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Identical twin Steiner triple systems

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Abstract

Two Steiner triple systems, each containing precisely one Pasch configuration which, when traded, switches one system to the other, are called twin Steiner triple systems. If the two systems are isomorphic the systems are called identical twins. Hitherto, identical twins were only known for orders 21, 27 and 33. In this paper we construct infinite families of identical twin Steiner triple systems.

1 Introduction

A Steiner triple system of order v, STS(v), is an ordered pair $\mathcal{S} = (V, \mathcal{B})$, where V is a v-element set (the *points*) and $\mathcal{B}$ is a set of $v(v-1)/6$ triples from V (the *blocks*), such that each pair from V appears in precisely one block. A necessary and sufficient condition for the existence of an STS(v) is that $v \equiv 1$ or 3 (mod 6) [7], and these values of v are said to be *admissible*. Our constructions below also involve *group divisible designs*, GDDs. A 3-GDD of type w^t is a triple $(V, \mathcal{G}, \mathcal{B})$, where V is a wt-element set (the *points*), $\mathcal{G}$ is a partition of V into t parts (the *groups*) each of size w, and $\mathcal{B}$ is a set of triples from V (the *blocks*), such that each pair from V appears in precisely one block or one group, but not both. Two Steiner triple systems (or two 3-GDDs) are said to be isomorphic if there is a bijection ϕ from the point set of one system to the point set of the other that maps blocks to blocks (in the 3-GDD case ϕ must also map groups to groups), and then ϕ itself is called an isomorphism. An automorphism is an isomorphism from an STS (or 3-GDD) to itself. In each case the set of all automorphisms forms a group.

A *Pasch configuration* or *quadrilateral* is a set of four triples on six distinct points having the form $P = \{\{a,b,c\},\{a,d,e\},\{b,d,f\},\{c,e,f\}\}$. For each admissible $v \neq 7, 13$, it is known that there is an STS(v) containing no Pasch configurations [6]. Such a system is called *anti-Pasch* or *quadrilateral-free* and is denoted by QFSTS(v). A 3-GDD whose blocks do not contain any Pasch configurations is similarly described.

Given a Pasch configuration P of the form above, a different Pasch configuration P' can be produced by exchanging a with f, or equivalently b with e, or c with d. Note that the same effect is achieved by applying all three transpositions $(a, f)(b, e)(c, d)$ simultaneously. Such an operation is called a *Pasch trade* or a *Pasch switch*. If P lies in a Steiner triple system $\mathcal{S}$, then replacing the blocks of P by those of P' results in a different Steiner triple system $\mathcal{S}'$. If each of $\mathcal{S}$ and $\mathcal{S}'$ contains just a single Pasch configuration which, when traded, switches one system to the other, then $\mathcal{S}$ and $\mathcal{S}'$ are called twin Steiner triple systems. This terminology was introduced by J. P. Murphy in his Ph.D. thesis [8] where twin systems were constructed for orders $v = 19, 21, 25, 27, 31$ and 33 (and there are no twins of orders $v < 19$). It was also shown therein (see also [5]) that if there exists a QFSTS(u), a pair of twin STS(u)s, and a QFSTS(v), then there exists a pair of twin STS(uv)s. As a consequence, it is known that there are infinite linear classes of twin Steiner triple systems.

Two twin systems necessarily have the same automorphism group but they need not be isomorphic. When twin systems $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic, they are called *identical twins*. Relative to the number of distinct Steiner triple systems of order v, twins are rarities and identical twins even more so. There are three orders v for which identical twin systems were known up to the present, namely $v = 21, 27$ and 33, all of which were found by Murphy and appear in [5]. Apart from the STS(7) (which is unique up to isomorphism), these were also the only known systems that contain Pasch configurations and where every Pasch switch results in an isomorphic copy of the original system. It was shown in [2] that there are no identical twins of order 19.

The identical twin systems constructed by Murphy have a common form: the two systems forming each pair can be mapped to one another by an isomorphism ϕ of order 2 having precisely three fixed points. We will call such systems *Murphy identical twins*. Thus, up to the present, all known identical twins were Murphy identical twins. By computer search, we have found Murphy identical twins for all admissible $v \in [21, 199]$. It is not feasible to list all of these in this paper, but we give examples for $v = 25, 31$ and 37 in Section 3, and the remaining cases may be obtained from the authors. In Section 2 we produce infinite classes of Murphy identical twins and linear classes of identical twins for $v \equiv 3 \pmod 6$.

2 Results

We present two recursive constructions. The first of these produces Murphy identical twins for certain orders. The second produces identical twins, that are not necessarily Murphy identical twins, for a larger spectrum of orders. Both constructions require a QFSTS(v) that has an automorphism of order 2 with precisely three fixed points. We start by proving that such a system can only exist if $v \equiv 3 \pmod 6$. When no confusion is likely we may write a block $\{a, b, c\}$ more simply as abc or, as in Table 1 (and in the listings in Section 3), as $a\ b\ c$.

Theorem 2.1 *Suppose that $\mathcal{S}$ is a QFSTS(v) having an automorphism of order* 2 *with precisely three fixed points. Then $v \equiv 3$ (mod* 6*).*

Proof Suppose that $\mathcal{S}$ has the point set

$$V = \{\infty_1, \infty_2, \infty_3, a_i, b_i : i = 1, 2, \ldots, (v-3)/2\},$$

with automorphism ϕ of order 2 such that $\phi(\infty_i) = \infty_i$ and $\phi(a_i) = b_i$. In any STS(v), the points fixed by an automorphism form a subsystem, so in this case $\infty_1\infty_2\infty_3$ is necessarily a block of $\mathcal{S}$. Let N_i denote the number of blocks of the form $\infty_i a_j b_j$ in $\mathcal{S}$. Since each of the $(v-3)/2$ pairs $\{a_j, b_j\}$ must appear in a block with one of $\infty_1, \infty_2, \infty_3$, we have

$$N_1 + N_2 + N_3 = \frac{v-3}{2}.$$

The point ∞_1 appears in the block $\infty_1\infty_2\infty_3$ and in $(v-3)/2$ other blocks. Hence it appears in $\frac{v-3}{2} - N_1$ blocks of the forms $\infty_1 a_i b_j, \infty_1 a_i a_j, \infty_1 b_i b_j$ $(i \neq j)$. Consider then pairs of such blocks $\{\infty_1 a_i b_j, \infty_1 b_i a_j\}$ and $\{\infty_1 a_i a_j, \infty_1 b_i b_j\}$. The number of such pairs is $(\frac{v-3}{2} - N_1)/2$. For each such pair of blocks the pairs of points $\{a_i, b_i\}$ and $\{a_j, b_j\}$ must appear one with ∞_2 and one with ∞_3, otherwise $\mathcal{S}$ has a Pasch configuration, which is not the case. Hence

$$N_2 \geq \left(\frac{v-3}{2} - N_1\right)/2 \quad \text{and} \quad N_3 \geq \left(\frac{v-3}{2} - N_1\right)/2.$$

Now suppose that $N_2 > (\frac{v-3}{2} - N_1)/2$. Then $N_2 + N_3 > \frac{v-3}{2} - N_1$, giving $N_1 + N_2 + N_3 > \frac{v-3}{2}$, a contradiction. Hence $N_2 = (\frac{v-3}{2} - N_1)/2$, and similarly $N_3 = (\frac{v-3}{2} - N_1)/2$. So $N_2 = N_3$ and, by repeating the argument, we obtain $N_1 = N_2 = N_3$. Hence $3N_1 = \frac{v-3}{2}$. Consequently $3 \mid v$, and so $v \equiv 3 \pmod 6$. □

The condition $v \equiv 3 \pmod 6$ of the previous theorem is in fact sufficient to ensure the existence of a QFSTS(v) having an automorphism of order 2 with precisely three fixed points. This is established by our next theorem in which we give a slightly more general result, where the system and the automorphism have additional properties.

Theorem 2.2 *For $v \equiv 3$ (mod* 6*) there exists a QFSTS(v) that has a parallel class $\mathcal{P}$ and an automorphism ϕ of order* 2 *such that ϕ fixes $\mathcal{P}$, ϕ has precisely three fixed points a, b, c and $\{a, b, c\} \in \mathcal{P}$.*

Proof We use the method of Brouwer [1]. Suppose initially that $7 \nmid v$, and that $v = 3w$ where w is odd. Then the Bose construction [3] gives the required QFSTS(v). The point set is $\mathbb{Z}_3 \times \mathbb{Z}_w$. The blocks consist of all triples of the form $\{(i, a), (i, b), (i+1, (a+b)/2)\}$ (where $i \in \mathbb{Z}_3$, $a, b \in \mathbb{Z}_w$ and $a \neq b$), together with all triples of the form $\{(0, a), (1, a), (2, a)\}$ (where $a \in \mathbb{Z}_w$). The second class of blocks forms a parallel class $\mathcal{P}$. The mapping $\phi : (i, a) \mapsto (i, -a)$ (where $i \in \mathbb{Z}_3$, $a \in \mathbb{Z}_w$) is of order

2, has precisely three fixed points $(0,0),(1,0),(2,0)$ and $\{(0,0),(1,0),(2,0)\} \in \mathcal{P}$. It is shown in [3] that the Bose construction gives a QFSTS(v) when $v \equiv 3 \pmod 6$ and $7 \nmid v$.

We next consider cases when $7 \mid v$. Table 1 gives a QFSTS(v) with the required properties in the smallest case, $v = 21$. The parallel class $\mathcal{P}$ consists of the blocks $\{1,2,3\},\{4,5,6\},\ldots,\{19,20,21\}$, while ϕ fixes the points $1,2,3$ and transposes i and $25-i$ for $i = 4,5,\ldots,12$.

1 2 3	1 4 21	1 5 20	1 6 19	1 7 13
1 8 10	1 9 14	1 11 16	1 12 18	1 15 17
2 4 12	2 5 15	2 6 11	2 7 18	2 8 17
2 9 16	2 10 20	2 13 21	2 14 19	3 4 9
3 5 18	3 6 8	3 7 20	3 10 15	3 11 14
3 12 13	3 16 21	3 17 19	4 5 6	4 7 15
4 8 19	4 10 16	4 11 17	4 13 20	4 14 18
5 7 19	5 8 16	5 9 11	5 10 14	5 12 21
5 13 17	6 7 14	6 9 12	6 10 13	6 15 16
6 17 21	6 18 20	7 8 9	7 10 17	7 11 21
7 12 16	8 11 13	8 12 20	8 14 21	8 15 18
9 10 19	9 13 18	9 15 21	9 17 20	10 11 12
10 18 21	11 15 20	11 18 19	12 14 17	12 15 19
13 14 15	13 16 19	14 16 20	16 17 18	19 20 21

Table 1. A QFSTS(21) with the required properties.

When $7|v$ and $v > 21$ we proceed inductively. Suppose that $\mathcal{S} = (V,\mathcal{B})$ is a QFSTS(v) having a parallel class $\mathcal{P}$ and automorphism ϕ with fixed points a,b,c, and having the required properties. Then a QFSTS($7v$), $\bar{\mathcal{S}} = (\bar{V},\bar{\mathcal{B}})$, also having the required properties may be constructed as follows. Take its point set $\bar{V} = \mathbb{Z}_7 \times V$ and its block set $\bar{\mathcal{B}}$ to consist of triples of two types:

1: triples of the form $\{(i,x),(j,y),(k,z)\}$, where $\{x,y,z\} \in \mathcal{B} \setminus \mathcal{P}$ and $i+j+k = 0$ in $\mathbb{Z}_7$,

2: for each $\{x,y,z\} \in \mathcal{P}$, blocks of an appropriate copy of the QFSTS(21) from Table 1 on the point set $\{(i,x),(i,y),(i,z) : i \in \mathbb{Z}_7\}$. We explain below what is meant by "an appropriate copy". The copy corresponding to a particular $\{x,y,z\} \in \mathcal{P}$ will be called the $\{x,y,z\}$-copy.

These triples cover every pair of points and there are $49(v(v-1)/6 - v/3)$ triples of type 1, and $70(v/3)$ triples of type 2, making a total of $7v(7v-1)/6$ triples altogether. Hence these form the blocks of an STS($7v$).

We next prove that this is a QFSTS($7v$). If there were a Pasch configuration present which contained two type 2 blocks, these would have to be from the same $\{x,y,z\}$-copy, otherwise they would be disjoint blocks. But then five of the six points, and hence all four blocks, would come from that copy, contradicting the fact that the copy is anti-Pasch. Hence any possible Pasch configuration must contain at least three blocks of type 1 and without loss of generality we can take these to be $\{(i,x),(j,y),(k,z)\}$, $\{(i,x),(\ell,d),(m,e)\}$ and $\{(j,y),(\ell,d),(n,f)\}$. The fourth block is then necessarily $\{(k,z),(m,e),(n,f)\}$. Projecting onto the second coordinate in each case gives four blocks in $\mathcal{B}$ that appear to form a Pasch configuration and, since this is impossible, we deduce that these four blocks of $\mathcal{B}$ are not distinct. Hence $f = x, e = y$ and $d = z$ and the fourth block is type 1. Then $i + j + k = 0$, $i + \ell + m = 0$, $j + \ell + n = 0$ and $k + m + n = 0$. From these we deduce that $n = i$, $m = j$ and $\ell = k$, so the four blocks of the STS($7v$) are in fact identical and they do not form a Pasch configuration. It follows that the design is a QFSTS($7v$).

Define the mapping $\bar{\phi}$ on $\bar{V}$ by $\bar{\phi} : (i,x) \mapsto (-i, \phi(x))$. Plainly $\bar{\phi}$ is of order 2 and (i,x) is fixed by $\bar{\phi}$ if and only if $i = 0$ and ϕ fixes x, so $\bar{\phi}$ has precisely three fixed points $(0,a),(0,b)$ and $(0,c)$. It should also be clear that $\bar{\phi}$ maps type 1 blocks to type 1 blocks. It remains to determine its action on type 2 blocks and to give a precise definition of these.

Suppose that $\{x,y,z\} \in \mathcal{P}$ but $\{x,y,z\} \neq \{a,b,c\}$. Then $\{\phi(x),\phi(y),\phi(z)\} \in \mathcal{P}$, but $\{\phi(x),\phi(y),\phi(z)\} \neq \{x,y,z\}$. This latter is because if $\phi(x) = x$, then $x = a, b$ or c, while if $\phi(x) = y$ then $\phi(y) = x$ and so $\phi(z) = z$, giving $z = a, b$ or c. Hence $\bar{\phi}$ maps the point set of the $\{x,y,z\}$-copy of the QFSTS(21) onto the distinct point set of the $\{\phi(x),\phi(y),\phi(z)\}$-copy. On the other hand, $\bar{\phi}$ fixes the point set of the $\{a,b,c\}$-copy of the QFSTS(21). It follows that the blocks of these QFSTS(21)s can be chosen (the "appropriate copy") so that $\bar{\phi}$ maps blocks to blocks, preserves a parallel class $\bar{\mathcal{P}} \subseteq \bar{\mathcal{B}}$ consisting of the union of parallel classes from each copy of the QFSTS(21), and $\bar{\mathcal{P}}$ contains the block $\{(0,a),(0,b),(0,c)\}$. For example, an appropriate $\{a,b,c\}$-copy of the QFSTS(21) may be obtained from Table 1 by applying the mapping

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ (0,a) & (0,b) & (0,c) & (1,a) & (1,b) & (1,c) & (2,a) & (2,b) & (2,c) & (3,a) & (3,b) \\ 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & \\ (3,c) & (4,c) & (4,b) & (4,a) & (5,c) & (5,b) & (5,a) & (6,c) & (6,b) & (6,a) & \end{pmatrix}.$$

And if $\{x,y,z\} \neq \{a,b,c\}$ and the $\{x,y,z\}$-copy is chosen with a point k of the QFSTS(21) mapped to (i,x), then the $\{\phi(x),\phi(y),\phi(z)\}$-copy must be chosen with k mapped to $(-i,\phi(x))$. Thus the constructed QFSTS($7v$) has the required properties, and the result follows by induction. □

Our first recursive construction is for Murphy identical twins. It requires a QFSTS(u) that has an automorphism of order 2 with a single fixed point. An STS(u) with such an automorphism is known as a *reverse* Steiner triple system, and these systems exist if and only if $u \equiv 1, 3, 9, 19 \pmod{24}$ [4, 9, 10]. Moreover, we require such a system that has the additional property of being anti-Pasch and it is apparently not known if such systems exist for all possible values of u. Reverse QFSTS(u) systems certainly exist for some values of u, such as affine systems of orders 3^n, and those known as Netto systems of orders $u = p^n$, where p is a prime, $p \equiv 19 \pmod{24}$ and $n \geq 1$ [3].

Theorem 2.3 *Suppose that $\mathcal{S}_u$ is a QFSTS(u) having an automorphism χ of order 2 with precisely one fixed point, and that $\mathcal{S}_v$ is a QFSTS(v) having an automorphism ϕ of order 2 with precisely three fixed points. If there there also exist Murphy identical twins $\mathcal{T}_1, \mathcal{T}_2$ of order v, then there exist Murphy identical twins of order uv.*

Proof Take the point set of $\mathcal{S}_u$ to be $U = \{0, 1, \ldots, u-1\}$ and the point sets of $\mathcal{S}_v, \mathcal{T}_1, \mathcal{T}_2$ to be $V = \{0, 1, \ldots, v-1\}$. We may assume that χ fixes $0 \in U$, that ϕ fixes $0, 1, 2 \in V$, and that ϕ takes $\mathcal{T}_1$ to $\mathcal{T}_2$. We will construct Steiner systems $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ of order uv on the point set $U \times V$. For $\ell = 1, 2$, the blocks of $\mathcal{T}_\ell^*$ are of four types:

(1) $\{(i,x),(j,y),(k,z)\}$ for all blocks ijk of $\mathcal{S}_u$ and all blocks xyz of $\mathcal{S}_v$,
(2) $\{(i,x),(j,x),(k,x)\}$ for all blocks ijk of $\mathcal{S}_u$ and all $x \in V$,
(3) $\{(i,x),(i,y),(i,z)\}$ for $i \in U$ with $i \neq 0$ and for all blocks xyz of $\mathcal{S}_v$,
(4) $\{(0,x),(0,y),(0,z)\}$ for all blocks xyz of $\mathcal{T}_\ell$.

For each pair of blocks ijk of $\mathcal{S}_u$ and xyz of $\mathcal{S}_v$, six type 1 blocks are formed. Thus there are $6 \times \frac{u(u-1)}{6} \times \frac{v(v-1)}{6}$ blocks of type 1, $\frac{u(u-1)}{6} \times v$ blocks of type 2, $(u-1) \times \frac{v(v-1)}{6}$ blocks of type 3, and $\frac{v(v-1)}{6}$ blocks of type 4, making a total of $\frac{uv(uv-1)}{6}$ blocks altogether, which cover every pair of points from $U \times V$. Hence the blocks form Steiner triple systems for $\ell = 1, 2$. For each ℓ we will show that $\mathcal{T}_\ell^*$ contains only one Pasch configuration. We will call blocks of types 1 and 2 *vertical*, blocks of types 3 and 4 *horizontal*, and points (i,x) will be said to be at *level* i.

Any Pasch configuration in $\mathcal{T}_\ell^*$ cannot contain blocks from two different levels since such blocks do not intersect. If a Pasch configuration has two blocks from the same level, then five of its points lie at that level, and so therefore does the sixth point. Since $\mathcal{S}_v$ has no Pasch configurations, and $\mathcal{T}_\ell$ has only one, there can be only one Pasch configuration in $\mathcal{T}_\ell^*$ that has two blocks from the same level. There remains the possibility that there are additional Pasch configurations containing either (a) 1 horizontal and 3 vertical blocks, or (b) 4 vertical blocks.

In case (a), suppose the horizontal block is $\{(i,x),(i,y),(i,z)\}$ of type 3 or 4 and an intersecting vertical block is $\{(i,x),(j,v),(k,w)\}$ of type 1 or 2. Without loss of generality, the remaining vertical blocks must be of the forms $\{(i,y),(j,v),(k,u)\}$ and $\{(i,z),(k,w),(j,t)\}$ for some u, t; but these blocks do not intersect, so case (a) is not possible.

In case (b), projecting onto the first coordinate of each point gives four blocks of $\mathcal{S}_u$ that appear to form a Pasch configuration and, since this is impossible, we deduce that these four blocks of $\mathcal{S}_u$ are not distinct. Hence the four blocks must span just three levels, say i, j and k. If these four vertical blocks are all of type 1 then, projecting onto the second coordinate gives four blocks of $\mathcal{S}_v$ that appear to form a Pasch configuration and, since this is impossible, we deduce that these four blocks of $\mathcal{S}_v$ are not distinct. So if the four vertical blocks are all of type 1, two of them must have the forms $\{(i,x),(j,y),(k,z)\}$ and $\{(i,x),(j,z),(k,y)\}$, and it is not possible to form a Pasch configuration containing two such blocks. A Pasch configuration spanning just three levels cannot contain more than one type 2 block, because any two such blocks do not intersect. So the only remaining possibility is a Pasch configuration containing two blocks of the form $\{(i,x),(j,x),(k,x)\}$ and $\{(i,x),(j,y),(k,z)\}$, where $y,z \neq x$. But again this cannot be completed to form a Pasch configuration.

Hence for $\ell = 1,2$, $\mathcal{T}_\ell^*$ contains only one Pasch configuration. From the type 4 blocks, it should also be clear that switching the unique Pasch configuration in each system transforms one to the other. Hence $\mathcal{T}_1^*$ and $\mathcal{T}_2^*$ are twin systems. It remains to prove that they are Murphy identical twins.

Define the mapping ψ on $U \times V$ by setting $\psi(i,x) = (\chi(i),\phi(x))$. Then ψ is of order 2 and (i,x) is a fixed point if and only if $\chi(i) = i$ and $\phi(x) = x$. So ψ has precisely three fixed points, namely $(0,0),(0,1)$ and $(0,2)$. Furthermore, ψ maps type 1 blocks to type 1 blocks, type 2 to type 2, and type 3 to type 3. As regards type 4 blocks, its action is to exchange $\mathcal{T}_1$ with $\mathcal{T}_2$. Hence χ maps $\mathcal{T}_1^*$ to $\mathcal{T}_2^*$ and the result follows. □

By using affine systems of order $u = 3^n$ and the systems we have constructed for $v < 200$, the following corollary follows immediately.

Corollary 2.1 *For $v \in [21,195]$ with $v \equiv 3 \pmod 6$, and for all $n \geq 0$, there exists a pair of Murphy identical twins of order $3^n v$.*

Our second construction takes Murphy identical twins of order v and produces identical twins (not necessarily Murphy identical twins) of order uv for $u \equiv 1,3 \pmod 6$ $(u \neq 7,13)$.

Theorem 2.4 *Suppose that $u \equiv 1$ or $3 \pmod 6$ and $u \neq 7,13$. If there exist Murphy identical twins $\mathcal{T}_1,\mathcal{T}_2$ of order $v \equiv 3 \pmod 6$ then there exist identical twins of order uv.*

Proof The strategy is first to construct an anti-Pasch 3-GDD of type v^u. Then on each group except one, place a QFSTS(v), and on the remaining group place a Murphy identical twin of order v. This can be done in such a way that no additional Pasch configurations are formed and the resulting system is an identical twin. The details follow.

Take $v = 3w$, so that w is odd. Let $\mathcal{S} = (U,\mathcal{B})$ be a QFSTS(u) on the point set $\mathbb{Z}_u$; as noted in the Introduction, such a system exists for all admissible $u \neq 7,13$. The point set of our 3-GDD is $G = \mathbb{Z}_u \times \mathbb{Z}_w \times \mathbb{Z}_3$ with the groups $G_i = \{i\} \times \mathbb{Z}_w \times \mathbb{Z}_3$ for $i \in \mathbb{Z}_u$. The block set $\mathcal{D}$ of the GDD consists of all triples of points $\{(i_1,j_1,k_1),(i_2,j_2,k_2),(i_3,j_3,k_3)\}$ where $\{i_1,i_2,i_3\} \in \mathcal{B}$, $j_1+j_2+j_3 = 0$ in $\mathbb{Z}_w$, and $k_1+k_2+k_3 = 0$ in $\mathbb{Z}_3$.

To see that this has no Pasch configurations, suppose it did have one with blocks $\{(i_1,j_1,k_1),(i_2,j_2,k_2),(i_3,j_3,k_3)\}$, $\{(i_1,j_1,k_1),(i_4,j_4,k_4),(i_5,j_5,k_5)\}$, $\{(i_2,j_2,k_2),(i_4,j_4,k_4),(i_6,j_6,k_6)\}$ and $\{(i_3,j_3,k_3),(i_5,j_5,k_5),(i_6,j_6,k_6)\}$. Projecting onto the first coordinate in each block gives four blocks in $\mathcal{B}$ that appear to form a Pasch configuration and, since this is impossible, we deduce that these four blocks of $\mathcal{B}$ are not distinct. Hence, $i_4 = i_3, i_5 = i_2$ and $i_6 = i_1$. But then with arithmetic in $\mathbb{Z}_w$, $j_1 + j_2 + j_3 = 0$, $j_1 + j_4 + j_5 = 0$, $j_2 + j_4 + j_6 = 0$ and $j_3 + j_5 + j_6 = 0$. From these we deduce that $j_4 = j_3, j_5 = j_2$ and $j_6 = j_1$. With arithmetic in $\mathbb{Z}_3$, we also have $k_1 + k_2 + k_3 = 0$, $k_1 + k_4 + k_5 = 0$, $k_2 + k_4 + k_6 = 0$ and $k_3 + k_5 + k_6 = 0$ and this gives $k_4 = k_3, k_5 = k_2$ and $k_6 = k_1$. Hence the four blocks of the putative Pasch configuration are in fact identical. It follows that the GDD must be anti-Pasch.

Subsequently we will place an STS(v) on each group of our 3-GDD. We now prove that, however this is done, the only Pasch configurations in the resulting STS(uv) will be those already present in the STS(v)s. First observe that no Pasch configuration can contain blocks from two different groups since these blocks would not intersect. A Pasch configuration containing two blocks from a single group would have five of its six points in that group and hence the entire configuration must lie in that group. The only remaining possibility is a Pasch configuration containing three blocks of the GDD and one block within a group. Without loss of generality the three blocks of the GDD are $\{(i_1, j_1, k_1), (i_2, j_2, k_2), (i_3, j_3, k_3)\}$, $\{(i_1, j_1, k_1), (i_4, j_4, k_4), (i_5, j_5, k_5)\}$, $\{(i_2, j_2, k_2), (i_4, j_4, k_4), (i_6, j_6, k_6)\}$, and the remaining block must be $\{(i_3, j_3, k_3), (i_5, j_5, k_5), (i_6, j_6, k_6)\}$, so that $i_3 = i_5 = i_6$. But the first two blocks then give $i_2 = i_4$, and hence $\{i_2, i_4, i_6\}$ is not a block of $\mathcal{B}$, a contradiction.

Next consider the mapping ψ defined on G by $\psi : (i, j, k) \mapsto (i, -j, k)$. Plainly ψ is an automorphism of order 2 of the GDD. Moreover, in each group G_i, ψ has precisely three fixed points, namely the points $(i, 0, k)$ for $k = 0, 1, 2$, together with $(v - 3)/2$ transpositions.

Now suppose that $\mathcal{T}_1$ and $\mathcal{T}_2$ form a pair of Murphy identical twin STS(v)s with isomorphism ϕ taking one to the other. On the group G_0 place a copy of $\mathcal{T}_1$ in such a way that if a point x of $\mathcal{T}_1$ is assigned to a point $(0, j, k)$ of the group, then $\phi(x)$ is assigned to $(0, -j, k)$, thereby ensuring that ψ switches the copy of $\mathcal{T}_1$ to a copy of $\mathcal{T}_2$. On each remaining group G_i we place a QFSTS(v) that has ψ as an automorphism; the existence of such a system is guaranteed by Theorem 2.2. The resulting STS(uv) has just one Pasch configuration. Switching this gives another STS(uv) also with one Pasch configuration (effectively replacing $\mathcal{T}_1$ by $\mathcal{T}_2$), and ψ maps one system to the other. □

By using the systems we have constructed for $v < 200$, the following corollary follows immediately.

Corollary 2.2 *For $v \in [21, 195]$ with $v \equiv 3$ (mod 6), and for $u \equiv 1$ or 3 (mod 6) (and $u \neq 7, 13$), there exists a pair of identical twins of order uv.*

3 Some new identical twin systems

Systems of orders $v = 25, 31$ and 37 are presented below. In each case we list one of a pair of Murphy identical twins on the point set $\{1, 2, \ldots, v\}$. The second twin is obtained from the listed system by applying the mapping ψ that fixes each of the points 1, 2 and 3 and transposes the points $4+i$ and $v-i$ for $i = 0, 1, \ldots (v-5)/2$. The Pasch configuration in each listed system comprises the four blocks $\{4, 5, v-2\}$, $\{4, 6, v\}$, $\{5, 6, v-1\}$ and $\{v-2, v-1, v\}$. The mapping ψ takes these four blocks respectively to $\{v, v-1, 6\}$, $\{v, v-2, 4\}$, $\{v-1, v-2, 5\}$ and $\{6, 5, 4\}$, which form the Pasch configuration in the twin system. The set of remaining blocks is stabilized by ψ.

Using a modified hill-climbing method, we have constructed Murphy identical twins for all admissible $v \in [21, 199]$, and a complete set is available from the authors. These systems have been checked by two independently written computer programs. There seems little reason to doubt that such systems exist for all admissible $v \geq 21$, but proof eludes us.

STS(25)

1 2 3	1 4 19	1 5 12	1 6 23	1 7 22
1 8 21	1 9 15	1 10 25	1 11 13	1 14 20
1 16 18	1 17 24	2 4 12	2 5 7	2 6 13
2 8 14	2 9 20	2 10 19	2 11 18	2 15 21
2 16 23	2 17 25	2 22 24	3 4 9	3 5 8
3 6 10	3 7 18	3 11 22	3 12 17	3 13 16
3 14 15	3 19 23	3 20 25	3 21 24	4 5 23
4 6 25	4 7 10	4 8 11	4 13 18	4 14 22
4 15 24	4 16 20	4 17 21	5 6 24	5 9 21
5 10 22	5 11 15	5 13 17	5 14 25	5 16 19
5 18 20	6 7 20	6 8 17	6 9 14	6 11 19
6 12 22	6 15 18	6 16 21	7 8 9	7 11 12
7 13 21	7 14 16	7 15 25	7 17 23	7 19 24
8 10 15	8 12 25	8 13 23	8 16 22	8 18 19
8 20 24	9 10 16	9 11 24	9 12 18	9 13 25
9 17 19	9 22 23	10 11 21	10 12 20	10 13 24
10 14 17	10 18 23	11 14 23	11 16 25	11 17 20
12 13 14	12 15 19	12 16 24	12 21 23	13 15 22
13 19 20	14 18 24	14 19 21	15 16 17	15 20 23
17 18 22	18 21 25	19 22 25	20 21 22	23 24 25

STS(31)

1 2 3	1 4 19	1 5 23	1 6 29	1 7 28
1 8 27	1 9 26	1 10 18	1 11 21	1 12 30
1 13 15	1 14 24	1 16 31	1 17 25	1 20 22
2 4 14	2 5 9	2 6 19	2 7 20	2 8 17
2 10 25	2 11 24	2 12 23	2 13 22	2 15 28
2 16 29	2 18 27	2 21 31	2 26 30	3 4 26
3 5 24	3 6 23	3 7 25	3 8 22	3 9 31
3 10 28	3 11 30	3 12 29	3 13 27	3 14 21
3 15 20	3 16 19	3 17 18	4 5 29	4 6 31
4 7 13	4 8 25	4 9 22	4 10 30	4 11 17
4 12 21	4 15 16	4 18 23	4 20 27	4 24 28
5 6 30	5 7 15	5 8 21	5 10 12	5 11 18
5 13 20	5 14 26	5 16 28	5 17 19	5 22 27
5 25 31	6 7 9	6 8 10	6 11 27	6 12 18
6 13 24	6 14 28	6 15 17	6 16 21	6 20 25
6 22 26	7 8 14	7 10 26	7 11 31	7 12 16
7 17 27	7 18 22	7 19 30	7 21 29	7 23 24
8 9 16	8 11 19	8 12 26	8 13 30	8 15 31
8 18 28	8 20 23	8 24 29	9 10 19	9 11 15
9 12 24	9 13 29	9 14 18	9 17 20	9 21 30
9 23 27	9 25 28	10 11 14	10 13 23	10 15 29
10 16 17	10 20 21	10 22 24	10 27 31	11 12 28
11 13 25	11 16 20	11 22 29	11 23 26	12 13 19
12 14 20	12 15 27	12 17 31	12 22 25	13 14 16
13 17 28	13 18 21	13 26 31	14 15 25	14 17 22
14 19 29	14 23 31	14 27 30	15 18 26	15 19 24
15 21 23	15 22 30	16 18 30	16 22 23	16 24 27
16 25 26	17 21 26	17 23 29	17 24 30	18 19 25
18 20 29	18 24 31	19 20 31	19 21 22	19 23 28
19 26 27	20 24 26	20 28 30	21 24 25	21 27 28
22 28 31	23 25 30	25 27 29	26 28 29	29 30 31

STS(37)

1 2 3	1 4 13	1 5 21	1 6 35	1 7 34
1 8 33	1 9 32	1 10 31	1 11 24	1 12 22
1 14 25	1 15 23	1 16 27	1 17 30	1 18 26
1 19 29	1 20 36	1 28 37	2 4 22	2 5 34
2 6 23	2 7 36	2 8 21	2 9 24	2 10 16
2 11 30	2 12 29	2 13 28	2 14 27	2 15 26
2 17 32	2 18 35	2 19 37	2 20 33	2 25 31
3 4 7	3 5 30	3 6 13	3 8 29	3 9 14
3 10 15	3 11 36	3 12 33	3 16 25	3 17 24
3 18 23	3 19 22	3 20 21	3 26 31	3 27 32
3 28 35	3 34 37	4 5 35	4 6 37	4 8 14
4 9 30	4 10 17	4 11 21	4 12 24	4 15 18
4 16 26	4 19 25	4 20 32	4 23 36	4 27 31
4 28 29	4 33 34	5 6 36	5 7 26	5 8 10
5 9 22	5 11 23	5 12 20	5 13 16	5 14 32
5 15 33	5 17 25	5 18 37	5 19 27	5 24 28
5 29 31	6 7 31	6 8 18	6 9 28	6 10 27
6 11 33	6 12 26	6 14 29	6 15 30	6 16 21
6 17 20	6 19 34	6 22 25	6 24 32	7 8 37
7 9 25	7 10 20	7 11 14	7 12 32	7 13 18
7 15 24	7 16 28	7 17 21	7 19 23	7 22 35
7 27 29	7 30 33	8 9 16	8 11 34	8 12 25
8 13 20	8 15 27	8 17 23	8 19 31	8 22 24
8 26 36	8 28 32	8 30 35	9 10 23	9 11 18
9 12 31	9 13 33	9 15 19	9 17 35	9 20 26
9 21 37	9 27 36	9 29 34	10 11 19	10 12 36
10 13 24	10 14 37	10 18 25	10 21 30	10 22 33
10 26 28	10 29 32	10 34 35	11 12 17	11 13 27
11 15 16	11 20 31	11 22 28	11 25 29	11 26 35
11 32 37	12 13 37	12 14 34	12 15 28	12 16 30
12 18 19	12 21 23	12 27 35	13 14 23	13 15 31
13 17 36	13 19 30	13 21 22	13 25 34	13 26 29
13 32 35	14 15 17	14 16 20	14 18 21	14 19 24
14 22 36	14 26 33	14 28 30	14 31 35	15 20 22
15 21 32	15 25 37	15 29 35	15 34 36	16 17 18
16 19 35	16 22 37	16 23 31	16 24 36	16 29 33
16 32 34	17 19 33	17 22 27	17 26 34	17 28 31
17 29 37	18 20 29	18 22 34	18 24 33	18 27 28
18 30 36	18 31 32	19 20 28	19 21 26	19 32 36
20 23 27	20 24 34	20 25 35	20 30 37	21 24 35
21 25 27	21 28 33	21 29 36	21 31 34	22 23 29
22 26 32	22 30 31	23 24 25	23 26 37	23 28 34
23 30 32	23 33 35	24 26 27	24 29 30	24 31 37
25 26 30	25 28 36	25 32 33	27 30 34	27 33 37
31 33 36	35 36 37			

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