by the ADC front ends. Another interesting problem relates to the implementation of zero-delay joint source-channel coding over fading channels with finiteresolution ADSs.

In chapter four, we have studied the problem of zero-delay transmission of a Gaussian source over an AWGN channel followed by a 1-bit ADC front end, in the presence of correlated side information at the receiver. We studied the problem with two performance criteria, namely, the MSE distortion and the DOP under an average power constraint on the channel input. For both the MSE distortion and the DOP, we derived a necessary condition for the optimality of an encoder and a decoder function, and then, based on this condition, and using gradient descent algorithm, we obtained a numerically optimized encoder mappings. For the MSE distortion we observed that encoder mapping is
periodic, with a period that depends on the correlation coefficient of the side information. This motivated us to propose two new periodic parameterized encoding schemes, referred to as PLT and PBT. We have shown through numerical simulations that, PLT and PBT perform close to the NOE in the low and high SNR regimes, respectively. For the DOP we observed that the numerically optimized encoder mapping is fluctuating between positive and negative values, and that the encoder mapping for the DOP is bounded, that is, the mapping is damped until it gets zero as we get further from the origin. For both the MSE distortion and the DOP, we considered the scenario where the encoder also has access to the side information. We obtained the optimal performance for this scenario, which is used as a lower bound for the case where there is no side at the transmitter.

#### Appendix A

# Preliminaries: Calculus of Variations

In the proofs of the propositions reported above, we leverage the standard method in variational calculus to obtain necessary optimality conditions [32, Section 7]. The following lemma presents the key result that will be used throughout the following appendices. For the sake of brevity, we drop the arguments of functions and functionals where no confusion can arise. We also use the notation  $F^{f}$  to denote the partial derivative of a functional F with respect to a function f.

Lemma A.0.1. Let  $G_i$ , i = 1, ..., n, be continuous functionals of (f, f', u) and have continuous partial derivatives with respect to (f, f'). Also, let F be a continuous functional of  $(f, f', r_1, ..., r_n, u)$ , where  $r_i$  is a functional of  $G_i$  given as

$$r_i = \int_{t_1}^{t_2} G_i(f(t), f'(t), t) dt, \quad i = 1, \dots, n.$$
 (A.1)

Let F has continuous partial derivatives with respect to  $(f, f', r_1, \ldots, r_n)$ . Consider the following minimization problem

minimize 
$$L(f) \triangleq \int_{t_1}^{t_2} F(f(t), f'(t), r_1, \dots, r_n, t) dt.$$
 (A.2)

Define the functional derivative  $\nabla L$  as

$$\nabla L \triangleq F^{f}(f(u), f'(u), r_{1}, \dots, r_{n}, u) - \frac{d}{du} F^{f'}(f(u), f'(u), r_{1}, \dots, r_{n}, u)$$
  
+ 
$$\sum_{i=1}^{n} \left( G^{f}_{i}(f(u), f'(u), u) - \frac{d}{du} G^{f'}_{i}(f(u), f'(u), u) \right)$$
  
$$\cdot \int F^{r_{i}}(f(t), f'(t), r_{1}, \dots, r_{n}, t) dt,$$
(A.3)

where  $F^{f}, F^{f'}, F^{r_i}$  are derivatives of the functional F with respect to  $f, f', r_i$ , respectively. Similarly,  $G_i^{f}, G_i^{f'}$  are derivatives of the functional  $G_i$  with respect to f, f', respectively. A necessary condition for a function f to be a solution of the problem (A.2) is

$$\nabla L = 0, \ \forall u \in [t_1, t_2]. \tag{A.4}$$

*Proof*: Following the conventional approach in the calculus of variations, we perturb the function f(t) by an arbitrary function  $\eta(t)$ , which vanishes on the boundary points  $t_1$  and  $t_2$  [32]. Let  $\delta_f L \triangleq \frac{dL(f+\alpha\eta)}{d\alpha}\Big|_{\alpha=0}$  be the Gateaux derivative of the functional L with respect to the parameter  $\alpha$ . We have

$$\delta_f L = \frac{d}{d\alpha} \int_{t_1}^{t_2} F(f + \alpha \eta, f' + \alpha \eta', r_1^{\alpha}, \dots, r_n^{\alpha}, t) dt \bigg|_{\alpha = 0},$$
(A.5)

where  $r_i^\alpha$  is defined as

$$r_{i}^{\alpha} = \int_{t_{1}}^{t_{2}} G_{i}(f + \alpha \eta, f' + \alpha \eta', u) du, \quad i = 1, \dots, n.$$
 (A.6)

Therefore, the Gateaux derivative  $\delta_f L$  can be written as

$$\delta_{f}L = \int_{t_{1}}^{t_{2}} \left[ \eta F^{f}(f, f', r_{1}, \dots, r_{n}, t) + \eta' F^{f'}(f, f', r_{1}, \dots, r_{n}, t) \right] dt$$

$$+ \int_{t_{1}}^{t_{2}} \sum_{i=1}^{n} \left. \frac{dr_{i}^{\alpha}}{d\alpha} \right|_{\alpha=0}^{\cdot} F^{r_{i}}(f, f', r_{1}, \dots, r_{n}, t) dt \qquad (A.7)$$

$$= \int_{t_{1}}^{t_{2}} \eta \left[ F^{f}(f, f', r_{1}, \dots, r_{n}, t) - \frac{d}{dt} F^{f'}(f, f', r_{1}, \dots, r_{n}, t) \right] dt$$

$$+\underbrace{F^{f'}(f,f',r_1,\ldots,r_n,t)\eta(t)\Big|_{t_1}^{t_2}}_{=0} + \int_{t_1}^{t_2} \sum_{i=1}^n \frac{dr_i^{\alpha}}{d\alpha}\Big|_{\alpha=0} \cdot F^{r_i}(f,f',r_1,\ldots,r_n,t)dt, \quad (A.8)$$

where (A.8) is due to integration by parts, and the fact that  $\eta(t)$  vanishes at  $t_1$  and  $t_2$  by construction. We compute

$$\frac{dr_{i}^{\alpha}}{d\alpha}\bigg|_{\alpha=0} = \frac{d}{d\alpha} \int_{t_{1}}^{t_{2}} G_{i}(f + \alpha\eta, f' + \alpha\eta', u) du\bigg|_{\alpha=0}$$
(A.9)

$$= \int_{t_1}^{t_2} \left[ \eta G_i^f(f, f', u) + \eta' G_i^{f'}(f, f', u) \right] du$$
(A.10)

$$= \int_{t_1}^{t_2} \eta \left[ G_i^f(f, f', u) - \frac{d}{du} G_i^{f'}(f, f', u) \right] du + G_i^{f'}(f, f', u) \eta(u) \bigg|_{t_1}^{t_2}$$
(A.11)

$$= \int_{t_1}^{t_2} \eta \left[ G_i^f(f, f', u) - \frac{d}{du} G_i^{f'}(f, f', u) \right] du, \ i = 1, \dots, n.$$
(A.12)

By plugging (A.12) into (A.8) we finally have

Since  $\eta(u)$  is an arbitrary function and the term multiplying  $\eta(u)$  is continuous, it must be zero everywhere on the interval  $[t_1, t_2]$ . Thus, the optimal solution must satisfy the following equality

$$F^{f}(f, f', r_{1}, \dots, r_{n}, u) - \frac{d}{du} F^{f'}(f, f', r_{1}, \dots, r_{n}, u) + \sum_{i=1}^{n} \left( \left( G_{i}^{f}(f, f', u) - \frac{d}{du} G_{i}^{f'}(f, f', u) \right) \right) \\ \cdot \int_{t_{1}}^{t_{2}} F^{r_{i}}(f, f', r_{1}, \dots, r_{n}, t) dt \right) = 0 , \ \forall u \in [t_{1}, t_{2}],$$
(A.15)

which yields (A.4).

Remark A.0.1. When the functional F is independent of  $r_i$ , i = 1, ..., n, then  $F^{r_i}(f, f', u) = 0$ , and therefore, we obtain the well known Euler-Lagrange equation

$$F^{f}(f, f', u) - \frac{d}{du}F^{f'}(f, f', u).$$

Remark A.0.2. As a special case of Lemma (A.0.1), it will be useful in Appendices B, C and G to consider the minimization of the functional

$$L(f) = \frac{1}{\sigma_v} \int_{t_1}^{t_2} \Phi\left(\frac{t}{\sigma_v}\right) \tilde{F}(f(t), r_1, \dots, r_n, t) dt,$$
(A.16)

where we recall that  $\Phi(\cdot)$  is the Gaussian probability density function with mean zero and variance one, and  $r_i$ , i = 1, ..., n, are of the form given by

$$r_i = \frac{1}{\sigma_v} \int_{t_1}^{t_2} \Phi\left(\frac{t}{\sigma_v}\right) \tilde{G}_i(f(t), t) dt, \quad i = 1, \dots, n.$$
(A.17)

From Lemma A.0.1, setting

$$F = \frac{1}{\sigma_v} \Phi\left(\frac{t}{\sigma_v}\right) \tilde{F}(f(t), r_1, \dots, r_n, t), \tag{A.18}$$

$$G_i = \frac{1}{\sigma_v} \Phi\left(\frac{t}{\sigma_v}\right) \tilde{G}_i(f(t), t), \ i = 1, ..., n,$$
(A.19)

the solution for this problem needs to satisfy  $\nabla L = 0$ , where the functional derivative  $\nabla L$  is found as

$$\nabla L \triangleq \tilde{F}^f(f(u), r_1, \dots, r_n, u)$$

$$+\sum_{i=1}^{n} \left( \tilde{G}_{i}^{f}(f(u), u) \int \tilde{F}^{r_{i}}(f(t), r_{1}, \dots, r_{n}, t) dt \right), \ u \in [t_{1}, t_{2}], \quad (A.20)$$

with  $\tilde{G}_i$ , i = 1, ..., n, and  $\tilde{F}$  being continuous functionals of (f, t) and  $(f, r_1, ..., r_n, t)$ , respectively, and having continuous partial derivatives with respect to f, and  $(f, r_1, ..., r_n)$ , respectively.

Instead, in Appendices B, D, F and H we will consider the minimization of the functional

$$L(f) = \frac{1}{\sigma_v} \int_{t_1}^{t_2} \Phi\left(\frac{t}{\sigma_v}\right) \tilde{F}(t, f(t)) dt.$$
(A.21)

From Remark A.0.1, setting

$$F(f,t) = \frac{1}{\sigma_v} \Phi\left(\frac{t}{\sigma_v}\right) \tilde{F}(f(t),t), \qquad (A.22)$$

the solution for the minimization of the problem (A.21) needs to satisfy

$$\nabla L \triangleq \tilde{F}^f(u, f(u)) = 0, \ u \in [t_1, t_2], \tag{A.23}$$

where  $\tilde{F}$  has continuous partial derivatives with respect to f.

In the proofs of the Propositions 4.4.1 and 4.5.1, we leverage the standard method in variational calculus [32, Section 7] to obtain necessary optimality conditions. The next theorem summarizes the key result that will be needed.

Theorem A.0.2. Let F,  $G_0$  and  $G_i$ , i = 1, ..., n, be continuous functionals of (f, H, t),  $(f, r_1, ..., r_n, u, t)$  and (f, t, u), respectively, where H and  $r_i$ , i = 1, ..., n, are given by

$$H(t) = \int_{t_1}^{t_2} G_0(f(t), r_1(u), \dots, r_n(u), u, t) du,$$
(A.24)

$$r_i(u) = \int_{t_1}^{t_2} G_i(f(v), v, u) dv, \quad i = 1, \dots, n.$$
 (A.25)

Also, let F,  $G_0$  and  $G_i$ , i = 1, ..., n, have continuous partial derivatives with respect to (f, H),  $(f, r_1, ..., r_n)$  and f, respectively. Consider the following minimization problem

$$\underset{f}{\text{minimize}} \quad L(f) \triangleq \int_{t_1}^{t_2} F(f(t), H(t), t) dt.$$
(A.26)

Define  $\nabla L$  as

$$\nabla L \triangleq F^{f}(f(t), H(t), t) + F^{H}(f(t), H(t), t) \int_{t_{1}}^{t_{2}} G_{0}^{f}(f(t), r_{1}(u), \dots, r_{n}(u), u, t) du$$
  
+ 
$$\int_{t_{1}}^{t_{2}} \int_{t_{1}}^{t_{2}} F^{H}(f(v), H(v), v) \sum_{i=1}^{n} G_{i}^{f}(f(t), t, u) G_{0}^{r_{i}}(f(v), r_{1}(u), \dots, r_{n}(u), u, v) dv du,$$
  
(A.27)

where  $F^f$  and  $F^H$  denote the partial derivatives of the functional F with respect to fand H, respectively; and  $G_0^f$  and  $G_0^{r_i}$  denote the partial derivatives of the functional  $G_0$ with respect to f and  $r_i$ , respectively. Similarly,  $G_i^f$  denotes the partial derivative of the functional  $G_i$  with respect to f. A necessary condition for a function f to be a solution to the minimization problem in (A.26) is

$$\nabla L = 0. \tag{A.28}$$

Proof: Following the conventional approach in the calculus of variations, we perturb the function f(t) by an arbitrary function  $\eta(t)$  that vanishes on the boundary points  $t_1$  and  $t_2$  [32]. Let  $\delta_f L \triangleq \frac{dL(f+\alpha\eta)}{d\alpha}\Big|_{\alpha=0}$  be the resulting Gateaux derivative of the functional L with respect to the parameter  $\alpha$ . We have

$$\delta_f L = \frac{d}{d\alpha} \int_{t_1}^{t_2} F(f(t) + \alpha \eta(t), H^{\alpha}(t), t) dt \bigg|_{\alpha = 0},$$
(A.29)

where  $H^{\alpha}(t)$  is defined as

$$H^{\alpha}(t) \triangleq \int_{t_1}^{t_2} G_0(f(t) + \alpha \eta(t), r_1^{\alpha}(u), \dots, r_n^{\alpha}(u), u, t) du,$$
(A.30)

and  $r_i^{\alpha}(u), \ i = 1, \dots, n$ , are defined as

$$r_i^{\alpha}(u) \triangleq \int_{t_1}^{t_2} G_i(f(v) + \alpha \eta(v), v, u) dv, \quad i = 1, \dots, n.$$
 (A.31)

Evaluating the derivative in (A.29), we have

$$\delta_f L = \int_{t_1}^{t_2} \left[ \eta(t) F^f(f(t), H(t), t) + \frac{dH^{\alpha}(t)}{d\alpha} F^H(f(t), H(t), t) \right] dt,$$
(A.32)

where  $\frac{dH^{\alpha}(t)}{d\alpha}$  is the Gateaux derivative of the functional H(t), which is computed as

$$\frac{dH^{\alpha}(t)}{d\alpha} = \frac{d}{d\alpha} \int_{t_1}^{t_2} G_0(f(t) + \alpha \eta(t), r_1^{\alpha}(u), \dots, r_n^{\alpha}(u), u, t) du \Big|_{\alpha=0}$$
(A.33)
$$= \int_{t_1}^{t_2} \left( \eta(t) G_0^f(f(t), r_1(u), \dots, r_n(u), u, t) + \sum_{i=1}^n \frac{dr_i^{\alpha}(u)}{d\alpha} G_0^{r_i}(f(t), r_1(u), \dots, r_n(u), u, t) \right) du,$$
(A.34)

where

$$\frac{dr_i^{\alpha}(u)}{d\alpha} \triangleq \frac{d}{d\alpha} \int_{t_1}^{t_2} G_i(f(v) + \alpha \eta(v), v, u) dv \bigg|_{\alpha = 0}$$
(A.35)

$$= \int_{t_1}^{t_2} \eta(v) G_i^f(f(v), v, u) dv.$$
 (A.36)

By plugging (A.36) into (A.34), we can write

$$\frac{dH^{\alpha}(t)}{d\alpha} = \int_{t_1}^{t_2} \eta(t) G_0^f(f(t), r_1(u), \dots, r_n(u), u, t) du, 
+ \int_{t_1}^{t_2} \int_{t_1}^{t_2} \sum_{i=1}^n \eta(v) G_i^f(f(v), v, u) G_0^{r_i}(f(t), r_1(u), \dots, r_n(u), u, t) dv du.$$
(A.37)

By substituting (A.37) into (A.32) we have

$$\begin{split} \delta_{f}L &= \int_{t_{1}}^{t_{2}} \eta(t)F^{f}(f(t),H(t),t)dt \\ &+ \int_{t_{1}}^{t_{2}}F^{H}(f(t),H(t),t) \left(\int_{t_{1}}^{t_{2}} \eta(t)G_{0}^{f}(f(t),r_{1}(u),\ldots,r_{n}(u),u,t)du \\ &+ \int_{t_{1}}^{t_{2}}\int_{t_{1}}^{t_{2}}\sum_{i=1}^{n} \eta(v)G_{i}^{f}(f(v),v,u)G_{0}^{r_{i}}(f(t),r_{1}(u),\ldots,r_{n}(u),u,t)dvdu\right)dt \quad (A.38) \\ &= \int_{t_{1}}^{t_{2}} \eta(t) \left(F^{f}(f(t),H(t),t) + F^{H}(f(t),H(t),t)\int_{t_{1}}^{t_{2}}G_{0}^{f}(f(t),r_{1}(u),\ldots,r_{n}(u),u,t)du\right)dt \\ &+ \int_{t_{1}}^{t_{2}}\int_{t_{1}}^{t_{2}}\int_{t_{1}}^{t_{2}} \eta(v)F^{H}(f(t),H(t),t) \sum_{i=1}^{n}G_{i}^{f}(f(v),v,u)G_{0}^{r_{i}}(f(t),r_{1}(u),\ldots,r_{n}(u),u,t)dvdudt \\ &\qquad (A.39) \\ &= \int_{t_{1}}^{t_{2}} \eta(t) \left(F^{f}(f(t),H(t),t) + F^{H}(f(t),H(t),t)\int_{t_{1}}^{t_{2}}G_{0}^{f}(f(t),r_{1}(u),\ldots,r_{n}(u),u,t)du\right)dt \\ &+ \int_{t_{1}}^{t_{2}}\int_{t_{1}}^{t_{2}}\int_{t_{1}}^{t_{2}} \eta(v)F^{H}(f(v),H(v),v)\sum_{i=1}^{n}G_{i}^{f}(f(t),t,u)G_{0}^{r_{i}}(f(v),r_{1}(u),\ldots,r_{n}(u),u,v)dvdudt \\ &\qquad (A.40) \\ &= \int_{t_{1}}^{t_{2}} \eta(t) \left(F^{f}(f(t),H(t),t) + F^{H}(f(t),H(t),t)\int_{t_{1}}^{t_{2}}G_{0}^{f}(f(t),r_{1}(u),\ldots,r_{n}(u),u,v)dvdudt \\ &\qquad (A.40) \\ &= \int_{t_{1}}^{t_{2}} \eta(t) \left(F^{f}(f(t),H(t),v) + \sum_{i=1}^{n}G_{i}^{f}(f(t),t,u)G_{0}^{r_{i}}(f(v),r_{1}(u),\ldots,r_{n}(u),u,v)dvdudt \\ &\qquad (A.40) \\ &= \int_{t_{1}}^{t_{2}} \int_{t_{1}}^{t_{2}} F^{H}(f(v),H(v),v) \sum_{i=1}^{n}G_{i}^{f}(f(t),t,u)G_{0}^{r_{i}}(f(v),r_{1}(u),\ldots,r_{n}(u),u,v)dvdud \\ &+ \int_{t_{1}}^{t_{2}} \int_{t_{1}}^{t_{2}} F^{H}(f(v),H(v),v) \sum_{i=1}^{n}G_{i}^{f}(f(t),t,u)G_{0}^{r_{i}}(f(v),r_{1}(u),\ldots,r_{n}(u),u,v)dvdud \\ &+ \int_{t_{1}}^{t_{2}} \int_{t_{1}}^{t_{2}} F^{H}(f(v),H(v),v) \sum_{i=1}^{n}G_{i}^{f}(f(t),t,u)G_{0}^{r_{i}}(f(v),r_{1}(u),\ldots,r_{n}(u),u,v)dvdud \\ &+ \int_{t_{1}}^{t_{2}} \int_{t_{1}}^{t_{2}} F^{H}(f(v),H(v),v) \sum_{i=1}^{n}G_{i}^{f}(f(t),t,u)G_{0}^{r_{i}}(f(v),r_{1}(u),\ldots,r_{n}(u),u,v)dvdu \\ &+ \int_{t_{1}}^{t_{2}} \int_{t_{1}}^{t_{2}} F^{H}(f(v),H(v),v) \sum_{i=1}^{n}G_{i}^{f}(f(v),t,u)G_{0}^{r_{i}}(f(v),r_{1}(u),\ldots,r_{n}(u),u,v)dvdu \\ &+ \int_{t_{1}}^{t_{1}} \int_{t_{1}}^{t_{2}} F^{H}(f(v),H(v),v) \sum_{i=1}^{n}G_{i}^{f}(f(v),t,u)G_{0}^{r_{i}}(f(v),r_{1}(v),\ldots,r_{n}(v),v)dvdu \\ &+ \int_{t_{1}}^{t_{1}} \int$$

Since  $\eta(t)$  in (A.41) is an arbitrary function, the necessary condition for f to be a solution is that the term inside the round brackets in (A.41) is zero. This concludes the proof.

104	

(A.41)

#### Appendix B

## Proof of Proposition 3.4.1

As discussed in Section 3.4, the MMSE estimator  $g(\cdot) = \mathbb{E}[V|\cdot]$  is optimal for any encoder mapping f. Due to the orthogonality principle of the MMSE estimation, we can write  $\bar{D} = \sigma_v^2 - \mathbb{E}[V\hat{V}]$ . Rewriting the Lagrangian in (3.9) for the MSE distortion criterion, and dropping the constants that are independent of f, we have

$$L(f, g, \lambda) = -\mathbb{E}[V\hat{V}] + \lambda \mathbb{E}[f(V)^2].$$
(B.1)

In the following, we prove the proposition by means of three key lemmas. In the first, using (A.20), we obtain a necessary condition for the optimal encoder mapping. Then, using this necessary condition, we show that the Lagrangian function in (B.1) takes its minimum value when f is odd. Finally, tackling the optimization problem in (3.8) over odd functions, and using (A.23), we obtain the result of Proposition 3.4.1.

Lemma B.0.1. The optimal mapping function f for the problem (3.8), has to satisfy

$$f(v)e^{\frac{f(v)^2}{2\sigma_w^2}} = a_f(v+b_f),$$
 (B.2)

where  $a_f$  and  $b_f$  are defined as

$$a_f \triangleq \frac{-r_1}{\sqrt{2\pi\sigma_w\lambda r_2(1-r_2)}},\tag{B.3a}$$

$$b_f \triangleq -\frac{(1-2r_2)r_1}{2r_2(1-r_2)},$$
 (B.3b)

with  $r_1$  and  $r_2$  defined as

$$r_1 \triangleq \frac{1}{\sigma_v} \int v \Phi\left(\frac{v}{\sigma_v}\right) Q\left(\frac{f(v)}{\sigma_w}\right) dv, \tag{B.4a}$$

$$r_2 \triangleq \frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) Q\left(\frac{f(v)}{\sigma_w}\right) dv.$$
(B.4b)

*Proof*: Expanding the objective function  $L(f, g, \lambda)$  in (B.1), we have

$$L(f,g,\lambda) = -\mathbb{E}_V \left[ V \mathbb{E}_{\hat{V}|V} [\hat{V}|V] - \lambda f(V)^2 \right]$$
(B.5a)

$$= \frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) \left(vQ\left(\frac{f(v)}{\sigma_w}\right) \left(\hat{v}_{(2)} - \hat{v}_{(1)}\right) + \lambda f(v)^2\right) dv$$
(B.5b)

$$= \frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) \left(vQ\left(\frac{f(v)}{\sigma_w}\right) \left(\frac{r_1}{r_2(1-r_2)}\right) + \lambda f(v)^2\right) dv, \qquad (B.5c)$$

where  $\hat{v}_{(1)} = \frac{r_1}{r_2 - 1}$  and  $\hat{v}_{(2)} = \frac{r_1}{r_2}$ . We observe that (B.5c) can be stated in the form in (A.16) by setting

$$\tilde{F} = vQ\left(\frac{f(v)}{\sigma_w}\right) \cdot \left(\frac{r_1}{r_2(1-r_2)}\right) + \lambda f(v)^2, \tag{B.6a}$$

$$\tilde{G}_1 = vQ\left(\frac{f(v)}{\sigma_w}\right),\tag{B.6b}$$

$$\tilde{G}_2 = Q\left(\frac{f(v)}{\sigma_w}\right). \tag{B.6c}$$

Therefore, from (A.20) we have the necessary condition

$$\nabla L = \frac{vr_1 e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w r_2(1-r_2)} + 2\lambda f(v) + \frac{vr_1 e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w r_2(1-r_2)} - \frac{(1-2r_2)r_1^2}{(r_2(1-r_2))^2} \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w}$$
$$= \frac{2r_1 e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w r_2(1-r_2)} \left(v - \frac{(1-2r_2)r_1}{2r_2(1-r_2)}\right) + 2\lambda f(v) = 0. \tag{B.7}$$

By solving (B.7) with respect to f(v), we obtain (B.2), which concludes the proof.

Based on (B.2), we restrict the minimization of the objective function over encoder mappings  $f_{a,b}(v)$  that satisfy

$$f_{a,b}(v)e^{\frac{f_{a,b}(v)^2}{2\sigma_w^2}} = \frac{1}{a}(v-b),$$
(B.8)



FIGURE B.1: The function  $f_o(v)$  in (B.9) and the function  $f_{a,b}(v)$  in (B.8) with a = 4, b = 10.

for some parameters a and b. Defining the function  $f_o(\cdot)$  as the unique solution of

$$f_o(v)e^{\frac{f_o(v)^2}{2}} = v,$$
 (B.9)

it will be convenient to write

$$f_{a,b}(v) = \sigma_w f_o\left(\frac{v-b}{a\sigma_w}\right). \tag{B.10}$$

This shows that any function  $f_{a,b}(v)$  can be seen as a scaled and shifted version of the function that satisfies (B.9). The function  $f_o(v)$  is plotted in Figure B.1, along with an example of the function  $f_{a,b}(v)$  for a = 4, b = 10. It can be easily verified that  $f_o(v)$  is odd, that is,  $f_o(-v) = -f_o(v)$ . Furthermore, functions of the form  $f_{a,0}(v)$ , that is, with b = 0, are odd as well.

In the following lemma, we show that the Lagrangian functional in (B.1) takes its minimum value only over functions of the form  $f_{a,b}(v)$  where b = 0 for any a, which shows that the optimal function  $f_{a,b}(v)$  is odd as summarized in Corollary B.0.3. Lemma B.0.2. The optimal solution of the problem

$$\underset{b}{\text{minimize}} \quad L(f_{a,b}, g, \lambda), \tag{B.11}$$

is achieved for b = 0 for any fixed a.

Proof: We prove the lemma assuming that  $a, b \ge 0$ . This is without loss of generality given the relationship  $L(f_{a,b}, g, \lambda) = L(f_{a,-b}, g, \lambda) = L(f_{-a,b}, g, \lambda) = L(f_{-a,-b}, g, \lambda)$ . In the following, we show that decreasing the value of  $b \ge 0$ , reduces the average power of the mapping function, i.e.,  $\mathbb{E}[f_{a,b}(V)^2]$ , and increases  $\mathbb{E}[V\hat{V}]$ . Therefore, the function  $L(f_{a,b}, g, \lambda)$  becomes smaller by decreasing the value of  $b \ge 0$ .

1)  $\mathbb{E}[f_{a,b}(V)^2]$  is a strictly increasing function of b: By writing the average power of the function  $f_{a,b}(v)$  we have

$$\mathbb{E}[f_{a,b}(V)^2] = \frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) f_{a,b}(v)^2 dv \tag{B.12}$$

$$= \frac{\sigma_w}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) f_o\left(\frac{v-b}{a\sigma_w}\right)^2 dv \tag{B.13}$$

$$=\frac{a\sigma_w^2}{\sigma_v}\int\Phi\left(\frac{a\sigma_wv+b}{\sigma_v}\right)f_o\left(v\right)^2dv.$$
(B.14)

Differentiating  $\mathbb{E}[f_{a,b}(V)^2]$  with respect to b, we have

$$-\frac{d\mathbb{E}[f_{a,b}(V)^2]}{db} = \frac{a\sigma_w^2}{\sigma_v^3} \int (a\sigma_w v + b)\Phi\left(\frac{a\sigma_w v + b}{\sigma_v}\right) f_o(v)^2 dv$$
$$\leq \frac{a\sigma_w^2}{\sigma_v^3} \int (a\sigma_w v + b)\Phi\left(\frac{a\sigma_w v + b}{\sigma_v}\right) \Psi(v) dv, \tag{B.15}$$

where  $\Psi(v)$  is defined as

$$\Psi(v) \triangleq \begin{cases} f_o(v)^2 & v \le -\frac{b}{a\sigma_w} \\ f_o\left(v + \frac{2b}{a\sigma_w}\right)^2 & v > -\frac{b}{a\sigma_w} \end{cases}.$$
 (B.16)

It is easy to see that the function on the right hand side of (B.15) is an odd function shifted to the left by  $\frac{b}{a\sigma_w}$ ; therefore, the integral is zero, completing the proof. Note that (B.15) holds with equality only when b = 0.



FIGURE B.2: Plot of the function  $\tilde{g}(b)$  in (B.19).

2)  $\mathbb{E}[V\hat{V}]$  is a decreasing function of b: We first consider the noiseless scenario, i.e., W = 0. In the noiseless scenario, for the reconstruction points we have

$$\hat{V} = \begin{cases}
\mathbb{E}[V|v \ge b] & Y = 0 \\
\mathbb{E}[V|v < b] & Y = 1
\end{cases}$$

$$= \begin{cases}
\frac{\sigma_v e^{-\frac{b^2}{2}}}{\sqrt{2\pi}Q\left(\frac{b}{\sigma_v}\right)} & Y = 0, \\
\frac{-\sigma_v e^{-\frac{b^2}{2}}}{\sqrt{2\pi}\left(1-Q\left(\frac{b}{\sigma_v}\right)\right)} & Y = 1.
\end{cases}$$
(B.17)
(B.17)

Therefore,  $\mathbb{E}[V\hat{V}]$  can be written as

$$\mathbb{E}[V\hat{V}] = \Pr(V \ge b)\hat{v}_{(1)}\mathbb{E}[V|\hat{V} = \hat{v}_{(1)}] + \Pr(V < b)\hat{v}_{(2)}\mathbb{E}[V|\hat{V} = \hat{v}_{(2)}]$$
(B.19)

$$= \frac{\sigma_v^2 e^{-b^2}}{2\pi Q\left(\frac{b}{\sigma_v}\right) \left(1 - Q\left(\frac{b}{\sigma_v}\right)\right)} \triangleq \tilde{g}(b).$$
(B.20)

It can be verified that the function  $\tilde{g}(b)$  is even, and that it takes its maximum at b = 0 as seen in Figure B.2.

Now, considering the noisy received signal, we expand  $\mathbb{E}[V\hat{V}]$  as

$$\mathbb{E}[V\hat{V}] = \frac{1}{\sigma_w} \int \Phi\left(\frac{w}{\sigma_w}\right) \mathbb{E}[V\hat{V}|W = -w]dw, \tag{B.21}$$

where

$$\mathbb{E}[V\hat{V}|W = -w] = \mathbb{E}[V|Y = y_{(1)}, W = -w]^2 \Pr(Y = y_{(1)}|W = -w) + \mathbb{E}[V|Y = y_{(2)}, W = -w]^2 \Pr(Y = y_{(2)}|W = -w)$$
(B.22)

$$=\frac{\sigma_{v}e^{-\left(f_{a,b}^{-1}(w)\right)^{2}}}{2\pi Q\left(\frac{f_{a,b}^{-1}(w)}{\sigma_{v}}\right)}+\frac{\sigma_{v}e^{-\left(f_{a,b}^{-1}(w)\right)^{2}}}{2\pi \left(1-Q\left(\frac{f_{a,b}^{-1}(w)}{\sigma_{v}}\right)\right)}$$
(B.23)

$$= \tilde{g}\left(f_{a,b}^{-1}(w)\right). \tag{B.24}$$

Notice that the inverse of the function  $f_{a,b}(v)$  exists because it is one-to-one and is defined on the whole real line. Therefore,

$$\mathbb{E}[V\hat{V}] = \frac{1}{\sigma_w} \int \Phi\left(\frac{w}{\sigma_w}\right) \tilde{g}\left(f_{a,b}^{-1}(w)\right) dw.$$
(B.25)

We want to show that

$$\int e^{-\frac{w^2}{2}} \tilde{g}\left(f_{a,b}^{-1}(w)\right) dw \le \int e^{-\frac{w^2}{2}} \tilde{g}\left(f_{a,0}^{-1}(w)\right) dw.$$
(B.26)

Using the change of variables  $w = f_{a,0}(v)$  and the equality  $f_{a,b}^{-1}(w) = b + f_{a,0}^{-1}(w)$ , inequality (B.26) can be written as

$$\int e^{-\frac{f_{a,0}^2(v)}{2}} \tilde{g}(b+v) f_{a,0}'(v) dv \le \int e^{-\frac{f_{a,0}^2(v)}{2}} \tilde{g}(v) f_{a,0}'(v) dv.$$
(B.27)

Inequality (B.27) is equivalent to

$$\int_{-\infty}^{-\frac{b}{2}} e^{-\frac{f_{a,0}^2(v)}{2}} \left(\tilde{g}(b+v) - \tilde{g}(v)\right) f_{a,0}'(v) dv \le \int_{-\frac{b}{2}}^{\infty} e^{-\frac{f_{a,0}^2(v)}{2}} \left(\tilde{g}(v) - \tilde{g}(b+v)\right) f_{a,0}'(v) dv.$$
(B.28)

Using the transformation -b - v = t for the left hand side of (B.28), we have

$$\int_{-\frac{b}{2}}^{\infty} e^{-\frac{f_{a,0}^{2}(-b-t)}{2}} \left(\tilde{g}(-t) - \tilde{g}(-b-t)\right) f_{a,0}'(-t-b) dt = \int_{-\frac{b}{2}}^{\infty} e^{-\frac{f_{a,0}^{2}(b+t)}{2}} \left(\tilde{g}(t) - \tilde{g}(b+t)\right) f_{a,0}'(t+b) dt$$
(B.29)

where the second equality is due to the fact that functions  $\tilde{g}$ ,  $f_{a,0}^2$ ,  $f_{a,0}'$  are even. Therefore, (B.28) is equivalent to

$$\int_{-\frac{b}{2}}^{\infty} \left[ e^{-\frac{f_{a,0}^{2}(b+t)}{2}} f_{a,0}^{'}(t+b) - e^{-\frac{f_{a,0}^{2}(t)}{2}} f_{a,0}^{'}(t) \right] \left[ \tilde{g}(t) - \tilde{g}(b+t) \right] dt \le 0.$$
(B.30)

The inequality in (B.30) concludes the proof and follows from the following facts:

- $\tilde{g}(t) \ge \tilde{g}(t+b) > 0$  for  $t \ge -\frac{b}{2}$ , and hence, the second bracket is nonnegative;
- $e^{-\frac{f_{a,0}^2(t)}{2}} \ge e^{-\frac{f_{a,0}^2(b+t)}{2}}$ , due to the fact that  $f_{a,0}(\cdot)$  is nondecreasing;
- $f_{a,0}^{'}(t) \ge f_{a,0}^{'}(t+b) > 0$  for  $t > -\frac{b}{2}$ .

Corollary B.0.3. The optimal mapping for problem (3.9) under the MSE criterion is odd.

*Proof*: By Lemma B.0.1, the optimal mapping can be written in the form  $f_{a,b}(v)$  without loss of optimality. By Lemma B.0.2, we conclude that the optimal mapping is in the form  $f_{a,0}(v)$ ; and hence, it is odd.

Lemma B.0.4. A necessary condition for an optimal solution to problem (3.8) is

$$\nabla L = \frac{-2v}{\sqrt{2\pi\sigma_w}} e^{-\frac{f(v)^2}{2\sigma_w^2}} + 2\lambda f(v) = 0.$$
(B.31)

*Proof*: Since the optimal mapping is odd by Corollary B.0.3, it can be easily verified that  $\hat{v}_{(1)} = -\hat{v}_{(2)}$ . Furthermore, without loss of optimality we can assume  $\hat{v}_{(1)} \ge 0$ . Therefore, using (3.13c) problem (3.8) can be restated as

$$\underset{f}{\text{minimize}} \quad -\hat{v}_{(1)} + \lambda E[f(V)^2], \tag{B.32}$$

where  $\hat{v}_{(1)}$  is obtained as in (3.12c). Expanding (B.32), we can write

$$\underset{f}{\text{minimize}} \quad \frac{-1}{\sigma_v} \int_{-\infty}^{\infty} \Phi\left(\frac{v}{\sigma_v}\right) \left(2vQ\left(\frac{f(v)}{\sigma_w}\right) - \lambda f(v)^2\right) dv. \tag{B.33}$$

Since (B.33) is of the form (A.21) with

$$\tilde{F} = -2vQ\left(\frac{f(v)}{\sigma_w}\right) + \lambda f(v)^2, \tag{B.34}$$

we have the necessary condition in (B.31) using (A.23). This concludes the proof.

#### Appendix C

# Proof of Proposition 3.4.2

Expanding the objective function  $L(f, g, \lambda)$  as in (B.5a) we have

where we recall that  $\hat{v}_{(j)}$  is the MMSE estimation of the source when the received signal is  $Y = y_{(j)}$ , i.e.,  $Y = f(V) + W \in [z_{(j-1)}, z_{(j)})$ . We have  $\hat{v}_{(j)} = \frac{r_{1j}}{r_{2j}}$  with

$$r_{1j} = \frac{1}{\sigma_v} \int v\Phi\left(\frac{v}{\sigma_v}\right) \left(Q\left(\frac{z_{(j-1)} - f(v)}{\sigma_w}\right) - Q\left(\frac{z_{(j)} - f(v)}{\sigma_w}\right)\right) dv, \ j = 1, ..., K,$$
(C.2a)

$$r_{2j} = \frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) \left(Q\left(\frac{z_{(j-1)} - f(v)}{\sigma_w}\right) - Q\left(\frac{z_{(j)} - f(v)}{\sigma_w}\right)\right) dv, \ j = 1, ..., K.$$
(C.2b)

Since (C.1b) is of the form (A.16) with

$$\tilde{F} = -v \sum_{j=1}^{K} \frac{r_{1j}}{r_{2j}} \left( Q\left(\frac{z_{(j-1)} - f(v)}{\sigma_w}\right) - Q\left(\frac{z_{(j)} - f(v)}{\sigma_w}\right) \right) + \lambda f(v)^2, \tag{C.3}$$

$$\tilde{G}_{1j} = v \left( Q \left( \frac{z_{(j-1)} - f(v)}{\sigma_w} \right) - Q \left( \frac{z_{(j)} - f(v)}{\sigma_w} \right) \right), \quad j = 1, \dots, K,$$
(C.4)

$$\tilde{G}_{2j} = \left(Q\left(\frac{z_{(j-1)} - f(v)}{\sigma_w}\right) - Q\left(\frac{z_{(j)} - f(v)}{\sigma_w}\right)\right), \ j = 1, \dots, K.$$
(C.5)

By writing the necessary condition in (A.20), we have

$$\nabla L = -\frac{v}{\sqrt{2\pi}\sigma_w} \sum_{j=1}^{K} \frac{r_{1j}}{r_{2j}} \left( e^{-\frac{\left(z_{(j-1)}-f(v)\right)^2}{2\sigma_w^2}} - e^{-\frac{\left(z_{(j)}-f(v)\right)^2}{2\sigma_w^2}} \right) + 2\lambda f(v) - \frac{v}{\sqrt{2\pi}\sigma_w} \sum_{j=1}^{K} \frac{r_{1j}}{r_{2j}} \left( e^{-\frac{\left(z_{(j-1)}-f(v)\right)^2}{2\sigma_w^2}} - e^{-\frac{\left(z_{(j)}-f(v)\right)^2}{2\sigma_w^2}} \right) + \frac{1}{\sqrt{2\pi}\sigma_w} \sum_{j=1}^{K} \frac{r_{1j}^2}{r_{2j}^2} \left( e^{-\frac{\left(z_{(j-1)}-f(v)\right)^2}{2\sigma_w^2}} - e^{-\frac{\left(z_{(j)}-f(v)\right)^2}{2\sigma_w^2}} \right) = 0.$$
(C.6)

Solving for f, we have

$$f(v) = \frac{1}{2\sqrt{2\pi}\sigma_w\lambda} \sum_{j=1}^{K} \left( e^{-\frac{\left(z_{(j-1)} - f(v)\right)^2}{2\sigma_w^2}} - e^{-\frac{\left(z_{(j)} - f(v)\right)^2}{2\sigma_w^2}} \right) \hat{v}_{(j)} \left(2v - \hat{v}_{(j)}\right).$$
(C.7)

#### Appendix D

# Proof of Proposition 3.5.1

We start by expanding the Lagrangian function (3.9) for the DOP criterion as

$$L(f, g, \lambda) = \epsilon(D) + \lambda \mathbb{E}[f(V)^2]$$
(D.1a)  
=  $\Pr(V \in (I_1 \cup I_2)^C) + \Pr(V \in (I_1 \setminus I_2), \hat{V} = \hat{v}_{(2)}) + \Pr(V \in (I_2 \setminus I_1), \hat{V} = \hat{v}_{(1)})$ (D.1b)

$$+\frac{\lambda}{\sigma_v}\int \Phi\left(\frac{v}{\sigma_v}\right)f(v)^2dv,\tag{D.1c}$$

where we have used the decomposition in (3.25). The probabilities in (D.1b) can be written as

$$\Pr\left(V \in (I_1 \cup I_2)^C\right) = \frac{1}{\sigma_v} \int_{\substack{v \in (I_1 \cup I_2)^C}} \Phi\left(\frac{v}{\sigma_v}\right) dv,$$
(D.2a)

$$\Pr\left(V \in (I_1 \setminus I_2), \hat{V} = \hat{v}_{(2)}\right) = \frac{1}{\sigma_v} \int_{v \in I_1 \setminus I_2} \Phi\left(\frac{v}{\sigma_v}\right) Q\left(\frac{f(v)}{\sigma_w}\right) dv, \tag{D.2b}$$

$$\Pr\left(V \in (I_2 \setminus I_1), \hat{V} = \hat{v}_{(1)}\right) = \frac{1}{\sigma_v} \int_{v \in I_2 \setminus I_1} \Phi\left(\frac{v}{\sigma_v}\right) Q\left(\frac{-f(v)}{\sigma_w}\right) dv.$$
(D.2c)

Since the intervals on which the integrals in (D.2) are taken do not overlap and span the real line, the Lagrangian in (D.1) can be written in the form of (A.21) with  $\tilde{F}(v, f)$  defined as

$$\tilde{F}(v,f) = \begin{cases} 1 + \lambda f(v)^2 & v \in (I_1 \cup I_2)^C, \\ Q\left(\frac{f(v)}{\sigma_w}\right) + \lambda f(v)^2 & v \in (I_1 \setminus I_2), \\ Q\left(\frac{-f(v)}{\sigma_w}\right) + \lambda f(v)^2 & v \in (I_2 \setminus I_1), \\ \lambda f(v)^2 & v \in (I_1 \cap I_2). \end{cases}$$
(D.3)

Using the optimality condition in (A.23), we have the necessary condition  $\tilde{F}^f = 0$  with

$$\tilde{F}^{f}(v,f) = \begin{cases} 2\lambda f(v) & v \in (I_1 \cup I_2)^C \cup (I_1 \cap I_2), \\ \frac{-1}{\sqrt{2\pi}\sigma_v \sigma_w} \Phi\left(\frac{v}{\sigma_v}\right) e^{\frac{-f(v)^2}{2\sigma_w^2}} + 2\lambda f(v) & v \in (I_1 \setminus I_2), \\ \frac{1}{\sqrt{2\pi}\sigma_v \sigma_w} \Phi\left(\frac{v}{\sigma_v}\right) e^{\frac{-f(v)^2}{2\sigma_w^2}} + 2\lambda f(v) & v \in (I_2 \setminus I_1). \end{cases}$$
(D.4)

#### Appendix E

## Proof of Proposition 3.5.2

Given  $\lambda$ , and obtaining the value of u from (3.27), we aim to minimize the objective function in (3.9) for the DOP with respect to the reconstruction points  $\hat{v}_{(1)}$  and  $\hat{v}_{(2)}$ . Since by Proposition 3.5.1 the non-zero optimal values of f(v) are opposite of one opposite of one another over  $I_2 \setminus I_1$  and  $I_1 \setminus I_2$ , we can rewrite (3.9) for the DOP as follows

$$L(f,g,\lambda) = \frac{1}{\sigma_v} \left( \int_{(I_1 \cup I_2)^C} \Phi\left(\frac{v}{\sigma_v}\right) dv + Q\left(\frac{u}{\sigma_w}\right) \left( \int_{I_2 \setminus I_1} \Phi\left(\frac{v}{\sigma_v}\right) dv + \int_{I_1 \setminus I_2} \Phi\left(\frac{v}{\sigma_v}\right) dv \right) + \lambda \left( \int_{I_2 \setminus I_1} \Phi\left(\frac{v}{\sigma_v}\right) u^2 dv + \int_{I_1 \setminus I_2} \Phi\left(\frac{v}{\sigma_v}\right) u^2 dv \right) \right),$$
(E.1)

where u is the solution of  $ue^{\frac{u^2}{2\sigma_w^2}} = \frac{1}{2\sigma_w\sqrt{2\pi\lambda}}$ . From (E.1), it can be verified that (3.9) for the DOP is minimized if  $I_1$  and  $I_2$  lie around the origin, and are symmetric. Define the positive parameter  $0 \le a \le \sqrt{D}$  such that  $I_2 = [-2\sqrt{D} + a, a]$  and  $I_1 = [-a, 2\sqrt{D} - a]$ , and  $I_1 \cap I_2 = [-a, a]$ . Therefore, (E.1) can be rewritten as

$$L(f,g,\lambda) = 2Q\left(\frac{2\sqrt{D}-a}{\sigma_v}\right) + 2\left(Q\left(\frac{u}{\sigma_w}\right) + \lambda u^2\right)\left(Q\left(\frac{a}{\sigma_v}\right) - Q\left(\frac{2\sqrt{D}-a}{\sigma_v}\right)\right).$$
(E.2)

Optimizing (E.2) over a completes the proof of Proposition 3.5.2.

#### Appendix F

# Proof of Proposition 3.5.3

Necessary optimality condition of an encoder mapping f for a given set of reconstruction points  $\{\hat{v}_{(1)}, \ldots, \hat{v}_{(K)}\}$ : We start by expanding  $\epsilon(D)$  with respect to the different intervals defined in (3.32). We have

$$\epsilon(D) = 1 - \Pr\left((V - \hat{V})^2 < D\right)$$
(F.1a)

$$= 1 - \sum_{\mathcal{V}: \mathcal{V} \subset \{\hat{v}_{(1)}, \dots, \hat{v}_{(K)}\}} \Pr\left( (V - \hat{V})^2 < D | V \in I_{\mathcal{V}} \right) \Pr\left( V \in I_{\mathcal{V}} \right)$$
(F.1b)

$$= 1 - \Pr\left((V - \hat{V})^{2} < D | V \in I_{\emptyset}\right) \Pr\left(V \in I_{\emptyset}\right) - \Pr\left((V - \hat{V})^{2} < D | V \in I_{\{\hat{v}_{(1)}, \dots, \hat{v}_{(K)}\}}\right) \Pr\left(V \in I_{\{\hat{v}_{(1)}, \dots, \hat{v}_{(K)}\}}\right) - \sum_{\substack{\mathcal{V}: \mathcal{V} \subset \{\hat{v}_{(1)}, \dots, \hat{v}_{(K)}\}\\|\mathcal{V}|=1, \dots, K-1}} \Pr\left((V - \hat{V})^{2} < D | V \in I_{\mathcal{V}}\right) \Pr\left(V \in I_{\mathcal{V}}\right)$$
(F.1c)

$$= 1 - \Pr\left(V \in I_{\{\hat{v}_{(1)},...,\hat{v}_{(K)}\}}\right) - \sum_{\substack{\mathcal{V}:\mathcal{V}\subset\{\hat{v}_{(1)},...,\hat{v}_{(K)}\}\\|\mathcal{V}|=1,...,K-1}} \Pr\left((V - \hat{V})^2 < D|V \in I_{\mathcal{V}}\right) \Pr\left(V \in I_{\mathcal{V}}\right)$$
(F.1d)

$$= 1 - \Pr\left(V \in I_{\{\hat{v}_{(1)},...,\hat{v}_{(K)}\}}\right) - \sum_{\substack{\mathcal{V}:\mathcal{V} \subset \{\hat{v}_{(1)},...,\hat{v}_{(K)}\}\\|\mathcal{V}|=1,...,K-1}} \frac{1}{\sigma_{v}} \int_{v \in I_{\mathcal{V}}} \Phi\left(\frac{v}{\sigma_{v}}\right) \Pr\left(W + f(v) \in \zeta_{\mathcal{V}}\right) dv$$
(F.1e)  
$$= 1 - \Pr\left(V \in I_{\{\hat{v}_{(1)},...,\hat{v}_{(K)}\}}\right)$$

$$-\frac{1}{\sigma_{v}}\sum_{\substack{\mathcal{V}:\mathcal{V}\subset\{\hat{v}_{(1)},\dots,\hat{v}_{(K)}\}\\|\mathcal{V}|=1,\dots,K-1}}\int_{v\in I_{\mathcal{V}}}\Phi\left(\frac{v}{\sigma_{v}}\right)\sum_{j:\hat{v}_{(j)}\in\mathcal{V}}\left(Q\left(\frac{z_{(j-1)}-f(v)}{\sigma_{w}}\right)-Q\left(\frac{z_{(j)}-f(v)}{\sigma_{w}}\right)\right)dv,$$
(F.1f)

where  $\zeta_{\mathcal{V}} \triangleq \bigcup_{j:\hat{v}_{(j)} \in \mathcal{V}} [z_{(j-1)}, z_{(j)})$ . Note that the different sets in (F.1f) partition the real line, that is, they are disjoint and their union is the real line. Substituting (F.1f) in (3.14), the Lagrangian  $L(f, g, \lambda)$  can be written in the form of (A.21) with  $\tilde{F}(v, f)$  defined as

$$\tilde{F}(v,f) \triangleq \begin{cases} 1 + \lambda f(v)^2 & v \in I_{\emptyset}, \\ \lambda f(v)^2 & v \in I_{\{\hat{v}_{(1)},\dots,\hat{v}_{(K)}\}}, \\ 1 - \sum_{j:\hat{v}_{(j)} \in \mathcal{V}} \left( Q\left(\frac{z_{(j-1)} - f(v)}{\sigma_w}\right) - Q\left(\frac{z_{(j)} - f(v)}{\sigma_w}\right) \right) + \lambda f(v)^2 & v \in I_{\mathcal{V}}, \ |\mathcal{V}| = 1,\dots, K-1. \end{cases}$$
(F.2)

Writing the necessary condition in (A.23) and setting to zero leads to (3.36).

Optimal decoder function g for a given encoder mapping f: For a given encoder mapping f, the optimal decoder can be obtained as

$$\hat{v}_{(j)} = \arg\min_{t} \Pr((V-t)^2 \ge D|Y=y_{(j)})$$
 (F.3a)

$$\equiv \arg \max_{t} \Pr\left( (V-t)^2 < D, Y = y_{(j)} \right)$$
(F.3b)

$$= \arg \max_{t} \int_{t-\sqrt{D}}^{t+\sqrt{D}} \int_{z_{(j-1)}-f(v)}^{z_{(j)}-f(v)} \Phi\left(\frac{v}{\sigma_{v}}\right) \Phi\left(\frac{w}{\sigma_{w}}\right) dwdv$$
(F.3c)

$$= \arg \max_{t} \int_{t-\sqrt{D}}^{t+\sqrt{D}} \Phi\left(\frac{v}{\sigma_{v}}\right) \left(Q\left(\frac{z_{(j-1)} - f(v)}{\sigma_{w}}\right) - Q\left(\frac{z_{(j)} - f(v)}{\sigma_{w}}\right)\right) dv. \quad (F.3d)$$

#### Appendix G

## Proof of Proposition 3.6.1

Define  $y_i \triangleq \mathcal{Q}(f(v) + w_i), w^N \triangleq [w_1, ..., w_N]^T, y^N \triangleq [y_1, ..., y_N]^T$ . Expanding  $L(f, g, \lambda)$ in (3.9) for the MSE we have

$$L(f,g,\lambda) = \frac{-1}{\sigma\sigma_v} \int_{v} \int_{w^N} vg\left(y^N\right) \Phi\left(\frac{v}{\sigma_v}\right) \prod_{i=1}^N \Phi\left(\frac{w_i}{\sigma_{w_i}}\right) dw^N dv + \lambda \frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) f(v)^2 dv$$
(G.1a)

$$= \frac{-1}{\sigma\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) \left(v \int\limits_{w^N} g\left(y^N\right) \prod_{i=1}^N \Phi\left(\frac{w_i}{\sigma_{w_i}}\right) dw^N - \lambda f(v)^2\right) dv$$
(G.1b)

$$= \frac{-1}{\sigma\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) \left(v \sum_{j=1}^{2^N} \hat{v}_{(j)} \operatorname{Pr}\left(\hat{V} = \hat{v}_{(j)} | V = v\right) - \lambda f(v)^2\right) dv, \quad (G.1c)$$

where  $\hat{v}_{(j)} = \mathbb{E}[V|Y^N = b_j^N] = \mathbb{E}[V|Y_1 = b_j^N(1), \dots, Y_N = b_j^N(N)]$  and  $\sigma = \prod_{i=1}^N \frac{1}{\sigma_{w_i}}$ . We can compute

$$\Pr\left(\hat{V} = \hat{v}_{(j)}|V = v\right) = \prod_{i=1}^{N} \Pr\left(\mathcal{Q}(f(v) + W_i) = b_j^N(i)\right)$$
(G.2a)

$$=\prod_{i=1}^{N} Q\left(\frac{(-1)^{b_{j}^{N}(i)+1}f(v)}{\sigma_{w_{i}}}\right), \ j=1,...,2^{N}.$$
 (G.2b)

Substituting (G.2b) in (G.1c), we have

$$L(f,g,\lambda) = \frac{-1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) \left(v \sum_{j=1}^{2^N} \hat{v}_{(j)} \prod_{i=1}^N Q\left(\frac{(-1)^{b_j^N(i)+1} f(v)}{\sigma_{w_i}}\right) - \lambda f(v)^2\right) dv, \quad (G.3)$$

where for  $\hat{v}_{(j)}$  we have

$$\hat{v}_{(j)} = \frac{\frac{1}{\sigma_v} \int v \Phi\left(\frac{v}{\sigma_v}\right) \prod_{i=1}^N Q\left(\frac{(-1)^{b_j^N(i)+1} f(v)}{\sigma_{w_i}}\right) dv}{\frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) \prod_{i=1}^N Q\left(\frac{(-1)^{b_j^N(i)+1} f(v)}{\sigma_{w_i}}\right) dv} \triangleq \frac{r_{1_j}}{r_{2_j}}, \ j = 1, \dots, 2^N.$$
(G.4)

Since (G.3) is of the form (A.16) with

$$\tilde{F} = -v \sum_{j=1}^{2^N} \hat{v}_{(j)} \prod_{i=1}^N Q\left(\frac{(-1)^{b_j^N(i)+1} f(v)}{\sigma_{w_i}}\right) + \lambda f(v)^2, \tag{G.5}$$

$$\tilde{G}_{1_j} = vQ\left(\frac{(-1)^{b_j^N(i)+1}f(v)}{\sigma_{w_i}}\right), \ j = 1,\dots,2^N,$$
(G.6)

$$\tilde{G}_{2_j} = Q\left(\frac{(-1)^{b_j^N(i)+1}f(v)}{\sigma_{w_i}}\right), \ j = 1, \dots, 2^N.$$
(G.7)

Writing down the optimality condition in (A.20), we have

$$\nabla L = -v \sum_{j=1}^{2^{N}} \frac{r_{1_{j}}}{r_{2_{j}}} \Theta\left(N, f(v), j\right) + 2\lambda f(v) - \sum_{j=1}^{2^{N}} \frac{v \cdot \Theta\left(N, f(v), j\right)}{r_{2_{j}}} \cdot r_{1_{j}} + \sum_{j=1}^{2^{N}} \frac{r_{1_{j}} \cdot \Theta\left(N, f(v), j\right)}{r_{2_{j}}^{2}} \cdot r_{1_{j}}$$
(G.8a)

$$= -\sum_{j=1}^{2^{N}} \frac{r_{1_{j}}}{r_{2_{j}}} \Theta\left(N, f(v), j\right) \left(2v - \frac{r_{1_{j}}}{r_{2_{j}}}\right) + 2\lambda f(v)$$
(G.8b)

$$= -\sum_{j=1}^{2^{N}} \Theta\left(N, f(v), j\right) \hat{v}_{(j)} \left(2v - \hat{v}_{(j)}\right) + 2\lambda f(v) = 0, \tag{G.8c}$$

where

$$\Theta(N, f(v), j) = \sum_{k=1}^{N} \left( \frac{(-1)^{b_j^N(k)} e^{-\frac{f(v)^2}{2\sigma_{w_k}^2}}}{\sqrt{2\pi}\sigma_{w_k}} \prod_{l=1, l \neq k}^{N} Q\left(\frac{(-1)^{b_j^N(l)+1} f(v)}{\sigma_{w_l}}\right) \right).$$
(G.9)

Therefore, the optimal mapping must be in the form given by (3.43).

#### Appendix H

# Proof of Proposition 3.6.2

Necessary optimality condition of an encoder mapping f for a given set of reconstruction points  $\{\hat{v}_{(1)}, \ldots, \hat{v}_{(K)}\}$ : We expand  $\epsilon(D)$ 

$$\epsilon(D) = 1 - \Pr\left((V - \hat{V})^2 < D\right) \tag{H.1a}$$

$$=1-\frac{1}{\sigma\sigma_{v}}\iint_{\substack{v,w^{N}:|v-g(y^{N})|^{2}(H.1b)$$

$$=1-\frac{1}{\sigma\sigma_{v}}\sum_{\mathcal{V}:\mathcal{V}\subset\left\{\hat{v}_{(1)},\ldots,\hat{v}_{(2^{N})}\right\}}\int_{v\in I_{\mathcal{V}}}\int_{w^{N}:g(y^{N})\in\mathcal{V}}\Phi\left(\frac{v}{\sigma_{v}}\right)\prod_{i=1}^{N}\Phi\left(\frac{w_{i}}{\sigma_{w_{i}}}\right)dw^{N}dv$$
(H.1c)

$$=1-\frac{1}{\sigma_{v}}\sum_{\mathcal{V}:\mathcal{V}\subset\left\{\hat{v}_{(1)},\ldots,\hat{v}_{(2^{N})}\right\}}\int_{v\in I_{\mathcal{V}}}\Phi\left(\frac{v}{\sigma_{v}}\right)\sum_{j:\hat{v}_{(j)}\in\mathcal{V}}\left(\prod_{i=1}^{N}Q\left(\frac{(-1)^{b_{j}^{N}(i)+1}f(v)}{\sigma_{w_{i}}}\right)\right)dv$$
(H.1d)

$$= 1 - \frac{1}{\sigma_{v}} \int_{\substack{v \in I_{\left\{\hat{v}_{(1)}, \dots, \hat{v}_{(2^{N})}\right\}}}} \Phi\left(\frac{v}{\sigma_{v}}\right) dv$$
$$- \frac{1}{\sigma_{v}} \sum_{\substack{\mathcal{V}: \mathcal{V} \subset \left\{\hat{v}_{(1)}, \dots, \hat{v}_{(2^{N})}\right\} \\ |\mathcal{V}|=1, \dots, 2^{N}-1}} \int_{\substack{v \in I_{\mathcal{V}}}} \Phi\left(\frac{v}{\sigma_{v}}\right) \sum_{j: \hat{v}_{(j)} \in \mathcal{V}} \left(\prod_{i=1}^{N} Q\left(\frac{(-1)^{b_{j}^{N}(i)+1}f(v)}{\sigma_{w_{i}}}\right)\right) dv,$$
(H.1e)

where  $\sigma = \prod_{i=1}^{N} \sigma_{w_i}$ . Substituting (H.1e) in (3.9), the Lagrangian  $L(f, g, \lambda)$  can be written in the form of (A.21) with  $\tilde{F}(v, f)$  defined as

$$\tilde{F}(v,f) \triangleq \begin{cases} 1 + \lambda f(v)^2 & v \in I_{\emptyset}, \\ \lambda f(v)^2 & v \in I_{\{\hat{v}_{(1)},\dots,\hat{v}_{(2^N)}\}}, \\ 1 - \sum_{j:\hat{v}_{(j)} \in \mathcal{V}} \left(\prod_{i=1}^N Q\left(\frac{(-1)^{b_j^N(i)+1} f(v)}{\sigma_{w_i}}\right)\right) + \lambda f(v)^2 & v \in I_{\mathcal{V}} : |\mathcal{V}| = 1,\dots,2^N - 1 \end{cases}$$
(H.2)

Writing the necessary condition in (A.23) and setting to zero yields the result in (3.45).

Optimal decoder function g for a given encoder mapping f: Assume that the encoder mapping function is given. Following a similar approach to the derivation of the optimal decoder for a K-level ADC front end, the optimal decoder at the receiver is obtained as

$$\hat{v}_{(j)} = \arg \max_{t} \int_{t-\sqrt{D}}^{t+\sqrt{D}} \Phi\left(\frac{v}{\sigma_{v}}\right) \Pr(Y_{1} = b_{j}^{N}(1), \dots, Y_{N} = b_{j}^{N}(N)) dv$$
(H.3a)

$$= \arg \max_{t} \int_{t-\sqrt{D}}^{t+\sqrt{D}} \Phi\left(\frac{v}{\sigma_{v}}\right) \prod_{i=1}^{N} Q\left(\frac{(-1)^{b_{j}^{N}(i)+1}f(v)}{\sigma_{w_{i}}}\right) dv.$$
(H.3b)

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#### Appendix I

# Proof of Proposition 4.4.1

Due to the orthogonality principle of the MMSE estimation, it can be easily verified that  $\overline{D} = \sigma_v^2 - \mathbb{E}[V\hat{V}]$ . Rewriting the Lagrangian  $L(f, g, \lambda)$  for the MSE distortion criterion and dropping constants that are independent of f, we have

$$\begin{array}{l} \text{minimize} \quad -\mathbb{E}[V\hat{V}] + \lambda \mathbb{E}[f(V)^2]. \\ f \end{array} \tag{I.1}$$

By expanding the objective function in (I.1), it can be written as

$$\begin{split} & \frac{-1}{\sigma_w \sigma_v \sigma_u} \int \int \int vg(y, u) \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \Phi\left(\frac{w}{\sigma_w}\right) dw du dv \\ & + \frac{\lambda}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) f(v)^2 dv \quad (I.2) \\ & = \frac{-1}{\sigma_v \sigma_u} \int \int v\left(g(1, u) Q\left(\frac{f(v)}{\sigma_w}\right) + g(0, u) Q\left(\frac{-f(v)}{\sigma_w}\right)\right) \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) du dv \\ & + \frac{\lambda}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) f(v)^2 dv \quad (I.3) \\ & = \frac{-1}{\sigma_v} \int \left(v \int \frac{1}{\sigma_u} \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \left(\frac{r_1(u)}{r_2(u)} Q\left(\frac{f(v)}{\sigma_w}\right) + \frac{r_3(u)}{r_4(u)} Q\left(\frac{-f(v)}{\sigma_w}\right)\right) du \\ & + \lambda \Phi\left(\frac{v}{\sigma_v}\right) f(v)^2 \right) dv, \quad (I.4) \end{split}$$

where

$$r_1(u) \triangleq \int v\Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{f(v)}{\sigma_w}\right) dv, \tag{I.5a}$$

$$r_2(u) \triangleq \int \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{f(v)}{\sigma_w}\right) dv,$$
 (I.5b)

$$r_{3}(u) \triangleq \int v\Phi\left(\frac{v}{\sigma_{v}}, \frac{u}{\sigma_{u}}\right) Q\left(\frac{-f(v)}{\sigma_{w}}\right) dv, \qquad (I.5c)$$

$$r_4(u) \triangleq \int \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{-f(v)}{\sigma_w}\right) dv.$$
 (I.5d)

Note that (I.4) is in the form of (A.26) with F(f, H(v), v) and H(v) defined as

$$F(f, H(v), v) = \frac{1}{\sigma_v} \left( -vH(v) + \lambda \Phi\left(\frac{v}{\sigma_v}\right) f(v)^2 \right),$$
(I.6a)

and 
$$H(v) = \int G_0(f(v), r_1(u), \dots, r_4(u), u, v) du,$$
 (I.6b)

where  $G_0\left(f(v), r_1(u), \dots, r_4(u), u, v\right), \ G_i, i = 1, \dots, 4$  are given by

$$G_0\left(f(v), r_1(u), \dots, r_4(u), u, v\right) = \frac{1}{\sigma_u} \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \left(\frac{r_1(u)}{r_2(u)} Q\left(\frac{f(v)}{\sigma_w}\right) + \frac{r_3(u)}{r_4(u)} Q\left(\frac{-f(v)}{\sigma_w}\right)\right),$$
(I.7a)

$$G_1 = v\Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{f(v)}{\sigma_w}\right),\tag{I.7b}$$

$$G_2 = \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{f(v)}{\sigma_w}\right),\tag{I.7c}$$

$$G_3 = v\Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{-f(v)}{\sigma_w}\right),\tag{I.7d}$$

$$G_4 = \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{-f(v)}{\sigma_w}\right).$$
(I.7e)

Now we can apply the necessary condition in (A.27). To this end, we compute

$$F^{f}(f(v), H(v), v) = \frac{2\lambda}{\sigma_{v}} \Phi\left(\frac{v}{\sigma_{v}}\right) f(v), \qquad (I.8a)$$

$$F^{H}(f(v), H(v), v) = \frac{-v}{\sigma_{v}},$$
(I.8b)

$$G_0^f(f(v), r_1(u), \dots, r_4(u), u, v) = \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sigma_w \sigma_u \sqrt{2\pi}} \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \left(\frac{r_3(u)}{r_4(u)} - \frac{r_1(u)}{r_2(u)}\right), \quad (I.8c)$$

$$G_0^{r_1}(f(v), r_1(u), \dots, r_4(u), u, v) = \frac{1}{\sigma_u r_2(u)} \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{f(v)}{\sigma_w}\right),$$
(I.8d)

$$G_0^{r_2}(f(v), r_1(u), \dots, r_4(u), u, v) = \frac{-r_1(u)}{\sigma_u r_2(u)^2} \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{f(v)}{\sigma_w}\right), \tag{I.8e}$$

$$G_0^{r_3}(f(v), r_1(u), \dots, r_4(u), u, v) = \frac{1}{\sigma_u r_4(u)} \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{-f(v)}{\sigma_w}\right), \tag{I.8f}$$

$$G_0^{r_4}(f(v), r_1(u), \dots, r_4(u), u, v) = \frac{-r_3(u)}{\sigma_u r_4(u)^2} \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{-f(v)}{\sigma_w}\right),$$
(I.8g)

$$G_1^f(f(v), v, u) = v\Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \frac{-e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w},$$
(I.8h)

$$G_2^f(f(v), v, u) = \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \frac{-e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi\sigma_w}},\tag{I.8i}$$

$$G_3^f(f(v), v, u) = v\Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi\sigma_w}},\tag{I.8j}$$

$$G_4^f(f(v), v, u) = \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w}.$$
 (I.8k)

Substituting (I.8) in (A.27), the necessary condition in (A.26) is obtained as

$$\begin{aligned} \nabla L &= \frac{2\lambda}{\sigma_v} \Phi\left(\frac{v}{\sigma_v}\right) f(v) - \frac{v e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sigma_v \sigma_w \sigma_u \sqrt{2\pi}} \int \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \left(\frac{r_3(u)}{r_4(u)} - \frac{r_1(u)}{r_2(u)}\right) du \\ &- \int \int \frac{t}{\sigma_v} \left( v \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \frac{-e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w} \cdot \frac{1}{\sigma_u r_2(u)} \Phi\left(\frac{t}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{f(t)}{\sigma_w}\right) \right. \\ &+ \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \frac{-e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w} \cdot \frac{-r_1(u)}{\sigma_u r_2(u)^2} \Phi\left(\frac{t}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{f(t)}{\sigma_w}\right) \\ &+ v \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w} \cdot \frac{1}{\sigma_u r_4} \Phi\left(\frac{t}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{-f(t)}{\sigma_w}\right) \\ &+ \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{\sqrt{2\pi}\sigma_w} \cdot \frac{-r_3(u)}{\sigma_u r_4(u)^2} \Phi\left(\frac{t}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{-f(t)}{\sigma_w}\right) \right) dt du = 0. \end{aligned}$$

Rewriting (I.9), we have

$$\begin{aligned} &2\sqrt{2\pi}\sigma_{w}\sigma_{u}\lambda\Phi\left(\frac{v}{\sigma_{v}}\right)f(v)e^{\frac{f(v)^{2}}{2\sigma_{w}^{2}}} = v\int\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\left(\frac{r_{3}(u)}{r_{4}(u)} - \frac{r_{1}(u)}{r_{2}(u)}\right)du \\ &-v\int\int t\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\cdot\frac{1}{r_{2}(u)}\Phi\left(\frac{t}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)Q\left(\frac{f(t)}{\sigma_{w}}\right)dtdu \\ &+\int\int t\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\cdot\frac{r_{1}(u)}{r_{2}(u)^{2}}\Phi\left(\frac{t}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)Q\left(\frac{f(t)}{\sigma_{w}}\right)dtdu \\ &+v\int\int t\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\cdot\frac{1}{r_{4}(u)}\Phi\left(\frac{t}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)Q\left(\frac{-f(t)}{\sigma_{w}}\right)dtdu \\ &-\int\int t\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\cdot\frac{r_{3}(u)}{r_{4}(u)^{2}}\Phi\left(\frac{t}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)Q\left(\frac{-f(t)}{\sigma_{w}}\right)dtdu \end{aligned} \tag{I.10} \\ &=v\int\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\left(\frac{r_{3}(u)}{r_{4}(u)}-\frac{r_{1}(u)}{r_{2}(u)}\right)du-v\int\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\cdot\frac{r_{1}(u)}{r_{2}(u)}du \\ &+\int\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\cdot\frac{r_{1}(u)^{2}}{r_{2}(u)^{2}}du+v\int\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\cdot\frac{r_{3}(u)}{r_{4}(u)}du-\int\Phi\left(\frac{v}{\sigma_{v}},\frac{u}{\sigma_{u}}\right)\cdot\frac{r_{3}(u)^{2}}{r_{4}(u)^{2}}du. \end{aligned} \tag{I.11}$$

Finally, by some elementary manipulations the result in (4.12) is obtained.

## Appendix J

# Proof of Remark 4.4.2

For brevity we set that  $\sigma_v^2 = \sigma_w^2 = 1$  in the following proof. The results are valid for any values of  $\sigma_v^2$  and  $\sigma_w^2$ . We have

$$A(-v) = \int \Phi(u|-v) (g(0,u) - g(1,u)) du$$
(J.1a)
$$\int \int f(t \Phi(t|u) Q(-f(t)) dt - \int t \Phi(t|u) Q(f(t)) dt]$$

$$= \int \Phi(u|-v) \left[ \frac{\int t \Phi(t|u) Q(-f(t)) dt}{\int \Phi(t|u) Q(-f(t)) dt} - \frac{\int t \Phi(t|u) Q(f(t)) dt}{\int \Phi(t|u) Q(f(t)) dt} \right] du$$
(J.1b)  
=  $\int \Phi(-u|-v) \left[ -\int t \Phi(-t|-u) Q(f(t)) dt - \int t \Phi(-t|-u) Q(-f(t)) dt \right]_{du}$ 

$$= \int \Phi(-u|-v) \left[ \frac{-\int t\Phi(-t|-u)Q(f(t))dt}{\int \Phi(-t|-u)Q(f(t))dt} - \frac{-\int t\Phi(-t|-u)Q(-f(t))dt}{\int \Phi(-t|-u)Q(-f(t))dt} \right] du$$
(J.1c)

$$= \int \Phi\left(u|v\right) \left[\frac{-\int t\Phi\left(t|u\right)Q\left(f(t)\right)dt}{\int \Phi\left(t|u\right)Q\left(f(t)\right)dt} + \frac{\int t\Phi\left(t|u\right)Q\left(-f(t)\right)dt}{\int \Phi\left(t|u\right)Q\left(-f(t)\right)dt}\right]du$$
(J.1d)

$$= \int \Phi(u|v) (g(0,u) - g(1,u)) du$$
 (J.1e)

$$=A(v).$$
 (J.1f)

Similarly, we have

$$\begin{split} B(-v) &= \int \Phi\left(u|-v\right) \left(g(0,u)^2 - g(1,u)^2\right) du & (J.2a) \\ &= \int \Phi\left(u|-v\right) \left[ \left(\frac{\int t\Phi\left(t|u\right) Q\left(-f(t)\right) dt}{\int \Phi\left(t|u\right) Q\left(-f(t)\right) dt}\right)^2 - \left(\frac{\int t\Phi\left(t|u\right) Q\left(f(t)\right) dt}{\int \Phi\left(t|u\right) Q\left(f(t)\right) dt}\right)^2 \right] du \\ & (J.2b) \\ &= \int \Phi\left(-u|-v\right) \left[ \left(\frac{\int t\Phi\left(-t|-u\right) Q\left(f(t)\right) dt}{\int \Phi\left(-t|-u\right) Q\left(f(t)\right) dt}\right)^2 - \left(\frac{\int t\Phi\left(-t|-u\right) Q\left(-f(t)\right) dt}{\int \Phi\left(-t|-u\right) Q\left(-f(t)\right) dt}\right)^2 \right] du \\ & (J.2c) \end{split}$$

$$= \int \Phi\left(u|v\right) \left[ \left( \frac{\int t\Phi\left(t|u\right)Q\left(f(t)\right)dt}{\int \Phi\left(t|u\right)Q\left(f(t)\right)dt} \right)^2 - \left( \frac{\int t\Phi\left(t|u\right)Q\left(-f(t)\right)dt}{\int \Phi\left(t|u\right)Q\left(-f(t)\right)dt} \right)^2 \right] du$$
(J.2d)

$$= \int \Phi(u|v) \left( \hat{v}_1^2(u) - \hat{v}_0^2(u) \right) du$$
 (J.2e)

$$= -B(v). \tag{J.2f}$$

#### Appendix K

# Proof of Proposition 4.5.1

Assume that a decoder function g(Y,U) is given. By expanding the Lagrangian function  $L(f,g,\lambda)$  for the DOP we have

$$L(f,g,\lambda) = \epsilon(D) + \lambda \mathbb{E}[f(V)^2]$$
(K.1)

$$= \frac{1}{\sigma_u} \int \epsilon(D|U=u) \Phi\left(\frac{u}{\sigma_u}\right) du + \frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) f(v)^2 dv.$$
(K.2)

Expanding  $\epsilon(D|U=u) = \Pr\left(|V-\hat{V}|^2 \ge D|U=u\right)$  we have

$$\begin{aligned} \epsilon(D|U=u) &= \Pr(V \in I_0(u) \setminus I_1(u), Y=1|U=u) \\ &+ \Pr(V \in I_1(u) \setminus I_0(u), Y=0|U=u) \\ &+ \Pr(V \in (I_0(u) \cup I_1(u))^C, |\hat{V}-V|^2 \ge D|U=u) \\ &+ \Pr(V \in (I_0(u) \cap I_1(u)), |\hat{V}-V|^2 \ge D|U=u) \end{aligned}$$
(K.3a)
$$&= \frac{1}{\sigma_v} \int_{v \in I_0(u) \setminus I_1(u)} \Phi\left(\frac{v}{\sigma_v} \middle| \frac{u}{\sigma_u}\right) Q\left(\frac{f(v)}{\sigma_w}\right) dv \\ &+ \frac{1}{\sigma_v} \int_{v \in I_1(u) \setminus I_0(u)} \Phi\left(\frac{v}{\sigma_v} \middle| \frac{u}{\sigma_u}\right) Q\left(\frac{-f(v)}{\sigma_w}\right) dv \\ &+ \frac{1}{\sigma_v} \int_{v \in (I_0(u) \cup I_1(u))^C} \Phi\left(\frac{v}{\sigma_v} \middle| \frac{u}{\sigma_u}\right) dv, \end{aligned}$$
(K.3b)

where we used the fact that no outage occurs when  $V \in I_0(U) \cap I_1(U)$ . Substituting (K.3b) in (K.2), we can write the Lagrangian  $L(f, g, \lambda)$  as

$$L(f,g,\lambda) = \frac{1}{\sigma_v \sigma_u} \int \Phi\left(\frac{u}{\sigma_u}\right) \int \Phi\left(\frac{v}{\sigma_v} \middle| \frac{u}{\sigma_u}\right) G\left(u,v,f(v)\right) dv du + \frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) \lambda f^2(v) dv$$
(K.4)

$$= \frac{1}{\sigma_v} \int \Phi\left(\frac{v}{\sigma_v}\right) \left[ \int \frac{1}{\sigma_u} \Phi\left(\frac{u}{\sigma_u} \middle| \frac{v}{\sigma_v}\right) G\left(u, v, f(v)\right) du + \lambda f^2(v) \right] dv, \quad (K.5)$$

with G(u, v, f(v)) defined as

$$G(u, v, f(v)) \triangleq \begin{cases} Q\left(\frac{f(v)}{\sigma_w}\right) & v \in (I_0(u) \setminus I_1(u)), \\ Q\left(\frac{-f(v)}{\sigma_w}\right) & v \in (I_1(u) \setminus I_0(u)), \\ 1 & v \in (I_0(u) \cup I_1(u))^C, \\ 0 & v \in (I_0(u) \cap I_1(u)). \end{cases}$$
(K.6)

Note that (K.5) is in the form of (A.26) with F(f, H(v), v) and H(v) given by

$$F(f, H(v), v) = \frac{1}{\sigma_v} \Phi\left(\frac{v}{\sigma_v}\right) \cdot \left(H(v) + \lambda f^2(v)\right), \qquad (K.7a)$$

$$H(v) = \int G_0\left(f(v), u, v\right) du, \qquad (K.7b)$$

respectively, where  $G_0(f(v), u, v)$  is given by

$$G_0(f(v), u, v) = \frac{1}{\sigma_u} \Phi\left(\frac{u}{\sigma_u} \middle| \frac{v}{\sigma_v}\right) G(u, v, f(v)).$$
(K.8)

Applying the necessary condition in (A.27) for the optimal solution, for different terms in (A.27) we have

$$F^{f}(f(v), H(v), v) = \frac{2\lambda}{\sigma_{v}} \Phi\left(\frac{v}{\sigma_{v}}\right) f(v), \qquad (K.9a)$$

$$F^{H}(f(v), H(v), v) = \frac{1}{\sigma_{v}} \Phi\left(\frac{v}{\sigma_{v}}\right),$$
(K.9b)

$$G_0^f(f(v), u, v) = \frac{1}{\sigma_u} \Phi\left(\frac{u}{\sigma_u} \Big| \frac{v}{\sigma_v}\right) G^f\left(u, v, f(v)\right),$$
(K.9c)

where  $G^{f}(u, v, f(v))$  is obtained as

$$G^{f}(u, v, f(v)) = \begin{cases} \frac{-1}{\sqrt{2\pi}} e^{-\frac{f(v)^{2}}{2\sigma_{w}^{2}}} & v \in (I_{0}(u) \setminus I_{1}(u)) \\ \frac{1}{\sqrt{2\pi}} e^{-\frac{f(v)^{2}}{2\sigma_{w}^{2}}} & v \in (I_{1}(u) \setminus I_{0}(u)) \\ 0 & v \in (I_{0}(u) \cap I_{1}(u)) \text{ or } v \in (I_{0}(u) \cup I_{1}(u))^{C} \end{cases}$$
(K.10)

Therefore, (A.27) can be written as

$$\nabla L = \frac{1}{\sigma_v} \Phi\left(\frac{v}{\sigma_v}\right) \left(2\lambda f(v) + \frac{1}{\sigma_u} \int \Phi\left(\frac{u}{\sigma_u} \left|\frac{v}{\sigma_v}\right) G^f\left(u, v, f(v)\right) du\right) = 0.$$
(K.11)

Note that the integration in (K.11) is over the side information u. In the following, we aim at identifying the boundaries of u, such that, for a given source output v we have  $G^{f}(u, v, f(v)) \neq 0$ . To do so, we characterize the intervals as

$$I_{0}(u) \setminus I_{1}(u) = (b_{0l}(u), b_{0r}(u)),$$
  

$$I_{1}(u) \setminus I_{0}(u) = (b_{1l}(u), b_{1r}(u)).$$
(K.12)

Note that for the other intervals in (K.10) we have  $G^f(u, v, f(v)) = 0$ . For a given side information realization u, g(0, u) and g(1, u) are two points. Hence, depending on the condition that g(0, u) is equal to, less than, or greater than g(1, u), we have different situations for  $I_0(u)$  and  $I_1(u)$  in (K.10).

Case 1) g(0, u) = g(1, u): In this case the two intervals  $I_0(u)$  and  $I_1(u)$  overlap completely, and therefore,  $I_0(u) \setminus I_1(u)$  and  $I_1(u) \setminus I_0(u)$  are both empty sets.

Case 2) g(0,u) > g(1,u): In this case  $b_{0l}(u)$ ,  $b_{0r}(u)$ ,  $b_{1r}(u)$  and  $b_{1l}(u)$  are obtained as

$$b_{0r}(u) = g(0, u) + \sqrt{D},$$
  

$$b_{0l}(u) = \max\left\{g(1, u) + \sqrt{D}, g(0, u) - \sqrt{D}\right\},$$
  

$$b_{1r}(u) = \min\left\{g(1, u) + \sqrt{D}, g(0, u) - \sqrt{D}\right\},$$
  

$$b_{1l}(u) = g(1, u) - \sqrt{D}.$$
(K.13)

Case 3) g(0,u) < g(1,u): In this case  $b_{0l}(u)$ ,  $b_{0r}(u)$ ,  $b_{1r}(u)$  and  $b_{1l}(u)$  are obtained as

$$b_{0r}(u) = \min\left\{g(0, u) + \sqrt{D}, g(1, u) - \sqrt{D}\right\},\$$
$$b_{0l}(u) = g(0, u) - \sqrt{D},$$
  

$$b_{1r}(u) = g(1, u) + \sqrt{D},$$
  

$$b_{1l}(u) = \max\left\{g(1, u) - \sqrt{D}, g(0, u) + \sqrt{D}\right\}.$$
(K.14)

It can be easily verified that for a given source output v, the side information range corresponding to  $G^f(u, v, f(v)) \neq 0$  can be obtained as  $S_{0\backslash 1}(v) \cup S_{1\backslash 0}(v)$  where  $S_{0\backslash 1}(v)$ and  $S_{1\backslash 0}(v)$  are defined as

$$S_{0\setminus 1}(v) \triangleq \{u: \ b_{0r}(u) \ge v \ge b_{0l}(u)\},$$
  

$$S_{1\setminus 0}(v) \triangleq \{u: \ b_{1r}(u) \ge v \ge b_{1l}(u) \le v\}.$$
(K.15)

Finally, we can simplify (K.11) as

$$\nabla L = \frac{1}{\sigma_v} \Phi\left(\frac{v}{\sigma_v}\right) \left[ \int_{u \in S_{0\backslash 1}(v)} \frac{-1}{\sqrt{2\pi\sigma_u}} e^{-\frac{f(v)^2}{2\sigma_w^2}} \Phi\left(\frac{u}{\sigma_u} \middle| \frac{v}{\sigma_v}\right) du \right] + \int_{u \in S_{1\backslash 0}(v)} \frac{1}{\sqrt{2\pi\sigma_u}} e^{-\frac{f(v)^2}{2\sigma_w^2}} \Phi\left(\frac{u}{\sigma_u} \middle| \frac{v}{\sigma_v}\right) du + 2\lambda f(v) \right]$$
(K.16)
$$= \frac{1}{\sigma_v} \Phi\left(\frac{v}{\sigma_v}\right) \left[ \frac{-1}{\sqrt{2\pi\sigma_u}} e^{-\frac{f(v)^2}{2\sigma_w^2}} \int_{u \in S_{0\backslash 1}(v)} \Phi\left(\frac{u}{\sigma_u} \middle| \frac{v}{\sigma_v}\right) du + 2\lambda f(v) \right] + \frac{1}{\sqrt{2\pi\sigma_u}} e^{-\frac{f(v)^2}{2\sigma_w^2}} \int_{u \in S_{0\backslash 1}(v)} \Phi\left(\frac{u}{\sigma_u} \middle| \frac{v}{\sigma_v}\right) du + 2\lambda f(v) \right].$$
(K.17)

Imposing (K.17) to be zero we have

$$f(v) = \frac{e^{-\frac{f(v)^2}{2\sigma_w^2}}}{2\lambda\sqrt{2\pi}} \left( \Pr\left(U \in S_{0\backslash 1}(v)\right) - \Pr\left(U \in S_{1\backslash 0}(v)\right) \right).$$
(K.18)

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#### Appendix L

### Proof of Proposition 4.5.2

The optimal decoder functions, i.e., g(0, u) and g(1, u) can be obtained as

$$g(0, u) = \arg\min_{\hat{v}} \Pr\left(|V - \hat{v}|^2 \ge D|U = u, Y = 0\right)$$
(L.1)

$$= \underset{\hat{v}}{\operatorname{arg max}} \operatorname{Pr}\left(|V - \hat{v}|^2 < D|U = u, Y = 0\right)$$
(L.2)

$$= \underset{\hat{v}}{\arg \max} \frac{1}{\sigma_v \sigma_u} \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} p_{V|U,Y}(t|u, Y=0) dt$$
(L.3)

$$= \arg \max_{\hat{v}} \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} \Phi\left(\frac{t}{\sigma_{v}} \Big| \frac{u}{\sigma_{u}}\right) Q\left(\frac{-f(t)}{\sigma_{w}}\right) dt.$$
(L.4)

We note that, since the mapping f(v) is given, it could be possible that for some encoder mapping f and side information realization u, more than one output is obtained in (L.4). From the DOP point of view, there is no difference in choosing either of these points. Therefore, we have

$$g^{*}(0,u) \in \arg\max_{\hat{v}} \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} \Phi\left(\frac{t}{\sigma_{v}},\frac{u}{\sigma_{u}}\right) Q\left(\frac{-f(t)}{\sigma_{w}}\right) dt, \qquad (L.5)$$

and similarly for g(1, u), we have

$$g^*(1,u) \in \arg\max_{\hat{v}} \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} \Phi\left(\frac{t}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{f(t)}{\sigma_w}\right) dt.$$
(L.6)

### Appendix M

## Proof of Remark 4.5.2

In the low SNR regime the encoder mapping function can be approximated as an all zero function. Hence, the DOP can be expanded as

$$\epsilon(D) = 1 - \Pr(|V - \hat{V}|^2 < D) \tag{M.1a}$$

$$= 1 - \frac{1}{\sigma_u} \int \Pr(|V - \hat{V}|^2 < D|U = u) \Phi\left(\frac{u}{\sigma_u}\right) du$$
 (M.1b)

$$= 1 - \frac{1}{\sigma_u} \int \Phi\left(\frac{u}{\sigma_u}\right) \Pr\left(\left|V - \frac{r\sigma_v}{\sigma_u}u\right|^2 < D|U = u\right) du \tag{M.1c}$$

$$=1-\frac{1}{\sigma_u\sigma_v}\int \Phi\left(\frac{u}{\sigma_u}\right) \int \int \frac{\sigma_u}{\sigma_v} \int \frac{du}{dv} \Phi\left(\frac{v}{\sigma_v} \left|\frac{u}{\sigma_u}\right|\right) du$$
(M.1d)

$$= 1 - \frac{1}{\sigma_u} \int \Phi\left(\frac{u}{\sigma_u}\right) \left(Q\left(\frac{-\sqrt{D}}{\sigma_v\sqrt{1-r^2}}\right) - Q\left(\frac{\sqrt{D}}{\sigma_v\sqrt{1-r^2}}\right)\right) du \qquad (M.1e)$$

$$=2Q\left(\frac{\sqrt{D}}{\sigma_v\sqrt{1-r^2}}\right).\tag{M.1f}$$

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### Appendix N

# Proof of Proposition 4.5.3

Assuming that the side information is available at both the encoder and the decoder, the decoder can be obtained as

$$g(y,u) = \arg\min_{\hat{v}} \Pr\left(|V - \hat{v}|^2 \ge D|Y = y, U = u\right) + \lambda \mathbb{E}[\tilde{f}(T)^2]$$
(N.1a)  
$$= \arg\max_{\hat{v}} \int_{\hat{v} - \sqrt{D}}^{\hat{v} + \sqrt{D}} \Phi\left(\frac{v}{\sigma_v}, \frac{u}{\sigma_u}\right) Q\left(\frac{(-1)^{y+1}\tilde{f}\left(v - \frac{r\sigma_v u}{\sigma_u}\right)}{\sigma_w}\right) dv - \lambda \mathbb{E}[\tilde{f}(T)^2]$$
(N.1b)

$$= \underset{\hat{v}}{\arg\max} \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} \Phi\left(\frac{v}{\sigma_{v}} \left|\frac{u}{\sigma_{u}}\right) Q\left(\frac{(-1)^{y+1}\tilde{f}\left(v - \frac{r\sigma_{v}u}{\sigma_{u}}\right)}{\sigma_{w}}\right) dv - \lambda \mathbb{E}[\tilde{f}(T)^{2}]$$
(N.1c)

$$= \arg\max_{\hat{v}} \int_{\hat{v}-\sqrt{D}}^{\hat{v}+\sqrt{D}} e^{-\frac{\left(v-\frac{r\sigma_v u}{\sigma_u}\right)^2}{2\sigma_v^2(1-r^2)}} Q\left(\frac{(-1)^{y+1}\tilde{f}\left(v-\frac{r\sigma_v u}{\sigma_u}\right)}{\sigma_w}\right) dv - \lambda \mathbb{E}[\tilde{f}(T)^2]$$
(N.1d)

$$= \arg\max_{\hat{t}} \int_{\hat{t}-\sqrt{D}-\frac{r\sigma_v u}{\sigma_u}}^{\hat{t}+\sqrt{D}-\frac{r\sigma_v u}{\sigma_u}} e^{-\frac{t^2}{2\sigma_v^2(1-r^2)}} Q\left(\frac{(-1)^{y+1}\tilde{f}(t)}{\sigma_w}\right) dt - \lambda \mathbb{E}[\tilde{f}(T)^2]$$
(N.1e)

$$= \frac{r\sigma_v u}{\sigma_u} + \arg\max_{\hat{t}} \int_{\hat{t}-\sqrt{D}}^{\hat{t}+\sqrt{D}} e^{-\frac{t^2}{2\sigma_v^2(1-r^2)}} Q\left(\frac{(-1)^{y+1}\tilde{f}(t)}{\sigma_w}\right) dt - \lambda \mathbb{E}[\tilde{f}(T)^2] \quad (N.1f)$$

$$=\frac{r\sigma_v u}{\sigma_u} + \hat{t}_Y,\tag{N.1g}$$

Also, the DOP can be expanded as

$$\epsilon(D) = \frac{1}{\sigma_v} \int \Phi\left(\frac{u}{\sigma_u}\right) \Pr\left(|V - \hat{V}|^2 \ge D|U = u\right) du \tag{N.2a}$$

$$= \frac{1}{\sigma_v} \int \Phi\left(\frac{u}{\sigma_u}\right) \Pr\left(\left|V - \frac{r\sigma_v u}{\sigma_u} - \hat{t}_Y\right|^2 \ge D \left|U = u\right) du$$
(N.2b)

$$= \frac{1}{\sigma_v} \int \Phi\left(\frac{u}{\sigma_u}\right) \left[ \Pr\left(V \in (I_0(u) \cup I_1(u))^C\right) + \right]$$
(N.2c)

$$\Pr\left(V \in I_0(u) \setminus I_1(u), \hat{t}_Y = t_1\right) + \Pr\left(V \in I_1(u) \setminus I_0(u), \hat{t}_Y = t_0\right) \left] du \quad (N.2d)$$

$$= \frac{1}{\sigma_u \sigma_v} \int \Phi\left(\frac{u}{\sigma_u}\right) \left| \int_{(I_0(u) \cup I_1(u))^C} \Phi\left(\frac{v}{\sigma_v} \middle| \frac{u}{\sigma_u}\right) dv + \right|$$
(N.2e)

$$Q\left(\frac{t}{\sigma_w}\right)\left(\int\limits_{I_1(u)\setminus I_0(u)} \Phi\left(\frac{v}{\sigma_v}\Big|\frac{u}{\sigma_u}\right)dv + \int\limits_{I_0(u)\setminus I_1(u)} \Phi\left(\frac{v}{\sigma_v}\Big|\frac{u}{\sigma_u}\right)dv\right)\right], \quad (N.2f)$$

where for different intervals we have

$$I_y(u) = \left\{ v : \left( v - \frac{r\sigma_v u}{\sigma_u} - \hat{t}_y \right)^2 \le D \right\} \ y = 0, 1.$$
 (N.3)

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