Orientably regular maps with given exponent group

Journal Item

Orientably regular maps with given exponent group

Marston D. E. Conder and Jozef Širáň

Abstract

We prove that for every integer $d \geqslant 3$ and every group U of units mod d, there exists an orientably regular map of valency d with exponent group U.

1. *Introduction*

An *orientably regular map* M is a 2-cell embedding of a connected graph in an orientable surface, such that the group of all orientation-preserving automorphisms Aut^+M of the embedding acts as regularly (sharply transitively) on the set of arcs of the graph. It follows that every vertex of M has the same valency, say d, and every face of M is bounded by a closed walk of the same length, say m.

If e is an arc at any vertex v of M, then regularity implies that Aut^+M contains an involution x acting like a 180-degree rotation of M about the centre of e, and an element y of order d acting like a d-fold rotation of M about v. Then by connectivity, the group Aut^+M is generated by x and y, and admits a presentation of the form $\mathrm{Aut}^+M = \langle x, y \mid x^2 = y^d = (xy)^m = \cdots = 1\rangle$. The pair (d, m) is called the *type* of the map. Conversely, given any generating pair (x, y) for a group G with the above form, one may construct an orientably regular map M with $\mathrm{Aut}^+M = G$ by taking edges, vertices and faces of M as the (right) cosets in G of the subgroups $\langle x\rangle$, $\langle y\rangle$ and $\langle xy\rangle$, respectively, and with incidence given by non-empty intersection of cosets. (Also the arcs may be taken as the elements of G.) Thus, orientably regular maps of valency d and face length m may be identified with 2-generator group presentations of the form $\langle x, y \mid x^2 = y^d = (xy)^m = \cdots = 1\rangle$.

Fundamentals of the theory of maps and orientably regular maps are explained in [**8**], some deep connections between such maps, Riemann surfaces and Galois groups are described in detail in [**9**], and a recent survey containing a large number of facts about regular maps is given in [**11**].

Next, let M and $G = \mathrm{Aut}^+M = \langle x, y\rangle$ be as above. An integer j relatively prime to d is said to be an *exponent* of M if the assignment $(x, y) \mapsto (x, y^j)$ extends to an automorphism of G. Algebraically, this means that (x, y) and (x, y^j) satisfy the relations as each other, while from the point of view of maps, it means that if a new map M^j is constructed from M by replacing the clockwise local cyclic order π_v of arcs at each vertex v by ${\pi_v}^j$, then resulting map M^j is isomorphic to M. Orientably regular maps admitting the exponent -1 are isomorphic to their mirror image, and are therefore called *reflexible*.

The collection of all exponents of M forms a subgroup of the group of units Z_d^*, and is called the *exponent group* of M. The notion of an exponent was introduced in [**10**], with applications in the classification of orientably regular maps with a given underlying graph. Previously, the

---

Received 25 February 2016; revised 7 July 2016.

2010 *Mathematics Subject Classification* 57M60 (primary), 05E18, 20B25, 20F05, 57M15 (secondary).

The first author was supported by New Zealand's Marsden Fund (grant no. UOA1323), and acknowledges the use of the Magma system [3] to discover and test the properties of various examples. The second author acknowledges support from the APVV Research Grants 0136/12 and 015-0220, and the VEGA Research Grants 1/0007/14 and 1/0026/16.

mapping $M \to M^j$ (even in the case when the two maps may not be isomorphic) was known as a *hole operator*, and studied by Wilson [14], but this mapping has also been attributed to Coxeter.

For the exponents of an orientably regular map of given valency d, there are two 'extremes': one where the exponent group is trivial, or consists only of 1 and -1, and the other where the map admits the 'full' exponent group $\mathbb{Z}_d^*$.

In [2], it was shown that for every $d \geqslant 3$ there are infinitely many finite orientably regular maps of valency d with trivial exponent group. This was done with the help of a method that allows one to forbid the creation of new automorphisms in lifted maps, but unfortunately the method offers no control over the face length. Also it was proved in [12] using residual finiteness of triangle groups that for every pair of positive integers d and m with $1/d + 1/m \leqslant 1/2$, there exist infinitely many finite orientably regular and reflexible maps of type (d, m) that admit no exponents other than 1 and -1.

At the other end of the spectrum, it was shown in [13] that for every integer $d \geqslant 3$ there exist infinitely many finite orientably regular maps with exponent group $\mathbb{Z}_d^*$. Again this was achieved using residual finiteness of triangle groups, but this time losing control over the face length of resulting maps. Such maps were called 'kaleidoscopic' in [1], where a covering construction was given for a kaleidoscopic d-valent regular map invariant also under duality and Petrie duality, for every even d. A different construction for such 'super-symmetric' d-valent maps was given for an infinite set of odd values of d in [6].

In this paper, we deal with the 'intermediate' cases, by considering arbitrary subgroups of the group of units modulo the valency d. We prove that for every $d \geqslant 3$ and every given subgroup U of $\mathbb{Z}_d^*$, there exist infinitely many finite orientably regular maps of valency d with exponent group *equal* to (and not just isomorphic to) U.

2. *The main result*

THEOREM 1. *For every $d \geqslant 3$ and every subgroup U of $\mathbb{Z}_d^*$, there are infinitely many finite orientably regular maps of degree d with exponent group equal to U.*

Proof. Let G be the free product $\mathbb{Z}_2 * \mathbb{Z}_d$ of the cyclic groups of order 2 and $d \geqslant 3$, with presentation $\langle X, Y \mid X^2 = Y^d = 1\rangle$, and let $D = G'$ be the derived subgroup of G, of index $2d$ in G, with quotient $G/D \cong \mathbb{Z}_2 \times \mathbb{Z}_d$. By Reidemeister–Schreier theory [5], the group D is free of rank $d-1$, generated by the commutators $W_j = [X, Y^j]$ for $j \in \{1, 2, \ldots, d-1\}$.

We will construct for any given subgroup U of $\mathbb{Z}_d^*$ an infinite family of quotients of G that give rise to orientably regular maps of degree d with exponent group U.

For any prime p, let $N_p = D'D^{(p)}$ be the subgroup of D generated by the commutators and pth powers of all elements of D. This subgroup is characteristic in D and hence normal in G, and the quotient D/N_p is isomorphic to the direct product $\mathbb{Z}_p^{d-1}$ of $d-1$ copies of $\mathbb{Z}_p$. Also G/N_p is an extension of $D/N_p \cong \mathbb{Z}_p^{d-1}$ by $(G/N_p)/(D/N_p) \cong G/D \cong \mathbb{Z}_2 \times \mathbb{Z}_d$, and hence G/N_p has order $2dp^{d-1}$.

Next, for any $u \in \mathbb{Z}_d^*$, let k_u be the automorphism of G that takes the generating pair (X, Y) to the generating pair (X, Y^u). Note that this permutes the generators $W_j = [X, Y^j]$ of D among themselves, and therefore preserves D, and its characteristic subgroup N_p, and so induces an automorphism h_u of $G_p = G/N_p$, with $(Ng)^{h_u} = N(g^{k_u})$ for all $g \in G$.

Now, let U be any subgroup of $\mathbb{Z}_d^*$. Then, $K_U = \{k_u : u \in U\}$ and $H_U = \{h_u : u \in U\}$ are groups of automorphisms of G and G_p (respectively), both isomorphic to U.

We will show that if the prime p is congruent to 1 mod d, then there exists a normal subgroup L_U of $G_p = G/N_p$ contained in D/N_p such that L_U is preserved by H_U, and furthermore, that L_U can be chosen so that it is not preserved by h_r for any $r \in \mathbb{Z}_d^* \setminus U$. Under these circumstances, the quotient G_p/L_U determines a finite orientably regular map M of valency d with exponent group containing U, and then finally, we will show that the exponent group of M is *equal* to U. We break this up into three steps below.

Step 1. Let x and y be the images of X and Y under the natural quotient homomorphism from G to $G/N_p = G_p$, and let $w_j = [x, y^j] = xy^{-j}xy^j$, which is the image of $W_j = [X, Y^j]$, for $j \in \{1, 2, \dots, d-1\}$. Then, these w_j are elements of the elementary abelian p-group $V_p = D/N_p \cong \mathrm{Z}_p^{d-1}$, and so commute with each other. Moreover, it is easy to see that $xw_jx = y^{-j}xy^jx = w_j^{-1}$ and $y^{-1}w_jy = y^{-1}xy^{-j}xy^{j+1} = y^{-1}xyxxy^{-(j+1)}xy^{j+1} = w_1^{-1}w_{j+1}$, for all $j \in \{1, 2, \dots, d-1\}$, if we define also $w_d = [x, y^d] = 1$.

Next, suppose $p \equiv 1 \bmod d$, and let t be any non-trivial dth root of 1 mod p, so that $1 + t + t^2 + \cdots + t^{d-1} \equiv 0 \bmod p$. Define

$$v_t = w_1^t w_2^{t^2} \cdots w_{d-2}^{t^{d-2}} w_{d-1}^{t^{d-1}},$$

which is an element of the abelian p-group $V_p = D/N$. Conjugation by x inverts v_t, while

$$\begin{aligned} y^{-1}v_ty &= (y^{-1}w_1y)^t(y^{-1}w_2y)^{t^2} \cdots (y^{-1}w_{d-2}y)^{t^{d-2}}(y^{-1}w_{d-1}y)^{t^{d-1}} \\ &= (w_1^{-1}w_2)^t(w_1^{-1}w_3)^{t^2} \cdots (w_1^{-1}w_{d-1})^{t^{d-2}}(w_1^{-1})^{t^{d-1}} \\ &= w_1^{-(t+t^2+\cdots+t^{d-2}+t^{d-1})} w_2^t w_3^{t^2} \cdots w_{d-2}^{t^{d-3}} w_{d-1}^{t^{d-2}} \\ &= w_1 w_2^t w_3^{t^2} \cdots w_{d-2}^{t^{d-3}} w_{d-1}^{t^{d-2}} \\ &= (v_t)^{t^{-1}}. \end{aligned}$$

It follows that the cyclic subgroup L_t of $V_p = D/N_p$ generated by v_t is normal in G_p.

Now, take $L_U = \langle L_t^{h_u} : u \in U \rangle$. Since L_t is a normal subgroup of G_p, the image $L_t^{h_u}$ of L_t under each automorphism h_u is also a normal subgroup of G_p, and hence L_U is normal in G_p. Moreover, L_U is clearly preserved by H_U, as required.

Step 2. Suppose further that t is a *primitive* dth root of 1 mod p, and for each $j \in \mathrm{Z}_d^*$, define the element $v_t^{(j)}$ of V_p by

$$v_t^{(j)} = h_{j^{-1}}(v_t) = \prod_{i \in \mathrm{Z}_d^*} h_{j^{-1}}\left(w_i^{\,t^i}\right) = \prod_{i \in \mathrm{Z}_d^*} (w_{j^{-1}i})^{t^i} = \prod_{\ell \in \mathrm{Z}_d^*} w_\ell^{(t^j)^\ell}.$$

We claim that these $\phi(d) = |\mathrm{Z}_d^*|$ elements $v_t^{(j)}$ generate a subgroup of order $p^{\phi(d)}$ in V_p, or equivalently, that they are linearly independent over Z_p when V_p is considered as a vector space over Z_p of dimension $d-1$. To see this, if we take the set $\{w_1, w_2, \dots, w_{d-1}\}$ as a basis for V_p, and write any element $w_1^{a_1}w_2^{a_2} \cdots w_{d-1}^{a_{d-1}}$ of V_p as a $(d-1)$-tuple $(a_1, a_2, \dots, a_{d-1})$, then by its definition above, $v_t^{(j)}$ can be written as the $(d-1)$-tuple $(t^j, t^{2j}, \dots, t^{(d-1)j})$. Hence, the set $\{v_t^{(j)} : j \in \mathrm{Z}_d^*\}$ can be represented by a $\phi(d) \times (d-1)$ sub-matrix of the Vandermonde matrix

$$\begin{pmatrix} t & t^2 & t^3 & \dots & t^{d-1} \\ t^2 & t^4 & t^6 & \dots & t^{2(d-1)} \\ t^3 & t^6 & t^9 & \dots & t^{3(d-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t^{d-1} & t^{2(d-1)} & t^{3(d-1)} & \dots & t^{(d-1)(d-1)} \end{pmatrix}.$$

This matrix has determinant $\prod_{1 \leqslant i < j \leqslant d-1}(t^j - t^i)$, which is non-zero in Z_p since t is a primitive dth root of 1 mod p, and it follows that for any subset S of Z_d^*, the rows with first entry t^j with $j \in S$ are linearly independent over Z_p. In particular, taking $S = \mathrm{Z}_d^*$, we see the above claim is true.

But also this shows that $h_r(L_U) \neq L_U$ for any $r \in \mathbb{Z}_d^* \setminus U$, because if $h_r(L_U) = L_U$, then $L_U = h_{r^{-1}}(L_U)$ and so the vector corresponding to $v_t^{(r)} = h_{r^{-1}}(v_t)$ is a linear combination of the vectors corresponding to the elements $v_t^{(u)}$ for $u \in U$, which is impossible.

Step 3. It remains to show that the exponent group of the orientable-regular map M arising from the quotient G_p/L_U of G is equal to U. By Step 1, we know that this exponent group contains U. To prove the reverse inclusion, suppose that j is any exponent of this map M. Then also j^{-1} is an exponent of M, and hence there exists an automorphism θ of G_p/L_U that fixes the element xL_U and takes yL_U to $y^{j^{-1}}L_U$. But now $v_t \in L_t \subseteq L_U$, so the coset v_tL_U is trivial in G_p/L_U, and it follows that the coset containing $v_t^{(j)} = h_{j^{-1}}(v_t)$ is trivial as well. Thus $v_t^{(j)}$ lies in L_U, and by Step 2, we deduce that $j \in U$.

This completes the proof.

3. *Concluding remarks*

The method we have used does not enable control over the face length of the resulting maps. This is no accident, as it is *not* true that there exist orientably regular maps of given type (d, m) with $1/d + 1/m \leqslant \frac{1}{2}$ and having a given exponent group. For example, in the case of triangulations (with $m = 3$), it was shown in [13] that an orientably regular map of type $(d, 3)$ with valency $d \equiv \pm 1 \bmod 6$ cannot have more than $\phi(d)/2$ exponents, and that if d is a prime such that $d \equiv -1 \bmod 8$ and $(d-1)/2$ is also prime, then such a triangulation cannot have exponents other than ± 1.

Finally, for completeness, we mention some interesting connections with the case where the exponent group U does not contain -1. Orientably regular maps with this property are known as *chiral*. In [4], it was shown by a direct permutation construction that for every pair (d, m) such that $1/d + 1/m \leqslant \frac{1}{2}$, there exist infinitely many finite orientably regular but chiral maps of type (d, m). The same thing was proved in [7] by a different method, with the help of holomorphic differentials.

References

1. D. Archdeacon, M. Conder and J. Širáň, 'Kaleidoscopic regular maps with trinity symmetry', *Trans. Amer. Math. Soc.* 366 (2014) 4491–4512.
2. D. Archdeacon, P. Gvozdjak and J. Širáň, 'Constructing and forbidding automorphisms in lifted maps', *Math. Slovaca* 47 (1997) 113–129.
3. W. Bosma, J. Cannon and C. Playoust, 'The Magma algebra system I: The user language', *J. Symbolic Comput.* 24 (1997) 235–265.
4. M. Conder, V. Hucíková, R. Nedela and J. Širáň, 'Chiral maps of given hyperbolic type', *Bull. Lond. Math. Soc.* 48 (2016) 38–52.
5. D. L. Johnson, *Presentations of groups*, 2nd edn (Cambridge University Press, 1997).
6. G. A. Jones, 'Combinatorial categories and permutation groups', Preprint, 2014.
7. G. A. Jones, 'Chiral covers of hypermaps', *Ars Math. Contemp.* 8 (2015) 425–431.
8. G. A. Jones and D. Singerman, 'Theory of maps on orientable surfaces', *Proc. Lond. Math. Soc.* 37 (1978) 273–307.
9. G. A. Jones and D. Singerman, 'Belyĭ functions, hypermaps, and Galois groups', *Bull. Lond. Math. Soc.* 28 (1996) 561–590.
10. R. Nedela and M. Škoviera, 'Exponents of orientable maps', *Proc. Lond. Math. Soc.* 75 (1997) 1–31.
11. J. Širáň, 'How symmetric can maps on surfaces be?', *Surveys in combinatorics*, London Mathematical Society, Lecture Note Series 409 (Cambridge University Press, Cambridge, 2013) 161–238.
12. J. Širáň, Ľ. Staneková and M. Olejár, 'Reflexible regular maps with no non-trivial exponents from residual finiteness', *Glasgow Math. J.* 53 (2011) 437–441.
13. J. Širáň and Y. Wang, 'Maps with highest level of symmetry that are even more symmetric than other such maps: regular maps with largest exponent groups', *Contemp. Math.* 531 (2010) 95–102.
14. S. Wilson, 'Operators over regular maps', *Pacific J. Math.* 81 (1979) 559–568.

Marston D. E. Conder
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland 1142
New Zealand

m.conder@auckland.ac.nz

Jozef Širáň
Department of Mathematics and Statistics
Open University
Milton Keynes
MK7 6AA
United Kingdom
and
Slovak University of Technology
Bratislava
Slovakia

jozef.siran@open.ac.uk