Resolvability of infinite designs

Journal Item

Resolvability of Infinite Designs

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Abstract

In this paper we examine the resolvability of infinite designs. We show that in stark contrast to the finite case, resolvability for infinite designs is fairly commonplace. We prove that every t-(v, k, Λ) design with t finite, v infinite and $k, \lambda < v$ is resolvable and, in fact, has α orthogonal resolutions for each $\alpha < v$. We also show that, while a t-(v, k, Λ) design with t and λ finite, v infinite and $k = v$ may or may not have a resolution, any resolution of such a design must have v parallel classes containing v blocks and at most $\lambda - 1$ parallel classes containing fewer than v blocks. Further, a resolution into parallel classes of any specified sizes obeying these conditions is realisable in some design. When $k < v$ and $\lambda = v$ and when $k = v$ and λ is infinite, we give various examples of resolvable and non-resolvable t-(v, k, Λ) designs.

Keywords: infinite design, resolvable, resolution, parallel class
Mathematics Subject Classification: 05B30

1. Introduction

We assume that the reader is familiar with the general concepts of design theory, and refer them to [9] for general definitions and notation. For finite t, v, k and λ, a t-(v, k, λ) design is a pair $(V, \mathcal{B})$ where V is a v-set of *points* and $\mathcal{B}$ is a collection

[1]supported by NSERC discovery grant
[2]supported by the Australian Research Council via grants DE120100040 and DP120103067

of k-subsets of V, called *blocks*, such that every t-subset of V is a subset of exactly λ blocks. A t-(v,k,λ) design $(V,\mathcal{B})$ is called *resolvable* if $\mathcal{B}$ can be partitioned into *parallel classes*, each of which is a partition of V.

Classically t, v, k and λ are taken to be finite. However, there has been considerable interest in designs with infinite parameter sets. Fundamental definitions and assumptions are given in [5]. Grannell, Griggs, and Phelan [14, 15] gave the first explicit constructions for infinite Steiner triple systems, and rigid, sparse and perfect countably infinite Steiner triple systems are constructed in [12], [8] and [6], respectively.

Extension results for Steiner systems can be found in Quackenbush [24] and Beutelspacher and Cameron [1]. Cameron has also considered the subject from a geometric point of view in [2], and in [4] he showed that large sets of Steiner systems exist for all finite t and k and infinite v. Orbits in infinite designs have been considered by Cameron [3], Webb [27, 28], Evans [11] and Camina [7]. The specific case of orbits on projective planes is considered by Moorhouse and Pentila [22]. Horsley, Pike and Sanaei [16, 23] have considered block intersection graphs of infinite designs. In this paper we consider the resolvability properties of infinite designs.

We use the set of assumptions laid out in [5]. In particular, we work in Zermelo–Fraenkel set theory with the axiom of choice. We use the following definition from [5].

Definition 1.1 ([5])**.** *Let t, v and k be cardinals with t finite and let $\Lambda = (\lambda_{i,j})_{0\le i,j\le t}$ be a matrix of cardinals, where $\lambda_{i,j}$ is undefined for $i+j>t$. A t-(v,k,Λ) design is a pair $(V,\mathcal{B})$ where V is a v-set of points and $\mathcal{B}$ is a collection of k-subsets of V, called blocks, with the properties that*

1. *no block is a strict subset of any other block;*
2. *the cardinality of the set of points missed by a block is non-zero, and is independent of the block;*
3. *for any disjoint subsets I and J of V with $|I|+|J|\le t$, exactly $\lambda_{|I|,|J|}$ blocks contain each point in I and no point in J.*

We will also need the notion of a *partial design*, which we define here.

Definition 1.2. *Let t, v and k be cardinals with t finite and let $\Lambda = (\lambda_{i,j})_{0\le i,j\le t}$ be a matrix of cardinals, where $\lambda_{i,j}$ is undefined for $i+j>t$. A partial t-(v,k,Λ) design is a pair $(V,\mathcal{B})$ where V is a v-set of points and $\mathcal{B}$ is a collection of k-subsets of V, called blocks, which satisfy 1 and 2 from Definition 1.1, and*

3′. for any disjoint subsets I and J of V with $|I|+|J|\le t$, at most $\lambda_{|I|,|J|}$ blocks contain each point in I and no point in J.

Throughout the paper we will assume that $t\ge 2$ is finite, that v is infinite, and that Λ is the matrix with entries $(\lambda_{i,j})_{0\le i,j\le t}$, where $\lambda_{i,j}$ is undefined for $i+j>t$. As in [5] we use b to denote $\lambda_{0,0}$, λ to denote $\lambda_{t,0}$ and r to denote $\lambda_{1,0}$. As we are considering resolvability of designs, the parameter r, which represents the cardinality of the set of blocks containing a given point x, is of particular relevance.

The following was shown in Theorem 3.1 of [5].

Theorem 1.3 ([5]). *Let t, v and k be cardinals such that t is finite and $t \geq 2$, and let V be a v-set and $\mathcal{B}$ be a collection of k-subsets of V. If $(V, \mathcal{B})$ satisfies 1 and 2 of Definition 1.1 and each t-subset of V is a subset of exactly λ sets in $\mathcal{B}$, then $(V, \mathcal{B})$ is a t-(v, k, Λ) design with $\lambda_{t,0} = \lambda$ and $\lambda_{i,j} = v$ for $0 \leq i \leq t-1$, $0 \leq j \leq t$ and $i + j \leq t$.*

As a consequence of this result, when t and λ are both finite we can replace Λ with λ and talk about a t-(v, k, λ) design.

The study of resolvability in designs has a long history, indeed in a geometric context it is related to the notion of parallelism and is a direct consequence of the affine property. In design theory, Kirkman introduced the concept in his original 1847 paper [17], posing the problem of the existence of resolvable triple systems, which are known as Kirkman systems in his honour. This existence problem was only settled in 1971 by Ray-Chadhuri and Wilson [25]. When $k = 2$, resolvable 2-$(v, 2, 1)$ designs are equivalent to one-factorizations of the complete graph, which are well studied (see for example [21]). Currently, for finite designs, the necessary conditions for the existence of a resolvable design are known to be sufficient only for $k = 2, 3, 4$ [9]. The sparsity of these values and the time it took to settle Kirkman's conjecture point to how difficult a problem resolvability is in the finite case. For sufficiently large orders, Lu [20] has shown that the necessary conditions for the existence of a resolvable design are also sufficient.

This paper is concerned with the resolvability of infinite designs. Köhler [18] showed that there exist cyclic resolvable designs with $v = \aleph_0$ and k, t finite. One of our results shows that *every* design with these parameter sets must be resolvable. Horsley, Pike and Sanaei [16] briefly considered the notion of resolvability in infinite designs, but refrained from formally defining resolvability of such designs for want of further study, which this paper now undertakes. To date, these appear to be the only mentions of resolvability for infinite designs in the literature.

Let $(V, \mathcal{B})$ be a t-(v, k, Λ) design with t finite and v infinite. It is clear that $\lambda_{0,0} \geq \lambda_{1,0} \geq \ldots \geq \lambda_{t,0}$, and so

$$\lambda \leq r \leq b. \tag{1}$$

Further, a block cannot contain more than v points, so

$$k \leq v. \tag{2}$$

A point x appears in $\binom{v-1}{t-1}$ t-sets and each block containing x contains $\binom{k-1}{t-1}$ t-sets containing x. Since each point appears in r blocks we have the classical equation

$$r\binom{k-1}{t-1} = \lambda\binom{v-1}{t-1}.$$

However, since t is finite and v is infinite we have $\binom{v-1}{t-1} = v$ and $r\binom{k-1}{t-1} = rk$, so the equation above becomes

$$rk = \lambda v. \tag{3}$$

(Note that when k is infinite we have $\binom{k-1}{t-1} = k$ and when k is finite we have that r is infinite and hence $r\binom{k-1}{t-1} = r = rk$.)

In the next section we define and discuss resolvability in the context of infinite designs. We then consider the case when $k < v$ in Section 3 and show in Theorem 3.3 that all such designs are necessarily resolvable when $\lambda < v$. We further show, in Theorem 3.7, that any such design has q orthogonal resolutions for each $q < v$. When $k < v$ and $\lambda = v$ we supply examples of both resolvable and non-resolvable designs.

In Section 4 we consider the case when $k = v$. Both resolvable and non-resolvable designs exist in this case, but we demonstrate in Theorem 4.3 that when λ is finite, any resolution of such a design has v parallel classes containing v blocks and at most $\lambda - 1$ parallel classes containing fewer than v blocks. Further, in Theorem 4.7 we show that a resolution into parallel classes of any specified sizes obeying these conditions is realisable in some design. When $k = v$ and $\lambda = v$, we give various examples of resolvable and non-resolvable designs.

2. Resolvability

For infinite designs we adopt a definition of resolvability which is unchanged from the finite case.

Definition 2.1. *A* parallel class *of a t-(v, k, Λ) design, $(V, \mathcal{B})$, is a collection of blocks $R \subseteq \mathcal{B}$ which forms a partition of V.*

Definition 2.2. *A t-(v, k, Λ) design, $(V, \mathcal{B})$, is said to be* resolvable *if $\mathcal{B}$ can be partitioned into parallel classes. Such a partition is called a* resolution *of the design.*

Thus in a resolvable t-(v, k, Λ) design, $(V, \mathcal{B})$, there exists a collection of pairwise disjoint parallel classes which between them contain every block in $\mathcal{B}$ exactly once. The number of parallel classes in any resolution of a design is the parameter r.

Resolvability in infinite designs behaves differently in the cases $k < v$ and $k = v$ and we will divide our investigation accordingly. The following theorem deals with the possible values of r in infinite designs with various parameters. It is essentially a restatement of Theorem 8.1 of [5], but with the emphasis on r rather than b.

Theorem 2.3. *Let t, v, k and λ be cardinals such that v is infinite, t is finite and $t \geq 2$.*

- *If $\lambda > v$, then a t-(v, k, Λ) design has $r = \lambda$.*
- *If either $k < v$ and $\lambda \leq v$ or $k = v$ and λ is finite, then a t-(v, k, Λ) design has $r = v$.*
- *If $k = v$, $\lambda \leq v$ and λ is infinite, then $\lambda \leq r \leq v$.*

Proof. If $\lambda > v$, then Equation (3) implies that $r = \lambda$. If $k < v$ and $\lambda \leq v$, then Equation (3) implies that $r = v$. If $k = v$ and λ is finite, then $r = v$ by Theorem 1.3 (recall that $r = \lambda_{1,0}$). If $k = v$, $\lambda \leq v$ and λ is infinite, then Equation (3) implies that $r \leq v$ and Equation (1) states that $\lambda \leq r$. $\square$

3. The case $k < v$

Clearly, any parallel class in a t-(v, k, Λ) design with $k < v$ must contain v blocks.

3.1. The case $k < v$, $\lambda < v$

Theorem 2.3 implies that a t-(v, k, Λ) design with $k < v$ and $\lambda < v$ has $r = v$. So any resolution of such a design is necessarily into v parallel classes each containing v blocks. In this section we show that every t-(v, k, Λ) design with $k < v$ and $\lambda < v$ is resolvable. As a consequence, every infinite triple system is a Kirkman system. We then strengthen our result to show that, in fact, every t-(v, k, Λ) design with $k < v$ and $\lambda < v$ has α orthogonal resolutions for each $\alpha < v$.

The following lemma is adapted from a similar lemma in [16].

Lemma 3.1. *Let t, v, k and λ be cardinals such that v is infinite, t is finite, $t \geq 2$, $k < v$ and $\lambda < v$. Let $(V, \mathcal{B})$ be a partial t-(v, k, λ) design such that every $(t-1)$-subset of V is a subset of v blocks of $\mathcal{B}$. Let S and S' be disjoint subsets of V such that $|S| \leq t-1$ and $|S'| < v$. Then there are v blocks in $\mathcal{B}$ which are supersets of S and which are disjoint from S'.*

Proof. Since $|S| \leq t-1$ and $|S'| < v$, it is easy to see that there is a subset $S^\dagger$ of V such that $S \subseteq S^\dagger$, $S^\dagger \cap S' = \emptyset$ and $|S^\dagger| = t-1$. Let $\mathcal{X}$ be the collection of all blocks in $\mathcal{B}$ which are supersets of $S^\dagger$ and let $\mathcal{Y}$ be the collection of all blocks of $\mathcal{B}$ which are supersets of $S^\dagger$ and contain at least one point in S'. It suffices to show that $|\mathcal{X} \setminus \mathcal{Y}| = v$. From our hypotheses $|\mathcal{X}| = v$ and thus it suffices to show that $|\mathcal{Y}| < v$.

Clearly, there are $|S'|$ t-subsets of V that are supersets of $S^\dagger$ and also contain a point of S'. Thus, since each block in $\mathcal{Y}$ is a superset of at least one of these t-sets and since each of these t-sets is a subset of at most λ blocks in $\mathcal{B}$, it follows that $|\mathcal{Y}| \leq \lambda |S'| < v$. □

Lemma 3.2. *Let t, v, k and λ be cardinals such that v is infinite, t is finite, $t \geq 2$, $k < v$ and $\lambda < v$. Let $(V, \mathcal{B})$ be a partial t-(v, k, Λ) design such that every $(t-1)$-subset of V is a subset of v blocks in $\mathcal{B}$, and let B^* be a block in $\mathcal{B}$. Then there is a parallel class of $(V, \mathcal{B})$ containing B^*.*

Proof. Let $V = \{x_\alpha\}_{\alpha < v}$. We will show, by transfinite induction on β, that for each ordinal $\beta \leq v$ there is a set $\mathcal{P}_\beta$ of pairwise disjoint blocks in $\mathcal{B}$ such that $B^* \in \mathcal{P}_\beta$, $|\mathcal{P}_\beta| \leq |\beta| + 1$, $x_\alpha \in \bigcup \mathcal{P}_\beta$ for each $\alpha < \beta$, and $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ for each $\alpha < \beta$. This will suffice to complete the proof, since $\mathcal{P}_v$ will be a parallel class of $(V, \mathcal{B})$ containing B^*.

Take $\mathcal{P}_0 = \{B^*\}$. Now let γ be an ordinal with $\gamma < v$ and assume that, for each $\beta < \gamma$, there is a set of pairwise disjoint blocks $\mathcal{P}_\beta$ of $(V, \mathcal{B})$ such that $B^* \in \mathcal{P}_\beta$, $|\mathcal{P}_\beta| \leq |\beta| + 1$, $x_\alpha \in \bigcup \mathcal{P}_\beta$ for each $\alpha < \beta$, and $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$ for each $\alpha < \beta$.

If γ is a limit ordinal take $\mathcal{P}_\gamma = \bigcup_{\beta < \gamma} \mathcal{P}_\beta$.

If γ is a successor ordinal and $x_{\gamma-1} \in \bigcup \mathcal{P}_{\gamma-1}$, then take $\mathcal{P}_\gamma = \mathcal{P}_{\gamma-1}$. If γ is a successor ordinal and $x_{\gamma-1} \notin \bigcup \mathcal{P}_{\gamma-1}$, then take $\mathcal{P}_\gamma = \mathcal{P}_{\gamma-1} \cup \{B\}$, where B is a block which contains $x_{\gamma-1}$ but is disjoint from $\bigcup \mathcal{P}_{\gamma-1}$ (v such blocks exist by Lemma 3.1, noting that $|\bigcup \mathcal{P}_{\gamma-1}| \leq k|\mathcal{P}_{\gamma-1}| \leq k(|\gamma - 1| + 1) < v$). □

Theorem 3.3. *Let t, v, k and λ be cardinals such that v is infinite, t is finite, $t \geq 2$, $\lambda < v$, and $k < v$. Then every t-(v, k, Λ) design is resolvable.*

Proof. Let $(V, \mathcal{B})$ be a t-(v, k, Λ) design. Let $\mathcal{B} = \{B_\alpha\}_{\alpha<v}$ (we have seen that $r = v$ and so $b = v$ by Equation (1)). We will show, by transfinite induction on β, that for each ordinal $\beta \leq v$ there is a collection $\mathfrak{R}_\beta$ of at most $|\beta|$ pairwise disjoint parallel classes of $(V, \mathcal{B})$ such that $B_\alpha \in \bigcup \mathfrak{R}_\beta$ for each $\alpha < \beta$ and $\mathfrak{R}_\alpha \subseteq \mathfrak{R}_\beta$ for each $\alpha < \beta$. This will suffice to complete the proof, since $\mathfrak{R}_v$ will be a resolution of $(V, \mathcal{B})$.

Take $\mathfrak{R}_0 = \emptyset$. Now let γ be an ordinal with $\gamma < v$ and assume that, for each $\beta < \gamma$, there is a collection $\mathfrak{R}_\beta$ of at most $|\beta|$ pairwise disjoint parallel classes of $(V, \mathcal{B})$ such that $B_\alpha \in \bigcup \mathfrak{R}_\beta$ for each $\alpha < \beta$ and $\mathfrak{R}_\alpha \subseteq \mathfrak{R}_\beta$ for each $\alpha < \beta$.

If γ is a limit ordinal take $\mathfrak{R}_\gamma = \bigcup_{\beta<\gamma} \mathfrak{R}_\beta$.

If γ is a successor ordinal and $B_{\gamma-1} \in \bigcup \mathfrak{R}_{\gamma-1}$ then take $\mathfrak{R}_\gamma = \mathfrak{R}_{\gamma-1}$. If γ is a successor ordinal and $B_{\gamma-1} \notin \bigcup \mathfrak{R}_{\gamma-1}$ then we take $\mathfrak{R}_\gamma = \mathfrak{R}_{\gamma-1} \cup \{\mathcal{P}\}$, where $\mathcal{P}$ is a parallel class of $(V, \mathcal{B})$ which contains $B_{\gamma-1}$ and is disjoint from $\bigcup \mathfrak{R}_{\gamma-1}$ (we will show that such a parallel class exists).

Clearly, $(V, \mathcal{B} \setminus \bigcup \mathfrak{R}_{\gamma-1})$ is a partial t-(v, k, Λ) design. If every $(t-1)$-subset of V is a subset of v blocks in $\mathcal{B} \setminus \bigcup \mathfrak{R}_{\gamma-1}$, then we can apply Lemma 3.2 to $(V, \mathcal{B} \setminus \bigcup \mathfrak{R}_{\gamma-1})$ to find our desired parallel class.

Let S be a $(t-1)$-subset of V. Let $\mathcal{X}$ be the collection of all blocks in $\mathcal{B}$ which are supersets of S and let $\mathcal{Y}$ be the collection of all blocks in $\bigcup \mathfrak{R}_{\gamma-1}$ which are supersets of S. We will complete the proof by showing that $|\mathcal{X}| - |\mathcal{Y}| = v$. Clearly there are exactly v t-subsets of V which are supersets of S. Each of these is a subset of exactly λ blocks in $\mathcal{B}$, all of which are in $\mathcal{X}$. Thus, since a block in $\mathcal{X}$ can be a superset of at most $\binom{k}{t}$ of these t-sets, it follows that $\binom{k}{t}|\mathcal{X}| \geq \lambda v$ and hence that $|\mathcal{X}| = v$. There are exactly $|\mathfrak{R}_{\gamma-1}|$ blocks in $\bigcup \mathfrak{R}_{\gamma-1}$ which contain any given point of S and hence at most $|\mathfrak{R}_{\gamma-1}|$ blocks in $\mathcal{Y}$. So $|\mathcal{Y}| \leq |\mathfrak{R}_{\gamma-1}| \leq |\gamma - 1| < v$, and hence $|\mathcal{X}| - |\mathcal{Y}| = v$. □

We now extend this result to cover orthogonal resolutions.

Definition 3.4. *Two resolutions $\mathfrak{R}_1$ and $\mathfrak{R}_2$ of a t-(v, k, Λ) design $(V, \{B_\alpha\}_{\alpha\in I})$ are called* orthogonal *if for any parallel classes $\{B_\alpha\}_{\alpha\in J_1} \in \mathfrak{R}_1$ and $\{B_\alpha\}_{\alpha\in J_2} \in \mathfrak{R}_2$ we have $|J_1 \cap J_2| \leq 1$.*

Note that this definition means that, in a non-simple design, two different blocks containing the same points are treated as distinct for the purposes of determining orthogonality.

The study of orthogonal resolutions of designs has a long history in the finite case [13, 26], culminating with Colbourn et al [10] showing the existence of doubly resolvable designs with $k = 3$ and Lamken [19] showing general asymptotic existence of designs with d mutually orthogonal classes for arbitrary finite k. We can strengthen our proof of Theorem 3.3 to show that any t-(v, k, Λ) design admits q orthogonal resolutions for any $q < v$.

Lemma 3.5. *Let t, q, v, k and λ be cardinals such that v is infinite, t is finite, $t \geq 2$, $q < v$, $k < v$ and $\lambda < v$. Let $(V, \mathcal{B})$ be a t-(v, k, Λ) design, let $\mathcal{B}'$ be a subset of $\mathcal{B}$ such* *that every $(t-1)$-subset of V is a subset of v blocks in $\mathcal{B}'$, let B^* be a block in $\mathcal{B}'$, and let $\{\mathfrak{R}_\alpha\}_{\alpha<q}$ be a collection of q pairwise orthogonal resolutions of $(V,\mathcal{B})$. Then there is a parallel class $\mathcal{P}$ of $(V,\mathcal{B}')$ which contains B^* and such that $\mathcal{P}$ shares at most one block with each parallel class in $\bigcup_{\alpha<q}\mathfrak{R}_\alpha$.*

Proof. We construct $\mathcal{P}$ exactly as the parallel class required in the proof of Lemma 3.2 is constructed, except that, when choosing a new block to add to our partial parallel class, we avoid blocks which are in parallel classes in $\bigcup_{\alpha<q}\mathfrak{R}_\alpha$ which already share a block with our partial parallel class. To see that we can always choose such a block, suppose we are choosing a block B which contains $x_{\gamma-1}$ but is disjoint from $\bigcup\mathcal{P}_{\gamma-1}$, and note that Lemma 3.1 ensures the existence of v such blocks. On the other hand, $\bigcup\mathcal{P}_{\gamma-1}$ contains fewer than v blocks and each of these is shared with at most one parallel class in each resolution in $\{\mathfrak{R}_\alpha\}_{\alpha<q}$. So, because $q<v$, it is clear that fewer than v parallel classes in $\bigcup_{\alpha<q}\mathfrak{R}_\alpha$ share a block with $\bigcup\mathcal{P}_{\gamma-1}$ and, since each of these contains exactly one block which contains $x_{\gamma-1}$, that there are fewer than v blocks which must be avoided.

It is routine to check that this construction produces the required parallel class. □

Lemma 3.6. *Let t, q, v, k and λ be cardinals such that v is infinite, t is finite, $t\geq 2$, $q<v$, $\lambda<v$ and $k<v$. Let $(V,\mathcal{B})$ be a t-(v,k,Λ) design, and let $\{\mathfrak{R}_\alpha\}_{\alpha<q}$ be a collection of q pairwise orthogonal resolutions of $(V,\mathcal{B})$. Then there is a resolution $\mathfrak{R}$ of $(V,\mathcal{B})$ which is orthogonal to each resolution in $\{\mathfrak{R}_\alpha\}_{\alpha<q}$.*

Proof. Construct $\mathfrak{R}$ exactly as the resolution required in the proof of Theorem 3.3 is constructed, except that, instead of applying Lemma 3.2 to find a new parallel class, we apply Lemma 3.5 to find a new parallel class which shares at most one block with each parallel class in $\bigcup_{\alpha<q}\mathfrak{R}_\alpha$.

It is routine to check that this construction produces the required resolution. □

Theorem 3.7. *Let t, v, k and λ be cardinals such that v is infinite, t is finite, $t\geq 2$, $k<v$ and $\lambda<v$. Then every t-(v,k,Λ) design has q orthogonal resolutions for each cardinal q such that $q<v$.*

Proof. This is easily proved by transfinite induction using Lemma 3.6. □

3.2. The case $k<v$, $\lambda=v$

We now show there exist both resolvable and non-resolvable t-(v,k,Λ) designs with $k<v$ and $\lambda=v$. In fact, the two results below also apply when $k=v$.

Lemma 3.8. *Let v, k and t be cardinals such that v is infinite, t is finite, $t\geq 2$ and $t+1\leq k\leq v$. There is a t-(v,k,Λ) design, with $\lambda_{i,j}=v$ for $i+j\leq t$, which has no resolution.*

Proof. Let V be a v-set. Let

$$\mathcal{I}=\{(I,J):I,J\subseteq V, I\cap J=\emptyset, |I|+|J|\leq t\}.$$

Because t is finite, it is clear that $|\mathcal{I}| = v$ and so we can write $\mathcal{I} = \{(I_\alpha, J_\alpha)\}_{\alpha<v}$. Let W be a subset of V with $|W| = t+1$. Since $|V \setminus W| = v$ we can find a set of v pairwise disjoint v-subsets of $V \setminus W$, $\{S^*\} \cup \{S'_\alpha\}_{\alpha<v}$ say. For each $\alpha < v$, let $S_\alpha = S'_\alpha \setminus (\bigcup_{\beta\leq\alpha}(I_\beta \cup J_\beta))$ and observe that $|S_\alpha| = v$ for each $\alpha < v$ (note that $|\bigcup_{\beta\leq\alpha}(I_\beta \cup J_\beta)| \leq t|\alpha+1| < v$).

Let B^* be a k-subset of $W \cup S^*$ such that $W \subseteq B^*$. For each $\alpha < v$, let B_α be a k-subset of $I_\alpha \cup (W \setminus J_\alpha) \cup S_\alpha$ such that $I_\alpha \subseteq B_\alpha$ and $|W \cap B_\alpha| \geq 1$ (such a subset exists because $|J_\alpha| \leq t < |W|$ and $k - |I_\alpha| \geq k - t \geq 1$). Let $\mathcal{B}$ be the collection containing B^* exactly once and each element of $\{B_\alpha : \alpha < v\}$ exactly v times. Note that $|\mathcal{B}| = v^2 + 1 = v$.

For each $(I, J) \in \mathcal{I}$, it follows from the definition of $\mathcal{B}$ that at least v sets in $\mathcal{B}$ contain each point in I and no point in J, and it follows from $|\mathcal{B}| = v$ that at most v sets in $\mathcal{B}$ contain each point in I and no point in J. If k is infinite, no set in $\mathcal{B}$ is a proper subset of any other because B^* is the only set in $\{B^*\} \cup \{B_\alpha : \alpha < v\}$ which contains infinitely many points in S^* and, for each $\alpha < v$, B_α is the only set in $\{B^*\} \cup \{B_\alpha : \alpha < v\}$ which contains infinitely many points of S_α. If k is finite, no set in $\mathcal{B}$ is a proper subset of any other trivially. Thus $(V, \mathcal{B})$ is a t-(v, k, Λ) design with $\lambda_{i,j} = v$ for $i + j \leq t$. Each block in $\mathcal{B}$ contains a point in W and $W \subseteq B^*$, so it follows that $(V, \mathcal{B})$ does not have a resolution. □

Lemma 3.9. *Let v, k and t be cardinals such that v is infinite, t is finite, $t \geq 2$ and $t + 1 \leq k \leq v$. There is a t-(v, k, Λ) design with $\lambda_{i,j} = v$ for $i + j \leq t$ which has a resolution into v parallel classes each containing v blocks.*

Proof. Let V be a v-set. Let

$$\mathcal{I} = \{(I, J) : I, J \subseteq V, I \cap J = \emptyset, |I| + |J| \leq t\}.$$

Because t is finite, it is clear that $|\mathcal{I}| = v$ and so we can write $\mathcal{I} = \{(I_\alpha, J_\alpha)\}_{\alpha<v}$. We can find a collection of v pairwise disjoint v-subsets of V, $\{S'_\alpha\}_{\alpha<v}$ say. For each $\alpha < v$ let $S_\alpha = S'_\alpha \setminus (\bigcup_{\beta\leq\alpha}(I_\beta \cup J_\beta))$ and observe that $|S_\alpha| = v$ for each $\alpha < v$ (note that $|\bigcup_{\beta\leq\alpha}(I_\beta \cup J_\beta)| \leq t|\alpha+1| < v$).

For each $\alpha < v$, let B^*_α be a k-subset of $V \setminus J_\alpha$ such that $I_\alpha \subseteq B^*_\alpha$, $B^*_\alpha \subseteq I_\alpha \cup S_\alpha$ and $|S_\alpha \setminus B^*_\alpha| = v$, and let $\mathcal{P}^*_\alpha$ be a partition of $V \setminus B^*_\alpha$ into v parts of size k such that each part contains exactly one point in $V \setminus (S_\alpha \cup B^*_\alpha)$ (such a partition exists because $|V \setminus (S_\alpha \cup B^*_\alpha)| = v$ and $S_\alpha \setminus B^*_\alpha$ can be partitioned into v parts of size $k-1$). Let $\mathcal{P}_\alpha = \{B^*_\alpha\} \cup \mathcal{P}^*_\alpha$ for each $\alpha < v$ and note that $\mathcal{P}_\alpha$ is a partition of V into v parts of size k. Let $\mathcal{B}$ be the collection containing each element of $\bigcup_{\alpha<v} \mathcal{P}_\alpha$ exactly v times. Note that $|\mathcal{B}| = v^2 = v$.

For each $(I, J) \in \mathcal{I}$, it follows from the definition of $\mathcal{B}$ that at least v sets in $\mathcal{B}$ contain each point in I and no point in J, and it follows from $|\mathcal{B}| = v$ that at most v sets in $\mathcal{B}$ contain each point in I and no point in J. If k is infinite, no set in $\mathcal{B}$ is a proper subset of any other because, for each $\alpha < v$, the only sets in $\mathcal{B}$ which contain infinitely many points of S_α are those in $\mathcal{P}_\alpha$. If k is finite, no set in $\mathcal{B}$ is a proper subset of any other trivially. Thus $(V, \mathcal{B})$ is a t-(v, k, Λ) design with $\lambda_{i,j} = v$ for $i + j \leq t$, and $\{\mathcal{P}_\alpha : \alpha < v\}$ is a resolution of $(V, \mathcal{B})$. □

4. The case $k = v$

A parallel class in a t-(v, k, Λ) design with $k = v$ may contain as few as two blocks or as many as v blocks. Further, a resolution of such a design may contain parallel classes containing different numbers of blocks. We call a parallel class of such a design with cardinality less than v a *short parallel class.*

4.1. The case $k = v$, λ finite

The Euclidean and extended Euclidean planes show that there exist both resolvable and non-resolvable t-(v, k, Λ) designs with $k = v$ and λ finite. The Euclidean plane, viewed as a 2-$(2^{\aleph_0}, 2^{\aleph_0}, 1)$ design has a resolution into $2^{\aleph_0}$ parallel classes each containing $2^{\aleph_0}$ blocks, where the parallel classes are all complete sets of lines of the same slope. The extended Euclidean plane, viewed as a 2-$(2^{\aleph_0}, 2^{\aleph_0}, 1)$ design, has the property that any two blocks intersect and so is not resolvable.

It is worth noting that we can "projectivise" any resolvable 2-$(v, v, 1)$ design by adding one new point to the design for each parallel class of some resolution of the design, adding to each existing block the new point corresponding to its parallel class, and adding a new block containing exactly the new points. This produces a 2-$(v, v, 1)$ design with no resolution.

By Theorem 2.3, a t-(v, k, Λ) design with $k = v$ and λ finite has $r = v$, and hence any resolution of such a design has v parallel classes. However, some of these parallel classes may be short. In this section we show that a resolution of a t-(v, k, Λ) design with $k = v$ and λ finite can have at most $\lambda - 1$ short parallel classes (and hence must have v parallel classes containing v blocks). We further show that for any $\ell < \lambda$ sizes less than v, it is possible to find a resolution of a design with short parallel classes of exactly these sizes. These results generalise Lemma 2.7 of [16] which shows that any resolution of a t-$(v, v, 1)$ design has v parallel classes, each of which contains v blocks.

We begin by constructing an array in which the columns represent the points of a design, the rows the parallel classes and the entries the blocks. Partial arrays correspond to partial designs. Let t, v, k, λ and r be cardinals such that v is infinite, t and λ are finite, and $t \geq 2$. Let $\mathcal{A}$ be an $r \times v$ array such that each cell of $\mathcal{A}$ contains exactly one symbol. For a particular row s of $\mathcal{A}$ and a symbol σ which appears in s, we call the set B of all columns of $\mathcal{A}$ which contain σ in row s a *block* of $\mathcal{A}$. We will say that B is a block in row s and induced by the symbol σ. If a set T of columns of $\mathcal{A}$ is a subset of a block of $\mathcal{A}$ in row s then we will say that T is *covered* by row s. If $\mathcal{A}$ satisfies

(i) each block of $\mathcal{A}$ has size k and no block of $\mathcal{A}$ contains every column of $\mathcal{A}$;

(ii) no block of $\mathcal{A}$ is a strict subset of another block of $\mathcal{A}$; and

(iii) each t-set of columns is covered by exactly λ rows of $\mathcal{A}$;

then we say that it is a t-(v, k, λ) *resolution array.* If $\mathcal{A}$ satisfies (i), (ii) and

(iii)$'$ each t-set of columns is covered by at most λ rows of $\mathcal{A}$;

then we say that it is a *partial* t-(v, k, λ) *resolution array.*

Clearly, a t-(v, k, λ) resolution array is equivalent to a resolution of a t-(v, k, λ) design, together with an ordering of its points and parallel classes. Similarly, a partial t-(v, k, λ) resolution array is equivalent to a resolution of a partial t-(v, k, λ) design, together with an ordering of its points and parallel classes. Note that permuting the rows or columns of a (partial) resolution array $\mathcal{A}$ does not affect the structure of the corresponding resolution.

For cardinals v and r such that v is infinite we call a row of an $r \times v$ array *special* if it contains fewer than v distinct symbols and *normal* if it contains v distinct symbols. Special rows correspond to short parallel classes in the corresponding resolution. In what follows we will often assume that any special rows occur above any normal ones. We call the $\ell \times 1$ array comprising the first ℓ cells of a column c of $\mathcal{A}$ the *ℓ-initial pattern* of c. We say an ℓ-initial pattern occurs in an array if there is at least one column of the array which has that initial pattern.

Lemma 4.1. *Let v, r and ℓ be cardinals such that v is infinite and ℓ is finite. Let $\mathcal{A}$ be an $r \times v$ array such that the first ℓ rows of $\mathcal{A}$ are special. Then fewer than v ℓ-initial patterns occur in $\mathcal{A}$, and any set of v columns of $\mathcal{A}$ has a subset of v columns which all have the same ℓ-initial pattern.*

Proof. For $i = 1, \ldots, \ell$, let b_i be the number of distinct symbols which occur in row i of $\mathcal{A}$. Because the first ℓ rows of $\mathcal{A}$ are special, $b_i < v$ for $i \in \{1, \ldots, \ell\}$. Thus the number of ℓ-initial patterns which occur in $\mathcal{A}$ is at most $b_1 b_2 \cdots b_\ell < v$. It follows that, if C is a set of v columns of $\mathcal{A}$, then there must be v columns in C which all have the same ℓ-initial pattern. □

Lemma 4.2. *Let t, v and λ be cardinals such that v is infinite, t and λ are finite, and $t \geq 2$, and let $\mathcal{A}$ be a t-(v, v, λ) resolution array. Then $\mathcal{A}$ has v normal rows and at most $\lambda - 1$ special rows.*

Proof. By Theorem 2.3, a t-(v, v, λ) design has $r = v$, and it follows that $\mathcal{A}$ must have v rows. So it suffices to show that $\mathcal{A}$ has at most $\lambda - 1$ special rows. Suppose for a contradiction that $\mathcal{A}$ has at least λ special rows and assume that the special rows of $\mathcal{A}$ occur above the normal rows. Choose any block B in row $\lambda + 1$ of $\mathcal{A}$ and note that $|B| = v$. By Lemma 4.1 there is certainly a t-subset T of B such that every column in T has the same λ-initial pattern. Then T is covered by each of the first $\lambda + 1$ rows in $\mathcal{A}$, yielding a contradiction. □

Since a resolution array corresponds to a resolution of a design with short parallel classes corresponding to special rows, we have the following theorem as a corollary.

Theorem 4.3. *Let t, v and λ be cardinals such that v is infinite, t and λ are finite, and $t \geq 2$. Any resolution of a t-(v, v, λ) has v parallel classes, at most $\lambda - 1$ of which are short.*

Given a partial t-(v, v, λ) resolution array $\mathcal{A}$, we will say that an ℓ-initial pattern is *common* if it is the ℓ-initial pattern of v columns in $\mathcal{A}$. We define an *ℓ-refinement* R of $\mathcal{A}$ as a set of columns of $\mathcal{A}$ such that

1. every common ℓ-initial pattern in $\mathcal{A}$ is the initial pattern of v columns of R;
2. the ℓ-initial pattern of every column in R is common;
3. each block in a normal row of $\mathcal{A}$ contains at most one column of R.

We note that the set of all columns of $\mathcal{A}$ with common ℓ-initial patterns will always satisfy Properties 1 and 2 above. Using Property 1 of the definition of ℓ-refinement, it can be seen that, given an ℓ-refinement Q of $\mathcal{A}$, there is a partition $\{Q_\alpha\}_{\alpha<v}$ of Q into v parts each of which is itself an ℓ-refinement.

Lemma 4.4. *Let t, v, λ and ℓ be cardinals such that v is infinite, t and λ are finite, $t \geq 2$ and $\ell \leq \lambda - 1$. Let $\mathcal{A}$ be a partial t-(v, v, λ) resolution array with fewer than v rows such that the first ℓ rows of $\mathcal{A}$ are special and the remainder are normal, and let T be a t-set of columns of $\mathcal{A}$ which is covered by at most $\lambda - 1$ rows of $\mathcal{A}$. If there is an ℓ-refinement D of $\mathcal{A}$ such that D is disjoint from T and there are v columns of $\mathcal{A}$ not in D, then there is a partial t-(v, v, λ) resolution array which is obtained from $\mathcal{A}$ by adding a normal row such that the new row covers T and such that each block in the new row contains at most one column in $V \setminus (T \cup D)$.*

Proof. Let V be the set of columns of $\mathcal{A}$ and let $\{D_\alpha\}_{\alpha<v}$ be a partition of D into v ℓ-refinements. For each column $c \in V \setminus D$, let

$$S_c = \{y \in D : c, y \in T \text{ for some } t\text{-subset } T \text{ of } V \text{ which is covered by } \lambda \text{ rows of } \mathcal{A}\}.$$

Note that $\ell \leq \lambda - 1$ and that each block in a normal row of $\mathcal{A}$ contains at most one column in D. Thus, because $\mathcal{A}$ has fewer than v rows, it can be seen that $|S_c| < v$ for each $c \in V \setminus D$. Let $S_T = \bigcup_{c \in T} S_c$ and note that $|S_T| < v$, since t is finite.

Index the columns in $V \setminus (T \cup D)$ with the ordinals greater than 1 and less than v as $\{x_\alpha\}_{2 \leq \alpha < v}$. Form an array $\mathcal{A}'$ by adding a new row to $\mathcal{A}$ in which the symbol σ_1 appears in every column in $T \cup (D_1 \setminus S_T)$, the symbol σ_α appears in every column in $\{x_\alpha\} \cup (D_\alpha \setminus S_{x_\alpha})$ for each ordinal $2 \leq \alpha < v$, and the symbol σ_0 appears in every other column in V (note that in particular σ_0 appears in every column in D_0). We claim that $\mathcal{A}'$ is the required partial t-(v, v, λ) resolution array.

Certainly the new row of $\mathcal{A}'$ covers T, and each block in the new row contains at most one column in $V \setminus (T \cup D)$. Furthermore, because $|D_\alpha| = v$ for each $\alpha < v$, $|S_T| < v$, and $|S_c| < v$ for each $c \in V \setminus D$, it can be seen that each block in the new row has size v. So if $\mathcal{A}'$ does not satisfy the conditions of the lemma, it must be the case that either there is a t-set of columns which is covered by the new row and by λ rows of $\mathcal{A}$, or that some block in the new row is a strict subset or a strict superset of some block in $\mathcal{A}$.

Suppose there is a t-subset U of V which is covered by the new row and by λ rows of $\mathcal{A}$. Because there are at most $\lambda - 1$ special rows, U is covered in some normal row s of $\mathcal{A}$. Thus, since every block in a normal row of $\mathcal{A}$ contains at most one column in D, U contains at most one column in D. It follows that U contains at least one column c in $V \setminus D$. If $c \notin T$, then $c = x_\alpha$ for some α such that $2 \leq \alpha < v$ and, because U is covered by the new row, $U \subseteq \{x_\alpha\} \cup (D_\alpha \setminus S_{x_\alpha})$. But then the fact that U is covered by λ rows of $\mathcal{A}$ implies that $U \cap D \subseteq S_{x_\alpha}$, contradicting $|U| = t \geq 2$. If $c \in T$, then $U \subseteq T \cup (D_1 \setminus S_T)$ because U is covered by the new row. But then the fact that U is covered by λ rows of $\mathcal{A}$ implies $U \cap D \subseteq S_T$ and $U \neq T$, contradicting $|U| = t$.

Suppose a block B' in the new row is a strict subset or a strict superset of a block B in some other row s of $\mathcal{A}$. Let σ_α be the symbol that induces B', where $\alpha < v$. It cannot be that s is a normal row of $\mathcal{A}$ because then B contains at most one column in D and v columns in $V \setminus D$ (by the definition of refinement), and B' contains v columns in D and at most t columns in $V \setminus D$ (by the construction of the new row). Thus s is a special row of $\mathcal{A}$. Let σ be the symbol which induces B. Because there are v columns which contain σ in row s, by Lemma 4.1 there is a common ℓ-initial pattern P in which σ appears in row s. Then, by the definition of refinement, $D_{\alpha+1}$ contains v columns with ℓ-initial pattern P and these columns are in B but not in B'. So $B \nsubseteq B'$. Let τ be a symbol other than σ which appears in row s. Because there are v columns which contain τ in row s, by Lemma 4.1 there is a common ℓ-initial pattern Q in which τ appears in row s. Then, by the definition of refinement, D_α contains v columns with ℓ-initial pattern Q and these columns are in B' but not in B. So $B' \nsubseteq B$. □

If an array $\mathcal{A}'$ can be formed by adding rows to an array $\mathcal{A}$, then we will say that $\mathcal{A}'$ is an *extension* of $\mathcal{A}$. The following lemma shows that a partial resolution array with ℓ special rows can be extended to a (complete) resolution array by adding normal rows. We then show that partial resolution arrays with special rows of any prescribed $\ell \leq \lambda - 1$ types exist. Given the correspondence between resolution arrays and resolutions of designs, these results show that resolutions of designs with short parallel classes of any prescribed $\ell \leq \lambda - 1$ sizes exist.

Lemma 4.5. *Let t, v, λ and ℓ be cardinals such that v is infinite, t and λ are finite, $t \geq 2$ and $\ell \leq \lambda - 1$. Let $\mathcal{A}'$ be a partial t-(v, v, λ) resolution array with ℓ rows, all of which are special. Then $\mathcal{A}'$ can be extended to a t-(v, v, λ) resolution array by adding normal rows.*

Proof. Let V be the set of columns of $\mathcal{A}'$. Let $(T_\alpha)_{\alpha<v}$ be a sequence which contains each t-subset of V exactly λ times. Let E be the set of all columns in $\mathcal{A}'$ with common ℓ-initial patterns. Since all the rows of $\mathcal{A}'$ are special, E is an ℓ-refinement of $\mathcal{A}'$, let $\{E_\alpha\}_{\alpha<v}$ be a partition of E into v ℓ-refinements. For each $\alpha < v$, let $C_\alpha = E_\alpha \setminus (\bigcup_{\beta\leq\alpha} T_\beta)$ and observe that $|C_\alpha| = v$ and that C_α is an ℓ-refinement because every common initial pattern in $\mathcal{A}'$ is the initial pattern of v columns in C_α (note that $|\bigcup_{\beta\leq\alpha} T_\beta| \leq t|\alpha + 1| < v$).

We will show, by transfinite induction on α, that for each ordinal $\alpha \leq v$ there is a partial t-(v, v, λ) resolution array $\mathcal{A}_\alpha$ such that

(i) $\mathcal{A}_\alpha$ is an extension of $\mathcal{A}_\beta$ for each $\beta < \alpha$;

(ii) $\mathcal{A}_\alpha$ has exactly ℓ special rows and at most $|\alpha|$ normal rows;

(iii) for each β such that $\alpha \leq \beta < v$, each block in a normal row of $\mathcal{A}_\alpha$ contains at most one column in C_β, so C_β is an ℓ-refinement of $\mathcal{A}_\alpha$; and

(iv) each t-subset T of V is covered by at least $\mu_\alpha(T)$ rows of $\mathcal{A}_\alpha$, where $\mu_\alpha(T)$ is the number of times T occurs in $(T_\beta)_{\beta<\alpha}$.

This will suffice to complete the proof, since $\mathcal{A}_v$ will be a t-(v, v, λ) resolution array which is an extension of $\mathcal{A}'$.

Take $\mathcal{A}_0 = \mathcal{A}'$. Now let γ be an ordinal with $\gamma < v$ and assume that, for each $\alpha < \gamma$, there is a partial t-(v, v, λ) resolution array $\mathcal{A}_\alpha$ satisfying (i)-(iv).

If γ is a limit ordinal then let $\mathcal{A}_\gamma = \bigcup_{\alpha<\gamma} \mathcal{A}_\alpha$.

If γ is a successor ordinal and $T_{\gamma-1}$ is covered by λ rows of $\mathcal{A}_{\gamma-1}$, then let $\mathcal{A}_\gamma = \mathcal{A}_{\gamma-1}$. If γ is a successor ordinal and $T_{\gamma-1}$ is covered by at most $\lambda - 1$ rows of $\mathcal{A}_{\gamma-1}$, then we construct $\mathcal{A}_\gamma$ from $\mathcal{A}_{\gamma-1}$ by applying Lemma 4.4 with $\mathcal{A} = \mathcal{A}_{\gamma-1}$, $T = T_{\gamma-1}$, and $D = C_{\gamma-1}$. Note that, by Lemma 4.4, each block in the new row contains at most one column in $V \setminus (T_{\gamma-1} \cup C_{\gamma-1})$, ensuring that C_β is an ℓ-refinement of $\mathcal{A}_\gamma$ for each β such that $\gamma \le \beta < v$. □

Lemma 4.6. *Let t, v, λ and ℓ be cardinals such that v is infinite, t and λ are finite, $t \ge 2$ and $\ell \le \lambda - 1$. Let $b_1, \ldots, b_\ell$ be cardinals each of which is less than v. Then there is a partial t-(v, v, λ) resolution array with ℓ rows such that row i contains b_i blocks for $i = 1, \ldots, \ell$.*

Proof. Let V be the Cartesian product $b_1 \times \cdots \times b_\ell \times v$, where we are considering $b_1, \ldots, b_\ell$ and v as sets of ordinals. Let V be the set of columns of our array and note that $|V| = v$. Let $\mathcal{A}$ be the array such that, for $i = 1, \ldots, \ell$ and $\beta < b_i$, the symbol σ_β appears in row i in the columns in $\{(\alpha_1, \ldots, \alpha_{\ell+1}) : \alpha_i = \beta\}$. Clearly, row i contains b_i blocks of size v for $i = 1, \ldots, \ell$. For any distinct $i, j \in \{1, \ldots, \ell\}$, $\beta < b_i$ and $\gamma < b_j$, the columns in $\{(\alpha_1, \ldots, \alpha_{\ell+1}) : \alpha_i = \beta, \alpha_j \ne \gamma\}$ are in the block induced by σ_β in row i but not in the block induced by σ_γ in row j. Thus no block of $\mathcal{A}$ is a strict subset of another. Obviously each t-set of columns is covered by at most λ rows of $\mathcal{A}$ because $\mathcal{A}$ contains less than λ rows. □

Lemmas 4.5 and 4.6 give us the following Theorem.

Theorem 4.7. *Let t, v, λ and ℓ be cardinals such that v is infinite, t and λ are finite, $t \ge 2$ and $\ell \le \lambda - 1$. Let $b_1, \ldots, b_\ell$ be cardinals each of which is less than v. Then there is a t-(v, v, λ) design with a resolution into v parallel classes each of size v and ℓ short parallel classes of sizes $b_1, \ldots, b_\ell$.*

Having shown that every feasible "type" of resolution for a t-(v, k, Λ) design with $k = v$ and λ finite is realisable in some such design, we conclude this section by presenting an example which demonstrates that a single design may admit two resolutions of different "types". Lemma 4.8 establishes the existence of a design which admits two different resolutions, one which contains short parallel classes and another which does not.

Lemma 4.8. *There exists a 2-$(\aleph_0, \aleph_0, 2)$ design admitting one resolution into $\aleph_0$ parallel classes each containing $\aleph_0$ blocks and another resolution into one parallel class containing four blocks and $\aleph_0$ parallel classes each containing $\aleph_0$ blocks.*

Proof. The example will be constructed by combining two copies of a 2-$(\aleph_0, \aleph_0, 1)$ design, $(V, \mathcal{B})$, admitting a resolution into $\aleph_0$ parallel classes each containing $\aleph_0$ blocks. We further require that $\mathcal{B}$ contains two blocks B_1 and B_2 in the same parallel class in this resolution such that there is a partition $\{D_1, D_2\}$ of the points outside these blocks such that $|D_1| = |D_2| = \aleph_0$ and each block other than B_1 and B_2 intersects both D_1 and D_2.

To see that such a design exists, note that we can take a 2-$(\aleph_0, \aleph_0, 1)$ design $(V, \mathcal{B})$ with point set $\mathbb{Z}^2$ whose block set is $\{L \cap \mathbb{Z}^2 : L \in \mathcal{L}\}$, where $\mathcal{L}$ is the set of all lines in $\mathbb{R}^2$ which contain at least one point in $\mathbb{Z}^2$ and which are either vertical or have rational slope (this is Example 2.6 from [16]). The design admits a resolution whose parallel classes are sets of blocks with the same slopes. We can choose any two blocks with the same slope as B_1 and B_2 and then let

$$D_1 = \{(x, y) \in V : 2^n < \sqrt{x^2 + y^2} \leq 2^{n+1} \text{ for some odd } n\} \setminus (B_1 \cup B_2), \text{ and}$$
$$D_2 = (\{(x, y) \in V : 2^n < \sqrt{x^2 + y^2} \leq 2^{n+1} \text{ for some even } n\} \cup \{(0, 0)\}) \setminus (B_1 \cup B_2).$$

(Intuitively, D_1 and D_2 are defined by forming concentric annuli centred on the origin, each time doubling the radii, and partitioning the points outside B_1 and B_2 according to whether they fall in an "odd" or "even" annulus.)

Let W be an $\aleph_0$-set of points and let $\{X_1, X_2, Y_1, Y_2\}$ be a partition of W into four parts of size $\aleph_0$. We now combine two copies of $(V, \mathcal{B})$ to form a 2-$(\aleph_0, \aleph_0, 2)$ design $(W, \mathcal{B}^\dagger)$ as follows. Place two copies of $(V, \mathcal{B})$, $(W, \mathcal{B}')$ and $(W, \mathcal{B}'')$ say, on W. Place $(W, \mathcal{B}')$ such that B_1 and B_2 map to X_1 and X_2 and D_1 and D_2 map to Y_1 and Y_2, so no block in $\mathcal{B}'$ is a subset of Y_1 or Y_2. Place $(W, \mathcal{B}'')$ in the same way, but with the roles of X and Y interchanged.

To see that no block in $\mathcal{B}^\dagger$ is a subset of another, note that no other block is a subset of X_1, X_2, Y_1 or Y_2, that each block in $\mathcal{B}'$ has at most two points in $X_1 \cup X_2$ and has $\aleph_0$ points in $Y_1 \cup Y_2$, and that each block in $\mathcal{B}''$ has at most two points in $Y_1 \cup Y_2$ and has $\aleph_0$ points in $X_1 \cup X_2$.

A resolution of $(W, \mathcal{B}^\dagger)$ into $\aleph_0$ parallel classes each containing $\aleph_0$ blocks is easily obtained by taking the union of the resolutions of $(W, \mathcal{B}')$ and $(W, \mathcal{B}'')$. A resolution of $(W, \mathcal{B}^\dagger)$ into one parallel class containing four blocks and $\aleph_0$ parallel classes each containing $\aleph_0$ blocks can be obtained by taking $\{X_1, X_2, Y_1, Y_2\}$ as one parallel class, the union of the remainders of the parallel classes of $(W, \mathcal{B}')$ and $(W, \mathcal{B}'')$ that X_1, X_2, Y_1, Y_2 were stolen from as another parallel class, and the other parallel class from $(W, \mathcal{B}')$ and $(W, \mathcal{B}'')$ as the remaining parallel classes. $\square$

4.2. The case $k = v$, λ infinite

A t-(v, v, Λ) design with $k = v$ and λ infinite may or may not be resolvable. Lemma 3.8 gives an example of a t-(v, v, Λ) design with $k = v$ and λ infinite which has no resolution, and Lemma 3.9 gives an example of a t-(v, v, Λ) design with $k = v$ and λ infinite which has a resolution into v parallel classes each containing v blocks. We conclude with Lemma 4.9 which shows that, at the other extreme, there is a t-(v, v, Λ) design with $k = v$ and λ infinite which has a resolution into v parallel classes each containing two blocks. It is also worth noting that Example 6 of [5] gives a 2-$(2^{\aleph_0}, 2^{\aleph_0}, \Lambda)$ design which has a resolution into $\aleph_0$ parallel classes of size two. Moreover, Example 7 of [5] gives, for any fixed finite t, a t-$(2^{\aleph_0}, 2^{\aleph_0}, \Lambda)$ design which has a resolution into $\aleph_0$ parallel classes of size two and one parallel class of size $2^{\aleph_0}$.

Lemma 4.9. *Let v and t be cardinals such that v is infinite and t is finite. There is a t-(v, v, Λ) design with $\lambda_{i,j} = v$ for $i + j \leq t$ with a resolution into v parallel classes each of size two.*

Proof. We first show there is a t-(v, v, Λ) design $(V, \mathcal{B})$, with $\lambda_{i,j} = v$ for $i + j < t$, having the property that no two blocks in $\mathcal{B}$ are disjoint and no two blocks in $\mathcal{B}$ have union V. To see this, repeat the construction in Lemma 3.8, except let $|W| = 2t + 1$ and for each $\alpha < v$ choose B_α to be a k-subset of $I_\alpha \cup (W \setminus J_\alpha) \cup S_\alpha$ such that $I_\alpha \cup (W \setminus J_\alpha) \subseteq B_\alpha$ and $|S_\alpha \setminus B_\alpha| = v$. Identical arguments to those in the proof of Lemma 3.8 above show that $(V, \mathcal{B})$ is a t-(v, v, Λ) design $(V, \mathcal{B})$ with $\lambda_{i,j} = v$ for $i + j < t$. Our new choices of W and B_α ensure that each block in $\mathcal{B}$ contains at least $t + 1$ points in W, which in turn ensures that no two blocks in $\mathcal{B}$ are disjoint (recall $|W| = 2t + 1$ and $|J_\alpha| \leq t$). The choice also ensures that, for each $\alpha < v$, the union of B_α and any other block in $\{B^*\} \cup \{B_\beta : \beta < v\}$ omits v points in S_α, because any other block in $\{B^*\} \cup \{B_\beta : \beta < v\}$ contains at most t points in S_α. So a design $(V, \mathcal{B})$ with the desired properties does indeed exist.

We now claim that $(V, \{B, V \setminus B : B \in \mathcal{B}\})$ is a t-(v, v, Λ) design with $\lambda_{i,j} = v$ for $i + j \leq t$. For any disjoint subsets I and J of V with $|I| + |J| \leq t$ it is clear that at least v sets in $\{B, V \setminus B : B \in \mathcal{B}\}$ contain each point in I and no point in J, and it follows from $|\{B, V \setminus B : B \in \mathcal{B}\}| = 2|\mathcal{B}| = 2v = v$ that at most v sets in $\mathcal{B}$ contain each point in I and no point in J. Also, the facts that no block in $\mathcal{B}$ is a proper subset of another, that no two blocks in $\mathcal{B}$ are disjoint, and that no two blocks in $\mathcal{B}$ have union V, together imply that no block in $\{B, V \setminus B : B \in \mathcal{B}\}$ is a proper subset of another. So $(V, \{B, V \setminus B : B \in \mathcal{B}\})$ is indeed a t-(v, v, Λ) design with $\lambda_{i,j} = v$ for $i + j \leq t$, and it is easy to see that $\{\{B, V \setminus B\} : B \in \mathcal{B}\}$ is a resolution of the design into v parallel classes each of size two. □

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