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We introduce a generalization of the Dobiński relation, through which we define a family of Bell-type numbers and polynomials. Such generalized Dobiński relations are coherent state matrix elements of expressions involving boson ladder operators. This may be used in order to obtain normally ordered forms of polynomials in creation and annihilation operators, both if the latter satisfy canonical and deformed commutation relations.

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1 Introduction

Operator ordering plays an exceptionally important role both in pure mathematics and in theoretical and mathematical physics. In particular, considering problems of quantum physics one has to fix the order of basic quantum mechanical operators, which in the framework of the canonical formalism are noncommuting elements of the Heisenberg algebra and its linear envelope. The order of operators involved in quantum mechanical calculations influences the results and hence it is very useful to know relations between differently ordered operator expressions, in particular to know relations that connect differently ordered but still equivalent operators. Examples of such relations are those between functions of the ladder operators satisfying

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$[a, a^\dagger] = 1$ and their normally ordered forms [1–3], whose theory has been recently pushed forward [4], and the so-called polynomial relations, like $(ABA)^n = A^n B^n A^n$ valid for any operators satisfying $[A, B] = 1$, see [5]. This note is devoted to a slightly different subject — our goal is to show how a class of expressions constructed from ladder boson operators can be given as products of infinite series. For some special cases these expressions reduce to polynomials which obey simple combinatorial interpretation. New relations have numerous interesting properties, [6], and generalize the long-time known remarkable Dobiński relation, [7], representing the integer sequence of *Bell numbers* $B(n) = 1, 1, 2, 5, 52, 203, 877, \ldots$; $n = 0, 1, 2, 3, \ldots$ as

$$B(n) = \mathrm{e}^{-1} \sum_{k=0}^{\infty} \frac{k^n}{k!} \,. \tag{1}$$

Closely related to the Bell numbers are other quantities, whose origin is enumerative combinatorics: *Stirling numbers of the second kind* $S(n, k)$, $k = 1, \ldots, n$, and the *Bell polynomials* defined as

$$B(n, x) = \sum_{k=1}^{n} S(n, k)\, x^k \,. \tag{2}$$

The latter are connected to $B(n)$ by $B(n) = B(n, 1) = \sum_{k=1}^{n} S(n, k)$, see [8]. For the Bell polynomials the Dobiński relation Eq. (1) generalizes to

$$B(n, x) = \mathrm{e}^{-x} \sum_{k=0}^{\infty} \frac{k^n}{k!}\, x^k \,. \tag{3}$$

These formulas may be derived using either combinatorial or purely analytical methods starting from the original interpretation of Bell and Stirling numbers given in enumerative combinatorics [8,9]. Accordingly, Stirling numbers $S(n, k)$ count the number of possible partitions of the n-element set into k subsets (none of them empty) and Bell numbers $B(n)$ count all such partitions.

2 Dobiński relations as an intrinsic property of the Fock space

Consider a one-mode boson system described by creation $a^\dagger$ and a annihilation operators, satisfying $[a, a^\dagger] = 1$. The Fock space of such a system is spanned by eigenstates of the number operator $\hat{n} = a^\dagger a$, $\hat{n}|n\rangle = n|n\rangle$, $n = 0, 1, 2, 3, \ldots$. Consider also an operator $\hat{F}$, which depends on the number operator only, *i.e.*, $\hat{F} := \hat{F}(\hat{n})$. It may be represented as

$$\hat{F}(\hat{n}) = \sum_{m=0}^{\infty} F_{mm} |m\rangle\langle m| \,, \tag{4}$$

where $F_{mm} = \langle m|\hat{F}(\hat{n})|m\rangle$. Because $|m\rangle = (a^\dagger)^m/\sqrt{m!}\,|0\rangle$ we have

$$\hat{F}(\hat{n}) = \sum_{m=0}^{\infty} \frac{F_{mm}}{m!}\, (a^\dagger)^m |0\rangle\langle 0| (a)^m \,. \tag{5}$$

Using the double dot symbol : : which denotes the normal ordering operation under which all creation operators are moved left to all anihilation operators as if they were commuting and the relation $|0\rangle\langle 0| =: \mathrm{e}^{-a^\dagger a} :$ (see [10, 11]) we can rewrite Eq. (5) as follows

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{F_{mm}}{m!}(a^\dagger)^m|0\rangle\langle 0|(a)^m &= :\sum_{m=0}^{\infty}\frac{F_{mm}}{m!}\,(a^\dagger)^m|0\rangle\langle 0|(a)^m := \\ &= :\sum_{m=0}^{\infty}\frac{F_{mm}}{m!}\,(a^\dagger)^m : \mathrm{e}^{-a^\dagger a} : (a)^m := \\ &= : \mathrm{e}^{-a^\dagger a}\sum_{m=0}^{\infty}\frac{F_{mm}}{m!}\,(a^\dagger)^m(a)^m : \;.\end{aligned} \tag{6}$$

This means that we have found normally ordered form of $\hat{F}$ and that it is given by the right-hand side of the formula above. The same result can be easily obtained if we calculate the coherent state matrix element $\langle z|\hat{F}(\hat{n})|z\rangle$, where $|z\rangle = \mathrm{e}^{-|z|^2/2}\sum_{m=0}^{\infty} z^m/\sqrt{m!}\,|m\rangle$ are Glauber–Klauder–Sudarshan coherent states. The coherent states form an overcomplete set of normalizable states $|z\rangle$ labelled by a complex number z, satisfy the overlapping relation $\langle z'|z\rangle = \mathrm{e}^{-\left(|z|^2+|z'|^2-2z'^*z\right)/2}$ and are eigenstates of the annihilation operator $a|z\rangle = z|z\rangle$, [12]. Simple calculation gives $\langle m|z\rangle = \mathrm{e}^{-|z|^2/2}z^m/\sqrt{m!}$ and

$$\langle z|\hat{F}(\hat{n})|z\rangle = \mathrm{e}^{-|z|^2}\sum_{m=0}^{\infty}F_{mm}\frac{|z|^{2m}}{m!}\,. \tag{7}$$

Properties of the coherent states imply that if one knows a functional form of the coherent state matrix element of an operator then one can find normal form of this operator replacing $z^* \to a^\dagger$ and $z \to a$ under the double dot symbol. Thus normally ordered form of $\hat{F}(\hat{n})$ is that given by Eq. (6), as it should be. If $\hat{F}_M(\hat{n})$ is an M-th order polynomial of $\hat{n}$, then its normally ordered form is an M-th order polynomial of $(a^\dagger)^k a^k$ and reads

$$\langle z|\hat{F}_M(\hat{n})|z\rangle = \sum_{k=0}^{M} S_F(M,k)(a^\dagger)^k a^k\,. \tag{8}$$

Consequently, $\langle z|\hat{F}_M(\hat{n})|z\rangle$ is a polynomial of $|z|^2$:

$$\langle z|\hat{F}_M(\hat{n})|z\rangle = \sum_{k=0}^{M} S_F(M,k)|z|^{2k} \tag{9}$$

with coefficients $S_F(M,k)$ uniquely determined for any operator F and calculable analytically from explicit multiplication of the series in the Eq. (7).

Equations (7) and (9) are very similar to Eqs. (2) and (3). Moreover, for $\hat{F}_M(\hat{n}) = \hat{n}^M$ the coefficients $S_F(M,k)$ are Stirling numbers of the second kind. Motivated by these facts we proposed to call the Eq. (7) the generalized Dobiński relation, while $S_F(m,k)$ and $\sum_{k=0}^{m} S_F(m,k)|z|^2$ we named generalized Stirling numbers of the second kind and generalized Bell polynomials, respectively, [4, 6].

Another example that may be analyzed within the same approach is the operator $a^r a^\dagger a$ (Laguerre derivative, [13]) and its n-th power $\left(a^r a^\dagger a\right)^n$. Using the relation

$$\left(a^r a^\dagger a\right)^n = (a^\dagger a + r)(a^\dagger a + 2r)\ldots(a^\dagger a + nr)a^{nr} \tag{10}$$

we get

$$\left(a^r a^\dagger a\right)^n = \sum_{l=nr}^{\infty} r^n \sqrt{(nr)!\binom{l}{nr}}\,\frac{l}{r}\left(\frac{l}{r}-1\right)\ldots\left(\frac{l}{r}-(n-2)\right)\left(\frac{l}{r}-(n-1)\right)|l-nr\rangle\langle l|\,. \tag{11}$$

The coherent state matrix element of Eq.(11) reads

$$\begin{aligned}\langle z|\left(a^r a^\dagger a\right)^n|z\rangle &= \mathrm{e}^{-|z|^2}\sum_{j=0}^{\infty} r^n\left(n+\frac{j}{r}\right)\left(n-1+\frac{j}{r}\right)\ldots\left(2+\frac{j}{r}\right)\left(1+\frac{j}{r}\right)\frac{|z|^{2j}}{j!}\,z^{nr}\\ &= \mathrm{e}^{-|z|^2}\sum_{j=0}^{\infty} r^n\frac{\Gamma\left(n+j/r\right)}{\Gamma\left(j/r\right)}\,\frac{|z|^{2j}}{j!}\,z^{nr}\,.\end{aligned} \tag{12}$$

Because of the identity

$$r^n\left(n+\frac{j}{r}\right)\left(n-1+\frac{j}{r}\right)\ldots\left(2+\frac{j}{r}\right)\left(1+\frac{j}{r}\right) = r^n\sum_{0\le k\le n}\sigma(n+1,k+1)\left(\frac{j}{r}\right)^k, \tag{13}$$

where $\sigma(n+1,k+1)$ denote *unsigned Stirling numbers of the first kind* (defined as absolute values of *Stirling numbers of the first kind* $s(n,k)$, $\sigma(n+1,k+1) = (-1)^{n+k}\,s(n+1,k+1)$, [8]), we finally arrive at

$$\begin{aligned}\langle z|\left(a^r a^\dagger a\right)^n|z\rangle &= \mathrm{e}^{-|z|^2}\sum_{j=0}^{\infty} r^n\sum_{0\le k\le n}\sigma(n+1,k+1)\left(\frac{j}{r}\right)^k\frac{|z|^{2j}}{j!}\,z^{nr}\\ &= \sum_{0\le k\le n}\sigma(n+1,k+1)r^{n-k}\left(\mathrm{e}^{-|z|^2}\sum_{j=0}^{k}\frac{|z|^{2j}}{j!}\,j^k\right)z^{nr}\\ &= \sum_{0\le k\le n}\sigma(n+1,k+1)r^{n-k}z^{nr}\sum_{l=0}^{\infty}S(k,l)|z|^{2l}\,.\end{aligned} \tag{14}$$

This may be used to get the normally ordered form

$$\begin{aligned}\left(a^r a^\dagger a\right)^n &= :\mathrm{e}^{-a^\dagger a}\sum_{j=0}^{\infty} r^n\sum_{0\le k\le n}\sigma(n+1,k+1)\left(\frac{j}{r}\right)^k\frac{(a^\dagger)^j a^j}{j!}\,a^{nr}:\\ &= :\sum_{0\le k\le n}\sigma(n+1,k+1)r^{n-k}a^{nr}\sum_{l=0}^{k}S(k,l)(a^\dagger)^l a^l:\,.\end{aligned} \tag{15}$$

Both results, Eqs. (6) and (15), are particular cases of general normal ordering formula derived in [4] using a different, more abstract approach. Recalling this alternative we would like to emphasize that the just shown derivation based solely on properties of the generalized Dobiński relations is not only easy to follow, but it does provide us with a very nice illustration of the Fock space methods. Moreover, it offers a possibility how to deal with noncanonical ladder operators. We shall describe this problem in the next section.

3 Generalized Dobiński relations − noncanonical case

Consider the general deformation of the boson algebra [14] in the form

$$[A, N] = A\,, \quad [A^\dagger, N] = -A^\dagger\,, \quad [A, A^\dagger] = [N+1] - [N]\,. \tag{16}$$

In the above A and $A^\dagger$ are annihilation and creation operators, respectively, while the number operator N counts particles. It is defined in the Fock basis as $N|n\rangle = n|n\rangle$ and commutes with $A^\dagger A$. Because of that in any representation of (16) N can be written in the form $A^\dagger A = [N]$, where $[N]$ denotes an arbitrary function of N, usually called the "box" function. For general considerations we do not assume any realization of the number operator N and we treat it as an independent element of the algebra. Moreover, we do not assume any particular form of the "box" function $[N]$. The action of operators defined by Eq.(16) in the Fock space is

$$N|n\rangle = n|n\rangle\,, \quad A|n\rangle = \sqrt{[n]}\,|n-1\rangle\,, \quad A^\dagger|n\rangle = \sqrt{[n+1]}\,|n+1\rangle\,. \tag{17}$$

Suitable generalized coherent states defined as eigenstates of the annihilation operator are (see [15])

$$|\lambda\rangle = \frac{1}{\sqrt{E(|\lambda|^2)}}\sum_{k=0}^{\infty}\frac{\lambda^k}{\sqrt{[k]!}}\,|k\rangle\,, \tag{18}$$

where $[k]! = [1][2]\ldots[k]$, $[0]! = 1$ and

$$E(|\lambda|^2) = \sum_{k=0}^{\infty}\frac{1}{[k]!}\,|\lambda|^{2k} \tag{19}$$

is a normalization factor. If $[k]!$ are represented as Stieltjes moments, *i.e.*, if for any $k = 0,\,1,\,2,\,\ldots$ there exists a nondecreasing function $\phi(x)$ defined on real positive semiaxis and such that $[k]! = \int_0^\infty x^k\,\mathrm{d}\phi(x)$, then the coherent states (18) satisfy the resolution of unity, *i.e.*, they span an overcomplete set in the Fock space. In what follows we assume that the "box" function is chosen in a way which fulfils this requirement. Under this condition normally ordered form of any operator expression may be extracted from its coherent state matrix element.

Calculation of the coherent state matrix element $\langle\lambda|\left(A^\dagger A\right)^m|\lambda\rangle$ is immediate

$$\langle\lambda|\left(A^\dagger A\right)^m|\lambda\rangle = \frac{1}{E(|\lambda|^2)}\sum_{k=0}^{\infty}\frac{[k]^m}{[k]!}\,|\lambda|^{2k}\,, \tag{20}$$

and implies that the normally ordered form of the operator $\left(A^\dagger A\right)^m$ reads

$$:\left(A^\dagger A\right)^m: = :\frac{1}{\sqrt{E(A^\dagger A)}}\sum_{k=0}^{\infty}\frac{[k]^m}{[k]!}\,(A^\dagger)^k A^k:\,. \tag{21}$$

Normal ordering of noncanonical ladder operators is not an easy task, in general leading to tedious calculations and technical difficulties, [16, 17]. So the simplicity of the formula (21) is astonishing and the formula itself looks very interesting. We are convinced that this confirms our previous statement concerning universality and usefulness of the generalized Dobiński formulas as a powerful calculational tool.

4 Summary

We showed that the generalized Dobiński formulas emerge in standard Fock space calculations and that they are directly related to the normal ordering problem. Standard procedures of normal ordering are governed by the Wick theorem and connecting them with combinatorial notions enables us to see and to understand the combinatorial content of the Wick theorem. Moreover, as proved in [6], the generalized Dobiński relations may be represented as moments which has important consequences — we can explicitly calculate generating functions of the coherent state matrix elements of numerous operator expressions.

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